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Asymptotics of Functions Represented by Potentials*

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Abstract. In the investigation of a boundary value problem (BVP) for an elliptic partial differential equation in a domain $\Omega \subset \mathbb{R}^n$ by the potential method, the solution is represented by means of potential operators, and the problem is reduced to finding the density of these potentials on the basis of the corresponding boundary integral equation (BIE) on the boundary $\mathcal{S} = \partial\Omega$ or on its part $\mathcal{S} \subset \partial\Omega$. If the BVP under consideration is of crack type or mixed, then the manifold $\mathcal{S}$ can have a boundary, $\partial\mathcal{S} = \Gamma$. After proving the unique solvability of the BIE, one can apply the Wiener–Hopf method provided that the manifold $\mathcal{S}$ and its boundary Γ are smooth, and find a complete asymptotic expansion of the solution on $\mathcal{S}$ near the boundary Γ (see the previous paper by the authors [CD1] and Section 1 below). It is quite natural that the next step is to find the complete asymptotics of the solution to BVP in Ω in a neighborhood of Γ. To this end, we must find the asymptotics of a potential-type function provided that the asymptotics of the density on $\mathcal{S}$ is known, and this problem is solved in the present paper in a rather explicit form.

INTRODUCTION

The main goal of the present paper is to study the asymptotics for a solution of an elliptic boundary value problem (BVP) represented by surface potentials (within the potential method). We continue the investigations of [CD1] that extend results of [Es1, Be1, CS1, DW1, DSW1]; these investigations led to the asymptotics of the elliptic pseudodifferential equation (PsDE)

$$\mathbf{a}(t, D_t)v(t) = g(t), \qquad t \in \mathcal{S}_0,$$

on a smooth manifold with smooth boundary $\partial\mathcal{S}_0$ (the Wiener–Hopf method), and the corresponding results are described in Section 1 of the present paper. To obtain the asymptotics of a solution of the BVP, one needs to investigate the asymptotics of functions represented by potentials whose densities have prescribed asymptotic expansions on the surface.

For example, consider the crack problem for an anisotropic elastic body

$$\mathbf{A}(D_x)u(x) = 0, \qquad x \in \mathbb{R}^3 \setminus \mathcal{S}_0\,, \tag{0.1}$$

$$u^{\pm}(t) = f_{\pm}(t), \qquad t \in \mathcal{S}_0, \tag{0.2}$$

with a given vector field $u^{\pm}$ of displacements on both sides of the crack surface $\mathcal{S}_0$, where

$$\mathbf{A}(D_x) := \| \sum_{k,m=1}^{3} a_{jklm}\partial_k\partial_m \|_{3\times 3}, \qquad a_{jklm} \in \mathbb{R}, \quad a_{jklm} = a_{lmjk} = a_{kjlm},$$

$$\sum_{j,k,l,m=1}^{3} a_{jklm}\xi_{jk}\xi_{lm} \geqslant C_0 \sum_{j,k=1}^{3} \xi_{jk}^2 \quad \text{for arbitrary} \quad \xi_{jk} \in \mathbb{R}\,, \quad \xi_{jk} = \xi_{kj}\,.$$

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A solution to an elliptic equation of the form (0.1) can be represented as

$$u(x) = \mathbf{W}_{\mathcal{S}_0} f_0(x) - \mathbf{V}_{\mathcal{S}_0}\varphi(x), \qquad x \in \mathbb{R}^n, \tag{0.3}$$

where $\mathbf{W}_{\mathcal{S}_0}$ and $\mathbf{V}_{\mathcal{S}_0}$ are the double layer potential and the single layer potential for (0.1), respectively, while the densities

$$f_0(t) := f_+(t) - f_-(t), \qquad \varphi(t) := \{\mathcal{T}(D_t, n(t))u\}^+(t) - \{\mathcal{T}(D_t, n(t))u\}^-(t)$$

represent the jumps of the displacement field and of the stress field across the crack surface $\mathcal{S}_0$.

Thus, we must find $\varphi(t)$ by solving a boundary PsDE of the form

$$\mathbf{V}_{-1}\varphi(t) = \mathbf{W}_0 f_0(t) - \frac{1}{2}[f_+(t) + f_-(t)]\,, \tag{0.4}$$

where $\mathbf{V}_{-1}$ and $\mathbf{W}_0$ are the direct values of the potentials $\mathbf{V}_{\mathcal{S}_0}$ and $\mathbf{W}_{\mathcal{S}_0}$ on $\mathcal{S}_0$; they are pseudodifferential operators of orders -1 and 0, respectively (see [CS1, DNS1, DNS2]).

Let $t = (s, \rho) \in \mathbf{Col}^2_\varepsilon := \partial\mathcal{S}_0 \times]0, \varepsilon[$ be the so-called local "plain collar" coordinate system, where $\rho = \rho(t) := \mathbf{dist}(t, \partial\mathcal{S}_0)$ stands for the distance and $s \in \partial\mathcal{S}_0$ for the abscissa on the arc.

The function φ is a solution of the elliptic PsDE (0.4), and hence φ belongs to $C^\infty(\mathcal{S}_0)$ provided that $\mathcal{S}_0$ is infinitely smooth. Moreover, if the data are smooth, i.e., if $f_\pm \in C^\infty(\mathcal{S}_0)$, then the complete asymptotic expansion of $\varphi(t)$ in a neighborhood of the boundary $\partial\mathcal{S}_0$ is

$$\varphi(\rho, s) = c_0(s)\ \rho^{-1/2} + \sum_{k=1}^{M}\sum_{m=0}^{k} c_{kj}(s)\ \rho^{k-1/2}\log^m\rho + \varphi_{M+1}(\rho, s)\,, \tag{0.5}$$
$$(\rho, s) \in \mathbf{Col}^2_\varepsilon\,, \qquad c_0, c_{1,0}, \dots, c_{MM} \in C^\infty(\partial\mathcal{S}_0), \qquad \varphi_{M+1} \in C^M(\overline{\mathcal{S}}_0)\,,$$

for arbitrary $M = 1, 2, \dots$.

Our next important step is to establish an asymptotics of $u(x)$ in (0.3) by using the asymptotics of the density in (0.5). This is performed in a rather general setting in Section 2 (see Theorem 2.2 of the present paper). Similar results for the canonical half-space case and for particular potentials can be found in [Es1, §13].

To state the main result of the present paper for the BVP of the form (0.1)–(0.2), we assume that $\mathcal{S}$ is a smooth closed manifold (without boundary) that contains $\mathcal{S}_0$ and introduce a local "thin collar" coordinate system in a neighborhood of $\partial\mathcal{S}_0 \subset \mathbb{R}^3$,

$$x = (s, \rho, r) \in \mathbf{Col}^3_\varepsilon := \partial\mathcal{S}_0 \times]-\varepsilon, \varepsilon[\times]-\varepsilon, \varepsilon[\,, \quad \mathbf{Col}^3_\varepsilon \cap \mathcal{S}_0 = \mathbf{Col}^2_\varepsilon\,.$$

Here $\rho = \mathbf{dist}(\widetilde{x}, \partial\mathcal{S}_0)$ stands for the distance between $\partial\mathcal{S}_0$ and the projection $\widetilde{x} \in \mathcal{S}$ of a point $x \in \mathbb{R}^3 \setminus \mathcal{S}$ along the outward pointing normal to $\mathcal{S}$, and thus takes negative values for $\widetilde{x} \in \mathcal{S} \setminus \mathcal{S}_0$, while $r = \mathbf{dist}(x, \mathcal{S}_0)$ takes positive (negative) values if x is inside (outside, respectively) of $\mathcal{S}$.

The function u is a solution of problem (0.1)–(0.2) and hence $u \in C^\infty(\mathbb{R}^3 \setminus \mathcal{S}_0)$ provided that $\mathcal{S}_0$ is infinitely smooth. Moreover, if the data are smooth, $f_\pm \in C^\infty(\mathcal{S}_0)$, then the complete asymptotic expansion of $u(x)$ in a neighborhood of the boundary $\partial\mathcal{S}_0$ is

$$\begin{aligned} u(s, \rho, r) = \sum_{m=1}^{\ell} \mathrm{Re}\Big\{ &\sum_{j=0}^{n_m-1} [d_{mj}(s, +1)\ r^j z_{m,+1}^{1/2-j} - d_{mj}(s, -1) r^j z_{m,-1}^{1/2-j}] \\ &+ \sum_{\theta=\pm1}\sum_{\substack{k,l=0\\ k+l+j+p\neq1}}^{M+1}\sum_{j+p=1}^{M+2-l} \rho^l r^j z_{m,\theta}^{-1/2+p+k} B_{mlkjp}\,(s, \log z_{m,\theta}) \Big\} + u_{M+1}(s, \rho, r)\,, \end{aligned} \tag{0.6}$$
$$u_{M+1} \in C^{M+1}\left(\mathbf{Col}^3_\varepsilon\right), \qquad (s, \rho, r) \in \mathbf{Col}^3_\varepsilon\,, \qquad d_{mj}(\cdot, \pm1) \in C^\infty(\partial\mathcal{S}_0)\,,$$

Here $z_{m,\pm 1} := \rho \pm r\tau_{m,\pm 1}$, $-\pi < \operatorname{Arg} z_{m,\pm 1} < \pi$, and $\{\tau_{m,\pm 1}\}_{m=1}^{\ell}$ stand for the different roots of the polynomial $\det A(\mathcal{J}_{æ}^{\top}(s,0)\cdot(0,\pm 1,\tau))$ (whose multiplicities are denoted by n_m, $m=1,2,\dots,\ell$) in the complex lower half-plane $\operatorname{Re}\tau < 0$; $A(\xi)$, $\xi \in \mathbb{R}^3$, is the symbol of the elliptic differential operator $\mathbf{A}(D_x)$ in (0.1), the matrix $\mathcal{J}_{æ}^{\top}(s,\rho)$ is transposed to the Jacobian matrix of the coordinate diffeomorphism æ from $\mathbb{R}^2$ to the surface $\mathcal{S}_0$ (see Subsection 1.3 below), and $B_{mlkjp}(s,\lambda)$ is a polynomial of order $\nu_{kjp} = k+p+j-1$ with respect to λ with vector coefficients depending on the abscissa s on the arc.

The roots $\tau_{m,\pm 1}(s)$ depend on the parameter $s \in \partial\mathcal{S}_0$ smoothly, $\tau_{m,\pm 1} \in C^{\infty}(\partial\mathcal{S}_0)$, where $m = 1,\dots,\ell$.

Moreover, explicit relations between the coefficients of expansions (0.5) and (0.6) are found. In particular, the leading coefficients $d_{mj}(s,\pm 1)$ and $c_0(s)$ of these expansions are related as follows:

$$d_{mj}(s,\pm 1) = \frac{1}{2\sqrt{\pi}}(\mp i)^{j+1}\Gamma\Big(j-1/2\Big)\mathcal{G}_{æ}(s,0)V_{-1,j}^{(m)}(s,0,0,\pm 1)c_0(s)\,,$$

where $\mathcal{G}_{æ}(s,0)$ is the Gram determinant of the coordinate diffeomorphism æ and

$$V_{-1,j}^{(m)}(s,0,0,\pm 1) = -\frac{i^{j+1}}{j!(n_m-1-j)!}\frac{d^{n_m-1-j}}{d\tau^{n_m-1-j}}(\tau-\tau_{m,\pm 1})^{n_m}\,\Big(A(\mathcal{J}_{æ}^{\top}(s,0)\cdot(0,\pm 1,\tau))\Big)^{-1}\Big|_{\tau=\tau_{m,\pm 1}}.$$

In the isotropic case, the Lamé operator is $\mathbf{A}(\partial_x) = \mu\Delta + (\lambda+\mu)\,\mathbf{grad\,div}$, which gives $\ell = 1$, $\tau_{1,\pm 1} = -i$, $n_1 = 3$, and $d_{12}(s,\pm 1) = 0$; the asymptotic expansion of the corresponding solution can be simplified as follows:

$$\begin{aligned} u(s,\rho,r) &= d_1(s)\operatorname{Im} z_{+1}^{1/2} + \operatorname{Re} d_2(s) r(z_{+1}^{-1/2} - z_{-1}^{1/2}) \\ &\quad + \operatorname{Re}\sum_{\theta=\pm 1}\sum_{\substack{k,l=0\\ k+l+j+p\neq 1}}^{M+1}\sum_{j+p=1}^{M+2-l}\rho^l r^j z_\theta^{-1/2+p+k} B_{1lkjp}(s,\log z_\theta) + u_{M+1}(s,\rho,r), \\ &\qquad (s,\rho,r)\in \mathbf{Col}_{\varepsilon}^3\,, \qquad z_{\pm 1} := \rho \mp ir, \end{aligned} \tag{0.7}$$

where

$$d_1(s) = \mathbf{diag}\left\{\frac{1}{\mu}, \frac{\lambda+3\mu}{2\mu(\lambda+2\mu)}, \frac{\lambda+3\mu}{2\mu(\lambda+2\mu)}\right\}\mathcal{G}_{æ}(s,0)c_0(s)$$

and the coefficient $d_2(s) = d_{11}(s,\pm 1)$ can be evaluated in a similar way.

The explicit relationships between the coefficients of expansions (0.5), (0.6), and (0.7) can be used in crack mechanics to develop fracture criteria. This problem will be discussed in a forthcoming paper.

For functions represented by potentials, the asymptotic expansion is found in Section 2 in the general case. Namely, we consider the representation

$$u(x) = \mathbf{V}\circ\mathbf{B}_q\varphi_0(x)\,,\quad \mathbf{supp}\,\varphi_0 \subset \overline{\mathcal{S}}_0\,,\quad x\in\mathbb{R}^n, \tag{0.8}$$

where $\mathbf{B}_q$ is a classical pseudodifferential operator of order $q\in\mathbb{R}$ on the manifold $\mathcal{S}_0$; we assume that an asymptotic expansion of the density $\varphi_0 = (\varphi_{01},\dots,\varphi_{0N})$ is known. We write out a complete asymptotic expansion for the function $u(x)$ in (0.8) (see Theorems 2.2 and 2.3).

The reason to introduce the pseudodifferential operator $\mathbf{B}_q$ is that such operators arise in some problems in mechanics, in particular, in crack problems of mixed type [Ch2, Ch3] and in problems of cracks on an interface for an anisotropic elastic body [DSW1]. In our forthcoming paper, the asymptotics obtained here will be applied to certain problems in elasticity.

The asymptotics of solutions of boundary value problems in domains with nonsmooth boundaries (e.g., with cones, edges, etc.) is intensively studied by using the apparatus suggested by Kondrat'ev in [Ko1]. Some results of these investigations can be found in [Da1, Gr1, MP1, NP1, RS1, Sc1].

We use here another approach, which is based on the Wiener–Hopf method; this approach was suggested by G. Eskin and exploited in [Be1, CD1, CS1, DW1, DSW1], but, to our knowledge, only the research [Es1, §13] is devoted to the asymptotics of functions represented by potentials in the case of the canonical half-space and for special potentials.

1. ASYMPTOTICS OF SOLUTIONS TO PSEUDODIFFERENTIAL EQUATIONS

For convenience, in this section we briefly recall results on pseudodifferential equations in Bessel potential spaces that are obtained in [CD1] (see also [Be1, DW1, DS1, Es1, Sh2, Sh3]).

1.1. *Spaces*

Let $\mathbb{S}(\mathbb{R}^n)$ be the Schwartz space of all rapidly decreasing functions and let $\mathbb{S}'(\mathbb{R}^n)$ be the dual space of tempered distributions. Since both the Fourier transform and its inverse, which are defined by the formulas

$$\mathcal{F}\varphi(\xi)=\int\limits_{\mathbb{R}^n} e^{i\xi x}\varphi(x)dx \quad\text{and}\quad \mathcal{F}^{-1}\psi(x)=\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} e^{-ix\xi}\varphi(\xi)d\xi, \quad x,\xi\in\mathbb{R}^n\,, \tag{1.1}$$

are bounded operators in the spaces $\mathbb{S}(\mathbb{R}^n)$ and $\mathbb{S}'(\mathbb{R}^n)$, it follows that the convolution operator

$$\mathbf{a}(D)\varphi=W_a^0\varphi:=\mathcal{F}^{-1}a\mathcal{F}\varphi \quad\text{with}\quad a\in\mathbb{S}'(\mathbb{R}^n),\quad \varphi\in\mathbb{S}(\mathbb{R}^n), \tag{1.2}$$

is a bounded transformation from $\mathbb{S}(\mathbb{R}^n)$ into $\mathbb{S}'(\mathbb{R}^n)$ [Du1, DS1].

The Bessel potential space $\mathbb{H}^s_p(\mathbb{R}^n)$ is defined as a subspace of $\mathbb{S}'(\mathbb{R}^n)$ endowed with the following norm [Tr1, Tr2]:

$$\|u\big|\mathbb{H}^s_p(\mathbb{R}^n)\|:=\|\lambda^s(D)u\big|L_p(\mathbb{R}^n)\|,\quad\text{where}\quad \lambda^s(\xi):=(1+|\xi|^2)^{s/2}=\langle\xi\rangle^s. \tag{1.3}$$

For the Hilbert space $\mathbb{H}^s_2(\mathbb{R}^n)$, the index 2 is usually dropped, and the notation is reduced to $\mathbb{H}^s(\mathbb{R}^n)$ (cf. [Es1]).

For any σ in $]0,\infty[$, denote by $C^\sigma(\mathbb{R}^n)$ the Hölder space of continuous functions equipped with the norm

$$\|\varphi|C^m(\mathbb{R}^n)\|=\sum_{|\alpha|\leqslant m}\sup\{|\partial^\alpha\varphi(x)|:x\in\mathbb{R}^n\}\quad\text{for}\quad \sigma=m\in\mathbb{N}_0,$$

$$\|\varphi|C^\sigma(\mathbb{R}^n)\|=\|\varphi|C^m(\mathbb{R}^n)\|+\sum_{|\alpha|=m}\sup\left\{|h|^{-\nu}\|\Delta_h\partial^\alpha\varphi|C(\mathbb{R}^n)\|:h\in\mathbb{R}^n\setminus\{0\}\right\}$$
$$\text{for}\quad \sigma=m+\nu\,,\quad m\in\mathbb{N}_0\,,\quad 0<\nu<1\,,$$

where $\mathbb{N}$ stands for the set of positive integers and $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$.

The space $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^n_+)$ is defined as the subspace of $\mathbb{H}^s_p(\mathbb{R}^n)$ consisting of functions $\varphi\in\mathbb{H}^s_p(\mathbb{R}^n)$ with support in the positive half-space, $\mathbf{supp}\,\varphi\subset\overline{\mathbb{R}^n_+}$. Let $\mathbb{H}^s_p(\mathbb{R}^n_+)$ be the quotient space $\mathbb{H}^s_p(\mathbb{R}^n_+)=\mathbb{H}^s_p(\mathbb{R}^n)/\widetilde{\mathbb{H}}^s_p(\mathbb{R}^n\setminus\mathbb{R}^n_+)$, which can be identified with the space of distributions φ on $\mathbb{R}^n_+$ that admit an extension $\ell\varphi\in\mathbb{H}^s_p(\mathbb{R}^n)$. Therefore, $r_{\mathbb{R}^n_+}\mathbb{H}^s_p(\mathbb{R}^n)=\mathbb{H}^s_p(\mathbb{R}^n_+)$.

Now let us define weighted anisotropic Bessel potential spaces similar to those in [Es1, Sects. 23 and 26].

For $\mu,s\in\mathbb{R}$, $m\in\mathbb{N}_0$, and $1<p<\infty$, by $\mathbb{H}^{(\mu,s),m}_p(\mathbb{R}^n)$ we denote the space of functions (or of distributions if $\mu<0$ or $\mu+s<0$) endowed with the norm

$$\|u\big|\mathbb{H}^{(\mu,s),m}_p(\mathbb{R}^n)\|:=\sum_{k=0}^m\|\lambda^\mu(D')\lambda^{s+k}(D)x_n^k u\big|L_p(\mathbb{R}^n)\|,\qquad x=(x',x_n)\in\mathbb{R}^n\,,$$
$$\lambda^s(\xi):=(1+|\xi|^2)^{s/2}\,,\quad \xi=(\xi',\xi_n)\,,\quad \xi'\in\mathbb{R}^{n-1}\,,\quad \xi_n\in\mathbb{R}\,. \tag{1.4}$$

The operator

$$\lambda^\nu(D')\lambda^r(D) : H_p^{(\mu,s),m}(\mathbb{R}^n) \to H_p^{(\mu-\nu,s-r),m}(\mathbb{R}^n)$$

is an isometric isomorphism and its inverse is $\lambda^{-\nu}(D')\lambda^{-r}(D)$ (where $\nu, r \in \mathbb{R}$).

We write $\mathbb{H}_p^{(\mu,s)}(\mathbb{R}^n)$ instead of $\mathbb{H}_p^{(\mu,s),0}(\mathbb{R}^n)$ and $\mathbb{H}_p^{s,m}(\mathbb{R}^n)$ instead of $\mathbb{H}_p^{(0,s),m}(\mathbb{R}^n)$; note that $\mathbb{H}_p^{(0,s)}(\mathbb{R}^n) = \mathbb{H}_p^s(\mathbb{R}^n)$. The definitions of the spaces $\widetilde{\mathbb{H}}_p^{(\mu,s),m}(\mathbb{R}^n_+)$ and $\mathbb{H}_p^{(\mu,s),m}(\mathbb{R}^n_+)$ are similar to those of $\widetilde{\mathbb{H}}_p^s(\mathbb{R}^n_+)$ and $\mathbb{H}_p^s(\mathbb{R}^n_+)$, respectively.

