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Pseudodifferential equations on manifolds with boundary: Fredholm property and asymptotic

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Abstract. The main purpose of the present paper is the investigation of systems of pseudodifferential equations (PsDEs) with symbols from extended Hörmander classes on a manifold with smooth boundary. Equations are treated in anisotropic BESSEL potential spaces with weight (BPSwW). Theorem about factorization of symbols, proved earlier by E.SHAMIR, R.DUDUCHAVA and E.SHARGORODSKY is revised and general criteria is obtained for PsDEs in BPSwW on manifolds with smooth boundary to possess the FREDHOLM property. It is proved that the criteria is invariant with respect to the weight exponents and the co–normal smoothness parameter, which participate in the definition of the spaces. In the second part of the paper results of G.ESKIN and J.BENNISH on asymptotic of solutions to systems of PsDEs (L_2–theory) are extended and complete asymptotic expansion of a solution to near the boundary is obtained (L_p–theory). More precise description of exponents and of logarithmic terms of the expansion is presented. Investigations are carried out in connection with problems arising in elasticity (crack problems) and some other fields of mathematical physics when the potential method is applied. In forthcoming papers asymptotic of a function represented by a potential will be presented when asymptotic of a density on the boundary of the domain is known.

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Introduction

Investigation of a system of boundary integral (or, more precisely, pseudodifferential) equations (BPsDEs) constitutes a crucial part of the potential method for studying boundary value problems (BVPs) in general and BVPs of the elasticity in particular. If boundary of the domain where elliptic equation is treated is smooth, solutions are smooth too provided the data of BVP are smooth. But dealing with crack–type, screen–type and mixed problems of the mathematical physics we arrive to necessity of investigating BPsDEs on manifolds (surfaces) with boundary. It is well–known that solutions to such problems have singularities on the boundary and other sub manifolds (points, curves) of geometrical and structural peculiarities of the manifold (e.g. at conical points, edges etc.) regardless smoothness properties of given data. Both, mathematicians (cf. V.Kondrat'ev [Ko1], V.Kozlov, V.Maz'ya, J.Rossmann [KMR1], [MP1], M.Dauge [Da1], P.Grisvard [Gr1], S.Nazarov and B.Plamenevski [NP1], B.W.Schulze [Sc3] etc.) and mechanist (cf. G.Cherepanov [Ch1], J.Lekhnitsky [Le1], J.Rabotnov [Ra1], L.Malvern [Ma1], J.Rice [Ri1, Ri2] etc.) have analysed local asymptotic expansions for the elliptic system of linear elasticity. The methods used were either Mellin transform, suggested by V.Kondrat'ev, or an appropriate ansatz (M.Williams [Wi1], T.Ting [Ti1] etc.). With the help of the Mellin transform a big number of interesting and important problems were investigated, including asymptotic of solutions to boundary value problems near edges, conical points, cracks etc. The ansatz was used mostly by mechanist to get formulae for exponents.

An alternative approach based on the Wiener–Hopf method seems to be much less exploited, especially in applications. The method was originally designed for the investigation of pseudodifferential equations (PsDE) on the half–line $\mathbb{R}^+$. Later on the method was applied to the investigation of **Fredholm properties** and solvability of systems of PsDEs on manifolds with boundary (see G.Eskin and M.Vishik [Es1], I.Simonenko [Si1] for L_2–theory and R.Duduchava [Du1], E.Shargorodsky [Sh2] for L_p–theory).

The results found ample applications in BVPs of mathematical physics (see e.g. [No1] for two–dimensional case and [BG2, Ch2, Ch3, CS1, DNS1, DSW1, DW1, KGBB1, MPr] etc. for multidimensional case of elasticity and diffraction theory, aero and hydrodynamics).

The method is based on the factorization of symbols and provides rather explicit results concerning the Fredholm criteria and solvability of BPsDEs on manifolds with smooth boundary.

The pioneering work in the application of factorization to boundary value problems for elliptic differential systems was carried out by Y.Lopatinskij [Lo1, Lo2].

It turned out that the same Wiener–Hopf method can successfully be applied to derive full asymptotic expansion of solutions to PsDEs. G.Eskin [Es1, § 7] had applied the method to the investigation of the leading term of asymptotic of a scalar pseudodifferential equation. M.Costabel & E.Stephan [CS1] applied the explicit factorization of the symbol, while R.Duduchava & W.Wendland [DW1]–implicit one to describe the leading term of asymptotic of the solution to a system of PsDEs (see also [Es1, § 26]). In [CS1, DW1] the results were applied to the crack problem of elasticity.