If the boundary $\partial\mathcal{M}$ of a manifold $\mathcal{M}$ is nonempty, then $\mathcal{M}$ can be extended to a manifold $\widetilde{\mathcal{M}} \supset \mathcal{M}$ without boundary and with the same smoothness.

If $\{Y_j\}_{j=1}^l$ is a sufficiently fine covering of $\mathcal{M}$, then the spaces $\mathbb{H}_p^s(\mathcal{M})$, $C^\sigma(\mathcal{M})$, $\widetilde{\mathbb{H}}_p^s(\mathcal{M})$, $\widetilde{C}^\sigma(\mathcal{M})$, $\mathbb{H}_p^{(\mu,s),m}(\mathcal{M})$, and $\widetilde{\mathbb{H}}_p^{(\mu,s),m}(\mathcal{M})$ can be defined with the help of a partition of unity $\{\psi_j\}_{j=1}^l$ (subordinated to the covering $\{Y_j\}_{j=1}^l$) and of local coordinate diffeomorphisms

$$æ_j : X_j \to Y_j, \qquad X_j \subset \mathbb{R}^n_+. \tag{1.5}$$

The space $\widetilde{\mathbb{H}}_p^{(\mu,s),m}(\mathcal{M})$ can also be defined as a subspace of $\mathbb{H}_p^{(\mu,s),m}(\widetilde{\mathcal{M}})$ consisting of functions $\varphi \in \mathbb{H}_p^{(\mu,s),m}(\widetilde{\mathcal{M}})$ for which $\mathbf{supp}\,\varphi \subset \overline{\mathcal{M}}$, and the space $\mathbb{H}_p^{(\mu,s),m}(\mathcal{M})$ can be realized as the quotient space $\mathbb{H}_p^{(\mu,s),m}(\mathcal{M}) = \mathbb{H}_p^{(\mu,s),m}(\widetilde{\mathcal{M}})/\widetilde{\mathbb{H}}_p^{(\mu,s),m}(\widetilde{\mathcal{M}} \setminus \mathcal{M})$. The latter space can be identified with the space of distributions φ on $\mathcal{M}$ that admit an extension $\ell\varphi \in \mathbb{H}_p^{(\mu,s),m}(\widetilde{\mathcal{M}})$. Therefore, $r_{\mathcal{M}}\mathbb{H}_p^{(\mu,s),m}(\widetilde{\mathcal{M}}) = \mathbb{H}_p^{(\mu,s),m}(\mathcal{M})$.

If $\mathcal{B}^*$ denotes the dual space to the space $\mathcal{B}$ and if $\partial\mathcal{M} \neq \varnothing$, then the following relations hold [Tr1]:

$$\left(\widetilde{\mathbb{H}}_p^s(\mathcal{M})\right)^* = \mathbb{H}_{p'}^{-s}(\mathcal{M}), \qquad \left(\mathbb{H}_p^r(M)\right)^* = \widetilde{\mathbb{H}}_{p'}^{-r}(\mathcal{M}), \tag{1.6}$$

provided that $s, r \in \mathbb{R}$, $r \geqslant 1/p$, $1 < p < \infty$, and $p' = p/(p-1)$. If $\mathcal{S}^m \subset \mathbb{R}^n$ is an m-dimensional C^∞-smooth submanifold, where $m < n$, then the trace operator

$$\gamma_{\mathcal{S}^m} : \mathbb{H}_p^s(\mathbb{R}^n) \to \mathbb{B}_{p,p}^{s-(n-m)/p}(\mathcal{S}^m) \tag{1.7}$$

is well defined and bounded for

$$1 < p < \infty\,, \qquad \frac{n-m}{p} < s\,.$$

Here $\mathbb{B}_{p,q}^s(\mathcal{S}^m)$ stands for the Besov space [Tr1].

1.2. *PsDOs on* $\mathbb{R}^n$

If the convolution operator defined in (1.2) has a bounded extension

$$W_a^0 \colon L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n),$$

then we write $a \in M_p(\mathbb{R}^n)$, and $a(\xi)$ is called a Fourier L_p-multiplier or simply L_p-*multiplier*. For $\nu \in \mathbb{R}$, we write

$$M_p^{(\nu)}(\mathbb{R}^n) := \left\{(1+|\xi|^2)^{\nu/2}a(\xi) : a \in M_p(\mathbb{R}^n)\right\}.$$

By using the isomorphism (1.3) and the obvious property

$$W_{a_1}^0 W_{a_2}^0 = W_{a_1 a_2}^0, \quad a_j \in M_p^{(\nu_j)}(\mathbb{R}_n), \quad j = 1, 2, \tag{1.8}$$

we see that the operator $W_a^0 \colon \mathbb{H}_p^s(\mathbb{R}^n) \to \mathbb{H}_p^{s-\nu}(\mathbb{R}^n)$ is bounded if and only if $a \in M_p^{(\nu)}(\mathbb{R}^n)$.

The following result is known as the Mikhlin–Hörmander–Lizorkin multiplier theorem. The proofs can be found in [Sr2] and [Hr1, Theorem 7.9.5].

Theorem 1.1. *If*

$$\sup\left\{|\xi^\beta\partial^\beta a(\xi)| : \xi\in\mathbb{R}^n,\ \ \beta\in\mathbb{N}_0^n,\ \ |\beta|\leqslant\frac{n+1}{2},\ \ 0\leqslant\beta\leqslant 1\right\}\leqslant M<\infty,$$

then $a\in\bigcap_{1<p<\infty}M_p(\mathbb{R}^n)$.

Let $a\in M_p^{(\nu)}(\mathbb{R}^n)$. Then the operator

$$W_a := r_+\mathbf{a}(D) : \widetilde{\mathbb{H}}_p^s(\mathbb{R}_+^n)\to\mathbb{H}_p^{s-r}(\mathbb{R}_+^n) \tag{1.9}$$

is bounded, where $r_+ := r_{\mathbb{R}_+^n}$ is the restriction operator.

In general, the composition rule (1.8) fails for half-space operators (1.9). However, if there is an analytic continuation $a_1(\xi',\xi_n-i\lambda)$ (or $a_2(\xi',\xi_n+i\lambda)$) for $\xi_n\in\mathbb{R}$ and $\lambda\in\mathbb{R}^+$ that belongs to $\mathbb{S}'(\mathbb{R}^{n-1}\times\mathbb{C}^-)$ (to $\mathbb{S}'(\mathbb{R}^{n-1}\times\mathbb{C}^+)$, respectively), where $\mathbb{C}^\pm=\mathbb{R}\times\mathbb{R}^\pm$, then

$$W_{a_1}W_{a_2}=W_{a_1a_2}. \tag{1.10}$$

If the symbol $a(x,\xi)$ depends on the variable x, then the corresponding convolution operator (see (1.2))

$$\mathbf{a}(x,D)\varphi(x)=W^0_{a(x,\cdot)}\varphi(x):=\left(\mathcal{F}^{-1}_{\xi\to x}a(x,\xi)\mathcal{F}_{y\to\xi}\varphi(y)\right)(x) \tag{1.11}$$

with the symbol $a\in C(\mathbb{R}^n,\mathbb{S}'(\mathbb{R}^n))$ is said to be a *general pseudodifferential operator* (PsDO) acting on $\varphi\in\mathbb{S}(\mathbb{R}^n)$. Here $C(\Omega,\mathcal{B})$ stands for the set of all continuous functions $a\colon\Omega\to\mathcal{B}$.

Let $M_p^{(\nu)}(\mathbb{R}^n\times\mathbb{R}^n)$ be the class of symbols $a(x,\xi)$ for which the operator in (1.11) can be extended to a bounded mapping

$$\mathbf{a}(x,D)\colon\mathbb{H}_p^s(\mathbb{R}^n)\to\mathbb{H}_p^{s-\nu}(\mathbb{R}^n)\quad\text{for all}\quad s\in\mathbb{R}\,.$$

Theorem 1.2 [Sh2, Theorem 5.3]. *Let $\Omega\subset\mathbb{R}^n$, $n\in\mathbb{N}$, and $\nu\in\mathbb{R}$. If, for a function $a(x,\xi)$, $x\in\Omega$, $\xi\in\mathbb{R}^n$, there exist constants $M_{\alpha,\beta}$ such that*

$$\int\limits_\Omega|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|\,dx\leqslant M_{\alpha,\beta}\langle\xi\rangle^{\nu-|\beta|} \tag{1.12}$$

$$\textit{for all}\quad \alpha,\beta=(\beta',\beta_n)\in\mathbb{N}_0^n,\quad |\beta'|\leqslant\left[\frac{n}{2}\right]+1\,,\quad \beta'\leqslant 1,$$

and for all $\beta_n\in\{0,1,\dots\}$ and $\xi\in\mathbb{R}^n$, then $a\in M_p^{(\nu)}(\mathbb{R}^n\times\mathbb{R}^n)$ for all $1<p<\infty$.

Definition. Let $\mathcal{S}_\nu^{\rm cl}(\Omega,\mathbb{R}^n)$ be the class of functions $a(x,\xi)$ satisfying condition (1.12) and admitting an asymptotic expansion of the form

$$a(x,\xi)\simeq a_0(x,\xi)+a_1(x,\xi)+\cdots, \tag{1.13}$$

where

(i) $a_k(x,\xi)$ is positive homogeneous of order $\nu-k$ with respect to ξ,

$$a_k(x,\lambda\xi)=\lambda^{\nu-k}a_k(x,\xi)\quad\text{for all}\quad\lambda>0,\quad\xi\in\mathbb{R}^n\,,\quad x\in\Omega,$$

and

$$\int\limits_\Omega|\partial_x^\alpha\partial_\xi^\beta a_k(x,\xi)|dx\leqslant M_{\alpha,\beta}|\xi|^{\nu-k-|\beta|},\ |\xi|\geqslant 1,\ \text{for all}\ \alpha,\beta=(\beta',\beta_n)\in\mathbb{N}_0^n,\ k\in\mathbb{N}_0\,. \tag{1.14}$$

(ii) For any $N\in\mathbb{N}_0$, the difference

$$\widetilde{a}_{N+1}(x,\xi):=a(x,\xi)-a_0(x,\xi)-\cdots-a_N(x,\xi)$$

satisfies the inequality

$$\int\limits_\Omega|\partial_x^\alpha\partial_\xi^\beta\widetilde{a}_{N+1}(x,\xi)|dx\leqslant M_{\alpha,\beta}|\xi|^{\nu-|\beta|-N-1}\quad\text{for all}\quad\xi\in\mathbb{R}^n,\quad\alpha,\beta\in\mathbb{N}_0^n\,. \tag{1.15}$$

The function $a_0(x,\xi)=a_{\rm pr}(x,\xi)$ in (1.13) is said to be the *homogeneous principal symbol* of $a(x,D)$.

Theorem 1.3 [CD1, Theorem 1.6]. *Let $m \in \mathbb{N}_0$ and $1 < p < \infty$. If $\partial^k_{\xi_n} a(x,\xi) \in M^{(\nu)}_p(\mathbb{R}^n, \mathbb{R}^n)$ for all $k = 0, 1, \dots, m$, then the operator*

$$\mathbf{a}(x, D) \colon \mathbb{H}^{(\mu,s),m}_p(\mathbb{R}^n) \to \mathbb{H}^{(\mu,s-\nu),m}_p(\mathbb{R}^n) \tag{1.16}$$

is bounded for any $\mu, s \in \mathbb{R}$. In particular, if $a \in \mathcal{S}^{\mathrm{cl}}_\nu(\mathbb{R}^n, \mathbb{R}^n)$, then $a(x, D)$ in (1.16) is bounded for any $m \in \mathbb{N}_0$ and any $\mu, s \in \mathbb{R}$.

Lemma 1.4 [DW1, Lemma 1.7]. *Let $a, b \in M^{(\nu)}_p(\mathbb{R}^n \times \mathbb{R}^n)$ and $\nu \in \mathbb{R}$. Suppose that the conditions of Theorem 1.3 hold for a and b. Let there exist analytic extensions $a(x, \xi', \xi_n + i\lambda)$ and $b(x, \xi', \xi_n - i\lambda)$ in the upper and lower half-planes, respectively ($x \in \mathbb{R}^n$, $\xi' \in \mathbb{R}^{n-1}$, $\xi_n \in \mathbb{R}$, $\lambda \in \mathbb{R}^+$) whose growth at infinity is polynomial, i.e., let $|a|$ and $|b|$ be majorized by $(|\xi'| + |\xi_n| + \lambda)^N$ for some N and for all $x \in \mathbb{R}^n$ (uniformly). Then the operators*

$$\begin{aligned} \mathbf{a}(x, D) &\colon \widetilde{\mathbb{H}}^{(\mu,s),m}_p(\mathbb{R}^n_+) \to \widetilde{\mathbb{H}}^{(\mu,s-\nu),m}_p(\mathbb{R}^n_+), \\ r_+\mathbf{b}(x, D)\ell_0 &\colon \mathbb{H}^{(\mu,s),m}_p(\mathbb{R}^n_+) \to \mathbb{H}^{(\mu,s-\nu),m}_p(\mathbb{R}^n_+) \end{aligned} \tag{1.17}$$

are bounded and

$$\begin{aligned} r_+\mathbf{a}(x, D)\varphi &= \mathbf{a}(x, D)\varphi, && \varphi \in \widetilde{\mathbb{H}}^{(\mu,s),m}_p(\mathbb{R}^n_+), \\ r_+\mathbf{b}(x, D)\ell_1\psi &= r_+\mathbf{b}(x, D)\ell_2\psi, && \psi \in \mathbb{H}^{(\mu,s),m}_p(\mathbb{R}^n_+)\,. \end{aligned} \tag{1.18}$$

Here ℓ_0, ℓ_1, and ℓ_2 are some extensions of $\varphi \in \mathbb{H}^s_p(\mathbb{R}^n_+)$ with $\ell_j\varphi \in \mathbb{H}^s_p(\mathbb{R}^n)$, $j = 0, 1, 2$.

1.3. *PsDOs on a Manifold*

Let $\mathcal{M}$ be an $(n-1)$-dimensional[1], C^∞-smooth compact manifold with smooth boundary $\Gamma := \partial\mathcal{M} \neq \varnothing$ and let $1 < p < \infty$ and $s, \nu \in \mathbb{R}$.

We can readily see that the symbols of class $\mathcal{S}^{\mathrm{cl}}_\nu(\mathcal{M}, \mathbb{R}^{n-1})$ are invariant with respect to the diffeomorphisms $(x, \xi) \mapsto (g_0(x, \xi), g_1(x, \xi))$ with positively homogeneous $g_k \in C^\infty(\mathcal{M}, S^{n-2})$ of order k with respect to ξ ($k = 0, 1$; cf. [Sb1, Lemma 1.2]). Therefore, the class of symbols $\mathcal{S}^{\mathrm{cl}}_\nu(\mathcal{T}^*\mathcal{M})$ is well defined on the cotangent manifold $\mathcal{T}^*\mathcal{M}$ (see [Sb1, Subsection 1.3]).

Moreover, the definition of the principal symbol $a_{\mathrm{pr}}(x, \xi)$ is invariant and does not depend on the chosen chart.

Definition (see [DS1, Hr1, Sb1], etc.). An operator

$$\mathbf{A} \colon \widetilde{\mathbb{H}}^{(\mu,s),m}_p(\mathcal{M}) \to \mathbb{H}^{(\mu,s-\nu),m}_p(\mathcal{M}) \tag{1.19}$$

is called a pseudodifferential operator with symbol $a \in \mathcal{S}^{\mathrm{cl}}_\nu(\mathcal{T}^*\mathcal{M})$ if

(i) the mappings $\chi_1\mathbf{A}\chi_2 I \colon \mathbb{H}^{(\mu,s),m}_p(\mathcal{M}) \to C^\infty(\mathcal{M})$ are continuous for all pairs $\chi_1, \chi_2 \in C^\infty(\mathcal{M})$ with disjoint supports $\mathbf{supp}\,\chi_1 \bigcap \mathbf{supp}\,\chi_2 = \varnothing$, i.e., $\chi_1\mathbf{A}\chi_2 I$ is of order $-\infty$,

(ii) the transformed operators

$$æ_{j,*}\mathbf{A}æ^{-1}_{j,*}u = \mathbf{a}^{(j)}(x, D)u, \qquad u \in C^\infty_0(\mathbb{R}^{n-1}_+), \qquad j = 1, \dots, l,$$

(cf. (1.5)), where

$$\begin{aligned} æ_{j,*}u(x) &:= \begin{cases} \psi^0_j(x)u(æ_j(x)) & \text{for} \quad x \in X_j, \\ 0 & \text{for} \quad x \neq X_j, \end{cases} \\ æ^{-1}_{j,*}\varphi(t) &:= \begin{cases} \psi_j(t)\varphi(æ^{-1}_j(t)) & \text{for} \quad t \in Y_j, \\ 0 & \text{for} \quad t \neq Y_j, \end{cases} \end{aligned} \tag{1.20}$$

and $\psi^0_j(x) := \psi_j\,(æ_j(x))$, $x \in X_j$, are pseudodifferential operators on $\mathbb{R}^{n-1}_+$ with the symbols $a^{(j)}\,(æ_j(x), \xi) = \psi^0_j(x)a\,(æ_j(x), \xi)\psi^0_j(x)\,.$

---

[1] Due to our needs (see applications in Section 2), from now on we prefer to consider $(n-1)$-dimensional manifolds rather than n-dimensional ones.

The homogeneous principal symbol is responsible for the Fredholm properties and for the index of the corresponding pseudodifferential equation $\mathbf{A}u = f$ with $f \in \mathbb{H}_p^{s-r}(\mathcal{M})$. Moreover, it determines the leading term of the asymptotic expansion of the solution $u \in \widetilde{\mathbb{H}}_p^s(\mathcal{M})$ in a neighborhood of the boundary Γ. Lower order terms of the asymptotic expansion of the solution are influenced by the complete symbol $a(x,\xi)$ (see [Es1, Sect. 26], [Be1, CD1], and Theorem 1.7 below).

As is well known, purely homogeneous symbols can cause problems related to the boundedness of the corresponding operators. For example, recall the operator $|\partial|^{-\nu}$ with the symbol $|\xi|^{-\nu}$ for $\nu > 0$, which is unbounded as a mapping $\mathbb{H}_p^s(\mathbb{R}^{n-1}) \to \mathbb{H}_p^{s+\nu}(\mathbb{R}^{n-1})$, and even in $L_p(\mathbb{R}^{n-1})$ (see [St1, Sect. VI]). However, if $a(x,\xi)$ satisfies (1.12) for $|\xi| \geqslant 1$ and is homogeneous of order $\nu \leqslant -n$, then the truncated symbol

$$\overset{\vee}{a}(x,\xi) := [1-\chi_0(\xi)]a(x,\xi) \tag{1.21}$$

(where $\chi_0 \in C_0^\infty(\mathbb{R}^{n-1})$ is a cut-off function with $\chi_0(\xi) = 0$ for $|\xi| \geqslant 1$, $\chi_0(\xi) = 1$ for $|\xi| \leqslant 1/2$) belongs to the class $\mathcal{S}_\nu^{\mathrm{cl}}(\mathcal{M}, \mathbb{R}^{n-1})$ because it satisfies both formula (1.12) and the relation

$$\int\limits_\Omega |\partial_x^\alpha \partial_\xi^\beta [a(x,\xi) - \overset{\vee}{a}(x,\xi)]|\, dx \leqslant M_{\alpha,\beta,N} |\xi|^{\nu-|\beta|-N}$$

for arbitrary $N \in \mathbb{N}$, $\xi \in \mathbb{R}^{n-1}$, $\alpha, \beta \in \mathbb{N}_0^{n-1}$, and $x \in \mathcal{M}$. The advantages of the truncated symbol are obvious; in particular, the difference

$$\mathbf{a}(x,D) - \overset{\vee}{\mathbf{a}}(x,D) \colon \widetilde{\mathbb{H}}_p^s(\mathcal{M}) \to C^\infty(\mathcal{M})$$

is a smoothing operator and does not influence in the singular asymptotics of solutions. For the same reason, the operators $\mathbf{a}(x,D)$ and $\overset{\vee}{\mathbf{a}}(x,D)$ have the same Fredholm properties and equal indices provided that $\mathcal{M}$ is compact.

A similar property concerning asymptotic behavior holds under stronger perturbations. Namely, let conditions of Theorem 1.3 hold for $|\xi| \geqslant 1$ and let a cut-off function depend only on the last variable, $\chi_0^{(0)}(\xi_{n-1}) = 0$ for $|\xi_{n-1}| \geqslant 1$, $\chi_0^{(0)}(\xi_{n-1}) = 1$ for $|\xi_{n-1}| \leqslant 1/2$, and $\chi_0^{(0)} \in C_0^\infty(\mathbb{R})$; then the operator with truncated symbol

$$\overset{\vee}{\mathbf{a}}{}^0(x,D) := [1-\chi_0^{(0)}(D_{n-1})]\mathbf{a}(x,D) : \ \widetilde{\mathbb{H}}_p^{(\infty,s),m}(\mathcal{M}) \to \mathbb{H}_p^{(\infty,s-\nu),m}(\mathcal{M})$$

is bounded, and the difference

$$\mathbf{a}(x,D) - \overset{\vee}{\mathbf{a}}{}^0(x,D) \colon \widetilde{\mathbb{H}}_p^{(\infty,s),m}(\mathcal{M}) \to C^\infty(\mathcal{M}) \tag{1.22}$$

is a smoothing operator. However, $\overset{\vee}{\mathbf{a}}{}^0$ is not a compact perturbation of $\mathbf{a}(x,D)$ for $n-1 \geqslant 2$, and it essentially influences in the Fredholm properties and in the index.

1.4. *Example*

Let $\mathcal{M} \subset \mathbb{R}^3$ be a smooth surface with smooth boundary $\Gamma = \partial\mathcal{M}$ and let

$$\mathcal{M} = \bigcup_{j=1}^N Y_j, \quad æ_j = (æ_{j1}, æ_{j2}, æ_{j3})^\top : X_j \to Y_j, \quad X_j \subset \mathbb{R}_+^2 = \mathbb{R} \times \mathbb{R}^+, \tag{1.23}$$

be a C^∞-smooth atlas of the surface $\mathcal{M}$; let the functions

$$\begin{gathered} \widetilde{æ}_j \colon \widetilde{X}_j \to \widetilde{Y}_j, \qquad \widetilde{X}_j \subset \mathbb{R}_+^3, \qquad \widetilde{Y}_j \subset \mathbb{R}^3, \qquad \widetilde{Y}_j \cap \mathcal{M} = Y_j, \\ \widetilde{X}_j = (-\varepsilon, \varepsilon) \times X_j, \qquad \widetilde{æ}_j \,|_{X_j} = \widetilde{æ}_j(0,x) = æ_j(x), \qquad x = (x_1, x_2), \qquad j = 1, 2, \ldots, N, \end{gathered} \tag{1.24}$$

be extensions of the diffeomorphisms in (1.23). For $x = (x_1, x_2) \in X_j$ and $\widetilde{x} = (x_0, x_1, x_2) \in \widetilde{X}_j$, by $\mathcal{J}_{æ_j}(x) = \|\partial_k æ_{jl}(x)\|_{2\times 3}$ and $\mathcal{J}_{\widetilde{æ}_j}(\widetilde{x}) = \|\partial_k \widetilde{æ}_{jl}(\widetilde{x})\|_{3\times 3}$ we denote the corresponding Jacobi matrices, respectively, and $æ_j'(x)$ coincides with $\widetilde{æ}_j'(0,x)$ for $x \in X_j$ after deleting the first column, i.e., after removing the entries $\partial_0 \widetilde{æ}_{jl}(0,x)$, $l = 1,2,3$; therefore, $\mathcal{J}_{\widetilde{æ}_j}(0,x)(0,y) = æ_j'(x)y$ for $x \in X_j$ and $y \in \mathbb{R}^2$. It is clear that

$$\mathcal{J}_{\widetilde{æ}_j}(0,x) = \left(e_0\left(æ_j(x)\right), e_1\left(æ_j(x)\right), e_2\left(æ_j(x)\right)\right), \tag{1.25}$$

where the vector columns $e_0(t)$, $e_1(t)$, and $e_2(t)$ on the boundary $t = æ_j(x') \in \Gamma$ can be chosen to be orthogonal to each other, $e_0(y') = \vec{n}(y')^\top$, while e_1 and e_2 are tangent to $\mathcal{M}$, e_1 is tangent to Γ, and e_2 is cotangent to Γ. The fields $e_1(t)$ and $e_2(t)$ of unit vectors on $\mathcal{M}$ are not orthogonal in general, in contrast to the pairs e_0, e_1 and e_0, e_2.

As a consequence, the Jacobi matrix $\mathcal{J}_{\widetilde{æ}_j}(\widetilde{x})$ becomes orthogonal on the boundary Γ,

$$[\widetilde{æ}_j'(0,x_1,0)]^\top = [\mathcal{J}_{\widetilde{æ}_j}(0,x_1,0)]^{-1} \quad \text{for all} \quad (x_1,0) \in X_j \cap \partial\mathbb{R}^2_+ . \tag{1.26}$$

Let $-\infty < \nu \leqslant -1$ and let

$$\begin{gathered} a(\xi) = \mathcal{F}k(\xi) \simeq a_\nu(\xi) + a_{\nu-1}(\xi) + \cdots + a_{\nu-k}(\xi) + \cdots , \\ a_{\nu-k}(\lambda\xi) = \lambda^{\nu-k} a_{\nu-k}(\xi), \quad \xi \in \mathbb{R}^3, \quad \lambda > 0 \end{gathered}$$

be a classical $N \times N$ matrix symbol $a \in \mathcal{S}^{\mathrm{cl}}_\nu(\mathbb{R}^3)$.

If $\nu \neq -1$, then the trace

$$\begin{aligned} \mathbf{a}_{\mathcal{M}}(t,D)\varphi(t) &= \gamma_{\mathcal{M}} \mathbf{a}(D)(\varphi \times \delta_{\mathcal{M}})(t) \\ &= \int\limits_{\mathbb{R}^3} k(t-y)(\varphi \times \delta_{\mathcal{M}})(y)dy = \int\limits_{\mathcal{M}} k(t-\tau)\varphi(\tau)d_\tau\mathcal{M} , \quad t \in \mathcal{M}, \end{aligned} \tag{1.27}$$

where $(\varphi \times \delta_{\mathcal{M}}, \psi) := \langle \varphi, \gamma_{\mathcal{M}}\psi \rangle$, $\psi \in \mathbb{S}(\mathbb{R}^3)$, by definition, is a pseudodifferential operator

$$\mathbf{a}_{\mathcal{M}}(t,D) \colon \widetilde{\mathbb{H}}^{(\mu,s),m}_p(\mathcal{M}) \to \mathbb{H}^{(\mu,s-\nu-1),m}_p(\mathcal{M}).$$

This operator has the classical symbol

$$\begin{aligned} a_{\mathcal{M}}(t,\xi') &\simeq \sum_{k=0}^{\infty} a_{\mathcal{M},\nu+1-k}(t,\xi') , \quad a_{\mathcal{M},\nu+1-k} \in \mathcal{S}^{\infty,0}_{\nu+1-k}(\mathcal{T}^*\mathcal{M}) , \quad \xi' \in \mathbb{R}^2 , \\ a_{\mathcal{M},\nu+1-k}(æ_j(x),\xi') &= \sum_{m=0}^{k} \sum_{\substack{|\beta|+|\gamma|-|\alpha|=k-m \\ 2\alpha \leqslant \beta}} \frac{(-i)^{|\alpha+\beta+\gamma|} b_{\alpha,\beta}(x) \partial_x^\gamma \mathcal{G}_{æ_j}(x)}{2\pi \det \mathcal{J}_{\widetilde{æ}_j}(0,x)\gamma!} \\ &\qquad \times (-\xi')^\alpha \int\limits_{-\infty}^{\infty} \partial_{\xi'}^{\beta+\gamma} a_{\nu-m}\left(\mathcal{J}^{-1}_{\widetilde{æ}_j}(0,x)^\top(\xi',\lambda)\right) d\lambda \ , \end{aligned} \tag{1.28}$$

where

$$\mathcal{G}_{æ_j} := (\det\|(\partial æ_\ell j, \partial æ_{Pj\ell})\|_{3\times 3})^{1/2} \quad \text{with} \quad \partial æ_{jk} := (\partial_1 æ_{jk}, \partial_2 æ_{jk})^\top \tag{1.29}$$

stands for the square root of the Gram determinant of the vector function $æ_j = (æ_{j1}, æ_{j2}, æ_{j3})^\top$ for $j = 1,2,\ldots,N$, $b_{0,\beta}(x) = 1$, and the coefficients $b_{\alpha,\beta}(x)$ for $|\alpha| > 0$ can be found from the following relation:

$$\frac{1}{\alpha!}\left[\sum_{|\delta|=2}^{m} \frac{(-1)^{|\delta|+1}}{\delta!} \partial^\delta æ_j(x)\tau^\delta\right]^\alpha = \sum_{|\beta|=2|\alpha|}^{m+2} b_{\alpha,\beta}(x)\tau^\beta + \sum_{|\beta|=m+3}^{m|\alpha|} g^{(m)}_{\alpha,\beta}(x)\tau^\beta , \qquad \alpha \in \mathbb{N}^n.$$

In particular, the homogeneous principal symbol is

$$a_{\mathcal{M},\mathrm{pr}}(æ_j(x),\xi) := a_{\mathcal{M},\nu+1}(æ_j(x),\xi') = \frac{\mathcal{G}_{æ_j}(x)}{2\pi\det\mathcal{J}_{\widetilde{æ}_j}(0,x)}\int\limits_{-\infty}^{\infty} a_\nu\left(\mathcal{J}^{-1}_{\widetilde{æ}_j}(0,x)^\top(\xi',\lambda)\right)d\lambda\,,\quad x\in X_j\,. \tag{1.30}$$

For $\nu=-1$, we cannot write (1.27). In this case, the formula

$$\mathbf{a}_\mathcal{M}(t,D)\varphi(t)=\gamma_\mathcal{M}\mathbf{a}(D)(\varphi\times\delta_\mathcal{M})(t)=c_0(t)\varphi(t)+\int\limits_\mathcal{M} k_0(t,t-\tau)\varphi(\tau)d_\tau\mathcal{M} \tag{1.31}$$

defines a pseudodifferential operator of order zero, $\mathbf{a}_\mathcal{M}(t,D)\colon \widetilde{\mathbb{H}}^{(\mu,s),m}_p(\mathcal{M})\to\mathbb{H}^{(\mu,s),m}_p(\mathcal{M})$, i.e., it is a singular integral operator; the integral in (1.31) is regarded as the Cauchy principal value. Moreover (see [Es1, (3.26)]),

$$c_0(t)=\frac{1}{2\pi}\int\limits_{|\omega|=1} a_{\mathcal{M},\mathrm{pr}}(t,\omega)d_\omega S\,,\quad k_0(t,\tau)=\mathcal{F}^{-1}_{\xi\to\tau}[a_{\mathcal{M},\mathrm{pr}}(t,\xi)-c_0(t)]\,,\qquad t,\tau\in\mathcal{M}\,. \tag{1.32}$$

1.5. *Fredholm Property*

Although many general results on the asymptotics and Fredholm properties of PsDEs can be found in [Be1, CD1, DS1, DW1, Es1], here we collect some information needed below.

Let us consider an $N\times N$ system of pseudodifferential equations on a compact smooth manifold with smooth boundary $\partial\mathcal{M}$ of the form

$$\mathbf{a}_\mathcal{M}(x,D)\varphi_0(x)=v(x)\,,\qquad x\in\mathcal{M}, \tag{1.33}$$

with symbol $a_\mathcal{M}\in\mathcal{S}^{\mathrm{cl}}_\nu(\mathcal{M},\mathbb{R}^{n-1})$ and principal symbol $a_{\mathrm{pr}}(x,\xi)$. Let us seek a solution $\varphi_0\in\widetilde{\mathbb{H}}^{(\mu,s),m}_p(\mathcal{M})$ for a given $v\in\mathbb{H}^{(\mu,s-\nu),m}_p(\mathcal{M})$, where $m\in\mathbb{N}$, $\mu,s,\nu\in\mathbb{R}$, and $1<p<\infty$.

Let the symbol $a_\mathcal{M}(x,\xi)$ in (1.31) be elliptic, i.e.,

$$\inf\{\det a_{\mathrm{pr}}(x,\xi): x\in\overline{\mathcal{M}},\ \xi\in S^{n-2}\}>0, \tag{1.34}$$

and let $\lambda_1(x'),\dots,\lambda_\ell(x')$ be the eigenvalues of the matrix function

$$a^0_\mathcal{M}(x')=[a_{\mathrm{pr}}(x',0,\dots,0,+1)]^{-1}a_{\mathrm{pr}}(x',0,\dots,0,-1) \tag{1.35}$$

on the boundary $x'\in\partial\mathcal{M}$ with algebraic multiplicities $m_1,\dots,m_\ell$, respectively (i.e., to any eigenvalue $\lambda_j(x')$, m_j linearly independent associated vectors correspond, and $m_1+\dots+m_\ell=N$). Since the geometric and algebraic multiplicities are different, it follows that the eigenvalues $\lambda_1(x'),\dots,\lambda_\ell(x')$ need not be all different. Further, let

$$\delta_j(x')=\frac{\log\lambda_j(x')}{2\pi i}\,,\qquad \frac{1}{p}-1<s-\mathrm{Re}\,\delta_j(x')-\frac{\nu}{2}\leqslant\frac{1}{p}\quad\text{for}\quad j=1,\dots,\ell\,. \tag{1.36}$$

Lemma 1.5 (see [La1, Theorem 2.10.2] and [DSW1, Lemma A.6]). *The matrix $a^0_\mathcal{M}(x')$ in* (1.33) *is normal (i.e., commutes with its conjugate matrix), $\left(a^0_\mathcal{M}(x')\right)^*a^0_\mathcal{M}(x')=a^0_\mathcal{M}(x')\left(a^0_\mathcal{M}(x')\right)^*$, if and only if it has no generalized associated eigenvectors, $\ell=N$, and $a^0_\mathcal{M}(x')$ is unitary equivalent to the diagonal matrix $\Lambda(x'):=\mathbf{diag}\,\{\lambda_1(x'),\dots,\lambda_N(x')\}$, that is,*

$$a^0_\mathcal{M}(x')=\mathcal{K}(x')\Lambda(x')\mathcal{K}^*(x')\quad\textit{with}\quad\det\mathcal{K}(x')\neq0\,,\quad\mathcal{K}^{-1}(x')=\mathcal{K}^*(x')\,,\quad\mathcal{K}\in C^\infty(\partial\mathcal{M}). \tag{1.37}$$

In particular, if two matrices $a^0_{\mathrm{pr}}(x',\pm1)$ are positive definite, then (1.37) *holds, and the numbers $\delta_j(x')$ in* (1.36) *are all purely imaginary*[2], *i.e.,*

$$\mathrm{Re}\,\delta_j(x')=0\quad\textit{for}\quad j=1,\dots,N\,. \tag{1.38}$$

[2]We stress the relationship $\nu_j(x')=i\delta_j(x')$ between $\delta_j(x')$ defined in (1.36) and $\nu_j(x')$ defined in [DSW1, (A.32)].

Theorem 1.6 (see [DW1, Theorem 2.7] and [DS1]). *Let the symbol* $a_{\mathcal{M}}(x,\xi)$ *in* (1.33) *be elliptic* (*see* (1.34)) *and let it be strongly elliptic on the boundary*

$$\operatorname{Re}\,(a_{\rm pr}(x',\xi)\eta,\eta) \geqslant M|\xi|^{\nu}|\eta|^2 \quad \text{for all} \quad x' \in \partial\mathcal{M}, \quad \xi \in \mathbb{R}^{n-1} \quad \text{and} \quad \eta \in \mathbb{C}^N \tag{1.39}$$

with some constant $M > 0$.

Then the system of equations (1.33) *is Fredholm if and only if*

$$\operatorname{Re}\delta_j(x') \neq s - \frac{1}{p} - \frac{\nu}{2} \quad \text{for all} \quad j = 1,\ldots,\ell\,, \;\; x' \in \partial\mathcal{M}\,. \tag{1.40}$$

If, for each interior point $x \in \mathcal{M}$, *there exists an* $\alpha_x \in (0, 2\pi]$ *such that the numerical range of the matrix symbol* $a_{\rm pr}(x,\xi)$, *i.e., the set*

$$\mathcal{R}_x(a) := \left\{(a_{\rm pr}(x,\theta)\eta,\eta) : \theta \in \mathbb{R}^{n-1},\; \eta \in \mathbb{C}^N,\; |\theta| = |\eta| = 1\right\} \tag{1.41}$$

is disjoint with the ray $\{z \in \mathbb{C} : \arg z = \alpha_x\}$, *then the index of* (1.33) *is zero,* $\mathbf{Ind}\,\mathbf{a}_{\mathcal{M}}(x,D) = 0$.

If, in addition, the homogeneous equation $\mathbf{a}_{\mathcal{M}}(x,D)\varphi_0 = 0$ *has only the trivial solution* $\varphi_0 = 0$ *in one of the spaces* $\widetilde{\mathbb{H}}_p^{(\mu,s),m}(\mathcal{M})$, *where* s *and* p *satisfy conditions* (1.36), *then* (1.33) *has a unique solution in each of these spaces.*

If conditions (1.40) *hold, then* (1.33) *has the same kernel in all spaces* $\widetilde{\mathbb{H}}_p^{(\mu,s),m}(\mathcal{M})$, $m \in \mathbb{N}_0$, $\mu \in \mathbb{R}$. *In particular,*

$$\varphi_0 \in \widetilde{\mathbb{H}}_p^{(\infty,s),\infty}(\mathcal{M}) := \bigcap_{\mu,m} \widetilde{\mathbb{H}}_p^{(\mu,s),m}(\mathcal{M}) \quad \text{provided that} \quad v \in \mathbb{H}_p^{(\infty,s),\infty}(\mathcal{M})\,.$$

Note that neither the Fredholm properties nor the index and the kernel $\mathbf{Ker}\,\mathbf{a}_{\mathcal{M}}(x,D)$ of (1.33) depend on the parameters $m \in \mathbb{N}_0$ and $\mu \in \mathbb{R}$.

1.6. *Asymptotics of a Solution*

In the present subsection we formulate results on the asymptotics for a solution $\varphi_0(x)$ of system (1.33). For our purposes, for $\mathcal{M} = \mathcal{S}_0$ we take an $(n-1)$-dimensional smooth surface (with C^∞-smooth boundary $\partial\mathcal{S}_0$) in $\mathbb{R}^n$.

Let us introduce a special local coordinate system (s.l.c.s.) $(\,x'',\; x_{n-1,+}) \in \mathcal{S}_\varepsilon^+ := \partial\mathcal{S}_0 \times [0,\varepsilon]$ on $\mathcal{S}_0$ in a neighborhood of $\partial\mathcal{S}_0$, where $x'' \in \partial\mathcal{S}_0$, $x_{n-1,+}$ measures the distance from the boundary $\partial\mathcal{S}_0$, and ε is sufficiently small.

Let $\lambda_1(x''),\ldots,\lambda_\ell(x'')$ be the eigenvalues of $a^0_{\mathcal{S}_0}(x'')$ (see (1.35)) and let $m_1,\ldots,m_\ell$ be their algebraic multiplicities (i.e., the lengths of the corresponding chains of associated vectors). Then $\sum\limits_{j=1}^{\ell} m_j = N$, and $a^0_{\mathcal{S}_0}(x'')$ has the following representation in the normal (Jordan) form

$$a^0_{\mathcal{S}_0}(x'') = \mathcal{K}(x'')\mathcal{J}_{a^0_{\mathcal{S}_0}}\mathcal{K}^{-1}(x''), \quad \det\mathcal{K}(x'') \neq 0, \quad x'' \in \partial\mathcal{S}_0 \tag{1.42}$$

(cf. (1.37)), where $\mathcal{J}_{a^0_{\mathcal{S}_0}} := \mathbf{diag}\{\lambda_1(x'')B^{m_1}(1),\ldots,\lambda_\ell(x'')B^{m_\ell}(1)\}$ and $B^{m_j}(t)$ are the Jordan blocks defined as follows:

$$B^m(t) := \|b_{jk}(t)\|_{m\times m}, \qquad b_{jk}(t) := \begin{cases} t^{k-j}/(k-j)!, & j < k\,, \\ 1, & j = k\,, \\ 0, & j > k\,. \end{cases} \tag{1.43}$$

These matrices are upper triangular (truncated Toeplitz matrices),

$$B^m(t) = \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{m-2}}{(m-2)!} & \frac{t^{m-1}}{(m-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{m-3}}{(m-3)!} & \frac{t^{m-2}}{(m-2)!} \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 1 & t \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Let

$$B^0_{a_{\rm pr}}(t) := \mathbf{diag}\,\{B^{m_1}(t), \dots, B^{m_\ell}(t)\}\,. \tag{1.44}$$

Note that $B^0_{a_{\rm pr}}(t) = I$ is the identity matrix, and $\mathcal{K} \in C^\infty(\partial\mathcal{S}_0)$ if all chains are trivial, $m_1 = \cdots = m_\ell = 1$. For instance, this is the case if the matrix $a^0_{\mathcal{S}_0}(x'')$ is normal (see Lemma 1.5) or if the matrices $a_{\rm pr}(x'', \pm 1)$ (the specific values of the symbol $a_{\rm pr}(x'', \xi)$) are positive definite (cf. Lemma 1.5).