Further progress was a full asymptotic expansion of the solution to a system of pseudodifferential equations on arbitrary but smooth manifold with smooth boundary, obtained by G.Eskin in [Es1, Sec. 23]. This result was extended by J.Bennish in [Be1] (L_2–theory). In the present investigation the results are extended further: equations are treated in the weighted anisotropic Bessel potential spaces, which are well–adapted to the analysis of smoothness and asymptotic of solutions to PsDEs on smooth manifolds with smooth boundary (L_p–theory). More transparent asymptotic expansion formula is presented, which demonstrates clear dependences of asymptotic (of exponents, of presence and disposition of logarithmic terms) on the geometry of manifold and on the symbol of PsDE. The obtained dependences can be applied, for example, to reveal connections between different elastic fields (the stress tensor field, the traction and the displacement vector fields) on crack faces and on the prolongation of the crack surface. The latter connections play a crucial role in rupture criteria for elastic materials and will be treated in one of forthcoming papers. The results on asymptotic are already applied to different BVPs of elasticity (see e.g. R.Duduchava, A.M.Sändig, W.L.Wendland [DSW1], R.Duduchava, D.Natroshvili [DN1], O.Chkadua [Ch1]–[Ch4]).

The paper is organised as follows.

First we present classes of symbols which provide boundedness of corresponding pseudodifferential operators (PsDOs) and are closed with respect to the factorization, i.e. together with elliptic matrices they contain their factors. It is well–known, that by factorization of C^∞–function with respect to one variable, we might get factors which fail to possess even one continuous derivative (moreover–factors can be unbounded). Therefore the most frequently used classes of symbols, e.g. the Hörmander classes $S^r_{\gamma,\delta}(\omega, \mathbb{R}^n)$, are not suitable for the Wiener–Hopf method. This requires to extend classes of symbols of PsDEs up to relevant ones with respect to the Wiener–Hopf factorization. In our investigations we rely on the generalisation of Mikhlin–Hörmander–Lizorkin multiplier theorem for PsDOs, given by E. Shargorodsky [Sh1] (see Subsection 1.2).

The factorization is important tool in obtaining the FREDHOLM properties and investigating the solvability of PsDEs. In most problems in applications solvability of BPsDEs can be derived from the strong ellipticity or even positive definiteness of the corresponding symbol, but this does not work in some cases. The FREDHOLM property criteria of PsDEs on manifolds with boundary is presented in Subsection 1.5. Equations are studied in the anisotropic BESSEL potential spaces with weight $H_p^{(\mu,s),m}(\mathcal{M})$, introduced in [Es1] for the case $p = 2$ (cf. also [Be1]) . Main features of this space can be described as follows: $\mu + s$ indicates the smoothness of a function $\varphi(t', \rho)\,, \quad \varphi \in H_p^{(\mu,s),m}(\mathcal{M})$, with respect to the local variable t' on the boundary of the manifold $\partial\mathcal{M}$, while s indicates the smoothness with respect to the variable ρ, implementing the distance to the boundary $\partial\mathcal{M}$; moreover, $\rho^k\varphi(t', \rho)$ becomes smoother and belongs to $H_p^{(\mu,s),m+k}(\mathcal{M})$ for arbitrary $k \le m$. Precise definition of the space in § 1.3 is followed by the theorem on FREDHOLM properties and the index of PsDEs. These properties and the index turn out to be invariant with respect to the weight parameter $m = 0, 1, \cdots$ and with respect of the co-normal smoothness parameter $\mu \in \mathbb{R}$. The weighted anisotropic BESSEL potential spaces play a crucial role in obtaining asymptotic of solutions to systems of BPsDEs (cf. § 2). Results of Section 1 are revising those from [Du1, Es1, Si1, Sh2, Sr2] and enrich them to comply with the purposes of the present investigation.

In Section 2, continuing the investigations of G.ESKIN AND J.BENNISH, complete asymptotic expansions of solutions to pseudodifferential equations on manifolds with boundaries are derived. The results demonstrate transparent dependence of exponents and of coefficients of the expansion on the symbol of PsDE and on the geometry of the underlying manifold $\mathcal{M}$.

Compared with a similar asymptotic obtained earlier by the method of V.KONDRAT'EV (see [Ko1, KMR1, MP1, Da1, Gr1, NP1] etc.), the WIENER-HOPF method provides more transparent formulae for the exponents and coefficients of the expansion. For example, exponents are found as eigenvalues of the symbol matrix at certain points. On the other hand by applying the obtained results we can get rigorous justification of asymptotic for solutions to BPsDEs, encountered in elasticity and other problems of mathematical physics (cf. [DSW1]), which were available before only by ansatz (cf. [Wi1, Ti1]).

Investigations of asymptotic started in the present paper is continued in the paper [CD1], where spatial asymptotic of a function represented by potentials will be derived provided the asymptotic of a density on the surface is known. The topic is important because after obtaining asymptotic of solutions $v(x)$ to boundary PsDE one needs asymptotic of the solution to the corresponding BVP, which is written as potential with the density $v(x)$. Applications to different BVPs and, especially, to crack problems of anisotropic elasticity, is the main purpose.