Theorem 1.7 ([CD1, Theorem 2.1]). *Let $\mu \in \mathbb{R}$ and $m, M \in \mathbb{N}_0$, and let, for some $s \in \mathbb{R}$ and $1 < p < \infty$, equation* (1.33) *have a unique solution $\varphi_0 \in \widetilde{\mathbb{H}}^{(\infty,s),m}_p(\mathcal{S}_0)$ for any given $v \in \mathbb{H}^{(\infty,s-\nu),m}_p(\mathcal{S}_0)$. Then $1/p - 1 < s - \nu/2 - \operatorname{Re}\delta_j(x'') < 1/p$ for all $j = 1, \dots, \ell$. Let $\mathcal{K} \in C^\infty(\partial\mathcal{S}_0)$.*

If $\operatorname{Re}[\nu/2 + \delta_j(x'')] > -1$ for all $j = 1, \dots, \ell$, $v \in \mathbb{H}^{(\infty,s-\nu+M+1),\infty}_p(\mathcal{S}_0)$, then the solution φ_0 has the asymptotic expansion

$$\begin{aligned}\varphi_0(x'', x_{n-1,+}) = \mathcal{K}(x'')\; x^{\nu/2+\Delta(x'')}_{n-1,+} B^0_{a_{\rm pr}}\left(-\frac{1}{2\pi i}\log\; x_{n-1,+}\right)\mathcal{K}^{-1}(x'')c_0(x'') \\ + \sum_{k=1}^{M}\mathcal{K}(x'')x^{\nu/2+\Delta(x'')+k}_{n-1,+}B_k(x'', \log x_{n-1,+}) + \varphi_{M+1}(x'', x_{n-1,+})\end{aligned} \tag{1.45}$$

(*with $\varphi_{M+1} \in \widetilde{\mathbb{H}}^{(\infty,s+M+1),\infty}_p(\mathcal{S}^+_\varepsilon)$*) *for all sufficiently small $x_{n-1,+} > 0$. Here the N vector functions $B_k(x'', t)$ belong to $C^\infty(\partial\mathcal{S}_0)$, and*

$$B_k(x'', t) = B^0_{a_{\rm pr}}\left(-\frac{1}{2\pi i}t\right)\sum_{j=0}^{k(2m_0-1)} t^j c_{kj}(x''),$$

where the N vector functions c_{kj} belong to $C^\infty(\partial S_0)$.

The components of the vector $\Delta := (\delta_1, \dots, \delta_\ell)^\top$ are defined in (1.36), *and*

$$x^{\theta+\Delta}_{n-1,+} := \mathbf{diag}\left\{\, x^{\theta+\delta_1}_{n-1,+}, \dots,\; x^{\theta+\delta_\ell}_{n-1,+}\right\}, \quad \theta \in \mathbb{R},$$

where it is assumed that any component δ_j is repeated m_j times in the vector Δ, according to its multiplicity m_j, and therefore Δ is an N vector.

Furthermore, for any $q = 0, 1, \dots, M$, the a priori *estimates*

$$\begin{aligned} C_0 \sum_{0\leqslant j\leqslant k\leqslant M} \|c_k|C^q(\partial\mathcal{S}_0)\| + C_0\|\varphi_{M+1}|\widetilde{\mathbb{H}}^{(\infty,s+M+1),m}_p(\mathcal{S}_0)\| \\ \leqslant \|\varphi_0|\widetilde{\mathbb{H}}^{(\infty,s),m}_p(\mathcal{S}_0)\| \leqslant C_1\|v|\mathbb{H}^{(\infty,s-\nu+M),m}_p(\mathcal{S}_0)\|\end{aligned} \tag{1.46}$$

hold with some constants C_0 and C_1 that do not depend on v (φ_{M+1} coincides with φ_0 outside of $\mathcal{S}^+_\varepsilon$).

Let the matrix $a^0_{\mathcal{S}_0}(x'')$ *be normal for all* $x'' \in \partial\mathcal{S}_0$ *(see Lemma* 1.5*). Therefore,* $\ell = N$ *and the leading term of the asymptotics in* (1.45) *contains no logarithms* $\log x_{n-1,+}$ *and becomes simpler,*

$$\varphi_0(x'',\ x_{n-1,+}) = \mathcal{K}(x'')x_{n-1,+}^{\nu/2+\Delta(x'')}\mathcal{K}^{-1}(x'')\Big[c_0(x'') + \sum_{k=1}^{M} x^k_{n-1,+}\sum_{j=0}^{k} c_{kj}(x'')\ \log^j x_{n-1,+}\Big] + \varphi_{M+1}(x'', x_{n-1,+})\,. \tag{1.47}$$

2. ASYMPTOTICS OF POTENTIAL-TYPE FUNCTIONS

2.1. *Statement of the Results*

Let Ω be a domain in the Euclidean space $\mathbb{R}^n$, $n \geqslant 2$, which is not necessarily compact and has a compact boundary $\partial\Omega = \mathcal{S}$ that is sufficiently smooth. We consider a homogeneous $N \times N$ system of differential equations

$$\mathbf{A}(D_x)u = 0 \quad \text{in} \quad \Omega, \tag{2.1}$$

of order $2r$, $r \in \mathbb{N}$,

$$\mathbf{A}(D_x) := \sum_{|\alpha|=2r} a_\alpha D^\alpha_x \quad \text{with} \quad D_{x_l} := i\partial_{x_l} = i\frac{\partial}{\partial x_l}, \tag{2.2}$$

with constant matrix coefficients $a_\alpha = \|a^{jk}_\alpha\|_{N\times N}$. We suppose that the homogeneous principal symbol of $\mathbf{A}(D_x)$ given by

$$A_{\rm pr}(\xi) = A(\xi) := \sum_{|\alpha|=2r} a_\alpha \xi^\alpha \tag{2.3}$$

(which coincides with the symbol in this case) is elliptic, i.e., $\det A_{\rm pr}(\xi) \neq 0$ for all ξ, $|\xi| = 1$. The fundamental matrix function (see [Hr1]) for (2.1) can be written as follows:

$$H_A(x) = \mathcal{F}^{-1}_{\xi'\to x'}\left[\pm\frac{1}{2\pi}\int_{\mathcal{L}_\pm}\big(A(\xi',\tau)\big)^{-1}e^{-i\tau x_n}d\tau\right], \tag{2.4}$$

where the signs "–" and "+" refer to the cases $x_n > 0$ and $x_n < 0$, respectively, $x = (x', x_n)$, $x' = (x_1, \dots, x_{n-1})$, $\xi' = (\xi_1, \dots, \xi_{n-1})$, and the contours $\mathcal{L}_\pm$ are located in the complex half-planes $\mathbb{C}^\pm := \mathbb{R} \oplus i\mathbb{R}^\pm$, are oriented counterclockwise, and surround all the roots of the polynomial $\det A(\xi', \tau)$ with respect to τ that belong to the corresponding half-planes $\tau \in \mathbb{C}^\pm$ (see [Ch1]).

The fundamental solution H_A has the following properties:

1. $H_A \in C^\infty(\mathbb{R}^n \setminus \{0\})$ (see, e.g., [Ch1] and [Hr1, Theorem 7.1.22]).
2. H_A is an even matrix function, i.e., $H_A(-x) = H_A(x)$.
3. For $n > 2r$, the matrix function H_A is positively homogeneous of order $2r - n$, i.e., for any $\lambda > 0$ and any $x \in \mathbb{R}^n \setminus \{0\}$ we have $H_A(\lambda x) = \lambda^{2r-n}H_A(x)$.
4. For $n \leqslant 2r$ we have $H_A(x) = P(x)\ln|x| + Q(x)$, where $P(x)$ and $Q(x)$ are positively homogeneous of order $2r - n$ (exact formulas for $P(x)$ and $Q(x)$ can be found in [Es1, formulas (2.90)–(2.92)] and [Hr1, Theorem 7.1.20]).

For the simple layer potential

$$\mathbf{V}g(x) = \int_{\mathcal{S}} H_A(x-y)g(y)d_y\mathcal{S}, \qquad x \notin \mathcal{S}, \tag{2.5}$$

the following theorem holds.

Theorem 2.1 (see [DNS1, NCS1, Sh1]). *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leqslant \omega \leqslant \infty$, and $r \in \mathbb{N}$. Then V can be extended to continuous operators*

$$\mathbf{V}\colon \mathbb{B}^s_{p,\omega}(\mathcal{S}) \to \mathbb{B}^{s+2r-1+\frac{1}{p}}_{p,\omega,\mathrm{loc}}(\Omega), \qquad \mathbf{V}\colon \mathbb{B}^s_{p,p}(\mathcal{S}) \to \mathbb{H}^{s+2r-1+\frac{1}{p}}_{p,\mathrm{loc}}(\Omega)\,,$$

where $\mathbb{B}^\nu_{p,\omega,\mathrm{loc}}(\Omega)$ and $\mathbb{B}^\nu_{p,\omega}(\mathcal{S})$ stand for the Besov spaces (cf. [Tr1, Tr2]*) and the subscript $_{\mathrm{loc}}$ can be omitted if Ω is compact.*

We introduce the notation

$$\mathbf{V}_{1-2r}g(x) = \int\limits_{\mathcal{S}} H_A(x-y)g(y)d_y\mathcal{S}, \qquad x \in \mathcal{S}, \tag{2.6}$$

for the direct value of $\mathbf{V}g(x)$ on the surface $\mathcal{S}$.

Let $\mathcal{S}_0$ be an infinitely differentiable submanifold of $\mathcal{S} = \partial\Omega$ of the same dimension $n-1$ and with C^∞-smooth boundary $\partial\mathcal{S}_0$.

Let $\mathbf{B}_q$ be a pseudodifferential operator of order $q \in \mathbb{R}$ on the manifold $\mathcal{S}_0$ with classical symbol $B_q \in \mathcal{S}^{\mathrm{cl}}_q(\mathcal{T}^*\mathcal{S}_0)$ and let

$$B_q(x,\xi) \simeq \sum_{k=0}^{M} B^0_{q-k}(x,\xi) + \widetilde{B}_{q-M-1}(x,\xi) \tag{2.7}$$

be its representation, for arbitrary $M \in \mathbb{N}_0$, with symbols $B^0_{q-k}(x,\xi)$ that are homogeneous of orders $q-k$ $(k = 0,1,\dots,M)$ and with remainder $\widetilde{B}_{q-M-1} \in \mathcal{S}^{\mathrm{cl}}_{q-M-1}(\mathcal{T}^*\mathcal{S}_0)$ (cf. (1.13) and (1.14)).

Let us investigate the asymptotics of the following potential-type function:

$$u(x) = \mathbf{V} \circ \mathbf{B}_q\varphi_0(x)\,, \quad \mathbf{supp}\,\varphi_0 \subset \overline{\mathcal{S}}_0\,, \quad x \in \Omega, \tag{2.8}$$

in a neighborhood of $\partial\mathcal{S}_0$ under the assumption that the known asymptotics of the density $\varphi_0 = (\varphi_{01},\dots,\varphi_{0N})$ is given in an s.l.c.s. by formula (1.45). To this end, we extend the s.l.c.s. $(x'', x_{n-1,+})$ to $(x', x_n) = (x'', x_{n-1}, x_n) \in \mathbb{R}^n$.

We introduce the notation

$$B^{\pm}_{a_{\mathrm{pr}}}(t) = \mathbf{diag}\left\{B^{m_1}_{\pm},\dots,B^{m_\ell}_{\pm}(t)\right\}, \quad \text{where} \quad B^m_{\pm}(t) = B^m\left(\pm\frac{1}{2\pi i}\partial_t\right)\left(\Gamma(t+1)e^{i\pi(t+1)/2}\right).$$

Let $\lambda_i = -N_i + \mu_i$ and $-1 < \operatorname{Re}\mu_i \leqslant 0$, where N_i is a positive integer and the coefficients $d^{m_i(M)}_{pl}(\lambda_i)$ are defined by the recurrence relations

$$\begin{aligned} d^{m_i(M)}_{pl}(\lambda_i) &= \sum_{q=l}^{p} d^{m_i(M-1)}_{pq}(\lambda_i)d^{m_i(1)}_{ql}(\lambda_i+M-1)\,,\\ d^{m_i(1)}_{ql}(\lambda_i+M-1) &= (-1)^{q-l}\frac{q!}{l!}\frac{1}{(\lambda_i+M)^{q-l+1}}\,, \qquad i=1,\dots,\ell, \quad M \in \mathbb{N}. \end{aligned} \tag{2.9}$$

Now we define the matrix

$$D^{m_i}(\lambda_i) = \|D^{m_i}_{jp}(\lambda_i)\|_{m_i\times m_i}\,,$$

$$D^{m_i}_{jp}(\lambda_i) = \begin{cases} i^{\lambda_i-\mu_i}\left(\dfrac{1}{2\pi i}\right)^{p-j}\dfrac{j!}{p!}d^{m_i(N_i)}_{pj}(\lambda_i) & \text{for } j \leqslant p,\\ 0 & \text{for } j > p. \end{cases}$$

Let us introduce the upper triangular matrix function

$$\widetilde{B}^0_{a_{\mathrm{pr}}}(t) = \mathbf{diag}\{\widetilde{B}^{m_1}(t), \dots, \widetilde{B}^{m_\ell}(t)\} \tag{2.10}$$

by the formulas (cf. (1.43))

$$\widetilde{B}^{m_i}(t) = \begin{cases} B^{m_i}(-t) & \text{for } \lambda_i \neq -1, -2, \dots, \\ \|h^{m_i}_{kp}(t)\|_{m_i \times m_i} & \text{for } \lambda_i = -1, -2, \dots, \end{cases}$$

$$\lambda_i := -\frac{\nu}{2} - \delta_i(x'') - 2r + q + j\,, \quad i = 1, \dots, \ell,$$

$$h^{m_i}_{kp}(t) := \begin{cases} \dfrac{1}{(p-k)!} \displaystyle\sum_{s=0}^{p-k+1} \left(\frac{1}{2\pi i}\right)^{p-k-s} \widetilde{c}^{\,m_i}_{p-k+1,s}(\lambda_i) t^s & \text{for } p \geqslant k\,, \\ 0 & \text{for } p < k\,, \end{cases}$$

$$\widetilde{c}^{\,m_i}_{p-k+1,s}(\lambda_i) := \begin{cases} i^{\lambda_i} \displaystyle\sum_{l=1}^{p-k+1} (-1)^l (l-1)! (2\pi i)^l d^{m_i(N_i)}_{p-k,l-1}(\lambda_i) b^{m_i}_{0l}(0) \\ \quad \text{for } s = 0\,, \\ i^{\lambda_i} \displaystyle\sum_{l=s}^{p-k+1} \frac{(-1)^l (l-1)!}{s!} (2\pi i)^{l-s} d^{m_i(N_i)}_{p-k,l-1}(\lambda_i) b^{m_i}_{sl}(0) \\ \quad \text{for } s \in \{1, 2, \dots, p-k+1\}, \ \ N_i = 0, 1, 2, \dots, \end{cases}$$

$$b^{m_i}_{kp}(t) := \begin{cases} \left(\dfrac{1}{2\pi i}\right)^{p-k} \dfrac{(-1)^{p+k}}{(p-k)!} \dfrac{d^{p-k}}{dt^{p-k}} \Big(\Gamma(t+1) e^{i\pi(t+1)/2}\Big) & \text{for } \ k \leqslant p\,, \\ 0 & \text{for } \ k > p\,, \end{cases}$$

$$p = 0, \dots, m_i - 1, \quad i = 1, \dots, \ell\,, \quad B^{m_i}_-(t) = \|b^{m_i}_{kp}(t)\|_{m_i \times m_i},$$

the coefficients $d^{m_i(0)}_{pl}$ are given by $d^{m_i(0)}_{pl}(-1) = \delta_{pl}$, $l = 0, \dots, p$ (δ_{pl} is the Kronecker delta), and the coefficients $d^{m_i(N_i)}_{pl}(\lambda_i)$, $N_i \in \mathbb{N}$, are defined by (2.9).

Theorem 2.2. *Let the conditions of Theorem* 1.7 *hold and let* $\varphi_0(x'', x_{n-1,+})$ *be as in* (1.45). *Suppose that* $q \in \mathbb{R}$ *and* $M \in \mathbb{N}_0$ *and*

$$M > \max\left\{\frac{n-1}{p} - s, \ 2r - 3 - [q], \ \frac{n-1}{p} - \min\{[s-q], 0\}, \ r - 1\right\}.$$

Then the potential-type function $u(x)$ *in* (2.8) *has the following asymptotic expansion*:

$$\begin{aligned} u(x'', x_{n-1}, x_n) = \sum_{s=1}^{\ell(N)} \Bigg\{ \sum_{j=0}^{n_s - 1} x_n^j \Bigg[& d_{sj}(x'', +1) z_{s,+1}^{\nu/2 + \Delta(x'') + 2r - 1 - q - j} \widetilde{B}^0_{a_{\mathrm{pr}}} \left(\frac{1}{2\pi i} \log z_{s,+1}\right) \\ & - d_{sj}(x'', -1) z_{s,-1}^{\nu/2 + \Delta(x'') + 2r - 1 - q - j} \widetilde{B}^0_{a_{\mathrm{pr}}} \left(\frac{1}{2\pi i} \log z_{s,-1}\right)\Bigg] c^{(j)}(x'') \\ & + \sum_{\theta = \pm 1} \sum_{\substack{k,l=0 \\ k+l+j+p \neq 2r-[q]-1}}^{M+3-2r+[q]} \sum_{j+p=2r-[q]-1}^{M+2-l} x_{n-1}^l x_n^j d_{sljp}(x'', \theta) z_{s,\theta}^{\nu/2 + \Delta(x'') + p + k - \{q\}} \\ & \times B_{skjp}\left(x'', \log z_{s,\theta}\right) \Bigg\} + u_{M+1}(x'', x_{n-1}, x_n) \quad \textit{for} \quad x_n > 0, \end{aligned} \tag{2.11}$$

with coefficients $d_{sj}(\cdot,\pm 1)$, $c^{(j)}$, *and* $d_{sljp}(\cdot,\pm 1)$ *belonging to* $C_0^\infty(\mathbb{R}^{n-2})$ *and with remainder* u_{M+1} *in* $C_0^{M+1}(\overline{\mathbb{R}^n_+})$, *where*

$$z_{s,+1}=-x_{n-1}-x_n\tau_{s,+1},\quad z_{s,-1}=x_{n-1}-x_n\tau_{s,-1},\quad \tau_{s,\pm 1}\in C_0^\infty(\mathbb{R}^{n-2}),$$

and $\{\tau_{s,\pm 1}\}_{s=1}^{\ell(N)}$ *is the set of different roots of the polynomial* $\det A(\mathcal{J}_{æ}^\top(x'',0)\cdot(0,\pm 1,\tau))$ *of multiplicities* n_s, $s=1,\dots,\ell(N)$, *that belong to the lower complex half-plane,* $[q]\in\mathbb{Z}$ *is the integral part of a number* q, *and* $\{q\}\in[0,1)$ *is the fractional part of the number* q, $q=[q]+\{q\}$.

The polynomial $B_{skjp}(x'',t)$ *is of order* $\nu_{kjp}=\nu_k+p+j-(2r-1-[q])$ *if*

$$\lambda_{m_0}\neq -1,-2,\dots\quad (\nu_k=k(2m_0-1)+m_0-1,\ m_0=\max\{m_1,\dots,m_\ell\})$$

with respect to the variable t *with vector coefficients depending on the variable* x'', *and* $B_{skjp}(x'',t)$ *is a polynomial of order* $\nu_{kjp}+1$ *for* $\lambda_{m_0}=-1,-2,\dots$ $(\lambda_{m_0}=-\nu/2-\delta_{m_0}(x'')-2r+q+j)$.

Note that the matrix $\widetilde{B}^0_{a_{\rm pr}}(t)$, for $\lambda_i=-\nu/2-\delta(x'')-2r+q+j=-1,-2,\dots$, depends on j (see 2.10).

Theorem 2.3. *For the leading (first) coefficients* $c_0(x'')$ *and* $d_{sj}(x'',\pm 1)$, $c^{(j)}(x'')$ *of the asymptotic expansions* (1.45) *and* (2.11), *respectively, we have the following relations*:

$$\begin{aligned}
d_{sj}(x'',-1)&=\frac{1}{2\pi}\mathcal{G}_{æ}(x'',0)V^{(s)}_{1-2r,j}(x'',0,0,-1)B^0_q(x'',0,0,-1)\mathcal{K}(x'')e^{i\pi\lambda_0},\\
d_{sj}(x'',+1)&=\frac{1}{2\pi}\mathcal{G}_{æ}(x'',0)V^{(s)}_{1-2r,j}(x'',0,0,+1)B^0_q(x'',0,0,+1)\mathcal{K}(x''),\\
\lambda_0&=-\frac{\nu}{2}-\Delta(x''),\quad s=1,\dots,\ell(N)\,,
\end{aligned}$$

where $\mathcal{G}_{æ}(x'',0)$ *is the square root of the Gram determinant* (*see* (1.29)), B^0_q *stands for the principal symbol of the pseudodifferential operator* $\mathbf{B}_q$, *and*

$$V^{(s)}_{1-2r,j}(x'',0,0,\pm 1)=-\frac{i^{j+1}}{j!(n_s-1-j)!}\frac{d^{n_s-1-j}}{d\tau^{n_s-1-j}}(\tau-\tau_{s,\pm 1})^{n_s}\left(A(\mathcal{J}_{æ}^\top(x'',0)\cdot(0,\pm 1,\tau))\right)^{-1}\Big|_{\tau=\tau_{s,\pm 1}}.$$

The coefficient $c^{(j)}(x'')$ *in* (2.10) *is given by* $c^{(j)}(x'')=a_j(x'')B^-_{a_{\rm pr}}\Big(\frac{\nu}{2}+\Delta(x'')\Big)\mathcal{K}^{-1}(x'')c_0(x'')$, *where*

$$\begin{aligned}
a_j(x'')&=\mathbf{diag}\{a^{m_1}(\lambda_1),\dots,a^{m_\ell}(\lambda_\ell)\}\,,\\
\lambda_i&=-\frac{\nu}{2}-\delta_i(x'')-2r+q+j,\qquad i=1,\dots,\ell\,,\\
a^{m_i}(\lambda_i)&=\begin{cases} B_+^{m_i}(\lambda_i) & \textit{for}\ \ \mathrm{Re}\,\lambda_i>-1,\\ B_+^{m_i}(\mu_i)\cdot D^{m_i}(\lambda_i) & \textit{for}\ \ \mathrm{Re}\,\lambda_i<-1,\quad \lambda_i\notin\mathbb{Z}\,,\\ I & \textit{for}\ \ \lambda_i=-1,-2,\dots\,. \end{cases}
\end{aligned}\tag{2.12}$$

We postpone the proofs of Theorems 2.2 and 2.3 to Subsection 2.3.

2.2. *Auxiliary Propositions*

To prove Theorems 2.2 and 2.3, we need some auxiliary propositions.

Lemma 2.4. *Let $\varphi_0 \in \widetilde{\mathbb{H}}_p^{(\infty,s),\infty}(\mathcal{S}_0)$ and $M > (n-1)/p - \min\{[s-q], 0\}$, where s is defined in Theorem 1.7 and $q \in \mathbb{R}$. Then the potential-type function (2.6) has the following expansion in the s.l.c.s.:*

$$u = \mathbf{V} \circ \mathbf{B}_q \varphi_0 = \sum_{k=2r-1}^{2M} \sum_{m=0}^{2M-k} \mathbf{A}_{-k}(x', x_n, D') \circ \mathbf{B}^0_{q-m}(x', D')\varphi_0 + \mathbf{R}_{2M+1}(x', x_n, D')\varphi_0 \,, \quad (2.13)$$

where $\mathbf{A}_{-k}(x', x_n, D')$ is a pseudodifferential operator, depending on the parameter x_n, that has a homogeneous symbol of the order $-k$ for $x_n = 0$, the operator $\mathbf{B}^0_{q-m}(x', D')$ is defined in (2.7), and $\mathbf{R}_{2M+1}(x', x_n, D')\varphi_0 \in C^{M+1}(\overline{\mathbb{R}^n_+})$.

Proof. For any j, we introduce a local coordinate system at each point $z_j \in \mathcal{S}_0$ so that the part of the surface $\mathcal{S}_0$ lying within some ball $\mathcal{B}(z_j, d)$ centered at z_j and having a radius d admits a representation $x_n^{(j)} = \gamma_j(x'_{(j)})$, $x'_{(j)} = (x_1^{(j)}, \dots, x_{n-1}^{(j)})$, where (cf. [KGBB1, Definition 1.6.9])

$$\gamma_j \in C^\infty(\Omega_j), \qquad \gamma_j(0) = \frac{\partial \gamma_j(0)}{\partial x_1^{(j)}} = \cdots = \frac{\partial \gamma_j(0)}{\partial x_{n-1}^{(j)}} = 0.$$

Let X_j be the projection of the set $Y_j := \mathcal{B}(z_j, d) \cap \mathcal{S}$ to the tangent plane to the surface $\mathcal{S}$ at z_j. Denote this tangent plane by $\mathbb{R}_j^{n-1}$.

Let $\{Y_j\}_{j=0}^l$ be a covering of $\overline{\Omega}$,

$$\overline{\Omega} \subset \bigcup_{j=0}^{l} Y_j \,, \quad \partial\Omega \cap Y_0 = \varnothing \,, \quad \partial\Omega \cap Y_j \neq 0 \quad \text{for} \quad j = 1, \dots, l,$$

and let $\{\psi_j\}_{j=1}^{l+1}$ be a partition of unity subordinated to the covering $\{Y_j\}_{j=1}^l$. For each j, we can find an infinitely smooth function φ_j that is equal to 1 in a neighborhood of $\mathbf{supp}\,\psi_j \cap \mathcal{S}$ and vanishes outside of a larger neighborhood contained in Y_j. Then the simple-layer potential can be represented as

$$\mathbf{V}g = \sum_{j=0}^{l} \psi_j \mathbf{V}g = \sum_{j=1}^{l} \psi_j \mathbf{V}\varphi_j g + \sum_{j=1}^{l} \psi_j \mathbf{V}(1-\varphi_j)g + \psi_0 \mathbf{V}g \,.$$

Obviously, $\psi_j \mathbf{V}(1-\varphi_j)g, \psi_0 \mathbf{V}g \in C^\infty(\mathcal{S})$, and it suffices to study the asymptotics of the potential

$$\psi_j \mathbf{V}\varphi_j g(x) = \int_{Y_j} \psi_j(x) H_A(x-y)(\varphi_j g)(y) d_y\mathcal{S} \,, \qquad x \in \Omega,$$

for $j = 1, \dots, l$.

Let $\mathcal{B}^+(z_j, d)$ be a half-ball (in the (j)th local coordinate system) of radius d centered at the origin, i.e., $\mathcal{B}^+(z_j, d) := \{x^{(j)} = (x'_{(j)}, x_n^{(j)}) : x'_{(j)} \in \mathbb{R}_j^{n-1},\ x_n^{(j)} > 0,\ |x^{(j)}| < d\}$. We define the mapping

$$æ_j \colon \mathcal{B}^+(z_j, d) \to \mathcal{B}(z_j, d) \cap \Omega \qquad (2.14)$$

by the formulas $æ_j(x_{(j)}) = (x^{(j)}, \gamma_j(x'_{(j)}) - x_n^{(j)})$ for $x_n^{(j)} > 0$. For the Jacobi matrix $\mathcal{J}_{æ_j}$ of the mapping $æ_j$ we obtain

$$\mathcal{J}_{æ_j}(x'_{(j)}) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \dfrac{\partial \gamma_j(x'_{(j)})}{\partial x_1^{(j)}} & \dfrac{\partial \gamma_j(x'_{(j)})}{\partial x_2^{(j)}} & \cdots & \dfrac{\partial \gamma_j(x'_{(j)})}{\partial x_{n-1}^{(j)}} & -1 \end{pmatrix}_{n \times n} .$$

Applying the Taylor formula in vector form, we see that

$$æ_j(x^{(j)}) - æ_j(y^{(j)}) = \mathcal{J}_{æ_j}(x'_{(j)})(x^{(j)} - y^{(j)}) + P_{(j)}(x'_{(j)}, y'_{(j)}), \tag{2.15}$$

where

$$P_{(j)}(x'_{(j)}, y'_{(j)}) = \Big(0, \dots, 0, P_{jn}(x'_{(j)}, y'_{(j)})\Big)\,,$$
$$P_{jn}(x'_{(j)}, y'_{(j)}) = \mathrm{O}\,(|x'_{(j)} - y'_{(j)}|^2) \quad \text{and} \quad P_{jn} \in C^\infty(X_j \times X_j)\,,$$

$x^{(j)} = (x'_{(j)}, x_n^{(j)})$, $y^{(j)} = (y'_{(j)}, 0)$ and $x'_{(j)} = (x_1^{(j)}, \dots, x_{n-1}^{(j)})$, $y'_{(j)} = (y_1^{(j)}, \dots, y_{n-1}^{(j)})$.

Denote by H_j and g_j the matrix function $\psi_j H_A$ and the vector function $\varphi_j g$, respectively, written in the local coordinates, and let $v_j = g_j(æ_j)$.

We can readily see that the square root of the Gram determinant of the mapping (2.12) (cf. (1.29)) becomes

$$\mathcal{G}_{æ_j}(y'_{(j)}) = \sqrt{\left(\frac{\partial\gamma_j(y'_{(j)})}{\partial y_1^{(j)}}\right)^2 + \dots + \left(\frac{\partial\gamma_j(y'_{(j)})}{\partial y_{n-1}^{(j)}}\right)^2 + 1} \tag{2.16}$$

(see [KGBB1, Proposition 4.7.3] and [Sl1, §3.6]), and we can represent the simple-layer potential $\psi\mathbf{V}\varphi I$ in a new local coordinate system. Let us substitute $æ_j$ into this system,

$$(\psi_j\mathbf{V}\varphi_j g)(x^{(j)}) = \int_{X_j} H_j\big(æ_j(x^{(j)}) - æ_j(y^{(j)})\big)\mathcal{G}_{æ_j}(y^{(j)})v_j(y^{(j)})dy^{(j)}\,.$$

The Taylor formula and relation (2.15) yield

$$H_j\big(æ_j(x^{(j)}) - æ_j(y^{(j)})\big) = \sum_{|\mu|\leqslant k}\frac{1}{\mu!}(\partial_x^\mu H_j)(\mathcal{J}_{æ_j}(x'_{(j)})(x^{(j)} - y^{(j)}))\,\Big(P_{(j)}(x'_{(j)}, y'_{(j)})\Big)^\mu$$
$$+\sum_{|\mu|=k+1}\left[\frac{k+1}{\mu!}\int_0^1(1-t)^k(\partial_x^\mu H_j)\Big(\mathcal{J}_{æ_j}(x'_{(j)})(x^{(j)}-y^{(j)})+tP_{(j)}(x'_{(j)}, y'_{(j)})\Big)\,dt\right](P_{(j)}(x'_{(j)}, y'_{(j)}))^\mu. \tag{2.17}$$

For convenience, in what follows, we omit the index j denoting the local coordinate system.

Taking into account (2.15) and the Taylor expansions

$$\mathcal{G}_æ(y') = \mathcal{G}_æ(x') + \sum_{1\leqslant|\alpha|\leqslant 2M}\frac{1}{\alpha!}\partial^\alpha\mathcal{G}_æ(x')(y'-x')^\alpha + R^{(1)}_{2M+1}(x', y'),$$
$$P_n(x', y') = \sum_{2\leqslant|\alpha|\leqslant 2M}\frac{1}{\alpha!}\partial^\alpha\gamma(x')(x'-y')^\alpha + R^{(2)}_{2M+1}(x', y'), \tag{2.18}$$

where $R^{(k)}_{2M+1}(x', y') = \mathrm{O}\,(|x'-y'|^{M+1})$, $\alpha = (\alpha_1, \dots, \alpha_{n-1})$, $x = (x', x_n)$, and $y = (y', 0)$, $k = 1, 2$, we obtain the following representation of the simple-layer potential:

$$\psi\mathbf{V}\varphi g = \sum_{k=2r-1}^{2M}\mathbf{A}_{-k}(x', x_n, D')v + \mathbf{R}_{2M+1}v, \qquad v \in \widetilde{\mathbb{H}}_p^{(\infty, s-q), \infty}(X)\,, \tag{2.19}$$

where

$$\mathbf{A}_{1-2r}(x',x_n,D')v = \int\limits_{\mathbb{R}^{n-1}} \mathcal{G}_{\ae}(x')H(\mathcal{J}_{\ae}(x')(x'-y',x_n))v(y')dy'\,,$$

$$\mathbf{A}_{-k}(x',x_n,D') = \sum_{|\beta|-|\alpha|-1=-k} a_{\alpha\beta}(x')\mathbf{A}_{\alpha\beta}(x',x_n,D') \quad \text{for} \quad k=2r,2r+1,\dots,2M\,,$$

$$\mathbf{A}_{\alpha\beta}(x',x_n,D')v = \int\limits_{\mathbb{R}^{n-1}} (x'-y')^{\alpha}(\partial_x^{\beta}H)(\mathcal{J}_{\ae}(x')(x'-y',x_n))v(y')dy'\,,$$

and $a_{\alpha\beta} \in C_0^{\infty}(\mathbb{R}^{n-1})$ are defined by the Taylor coefficients in (2.18) ($\alpha = (\alpha_1,\dots,\alpha_{n-1})$ and $\beta = (\beta_1,\dots,\beta_n)$).

Since $M > (n-1)/p - \min\{[s-q],0\}$ in (2.20), it follows that $\mathbf{R}_{2M+1}v \in C_0^{M+1}(\overline{\mathbb{R}^n_+})$. Indeed, let $s-q \geqslant 0$. Then $v \in \widetilde{\mathbb{H}}_p^{s-q}(X) \subset L_p(X)$, where $X = X_j \subset \mathbb{R}^{n-1}$; therefore, the kernel $r_{2M+1}(x,y')$ of the integral operator R_{2M+1} admits the estimate $|\partial_x^{\alpha} r_{2M+1}(x,y')| \leqslant C_{\alpha}|x-y'|^{2M+2-n-|\alpha|}$, $|\alpha| \leqslant M+1$. Applying the Hölder inequality, we obtain $|\partial_x^{\alpha} R_{2M+1}v(x)| \leqslant C'_{\alpha} < \infty$, $|\alpha| \leqslant M+1$. For $s-q<0$ we have

$$v = \sum_{|\alpha|\leqslant m} \partial^{\alpha}u_{\alpha}, \quad u_{\alpha} \in L_p(X), \quad m = -[s-q]$$

(see [Es1]), and recalling the definition of the derivative of a distribution, we see that

$$|\partial_x^{\alpha} R_{2M+1}v(x)| = \sum_{|\alpha|\leqslant m} \int\limits_X \partial_x^{\alpha} r_{2M+1}(x,y')u_{\alpha}(y')dy'\,.$$

The desired inclusion follows as in the previous case.

Since

$$\begin{aligned}(x'-y')^{\alpha}&(\partial_x^{\beta}H)(\mathcal{J}_{\ae}(x')(x'-y',x_n)) \\ &= \mathcal{F}^{-1}_{\xi'\to x'-y'}\Big(-\frac{1}{2\pi}\int\limits_{\mathcal{L}_-}(-i\partial_{\xi'})^{\alpha}\left\{(-i\xi')^{\beta'}\left[A(\mathcal{J}_{\ae}^{\top}(x')(\xi',\tau))\right]^{-1}\right\}(-i\tau)^{\beta_n}e^{-i\tau x_n}d\tau\Big),\end{aligned}$$

where $\beta' = (\beta_1,\dots,\beta_{n-1})$, $\mathcal{A}^{\top}$ stands for the transposed matrix to $\mathcal{A}$, and the operator of the form $\mathbf{A}_{\alpha\beta}(x',x_n,D')$ can be represented as a parameter-dependent $x_n > 0$ pseudodifferential operator (or a potential operator)

$$\mathbf{A}_{\alpha\beta}(x',x_n,D')v = \mathcal{F}^{-1}_{\xi'\to x'}\left[A_{\alpha\beta}(x',x_n,\xi')\mathcal{F}_{y'\to\xi'}[v(y')]\right]$$

(cf (1.11)) with the symbol

$$A_{\alpha\beta}(x',x_n,\xi') = -\frac{1}{2\pi}\int\limits_{\mathcal{L}_-}(-i\partial_{\xi'})^{\alpha}\left\{(-i\xi')^{\beta'}\left[A(\mathcal{J}_{\ae}^{\top}(x')(\xi',\tau))\right]^{-1}\right\}(-i\tau)^{\beta_n}e^{-i\tau x_n}d\tau$$

depending on the parameter $x_n > 0$.

Relations (2.7) and (2.19) prove the desired expansion (2.13). □

We introduce the following classes of symbols.

Definition 2.5. For $k \in \mathbb{R}$ by $\mathcal{R}^{(1)}_{-k,-\infty}$, denote by $R(x', x_n, \xi')$ the class of (matrix) functions that vanish for sufficiently large $|(x', x_n)|$ ($|(x', x_n)| > C_0$) and admit the following estimates:

$$\left|\partial_x^\alpha \partial_{\xi'}^\beta R(x', x_n, \xi')\right| \leqslant C_{\alpha,\beta} \langle \xi' \rangle^{-k+\alpha_n-|\beta|} e^{-\gamma_0 x_n |\xi'|}$$

for some $\gamma_0 > 0$, for any multi-indices $\alpha \in \mathbb{N}_0^n$ and $\beta \in \mathbb{N}_0^{n-1}$, and for all $x_n > 0$ and $x', \xi' \in \mathbb{R}^{n-1}$.

Definition 2.6. For $r > 0$ and $m > 0, k \in \mathbb{R}$, denote by $\mathcal{R}^{(2)}_{r,-m,-k,-\infty}$ the class of (matrix) distributions $R(x', x_n, \xi')$ of class $S'(\mathbb{R}^{n-1})$ with respect to the variable ξ' that vanish for sufficiently large $|(x', x_n)|$ ($|(x', x_n)| > C_0$) and admit the following estimates:

$$|\partial_x^\alpha R(x', x_n, \xi')| \leqslant C_\alpha \frac{|\xi''|^r |\xi'|^{\alpha_n - k}}{|\xi_{n-1}|^m} e^{-\gamma_0 x_n |\xi'|}$$

for some $\gamma_0 > 0$, for any multi-indices $\alpha \in \mathbb{N}_0^n$, and for all $x_n > 0$, $x' \in \mathbb{R}^{n-1}$, and $\xi_{n-1} \in \mathbb{R} \setminus \{0\}$.

Let us prove the following assertion.

Lemma 2.7. *Let $\varphi_0(x'', x_{n-1,+})$ be as in* (1.45) (*or as in* (1.47)), *let $\varphi_0 \in \widetilde{\mathbb{H}}_p^{(\infty,s),m}(\mathcal{S}_0)$ and $M > (n-1)/p - s$, and let the conditions of Theorem* 1.7 *hold.*

Then $\mathbf{R}(x', x_n, D')\varphi_0 \in C^{M+1}(\overline{\mathbb{R}^n_+})$ for any pseudodifferential operator $\mathbf{R}(x', x_n, D')$ whose symbol $R(x', x_n, \xi')$ belongs to the class $\mathcal{R}^{(1)}_{-M-2,-\infty}$.

Proof. We have

$$\partial_x^\beta \mathbf{R}(x', x_n, D')\varphi_0 = \int\limits_{\mathbb{R}^{n-1}} R^{(\beta)}(x', x_n, \xi') e^{-i(x',\xi')} \overset{\wedge}{\varphi}_0(\xi')\, d\xi', \tag{2.20}$$

$$R^{(\beta)}(x', x_n, \xi') := \sum_{\alpha' \leqslant \beta'} c_{\alpha',\beta'} \partial_{x'}^{\alpha'} \partial_{x_n}^{\beta_n} R(x', x_n, \xi')(-i\xi')^{\beta'-\alpha'}$$

(where $|\beta| \leqslant M+1$, $\beta = (\beta', \beta_n)$, $\beta' = (\beta_1, \ldots, \beta_{n-1})$, and $\alpha' = (\alpha_1, \ldots, \alpha_{n-1})$).

The symbol $R^{(\beta)}(x', x_n, \xi')$ is infinitely differentiable with respect to the variable ξ', and

$$\left|R^{(\beta)}(x', x_n, \xi')\right| \leqslant C(1+|\xi'|)^{-1} e^{-\gamma_0 x_n |\xi'|} \qquad \text{for all} \quad x_n > 0 \quad \text{and some} \quad \gamma_0 > 0\,.$$

Now, taking into account the asymptotic expansion (1.45) for the function φ_0,

$$\varphi_0 = \sum_{k=0}^{M} \varphi_{0k} + \varphi_{M+1}\,,$$

$$\varphi_{00}(x'', x_{n-1,+}) = \mathcal{K}(x'')\; x_{n-1,+}^{\nu/2+\Delta(x'')} B^0_{a_{\rm pr}} \left(-\frac{1}{2\pi i} \log x_{n-1,+}\right) \mathcal{K}^{-1}(x'') c_0(x'')$$

$$\varphi_{0k}(x'', x_{n-1,+}) = \mathcal{K}(x'') x_{n-1,+}^{\nu/2+\Delta(x'')+k} B_k(x'', \log x_{n-1,+}), \qquad k = 1, \ldots, M\,,$$

$$\mathrm{Re}\left(\frac{\nu}{2} + \delta_j(x'')\right) > -1 \quad \text{for all} \quad j = 1, \ldots, \ell$$

and using the Fourier transform formulas

$$t_+^\lambda \log^p t_+ = \sum_{k=0}^{p} b^{(1)}_{pk}(\lambda) \mathcal{F}^{-1}((\sigma+i0)^{-1-\lambda} \log^k(\sigma+i0))\,, \tag{2.21}$$

$$\mathrm{Re}\,\lambda > -1, \qquad b^{(1)}_{pk}(\lambda) := \frac{(-1)^k p!}{k!(p-k)!} \frac{d^{p-k}}{d\lambda^{p-k}} \left(\Gamma(\lambda+1) e^{i\pi(\lambda+1)/2}\right)$$

(see [Es1, Example 2.3]), we readily obtain the estimates

$$\Big|(1-\chi_1(\xi_{n-1}))R^{(\beta)}(x',x_n,\xi')\widehat{\varphi}_{0k}(\xi')\Big| \leqslant C\,\frac{(1+|\xi''|)^{-N}}{(1+|\xi_{n-1}|)^{1+k+\varepsilon}}e^{-\gamma_0 x_n|\xi'|}$$
$$\text{for some } \gamma_0>0, \quad \text{for all } N>0, \ |\beta|\leqslant M+1, \quad x_n>0, \quad \varepsilon>0, \quad k=0,\dots,M,$$

where $\chi_1 \in C_0^\infty(\mathbb{R})$ is a cut-off function with $\chi_1(\xi_{n-1})=0$ for $|\xi_{n-1}|\geqslant 1$ and $\chi_1(\xi_{n-1})=1$ for $|\xi_{n-1}|\leqslant 1/2$, and

$$\mathcal{F}^{-1}_{\xi'\to x'}\big[\chi_1(\xi_{n-1})R^{(\beta)}(x',x_n,\xi')\widehat{\varphi}_{0k}(\xi')\big]$$
$$=\sum_{p=0}^{\nu_k}\mathcal{F}^{-1}_{\xi_{n-1}\to x_{n-1}}[\chi_1(\xi_{n-1})(\xi_{n-1}+i0)^{-\nu/2-\Delta(x'')-1-k}\log^p(\xi_{n-1}+i0)\Phi_{pk}(x',x_n,\xi_{n-1})];$$

(with $\nu_k=k(2m_0-1)+m_0-1$ and $m_0=\max\{m_1,\dots,m_\ell\}$). Here

$$\Phi_{pk}(x',x_n,\xi_{n-1})=\mathcal{F}^{-1}_{\xi''\to x''}[R^{(\beta)}(x',x_n,\xi')c_{pk}(\xi'')], \qquad |c_{pk}(\xi'')|\leqslant C(1+|\xi''|)^{-N} \quad \forall N>0$$

and $\Phi_{pk}(x',x_n,\xi_{n-1})$ is a C^∞-function with respect to the variables $x'\in\mathbb{R}^{n-1}$ and $x_n\in\overline{\mathbb{R}}^n_+$ for all $\xi_{n-1}\in\mathbb{R}$.

Therefore, we can readily see that the integrals on the right-hand side of (2.20) exist and that $\mathbf{R}(x',x_n,D')\varphi_{0k}\in C^{M+1}(\overline{\mathbb{R}^n_+})$ $(k=0,\dots,M)$.

It remains to show that $\mathbf{R}^{(\beta)}(x',x_n,D')\varphi_{M+1}\in C^{M+1}(\overline{\mathbb{R}^n_+})$.

Indeed, if $x_n>0$, then $\mathbf{R}^{(\beta)}(x',x_n,D')\varphi_{M+1}\in C^{\infty}(\overline{\mathbb{R}^n_+})$.

Here as usual, $C_0^\infty(\mathbb{R}^{n-1})$ stands for the class of all compactly supported infinitely differentiable functions, and $\chi_0(\xi')\in C_0^\infty(\mathbb{R}^{n-1})$ is a cut-off function with $\chi_0(\xi')=1$ for $|\xi'|\leqslant 1$.

If $x_n=0$ for $M>(n-1)/p-s$, then we obtain the representation

$$\begin{aligned}
\mathbf{R}^{(\beta)}(x',0,D')\varphi_{M+1}(x) &= \frac{1}{(2\pi)^{n-1}}\int\limits_{\mathbb{R}^{n-1}} R^{(\beta)}(x',0,\xi')\sum_{j=1}^{n-1}\frac{\xi_j^2}{|\xi'|^2}e^{i(x',\xi')}\widehat{\varphi}_{M+1}(\xi')d\xi' \\
&= \frac{1}{(2\pi)^{n-1}}\int\limits_{\mathbb{R}^{n-1}}\sum_{j=1}^{n-1}\frac{\xi_j}{|\xi'|^2}R^{(\beta)}(x',0,\xi')e^{-i(x',\xi')}(\widehat{D_j\varphi_{M+1}})(\xi')d\xi' \\
&= \sum_{j=1}^{n-1}\frac{1}{(2\pi)^{n-1}}\int\limits_{\mathbb{R}^{n-1}} B^{(1)}_{\beta j}(x',\xi')e^{-i(x',\xi')}(\widehat{D_j\varphi_{M+1}})(\xi')d\xi' \\
&\quad + \sum_{j=1}^{n-1}\frac{1}{(2\pi)^{n-1}}\int\limits_{\mathbb{R}^{n-1}} B^{(2)}_{\beta j}(x',\xi')e^{-i(x',\xi')}(\widehat{D_j\varphi_{M+1}})(\xi')d\xi',
\end{aligned} \tag{2.22}$$

where $B^{(1)}_{\beta j}(x',\xi')=(1-\chi_0(\xi'))\frac{\xi_j}{|\xi'|^2}R^{(\beta)}(x',0,\xi')$ and $B^{(2)}_{\beta j}(x',\xi')=\chi_0(\xi')\frac{\xi_j}{|\xi'|^2}R^{(\beta)}(x',0,\xi')$, $j=1,\dots,n-1$. The second sum in (2.21) is a C^∞-smoothing operator.

Since $B^{(1)}_{\beta j}\in\mathcal{S}^\infty_{-2}$, it follows that $|D^\mu_{\xi'}B^{(1)}_{\beta j}(x',\xi')|\leqslant C(1+|\xi'|)^{-2-|\mu|}$ for all μ and

$$\mathcal{F}^{-1}_{\xi'\to z'}[D^\mu_{\xi'}B^{(1)}_{\beta j}(x',\xi')]=(z')^\mu\mathcal{F}^{-1}_{\xi'\to z'}[B^{(1)}_{\beta j}(x',\xi')]=(z')^\mu K^{(1)}_{\beta j}(x',z').$$

Therefore, $|K^{(1)}_{\beta j}(x',z')|\leqslant C|z'|^{-|\mu|}$ for all $|\mu|>n-3$, and it becomes obvious that the kernel $K^{(1)}_{\beta j}(x',z')$ has a weak singularity at the point $z'=0$. Therefore, setting $|\mu|=n-2$, we see that

$$\frac{1}{(2\pi)^{n-1}}\int\limits_{\mathbb{R}^{n-1}} B^{(1)}_{\beta j}(x',\xi')e^{-i(x',\xi')}(\widehat{D_j\varphi_{M+1}})(\xi')d\xi'=\int\limits_X K^{(1)}_{\beta j}(x',x'-y')(D_j\varphi_{M+1})(y')dy',$$

where

$$|K^{(1)}_{\beta j}(x', x'-y')| \leqslant C|x'-y'|^{-(n-2)}, \qquad j=1,\dots,n-1.$$

The resulting representation shows that the first integral in (2.22) exists and defines a continuous mapping. Therefore, $\mathbf{R}^{(\beta)}(x',x_n,D')\varphi_{M+1}\in C(\overline{\mathbb{R}^n_+})$ and $\mathbf{R}(x',x_n,D')\varphi_0\in C^{M+1}(\overline{\mathbb{R}^n_+})$. □

Lemma 2.8. *Suppose $\varphi_0(x'',x_{n-1,+})$ is as in* (1.45) (*or in* (1.47)), *$\varphi_0\in\widetilde{\mathbb{H}}^{(\infty,s),m}_p(\mathcal{S}_0)$, $M>(n-1)/p-s$, and let conditions of Theorem* 1.7 *hold.*

Then $x^{M+2-[k]}_{n-1}\,\mathbf{R}\,(x',x_n,D')\,\varphi_0\in C^{M+1}(\overline{\mathbb{R}^n_+})$ for a pseudodifferential operator $\mathbf{R}(x',x_n,D')$ whose symbol $R(x',x_n,\xi')$ belongs to the class $\mathcal{R}^{(1)}_{-k,-\infty}$, where $[k]<M+2$.

Proof. We have

$$\begin{aligned} x^{M+2-[k]}_{n-1}\mathbf{R}(x',x_n,D')\varphi_0 &= \frac{x^{M+2-[k]}_{n-1}}{(2\pi)^{n-1}}\int\limits_{\mathbb{R}^{n-1}} e^{-i(x',\xi')}R(x',x_n,\xi')\overset{\wedge}{\varphi}_0(\xi')\,d\xi' \\ &= \frac{1}{(2\pi)^{n-1}}\int\limits_{\mathbb{R}^{n-1}} e^{-i(x',\xi')}(-i)^{M+2-[k]}\partial^{M+2-[k]}_{\xi_{n-1}}\Big[R(x',x_n,\xi')\overset{\wedge}{\varphi}_0(\xi')\Big]\,d\xi' \\ &= \sum_{p_1+p_2=M+2-[k]} c_{p_1,p_2}\int\limits_{\mathbb{R}^{n-1}} e^{-i(x',\xi')}\partial^{p_1}_{\xi_{n-1}}R(x',x_n,\xi')\partial^{p_2}_{\xi_{n-1}}\overset{\wedge}{\varphi}_0(\xi')d\xi', \end{aligned}$$

where the symbols $\partial^{p_1}_{\xi_{n-1}}R(x',x_n,\xi')$ belong to the class $\mathcal{R}^{(1)}_{-k-p_1,-\infty}$.

Now, taking into account the asymptotic expansion (1.45) of the function φ_0 and the Fourier transform formulas (2.20), we complete the proof as in Lemma 2.7. □

Lemma 2.9. *Let $\varphi_0(x'',x_{n-1,+})$ be as in* (1.45), *let $\varphi_0\in\widetilde{\mathbb{H}}^{(\infty,s),m}_p(\mathcal{S}_0)$, let $M>(n-1)/p-s$, and let the conditions of Theorem* 1.7 *hold.*

Then $\mathbf{R}(x',x_n,D')\varphi_0\in C^{M+1}(\overline{\mathbb{R}^n_+})$ for a pseudodifferential operator $\mathbf{R}(x',x_n,D')$ whose symbol $R(x',x_n,\xi')$ belongs to the class $\mathcal{R}^{(2)}_{r,-M-2+[k],-k,-\infty}$, where $r\geqslant 0$ and $[k]\leqslant M+2$.

Proof. Consider two cut-off functions $\chi_0(\xi')\in C^\infty_0(\mathbb{R}^{n-1})$ as in Lemma 2.7. Let $\chi_1(\xi_{n-1})\in C^\infty_0(\mathbb{R})$, and $\mathbf{R}(x',x_n,D')\varphi_0=\mathbf{R}^{(1)}(x',x_n,D')\varphi_0+\mathbf{R}^{(2)}(x',x_n,D')\varphi_0+\mathbf{R}^{(3)}(x',x_n,D')\varphi_0$, where

$$\mathbf{R}^{(1)}(x',x_n,D')\varphi_0=\frac{1}{(2\pi)^{n-1}}\int\limits_{\mathbb{R}^{n-1}}(1-\chi_1(\xi_{n-1}))(1-\chi_0(\xi'))R(x',x_n,\xi')e^{-i(x',\xi')}\overset{\wedge}{\varphi}_0(\xi')\,d\xi',$$

$$\mathbf{R}^{(2)}(x',x_n,D')\varphi_0=\frac{1}{(2\pi)^{n-1}}\int\limits_{\mathbb{R}^{n-1}}(1-\chi_1(\xi_{n-1}))\chi_0(\xi')R(x',x_n,\xi')e^{-i(x',\xi')}\overset{\wedge}{\varphi}_0(\xi')\,d\xi',$$

$$\mathbf{R}^{(3)}(x',x_n,D')\varphi_0=\mathcal{F}^{-1}_{\xi'\to x'}[\chi_1(\xi_{n-1})R(x',x_n,\xi')\overset{\wedge}{\varphi}_0(\xi')].$$

The kernel of the operator $\mathbf{R}^{(2)}(x',x_n,D')$ is infinitely differentiable, and we obviously have the relation $\mathbf{R}^{(2)}(x',x_n,D')\varphi_0\in C^\infty(\overline{\mathbb{R}^n_+})$.

As in the proof of Lemma 2.7, we obtain $\mathbf{R}^{(1)}(x',x_n,D')\varphi_0\in C^{M+1}(\overline{\mathbb{R}^n_+})$. Moreover,

$$\mathbf{R}^{(3)}(x',x_n,D')\varphi_0=\mathcal{F}^{-1}_{\xi'\to x'}(\chi_0(\xi_{n-1})R(x',x_n,\xi')\overset{\wedge}{\varphi}_0(\xi'))=\mathcal{F}^{-1}_{\xi_{n-1}\to x_{n-1}}(\chi_0(\xi_{n-1})\Phi(x',x_n,\xi_{n-1})),$$

where $\Phi(x',x_n,\xi_{n-1}):=\mathcal{F}^{-1}_{\xi''\to x''}\Big(R(x',x_n,\xi')\overset{\wedge}{\varphi}_0(\xi')\Big)$, and $\Phi(x',x_n,\xi_{n-1})$ is a C^∞-function with respect to the variables $x'\in\mathbb{R}^{n-1}$ and $x_n\in\overline{\mathbb{R}_+}$ for all $\xi_{n-1}\in\mathbb{R}\setminus\{0\}$, and this proves that $\mathbf{R}^{(3)}(x',x_n,D')\varphi_0\in C^\infty(\overline{\mathbb{R}^n_+})$ and $\mathbf{R}(x',x_n,D')\varphi_0\in C^{M+1}(\overline{\mathbb{R}^n_+})$. □

2.3. *Proof of Theorems* 2.2 *and* 2.3

Consider the composition

$$\mathbf{A}_{-k}(x',x_n,D')\circ\mathbf{B}^0_{q-m}(x',D')\varphi_0=\mathcal{F}^{-1}_{\xi'\to x'}\left(C_{km}(x',x_n,\xi')\overset{\wedge}{\varphi}_0(\xi')\right)+\mathbf{R}^{(1)}_{km,-\infty}(x',x_n,D')\varphi_0,$$

where $\mathbf{R}^{(1)}_{km,-\infty}(x',x_n,D')\varphi_0\in C^\infty(\overline{\mathbb{R}^n_+})$,

$$C_{km}(x',x_n,\xi')=\sum_{|\mu|=0}^{M+2+[q]-(k+m)}\frac{1}{\mu!}\partial^\mu_{\xi'}A_{-k}(x',x_n,\xi')\partial^\mu_{x'}B^0_{q-m}(x',\xi')+R_{km,M+3-\{q\}}(x',x_n,\xi'),$$

and the symbol $R_{km,M+3-\{q\}}(x',x_n,\xi')$ belongs to the class $\mathcal{R}^{(1)}_{-(M+3-\{q\}),-\infty}$ (for a similar expansion, see [Es1, Theorem 18.3] and [Sb1, Theorem 3.4]). Therefore,

$$\begin{aligned}\mathbf{A}_{-k}(x',x_n,D')&\circ\mathbf{B}^0_{q-m}(x',D')\varphi_0\\&=\sum_{|\mu|=0}^{M+2+[q]-(k+m)}\mathcal{F}^{-1}_{\xi'\to x'}\left(\frac{1}{\mu!}\partial^\mu_{\xi'}A_{-k}(x',x_n,\xi')\partial^\mu_{x'}B^0_{q-m}(x',\xi')\overset{\wedge}{\varphi}_0(\xi')\right)\\&\quad+\mathbf{R}_{km,M+3-\{q\}}(x',x_n,D')\varphi_0+\mathbf{R}^{(1)}_{km,-\infty}(x',x_n,D')\varphi_0.\end{aligned}$$

By Lemma 2.7, we obtain $\mathbf{R}_{km,M+3-\{q\}}(x',x_n,D')\varphi_0\in C^{M+1}(\overline{\mathbb{R}^n_+})$. Consider the function

$$C_{km\mu}(x',x_n,\xi')=\frac{1}{\mu!}\partial^\mu_{\xi'}A_{-k}(x',x_n,\xi')\partial^\mu_{x'}B^0_{q-m}(x',\xi')=\int_{\mathcal{L}_-}A_{km\mu}(x',\xi',\tau)e^{-i\tau x_n}d\tau\,,\tag{2.23}$$

$$\begin{aligned}A_{2r-1,m\mu}(x',\xi',\tau)&:=-\frac{1}{2\pi\mu!}\mathcal{G}_{æ}(x')\partial^\mu_{\xi'}[A(\mathcal{J}^\top_{æ}(x')(\xi',\tau))]^{-1}\partial^\mu_{x'}B^0_{q-m}(x',\xi'),\\A_{km\mu}(x',\xi',\tau)&:=-\frac{1}{2\pi}\sum_{|\alpha|-|\beta|+1=k}\frac{1}{\mu!}a_{\alpha\beta}(x')(-i\tau)^{\beta_n}\\&\qquad\times\partial^{\mu+\alpha}_{\xi'}\{(-i\xi')^{\beta'}[A(\mathcal{J}^\top_{æ}(x')(\xi',\tau))]^{-1}\}\partial^\mu_{x'}B^0_{q-m}(x',\xi')\end{aligned}$$

for $k=2r,2r+1,\dots,2M$, $\beta\in\mathbb{N}^n_0$, and $\mu,\alpha\in\mathbb{N}^{n-1}$ and the symbol $C_{km\mu}(x',\xi',\tau)$ is homogeneous of order $q-(m+k+|\mu|)=-\kappa$ with respect to the variable ξ'. Moreover,

$$[A(\mathcal{J}^\top_{æ}(x')(\xi',\tau))]^{-1}=\left\|\frac{\Delta_{ij}(\mathcal{J}^\top_{æ}(x')(\xi',\tau))}{\Delta(\mathcal{J}^\top_{æ}(x')(\xi',\tau))}\right\|_{N\times N},$$

where $\Delta(\mathcal{J}^\top_{æ}(x')(\xi',\tau))=\det\|A(\mathcal{J}^\top_{æ}(x')(\xi',\tau))\|_{N\times N}$, and $\Delta_{ij}(\mathcal{J}^\top_{æ}(x')(\xi',\tau))$ is the cofactor of the corresponding element.

Applying the Taylor formula for x_{n-1} at the point $(x'',0,x_n,\xi')$, we see that

$$\begin{aligned}C_{km\mu}(x',x_n,\xi')&=\sum_{l=0}^{M+1-[\kappa]}\frac{1}{l!}(\partial^l_{x_{n-1}}C_{km\mu})(x'',0,x_n,\xi')\cdot x^l_{n-1}\\&\quad+x^{M+2-[\kappa]}_{n-1}\left[R^{(1)}_{km\mu,\kappa}(x',x_n,\xi')+R^{(2)}_{km\mu,\kappa}(x',x_n,\xi')\right],\\R^{(1)}_{km\mu,\kappa}(x',x_n,\xi')&:=(1-\chi_0(\xi'))R_{km\mu,\kappa}(x',x_n,\xi'),\\R^{(2)}_{km\mu,\kappa}(x',x_n,\xi')&:=\chi_0(\xi')R_{km\mu,\kappa}(x',x_n,\xi')\,.\end{aligned}$$

Obviously, $x_{n-1}^{M+2-[\kappa]}\mathbf{R}^{(2)}_{km\mu,\kappa}(x',x_n,D')\varphi_0 \in C^\infty(\overline{\mathbb{R}^n_+})$.

We have $R^{(1)}_{km\mu,\kappa} \in \mathcal{R}^{(1)}_{-\kappa,-\infty}$ for $\kappa = 2r-1-q,\dots,M+2-\{q\}$, and by Lemma 2.8 we obtain $x_{n-1}^{M+2-[\kappa]}R^{(1)}_{km\mu,\kappa}(x',x_n,D')\varphi_0 \in C^{M+1}(\overline{\mathbb{R}^n_+})$.

Since the symbol $A_{km\mu}(x',\xi',\tau)$ is homogeneous of order $-\kappa-1$ with respect to (ξ',τ), it follows that (cf. (2.23))

$$C_{km\mu}(x',x_n,\xi') = |\xi_{n-1}|^{-\kappa}\int\limits_{\mathcal{L}_-} A_{km\mu}\Big(x', \frac{\xi''}{|\xi_{n-1}|}, \mathbf{sign}\,\xi_{n-1}, \tau\Big)e^{-ix_n\tau|\xi_{n-1}|}\,d\tau.$$

Introducing the notation

$$\widetilde{C}_{km\mu}\Big(x',x_n,\frac{\xi''}{|\xi_{n-1}|},\xi_{n-1}\Big) = \int\limits_{\mathcal{L}_-} A_{km\mu}\Big(x', \frac{\xi''}{|\xi_{n-1}|}, \mathbf{sign}\,\xi_{n-1}, \tau\Big)e^{-ix_n\tau|\xi_{n-1}|}\,d\tau$$

and applying the Taylor formula again, we proceed as follows:

$$\begin{aligned}
&\sum_{l=0}^{M+1-[\kappa]}\frac{1}{l!}(\partial^l_{x_{n-1}}C_{km\mu})(x'',0,x_n,\xi')\cdot x^l_{n-1}\\
&\quad= \sum_{l=0}^{M+1-[\kappa]}\frac{1}{l!}(\partial^l_{x_{n-1}}\widetilde{C}_{km\mu})\Big(x'',0,x_n,\frac{\xi''}{|\xi_{n-1}|},\mathbf{sign}\,\xi_{n-1}\Big)|\xi_{n-1}|^{-\kappa}x^l_{n-1}\\
&\quad= \sum_{l=0}^{M+1-[\kappa]}\ \sum_{p=0}^{M-[\kappa]-l+1}\frac{1}{l!}\frac{1}{p!}\sum_{|\gamma|=p}(\xi'')^\gamma(\partial^\gamma_{\xi''}\partial^l_{x_{n-1}}\widetilde{C}_{km\mu})(x'',0,x_n,0,\mathbf{sign}\,\xi_{n-1})\\
&\qquad\qquad\times|\xi_{n-1}|^{-\kappa-|\gamma|}x^l_{n-1} + \widetilde{R}_{km\mu}(x',x_n,\xi'),
\end{aligned}$$

where the symbol $\widetilde{R}_{km\mu}(x',x_n,\xi')$ belongs to the class $\mathcal{R}^{(2)}_{M_1,-M_1,-\kappa,-\infty}$, where $M_1 = M+2-[\kappa]$. Lemma 2.8 implies the relation $\widetilde{R}_{km\mu}(x',x_n,D')\varphi_0 \in C^{M+1}(\overline{\mathbb{R}^n_+})$.

We have

$$\frac{1}{l!}\frac{1}{|\gamma|!}(\xi'')^\gamma(\partial^\gamma_{\xi''}\partial^l_{x_{n-1}}C_{km\mu})(x'',0,x_n,0,\mathbf{sign}\,\xi_{n-1}) = |\xi_{n-1}|\int\limits_{\mathcal{L}_-}\Phi^{\gamma l}_{km\mu}(x'',\mathbf{sign}\,\xi_{n-1},\tau)e^{-i\tau x_n|\xi_{n-1}|}d\tau$$

(cf. (2.23)), and $\Phi^{\gamma l}_{km\mu}(x'',\mathbf{sign}\,\xi_{n-1},\tau) = (l!|\gamma|!)^{-1}(\xi'')^\gamma(\partial^\gamma_{\xi''}\partial^l_{x_{n-1}}A_{km\mu})(x'',0,0,\mathbf{sign}\,\xi_{n-1},\tau)$. Let all roots of the polynomial $\Delta(\mathcal{J}^T_{æ}(x'',0)(0,\mathbf{sign}\,\xi_{n-1},\tau))$ belonging to the lower complex half-plane be $\tau_s = \tau_s(x'',\mathbf{sign}\,\xi_{n-1})$, $s = 1,\dots,\ell(N)$, and let their multiplicities be n_s, $s = 1,\dots,\ell(N)$.

Since $A(\xi)$ is real, homogeneous, and elliptic ($\mathbf{det}\,A(\xi) = 0$ if and only if $\xi = 0$), it follows that the polynomial $\mathbf{det}\,A(\mathcal{J}^\top_{æ}(x'',0)\cdot(0,\pm1,\tau))$ has real $C^\infty_0(\mathbb{R}^{n-2})$-smooth coefficients, and the leading coefficient (at τ^{2rN}) is nonzero for any $x''\in\mathbb{R}^{n-2}$. Therefore, all the roots must be purely imaginary, and $\tau_{s,\pm1}\in C^\infty_0(\mathbb{R}^{n-2})$ for $s = 1,\dots,\ell(N)$ (see [Le1]).

Let $\Phi^{\gamma l}_{km\mu}(x'',\mathbf{sign}\,\xi_{n-1},\tau)$ have at a point τ_s a pole of order p_s ($s = 1,\dots,\ell(N)$). Note that $x_n > 0$, apply the residue formula

$$\begin{aligned}
&\int\limits_{\mathcal{L}_-}\Phi^{\gamma l}_{km\mu}(x'',\mathbf{sign}\,\xi_{n-1},\tau)e^{-i\tau x_n|\xi_{n-1}|}d\tau\\
&\qquad= \sum_{s=1}^{\ell(N)}\lim_{\tau\to\tau_s}\frac{d^{p_s-1}}{d\tau^{p_s-1}}\{(\tau-\tau_s)^{p_s}\Psi^{\gamma l}_{km\mu}(x'',\mathbf{sign}\,\xi_{n-1},\tau)e^{-i\tau x_n|\xi_{n-1}|}\},
\end{aligned}$$

where $\Psi^{\gamma ls}_{km\mu}(x'', \mathbf{sign}\,\xi_{n-1}, \tau) = \frac{2\pi i}{(p_s-1)!}(\tau-\tau_s)^{p_s}\Phi^{\gamma l}_{km\mu}(x'', \mathbf{sign}\,\xi_{n-1}, \tau)$, and obtain

$$\begin{aligned}
&\int_{\mathcal{L}_-} \Phi^{\gamma l}_{km\mu}(x'', \mathbf{sign}\,\xi_{n-1}, \tau) e^{i\tau x_n|\xi_{n-1}|} d\tau \\
&= \sum_{s=1}^{\ell(N)} \sum_{j=0}^{p_s-1} \frac{(-1)^j (p_s-1)!}{j!(p_s-1-j)!} \lim_{\tau\to\tau_s} \left\{ \frac{d^{p_s-1-j}}{d\tau^{p_s-1-j}} \Psi^{\gamma ls}_{km\mu}(x'', \mathbf{sign}\,\xi_{n-1}, \tau)((-ix_n)|\xi_{n-1}|)^j e^{-i\tau x_n|\xi_{n-1}|} \right\} \\
&= \sum_{s=1}^{\ell(N)} \sum_{j=0}^{p_s-1} d^{\gamma ls}_{km\mu j}(x'', \mathbf{sign}\,\xi_{n-1})(x_n|\xi_{n-1}|)^j e^{-i\tau_s x_n|\xi_{n-1}|}
\end{aligned}$$

with

$$d^{\gamma ls}_{km\mu j}(x'', \mathbf{sign}\,\xi_{n-1}) = \frac{(i)^j (p_s-1)!}{j!(p_s-1-j)!} \frac{d^{p_s-1-j}}{d\tau^{p_s-1-j}} \Psi^{\gamma ls}_{km\mu}(x'', \mathbf{sign}\,\xi_{n-1}, \tau_s).$$

Therefore,

$$\begin{aligned}
&\mathcal{F}^{-1}_{\xi'\to x'}\Big(\frac{1}{\mu!}\partial^\mu_{\xi'} A_{-k}(x', x_n, \xi')\partial^\mu_{x'} B^0_{q-m}(x', \xi')\widehat{\varphi}_0(\xi')\Big) \\
&= \sum_{l=0}^{M-[\kappa]+1} \sum_{|\gamma|=0}^{M-[\kappa]+1-l} \sum_{s=1}^{\ell(N)} \sum_{j=0}^{p_s-1} \mathcal{F}^{-1}_{\xi'\to x'}\Big(d^{\gamma ls}_{km\mu j}(x'', \mathbf{sign}\,\xi_{n-1}) \\
&\qquad\qquad \times (\xi'')^\gamma x^l_{n-1} x^j_n |\xi_{n-1}|^{j-|\gamma|-\kappa} e^{-i\tau_s x_n|\xi_{n-1}|}\widehat{\varphi}_0(\xi')\Big) + (R_{M+1}\varphi_0)(x'),
\end{aligned}$$

where $R_{M+1}\varphi_0 \in C^{M+1}_0(\overline{\mathbb{R}}^n_+)$ and $\kappa = -q + (m + k + |\mu|)$.

For simplicity, we introduce the following notation:

$$\begin{aligned}
\widetilde{d}(x'', \mathbf{sign}\,\xi_{n-1}) &= d^{\gamma ls}_{km\mu j}(x'', \mathbf{sign}\,\xi_{n-1}), & \widetilde{d}_0(x'', \mathbf{sign}\,\xi_{n-1}) &= d^{00s}_{km\mu j}(x'', \mathbf{sign}\,\xi_{n-1}), \\
\tau_{s,+1} &= \tau_s(x'', +1), & z_{s,+1} &= -(x_{n-1} + x_n\tau_{s,+1}), \\
\tau_{s,-1} &= \tau_s(x'', -1), & z_{s,-1} &= x_{n-1} - x_n\tau_{s,-1},
\end{aligned}$$

$$\vartheta_+(\xi_{n-1}) = \begin{cases} 1, & \xi_{n-1} > 0, \\ 0, & \xi_{n-1} < 0. \end{cases}$$

Taking into account the asymptotic expansion of the function φ_0 (see (1.45)), we see that

$$\begin{aligned}
&\mathcal{F}^{-1}_{\xi'\to x'}\Big(\widetilde{d}(x'', \mathbf{sign}\,\xi_{n-1})(\xi'')^\gamma |\xi_{n-1}|^{j-|\gamma|-\kappa} \cdot x^l_{n-1} \cdot x^j_n \cdot e^{-i\tau_s x_n|\xi_{n-1}|}\widehat{\varphi}_0(\xi')\Big) \\
&\qquad = \sum_{k=0}^{M_0} \mathcal{F}^{-1}_{\xi'\to x'}\Big(\widetilde{d}(x'', \mathbf{sign}\,\xi_{n-1})|\xi_{n-1}|^{j-|\gamma|-\kappa} \cdot x^l_{n-1} \cdot x^j_n \cdot e^{-i\tau_s x_n|\xi_{n-1}|}(\widehat{D^\gamma_{x''}\varphi_{0k}})(\xi')\Big) \\
&\qquad\quad + R(x', x_n, D')(D^\gamma_{x''}\varphi_{M_0+1}), \\
&\qquad \varphi_{00}(x'', x_{n-1,+}) := \mathcal{K}(x'') x^{\nu/2+\Delta(x'')}_{n-1,+} B^0_{a_{\mathrm{pr}}}\left(-\frac{1}{2\pi i}\log x_{n-1,+}\right)\mathcal{K}^{-1}(x'')c_0(x''), \\
&\qquad \varphi_{0k}(x'', x_{n-1,+}) := \mathcal{K}(x'') x^{\nu/2+\Delta(x'')+k}_{n-1,+} B_k(x'', \log x_{n-1,+}), \\
&\qquad\qquad k = 1, \dots, M_0, \qquad M_0 = M + 3 - 2r + [q];
\end{aligned}$$

here $B_k(x'',t)$ is a polynomial in t of order $\nu_k = k(2m_0-1)+m_0-1$ with vector coefficients depending on the variable x''.

Let $R(x',x_n,D')$ be a pseudodifferential operator with symbol

$$R(x',x_n,\xi') = \widetilde{d}(x'',\mathbf{sign}\,\xi_{n-1})|\xi_{n-1}|^{j-|\gamma|-\kappa}\cdot x_{n-1}^l\cdot x_n^j\cdot e^{-i\tau_s x_n|\xi_{n-1}|},$$

and let $c_0(x'')$ be the leading coefficient in asymptotics (1.45).

Since

$$|\gamma|+\kappa = 2r-1-q,\dots,M+2-\{q\},$$

it follows that, taking into account $\varphi_{M_0+1}\in\widetilde{\mathbb{H}}_p^{(\infty,s+M_0+1),\infty}(\mathbb{R}_+^{n-1})$ $(M>(n-1)/p-s)$ and using the method of proving the smoothness of a pseudodifferential operator acting on the remainder term that was used in Lemmas 2.7 and 2.9, we readily see that

$$R(x',x_n,D')D_{x''}^\gamma\varphi_{M_0+1}\in C^{M+1}(\overline{\mathbb{R}}_+^n).$$

Since

$$(\xi_{n-1}+i0)^\lambda = \begin{cases}\xi_{n-1}^\lambda, & \xi_{n-1}>0,\\ e^{i\pi\lambda}|\xi_{n-1}|^\lambda, & \xi_{n-1}<0,\end{cases} \tag{2.24}$$

we proceed as follows:

$$\begin{aligned}
&\mathcal{F}^{-1}_{\xi'\to x'}\Big(\widetilde{d}(x'',\mathbf{sign}\,\xi_{n-1})|\xi_{n-1}|^{j-|\gamma|-\kappa}\cdot x_{n-1}^l\cdot x_n^j\cdot e^{-\tau_s x_n|\xi_{n-1}|}(\widehat{D_{x''}^\gamma\varphi_{0k}})(\xi')\Big)\\
&= x_{n-1}^l\cdot x_n^j\Big\{\widetilde{d}(x'',+1)\mathcal{F}^{-1}_{\xi''\to x''}\Big(\mathcal{F}^{-1}_{\xi_{n-1}\to x_{n-1}}\vartheta_+(\xi_{n-1})|\xi_{n-1}|^{j-|\gamma|-\kappa}e^{-i\tau_{s,+1}x_n|\xi_{n-1}|}(\widehat{D_{x''}^\gamma\varphi_{0k}})(\xi')\Big)\\
&\quad+\widetilde{d}(x'',-1)\mathcal{F}^{-1}_{\xi''\to x''}\Big(\mathcal{F}^{-1}_{\xi_{n-1}\to x_{n-1}}\vartheta_+(-\xi_{n-1})|\xi_{n-1}|^{j-|\gamma|-\kappa}e^{-i\tau_{s,-1}x_n|\xi_{n-1}|}(\widehat{D_{x''}^\gamma\varphi_{0k}})(\xi')\Big)\Big\}\\
&= x_{n-1}^l\cdot x_n^j\Big\{\frac{1}{2\pi}\widetilde{d}(x'',+1)\mathcal{F}^{-1}_{\xi''\to x''}\Big(\mathcal{F}_{\xi_{n-1}\to z_{s,+1}}\xi_{n-1,+}^{j-|\gamma|-\kappa}(\widehat{D_{x''}^\gamma\varphi_{0k}})(\xi')\Big)\\
&\quad+\frac{1}{2\pi}\widetilde{d}(x'',-1)\mathcal{F}^{-1}_{\xi''\to x''}\Big(\mathcal{F}_{\xi_{n-1}\to z_{s,-1}}\xi_{n-1,+}^{j-|\gamma|-\kappa}(\widehat{D_{x''}^\gamma\varphi_{0k}})(\xi')\Big)\Big\}\\
&= x_{n-1}^l\cdot x_n^j\Big\{\frac{1}{2\pi}\widetilde{d}(x'',+1)\mathcal{F}_{\xi_{n-1}\to z_{s,+1}}\Big(\xi_{n-1,+}^{j-|\gamma|-\kappa}\mathcal{F}_{x_{n-1}\to\xi_{n-1}}(D_{x''}^\gamma\varphi_{0k})(x'',x_{n-1,+})\Big)\\
&\quad+\frac{1}{2\pi}\widetilde{d}(x'',-1)\mathcal{F}_{\xi_{n-1}\to z_{s,-1}}\Big(\xi_{n-1,+}^{j-|\gamma|-\kappa}\mathcal{F}_{x_{n-1}\to-\xi_{n-1}}(D_{x''}^\gamma\varphi_{0k})(x'',x_{n-1,+})\Big)\Big\}.
\end{aligned}$$

Obviously,

$$\begin{aligned}
&\mathcal{F}^{-1}_{\xi'\to x'}\Big(\frac{1}{\mu!}\,\partial_{\xi'}^\mu A_{-k}(x',x_n,\xi')\partial_{x'}^\mu B_{q-m}^0(x',\xi')\widehat{\varphi}_0(\xi')\Big) \qquad (2.25)\\
&=\sum_{s=1}^{\ell(N)}\Big\{\sum_{j=0}^{n_s-1}x_n^j\Big[\frac{1}{2\pi}\widetilde{d}_0(x'',+1)\mathcal{F}_{\xi_{n-1}\to z_{s,+1}}\Big(\xi_{n-1,+}^{j-\kappa}\mathcal{F}_{x_{n-1}\to\xi_{n-1}}(\varphi_{00}(x'',x_{n-1,+}))\Big)\\
&\qquad+\frac{1}{2\pi}\widetilde{d}_0(x'',-1)\mathcal{F}_{\xi_{n-1}\to z_{s,-1}}\Big(\xi_{n-1,+}^{j-\kappa}\mathcal{F}_{x_{n-1}\to-\xi_{n-1}}(\varphi_{00}(x'',x_{n-1,+}))\Big)\Big]\\
&\qquad+\sum_{\vartheta=\pm1}\sum_{l=0}^{M-[\kappa]+1}\sum_{\substack{|\gamma|=0\\ l+|\gamma|+k\neq0}}^{M-[\kappa]+1-l}\sum_{k=0}^{M_0}\sum_{j=0}^{p_s-1}x_{n-1}^l\cdot x_n^j\cdot\frac{1}{2\pi}\widetilde{d}(x'',\vartheta)\\
&\qquad\times\mathcal{F}_{\xi_{n-1}\to z_{s,\vartheta}}\Big(\xi_{n-1,+}^{j-|\gamma|-\kappa}\mathcal{F}_{x_{n-1}\to\theta\xi_{n-1}}(D_{x''}^\gamma\varphi_{0k})(x'',x_{n-1,+})\Big)\Big\}+R(x'',x_{n-1},x_n),
\end{aligned}$$

where all $R(x'', x_{n-1}, x_n)$ belong to the class $C^{M+1}(\overline{\mathbb{R}}^n_+)$.

Furthermore, for $\operatorname{Re}\mu > -1$, the following relations hold:

$$\mathcal{F}_{t\to\sigma}(t_+^\mu \log^p t_+ e^{-\tau t}) = \sum_{k=0}^{p} b_{pk}^{(1)}(\mu)(\sigma + i\tau)^{-\mu-1} \log^k(\sigma + i\tau), \tag{2.26}$$

where

$$b_{pk}^{(1)}(\mu) = \frac{(-1)^k p!}{k!(p-k)!} \frac{d^{p-k}}{d\mu^{p-k}} \Big(\Gamma(\mu+1)e^{i\pi(\mu+1)/2}\Big);$$

for $\operatorname{Re}\lambda \leqslant -1$ we set $\lambda = -m + \mu$, $-1 < \operatorname{Re}\mu \leqslant 0$, and $\lambda \notin \mathbb{Z}$, where $m > 0$ is an integer, which gives

$$\mathcal{F}_{t\to\sigma}(t_+^\lambda \log^p t_+ e^{-\tau t}) = \sum_{k=0}^{p} c_{pk}(\lambda)(\sigma + i\tau)^{-\lambda-1} \log^k(\sigma + i\tau), \tag{2.27}$$

$$c_{pk}(\lambda) := i^{\lambda-\mu} \sum_{j=k}^{p} d_{pj}^{(m)}(\lambda) b_{jk}^{(1)}(\mu). \tag{2.28}$$

Here $b_{jk}^{(1)}(\mu)$ is defined by relation (2.26), and the coefficients $d_{pj}^{(M)}(\lambda)$ are given by means of the recurrence relations

$$d_{pj}^{(M)}(\lambda) = \sum_{q=j}^{p} d_{pq}^{(M-1)}(\lambda) \cdot d_{qj}^{(1)}(\lambda + M - 1), \qquad d_{qj}^{(1)}(\lambda + M - 1) = (-1)^{q-j} \frac{q!}{j!} \frac{1}{(\lambda + M)^{q-j+1}},$$

where $M \in \mathbb{N}$. If $\lambda = -m - 1 = -1, -2, \dots$, $m \in \mathbb{N}_0$, then

$$\mathcal{F}_{t\to\sigma}(t_+^\lambda \log^p t_+ e^{-\tau t}) = \Big(\sum_{k=0}^{p+1} \widetilde{c}_{p+1,k}(\lambda) \log^k(\sigma + i\tau)\Big)(\sigma + i\tau)^{-\lambda-1}, \tag{2.29}$$

where

$$\widetilde{c}_{p+1,k}(\lambda) = \begin{cases} i^\lambda \displaystyle\sum_{l=1}^{p+1} d_{p,l-1}^{(m)}(\lambda) \frac{b_{l0}^{(1)}(0)}{l} & \text{for} \quad k = 0\,, \\ i^\lambda \displaystyle\sum_{l=k}^{p+1} d_{p,l-1}^{(m)}(\lambda) \frac{b_{lk}^{(1)}(0)}{l} & \text{for} \quad k \in \{1, 2, \dots, p+1\}, \end{cases} \tag{2.30}$$

the coefficients $d_{pl}^{(m)}(\lambda)$, $m \in \mathbb{N}$, are defined as above, and the coefficients $d_{pl}^{(0)}$ are given by $d_{pl}^{(0)}(-1) = \delta_{pl}$, $l = 0, \dots, p$ (δ_{pl} is the Kronecker delta).

Indeed, relation (2.25) follows from the formula (see [Es1, (2.36)])

$$\mathcal{F}_{t\to\sigma}(t_+^\mu e^{-\tau t}) = \Gamma(\mu+1)e^{i\pi(\mu+1)/2}(\sigma + i\tau)^{-\mu-1}\,, \qquad \operatorname{Re}\mu > -1\,,$$

after differentiating p times with respect to the parameter μ.

For $\operatorname{Re}\lambda \leqslant -1$, the function $t_+^\lambda \log^p t_+$ is not integrable on $\mathbb{R}$; however, it can be expressed as a linear combination of derivatives of functions for which $\operatorname{Re}\lambda > -1$, i.e., if $\lambda = -m + \mu$, $-1 < \operatorname{Re}\mu \leqslant 0$, and $\lambda \notin \mathbb{Z}$, where $m > 0$ is an integer, then

$$t^\lambda \log^p t = \sum_{k=0}^{p} d_{pk}^{(m)}(\lambda) \left(\frac{d}{dt}\right)^m (t^\mu \log^k t)\,.$$

If $\lambda = -m-1 = -1, -2, \dots, m \in \mathbb{N}_0$, then

$$t^\lambda \log^p t = \sum_{k=0}^{p} \frac{d_{pk}^{(m)}(\lambda)}{k+1} \left(\frac{d}{dt}\right)^{m+1} (\log^{k+1} t).$$

Applying the Fourier transform, we obtain relations (2.26), (2.27), and (2.29).

Thus, substituting relations (2.26), (2.27), (2.29), and (2.24) into (2.25) and taking into account the formula

$$|\gamma| + \kappa + \{q\} = 2r - 1 - [q], \dots, M + 2 - l, \qquad l = 0, \dots, M + 3 - 2r + [q],$$

we obtain the asymptotic expansion (2.11).

This completes the proof of Theorem 2.2. □

Now let us prove Theorem 2.3.

First we shall find out how the leading coefficients of the expansions (2.11) and (1.45) are related. To this end, we perform a detailed calculation of the leading term of the asymptotic expansion (2.11). By (2.24), we can readily see that

$$\begin{aligned} d_{sj}(x'', +1) &= \frac{1}{2\pi} \mathcal{G}_{æ}(x'', 0) V_{1-2r,j}^{(s)}(x'', 0, 0, +1) B_q^0(x'', 0, 0, +1) \mathcal{K}(x''), \\ d_{sj}(x'', -1) &= \frac{1}{2\pi} \mathcal{G}_{æ}(x'', 0) V_{1-2r,j}^{(s)}(x'', 0, 0, -1) B_q^0(x'', 0, 0, -1) \mathcal{K}(x'') e^{i\pi\lambda_0}, \\ \lambda_0 &= -\frac{\nu}{2} - \Delta(x''), \qquad s = 1, \dots, \ell(N), \quad j = 0, \dots, n_s - 1, \end{aligned}$$

where $\mathcal{G}_{æ}(x'', 0)$ is the square root of the Gram determinant and

$$\begin{aligned} &V_{1-2r,j}^{(s)}(x'', 0, 0, \pm 1) \\ &\quad = -\frac{i^{j+1}}{j!(n_s - 1 - j)!} \frac{d^{n_s - 1 - j}}{d\tau^{n_s - 1 - j}} (\tau - \tau_{s,\pm 1})^{n_s} \left(A(\mathcal{J}_{æ}^\top(x'', 0) \cdot (0, \pm 1, \tau)) \right)^{-1} \Big|_{\tau = \tau_{s,\pm 1}} . \end{aligned}$$

Note that, if $\ell(N) = 1$ and $j = 0$, then

$$V_{1-2r,0}^{(s)}(x'', 0, 0, \pm 1) = V_{1-2r,0}^{(1)}(x'', 0, 0, \pm 1) = V_{1-2r}(x'', 0, 0, +1),$$

where $V_{1-2r}(x', \xi')$ is the principal symbol of the pseudodifferential operator $\mathbf{V}_{1-2r}$.

Now let us calculate the coefficient $c^{(j)}(x'')$, on the right-hand side, of the leading term of asymptotic expansion (2.10). We start from the relation

$$\begin{aligned} &\mathcal{F}_{x_{n-1} \to \xi_{n-1}} \left(x_{n-1,+}^{\mu} B_{a_{\rm pr}}^0 \left(-\frac{1}{2\pi i} \log x_{n-1,+} \right) \right) \\ &\qquad = (\xi_{n-1} + i0)^{-\mu - 1} B_{a_{\rm pr}}^0 \left(\frac{1}{2\pi i} \log(\xi_{n-1} + i0) \right) \cdot B_{a_{\rm pr}}^-(\mu), \end{aligned} \tag{2.31}$$

where $\operatorname{Re} \mu_i > -1$ for all $i = 1, \dots, \ell$, and

$$\mu = (\underbrace{\mu_1, \dots, \mu_1}_{m_1-times}, \dots, \underbrace{\mu_\ell, \dots, \mu_\ell}_{m_\ell-times}), \quad B_{a_{\rm pr}}^-(\mu) = \mathbf{diag}\{B_-^{m_1}(\mu_1), \dots, B_-^{m_\ell}(\mu_\ell)\},$$

$$\begin{aligned} B_-^{m_i}(t) &:= B^{m_i}\left(-\frac{1}{2\pi i} \partial_t \right) \left(\Gamma(t+1) e^{i\pi(t+1)/2} \right), \\ B_-^{m_i}(\mu_i) &:= \| b_{kp}^{m_i}(\mu_i) \|_{m_i \times m_i}, \quad p = 0, \dots, m_i - 1, \end{aligned}$$

$$b_{kp}^{m_i}(\mu_i) := \begin{cases} \left(\dfrac{1}{2\pi i} \right)^{p-k} \dfrac{(-1)^{p+k}}{(p-k)!} \dfrac{d^{p-k}}{dt^{p-k}} \left(\Gamma(t+1) e^{i\pi(t+1)/2} \right) \Big|_{t = \mu_i}, & k \leqslant p, \\ 0, & k > p, \end{cases} \tag{2.32}$$

$i = 1, \dots, \ell$.

Indeed, since $\operatorname{Re}\mu_i > -1$ for all $i = 1, \ldots, \ell$, by using relation (2.25), as well as the relation

$$b^{m_i}_{l-r,p-r}(\mu_i) = b^{m_i}_{lp}(\mu_i),$$

we obtain

$$\begin{aligned}\mathcal{F}_{x_{n-1}\to\xi_{n-1}}\left(x^{\mu_i}_{n-1,+}\frac{\left(-\dfrac{1}{2\pi i}\log x_{n-1,+}\right)^{p-r}}{(p-r)!}\right)&= (\xi_{n-1}+i0)^{-\mu_i-1}\sum_{k=0}^{p-r} b^{m_i}_{k,p-r}(\mu_i)\frac{\left(\dfrac{1}{2\pi i}\log(\xi_{n-1}+i0)\right)^{k}}{k!}\\&= (\xi_{n-1}+i0)^{-\mu_i-1}\sum_{l=r}^{p} b^{m_i}_{l-r,p-r}(\mu_i)\frac{\left(\dfrac{1}{2\pi i}\log(\xi_{n-1}+i0)\right)^{l-r}}{(l-r)!}\\&= (\xi_{n-1}+i0)^{-\mu_i-1}\sum_{l=r}^{p} \frac{\left(\dfrac{1}{2\pi i}\log(\xi_{n-1}+i0)\right)^{l-r}}{(l-r)!}b^{m_i}_{lp}(\mu_i).\end{aligned}$$

This implies the formula

$$\begin{aligned}&\mathcal{F}_{x_{n-1}\to\xi_{n-1}}\left(x^{\mu_i}_{n-1,+}B^{m_i}\left(-\frac{1}{2\pi i}\log x_{n-1,+}\right)\right)\\&\qquad= (\xi_{n-1}+i0)^{-\mu_i-1}B^{m_i}\left(\frac{1}{2\pi i}\log(\xi_{n-1}+i0)\right)B^{m_i}_{-}(\mu_i)\end{aligned}$$

(see (2.32)). In turn, this proves relation (2.31).

Since $\operatorname{Re}(\nu/2+\delta_i(x'')) > -1$ for all $i = 1, \ldots, \ell$, it follows that relation (2.29) holds whenever $\mu = \nu/2+\Delta(x'')$, i.e.,

$$\begin{aligned}&\mathcal{F}_{x_{n-1}\to\xi_{n-1}}\left(x^{\nu/2+\Delta(x'')}_{n-1,+}B^{0}_{a_{\mathrm{pr}}}\left(-\frac{1}{2\pi i}\log x_{n-1,+}\right)\right)\\&\qquad= (\xi_{n-1}+i0)^{-\nu/2-\Delta(x'')-1}B^{0}_{a_{\mathrm{pr}}}\left(\frac{1}{2\pi i}\log(\xi_{n-1}+i0)\right)B^{-}_{a_{\mathrm{pr}}}\left(\frac{\nu}{2}+\Delta(x'')\right).\end{aligned}\tag{2.33}$$

For $\lambda_i = -\dfrac{\nu}{2} - \delta_i(x'') - 2r + q + j$ $(i = 1, \ldots, \ell)$, we can use relations (2.26), (2.27) and (2.29), as well as the relation

$$(-1)^{p-r}\frac{(k-r)!}{(p-r)!}c^{m_i}_{p-r,k-r}(\lambda_i) = (-1)^p\frac{k!}{p!}c^{m_i}_{pk}(\lambda_i)$$

(for the definition of $c^{m_i}_{kp}(\lambda_i)$, see (2.28)), and similarly obtain the formula

$$\mathcal{F}_{\xi_{n-1}\to z_{s,\theta}}\left(\xi^{\lambda_i}_{n-1,+}B^{m_i}\left(\frac{1}{2\pi i}\log\xi_{n-1,+}\right)\right) = z^{-\lambda_i-1}_{s,\theta}B^{m_i}\left(-\frac{1}{2\pi i}\log z_{s,\theta}\right)B^{m_i}_{+}(\lambda_i)$$

for $\operatorname{Re}\lambda_i > -1$.

If $\operatorname{Re}\lambda_i \leqslant -1$, $\lambda_i \notin \mathbb{Z}$, $\lambda_i = -N_i + \mu_i$, $1 < \operatorname{Re}\mu_i \leqslant 0$, and $N_i > 0$ is an integer, then

$$\mathcal{F}_{\xi_{n-1}\to z_{s,\theta}}\left(\xi_{n-1,+}^{\lambda_i} B^{m_i}\left(\frac{1}{2\pi i}\log \xi_{n-1,+}\right)\right) = z_{s,\theta}^{-\lambda_i-1} B^{m_i}\left(-\frac{1}{2\pi i}\log z_{s,\theta}\right) a^{m_i}(\lambda_i),$$
$$a^{m_i}(\lambda_i) := B_+^{m_i}(\mu_i)\cdot D^{m_i}(\lambda_i), \qquad D^{m_i}(\lambda_i) = \|D_{jp}^{m_i}(\lambda_i)\|_{m_i\times m_i},$$
$$D_{jp}^{m_i}(\lambda_i) := \begin{cases} i^{\lambda_i-\mu_i}\dfrac{j!}{p!}\left(\dfrac{1}{2\pi i}\right)^{p-j} d_{pj}^{m_i(N_i)}(\lambda_i), & j \leqslant p, \\ 0, & j > p \end{cases}$$

and $d_{pj}^{m(N_i)}(\lambda_i)$ are defined by (2.9) and (2.28).

Indeed, proceeding as above, we transform the matrix $a^{m_i}(\lambda_i)$ to the form

$$a^{m_i}(\lambda_i) = \|a_{kp}^{m_i}(\lambda_i)\|_{m_i\times m_i}, \qquad a_{kp}^{m_i}(\lambda_i) := \begin{cases} (-1)^k\left(\dfrac{1}{2\pi i}\right)^{p-k}\dfrac{k!}{p!}\,c_{pk}^{m_i}(\lambda_i), & k \leqslant p, \\ 0, & k > p. \end{cases}$$

Further, taking into account the expression $c_{pk}^{m_i}(\lambda_i)$ (see (2.28)), we see that

$$(-1)^k\left(\frac{1}{2\pi i}\right)^{p-k}\frac{k!}{p!}\,c_{pk}^{m_i}(\lambda_i)$$
$$= \sum_{j=k}^{p}\left(\frac{1}{2\pi i}\right)^{j-k}\frac{1}{(j-k)!}\frac{d^{j-k}}{dt^{j-k}}\left(\Gamma(t+1)e^{i\pi(t+1)/2}\right)\Big|_{t=\mu_i}\cdot i^{\lambda_i-\mu_i}\left(\frac{1}{2\pi i}\right)^{p-j}\frac{j!}{p!}\,d_{pj}^{m_i(N_i)}(\lambda_i).$$

Then

$$a_{kp}^{m_i}(\lambda_i) = \sum_{j=0}^{m_i-1} b_{kj}^{m_i}(\mu_i) D_{jp}^{m_i}(\lambda_i).$$

Therefore, $a^{m_i}(\lambda_i) = B_+^{m_i}(\mu_i)\cdot D^{m_i}(\lambda_i)$.

For $\lambda_i = -1, -2, \ldots$, using relations (2.29) and (2.30), we obtain

$$\mathcal{F}_{\xi_{n-1}\to z_{s,\theta}}\left(\xi_{n-1,+}^{\lambda_i} B^{m_i}\left(\frac{1}{2\pi i}\log z_{s,\theta}\right)\right) = z_{s,\theta}^{-\lambda_i-1}\widetilde{B}^{m_i}\left(\frac{1}{2\pi i}\log z_{s,\theta}\right),$$

where $\widetilde{B}^{m_i}(t)$ is defined by (2.10).

This yields the relation

$$\mathcal{F}_{\xi_{n-1}\to z_{s,\vartheta}}\left(\xi_{n-1,+}^{-\nu/2-\Delta(x'')-2r+q+j} B_{a_{\mathrm{pr}}}^0\left(\frac{1}{2\pi i}\log \xi_{n-1,+}\right)\right)$$
$$= z_{s,\vartheta}^{\nu/2+\Delta(x'')+2r-1-q-j}\widetilde{B}_{a_{\mathrm{pr}}}^0\left(\frac{1}{2\pi i}\log z_{s,\vartheta}\right) a_j(x''), \qquad (2.34)$$

where

$$a_j(x'') = \mathbf{diag}\left\{a^{m_1}(\lambda_1), \ldots, a^{m_\ell}(\lambda_\ell\right\},$$
$$a^{m_i}(\lambda_i) = \begin{cases} B_+^{m_i}(\lambda_i), & \operatorname{Re}\lambda_i > -1, \\ B_+^{m_i}(\mu_i)\cdot D^{m_i}(\lambda_i), & \operatorname{Re}\lambda_i < -1, \quad \lambda_i \notin \mathbb{Z}, \\ I, & \lambda_i = -1, -2, \ldots. \end{cases}$$

Thus, it follows from (2.32) and (2.33) that

$$c^{(j)}(x'') = a_j(x'')B^-_{a_{\rm pr}}\Big(\frac{\nu}{2} + \Delta(x'')\Big)\mathcal{K}^{-1}(x'')c_0(x''),$$

where $c_0(x'')$ is the leading coefficient of the asymptotic expansion (1.45). □

Remark 2.10. If, in Theorem 2.2, the density φ_0 has the expansion (1.47), i.e., if $B^0_{a_{\rm pr}} = I$ is the identity matrix, then the asymptotic expansion (2.11) becomes simpler for $\lambda_i = -\frac{\nu}{2} - \delta_i(x'') - 2r + q + j \neq -1, -2, \dots$, namely,

$$\begin{aligned} u(x'', x_{n-1}, x_n) = \sum_{\theta=\pm1}\sum_{s=1}^{\ell(N)}\Bigg\{&\sum_{j=0}^{n_s-1}\theta x_n^j d_{sj}(x'',\theta) z_{s,\theta}^{\nu/2+\Delta(x'')+2r-1-q-j} c^{(j)}(x'') \\ &+ \sum_{\substack{k,l=0\\ k+l+j+p\neq 2r-[q]-1}}^{M+3-2r+[q]}\ \sum_{j+p=2r-[q]-1}^{M+2-l} x_{n-1}^l x_n^j d_{sljp}(x'',\theta) z_{s,\theta}^{\nu/2+\Delta(x'')+p+k-\{q\}} \\ &\times B_{skjp}(x'', \log z_{s,\theta})\Bigg\} + u_{M+1}(x'', x_{n-1}, x_n) \end{aligned}$$

for $x_n > 0$ and $u_{M+1} \in C_0^{M+1}(\overline{\mathbb{R}^n_+})$, where $B_{skjp}(x'', t)$ is a polynomial in t of degree $\nu_{kjp} = k + p + j - 2r + [q] + 1$ with vector coefficients depending on the variable x''. The coefficients $d_{sj}(x'', \pm1)$ have the same form as in (2.10), and

$$\begin{aligned} &c^{(j)}(x'') = a_j(x'')b_0(x'')\mathcal{K}^{-1}(x'')c_0(x''), \quad j = 0, \dots, n_s - 1, \\ &b_0(x'') := \Gamma(\mu + 1)e^{i\pi(\mu+1)/2}\Big|_{\mu=\nu/2+\Delta(x'')}, \\ &a_j(x'') := \mathbf{diag}\{a^{m_1}_{00}(\lambda_1), \dots, a^{m_1}_{m_1-1,m_1-1}(\lambda_1), \dots, a^{m_\ell}_{00}(\lambda_\ell), \dots, a^{m_\ell}_{m_\ell-1,m_\ell-1}(\lambda_\ell)\}, \\ &a^{m_k}_{pp}(\lambda_k) = \begin{cases} \Gamma(\lambda_k + 1)e^{i\pi(\lambda_k+1)/2} & \text{for } \operatorname{Re}\lambda_k > -1, \\ i^{\lambda_k-\mu_k}\Gamma(\mu_k + 1)e^{i\pi(\mu_k+1)/2}\displaystyle\prod_{m=1}^{N_k}\frac{1}{\lambda_k + m} & \text{for } \operatorname{Re}\lambda_k < -1, \ \lambda_k \notin \mathbb{Z}, \end{cases} \\ &k = 1, \dots, \ell; \end{aligned}$$

here $\lambda_k = -N_k + \mu_k$, $-1 < \operatorname{Re}\mu_k \leqslant 0$, and $N_k > 0$ is an integer.

In (1.47), $c_0(x'')$ is obviously the leading coefficient.

For $\lambda_i = -\nu/2 - \delta_i(x'') - 2r + q + j = -1, -2, \dots$ we obtain $\lambda_i = -m_j - 1 = \lambda^{(j)}$, $j = 0, \dots, n_s - 1$, these values do not depend on $i = 1, \dots, \ell$ (for some nonnegative integers $m_j = 0, 1, \dots$) and, in the asymptotic expansion of the density φ_0 (see (1.47)), the matrix $B^0_{a_{\rm pr}}(t)$ is equal to I. Then the asymptotic expansion (2.11) becomes

$$\begin{aligned} u(x'', x_{n-1}, x_n) = \sum_{s=1}^{\ell(N)}\Bigg\{&\sum_{j=0}^{n_s-1} x_n^j\Bigg[d_{sj}(x'', +1) z_{s,+1}^{\nu/2+\Delta(x'')+2r-1-q-j}\widetilde{B}^0_{a_{\rm pr}}\Big(\frac{1}{2\pi i}\log z_{s,+1}\Big) \\ &- d_{sj}(x'', -1) z_{s,-1}^{\nu/2+\Delta(x'')+2r-1-q-j}\widetilde{B}^0_{a_{\rm pr}}\Big(\frac{1}{2\pi i}\log z_{s,-1}\Big)\Bigg] c(x'') \\ &+ \sum_{\theta=\pm1}\sum_{\substack{k,l=0\\ k+l+j+p\neq 2r-[q]-1}}^{M+3-2r+[q]}\ \sum_{j+p=2r-[q]-1}^{M+2-l} x_{n-1}^l x_n^j z_{s,\theta}^{\nu/2+\Delta(x'')+p+k-\{q\}} B_{slkjp}(x'', \log z_{s,\theta})\Bigg\} \\ &+ u_{M+1}(x'', x_{n-1}, x_n) \qquad \text{for} \quad x_n > 0, \quad u_{M+1} \in C_0^{M+1}(\overline{\mathbb{R}^n_+}), \end{aligned}$$

where $B_{slkjp}(x'', t)$ is a polynomial in t of degree $\nu_{kjp} = k+p+j-2r+[q]+2$ with vector coefficients depending on the variable x'', and

$$\widetilde{B}^0_{a_{\rm pr}}\left(\frac{1}{2\pi i}\log z_{s,\theta}\right) = \left(\widetilde{c}_{10}(\lambda^{(j)}) + \widetilde{c}_{11}(\lambda^{(j)})\log z_{s,\theta}\right)I\,;$$

here

$$\widetilde{c}_{11}(\lambda^{(j)}) = i^{\lambda^{(j)}}\frac{(-1)^{m_j+1}}{m_j!}, \qquad \widetilde{c}_{10}(\lambda^{(j)}) = i^{\lambda^{(j)}}\frac{(-1)^{m_j}}{m_j!}\left[i\Gamma'(1) - \frac{\pi}{2}\right].$$

The coefficients $d_{sj}(x'', \pm 1)$ are as in (2.9), and $c(x'')$ becomes $c(x'') = b_0(x'')\mathcal{K}^{-1}(x'')c_0(x'')$.

Remark 2.11. Theorems 2.2 and 2.3 hold in more general cases. Indeed, let $P_l(x, D_x)$ be a differential operator of order l. Then, using the scheme of the proof of Theorems 2.2 and 2.3, we can obtain a similar asymptotic expansion for the functions $u = \mathbf{V}^{(l)} \circ \mathbf{B}_q\varphi_0$ and $\widetilde{u} = \widetilde{\mathbf{V}}^{(l)} \circ \mathbf{B}_q\varphi_0$, where $\mathbf{V}^{(l)}$ and $\widetilde{\mathbf{V}}^{(l)}$ are potential-type operators with kernels

$$V^{(l)}(x, y) = P_l(x, D_x)H_A(x-y) \quad \text{and} \quad \widetilde{V}^{(l)}(x, y) = P_l(y, D_y)H_A(x-y),$$

respectively.

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