1. Fredholm criteria

We recall definitions and some important results on Pseudodifferential operators, which are main tools in our investigations.

1.1. Spaces

We recall results mostly from [Tr1, Tr2] (see also [DW1], [Sr2, §1.1]).

$S(\mathbb{R}^n)$ denotes the SCHWARTZ space of all fast decaying functions and $S'(\mathbb{R}^n)$ – the dual space of tempered distributions. Since the FOURIER transform and its inverse, defined by

$$\mathcal{F}\varphi(\xi) = \int\limits_{\mathbb{R}^n} e^{i\xi x}\varphi(x)dx \qquad \text{and}$$
$$(1.1)\qquad \mathcal{F}^{-1}\psi(x) = \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} e^{-ix\xi}\varphi(\xi)d\xi, \quad x,\xi\in\mathbb{R}^n$$

are bounded operators in both spaces $S(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$, the **convolution operator**

$$(1.2)\qquad \mathbf{a}(D)\varphi = W_a^0\varphi := \mathcal{F}^{-1}a\mathcal{F}\varphi \quad \text{with} \quad a\in S'(\mathbb{R}^n), \quad \varphi\in S(\mathbb{R}^n)$$

is a bounded transformation from $S(\mathbb{R}^n)$ into $S'(\mathbb{R}^n)$ (see [Du1, DS1]).

The BESSEL potential space $\mathbb{H}_p^s(\mathbb{R}^n)$ is defined as a subset of $S'(\mathbb{R}^n)$ endowed with the norm (see [Tr1, Tr2])

$$(1.3)\qquad ||u\big|\mathbb{H}_p^s(\mathbb{R}^n)|| := ||\langle D\rangle^s u\big|L_p(\mathbb{R}^n)||, \text{ where } \langle\xi\rangle^s := (1+|\xi|^2)^{\frac{s}{2}}\,.$$

For the definitions of the BESOV spaces $\mathbb{B}_{p,q}^s(\mathbb{R}^n)$ ($1\le p\le\infty$, $1\le q\le\infty$, $s\in\mathbb{R}$) see [Sh1, Tr1]: the space $\mathbb{B}_{p,p}^s(\mathbb{R}^n)$ ($1<p<\infty$, $s>0$) coincides with the trace space $\gamma_{\mathbb{R}^n}\mathbb{H}_p^{s+\frac{1}{p}}(\mathbb{R}_+^{n+1})$ ($\mathbb{R}_+^{n+1} := \mathbb{R}^n\times\mathbb{R}^+$) and is known also as the SOBOLEV–SLOBODEČKII space $W_p^s(\mathbb{R}^n)$.

The space $\mathbb{B}_{\infty,\infty}^s(\mathbb{R}^n)$ coincides with the well–known ZYGMUND space $\mathbb{Z}^s(\mathbb{R}^n)$:

$$||f|\mathbb{Z}^s(\mathbb{R}^n)|| = ||f|C^{[s]^-}(\mathbb{R}^n)|| + \sum_{|\alpha|=[s]^-}\sup_{h\in\mathbb{R}^n\setminus\{0\}}\left\{|h|^{-\{s\}^+}||\Delta_h^2\partial^\alpha f|C(\mathbb{R}^n)||\right\},$$
$$s=[s]^- + \{s\}^+ \quad [s]^-\in\mathbb{N}_0, \quad 0<\{s\}^+\le 1,$$

where $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$, $\mathbb{N}$ denotes the set of positive integers, $\Delta_h f(x) := f(x+h)-f(x)$, $\Delta_h^2 = \Delta_h\Delta_h$ and

$$||f|C^m(\mathbb{R}^n)|| = \sum_{|\alpha|\le m}\sup\{|\partial^\alpha f(x)| : x\in\mathbb{R}^n\}.$$

For $s\in\mathbb{R}^+\setminus\mathbb{N}$ the space $\mathbb{B}_{\infty,\infty}^s(\mathbb{R}^n)$ (and $\mathbb{Z}^s(\mathbb{R}^n)$) coincide with the HÖLDER space $C^s(\mathbb{R}^n)$

$$||f|C^s(\mathbb{R}^n)|| = ||f|C^{[s]}(\mathbb{R}^n)|| + \sum_{|\alpha|=[s]}\sup_{h\in\mathbb{R}^n\setminus\{0\}}\left\{|h|^{-\{s\}}||\Delta_h\partial^\alpha f|C(\mathbb{R}^n)||\right\},$$
$$s=[s]+\{s\}, \quad [s]\in\mathbb{N}_0, \qquad 0<\{s\}<1.$$

The space $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^n_+)$ is defined as the subspace of $\mathbb{H}^s_p(\mathbb{R}^n)$ of those functions $\varphi \in \mathbb{H}^s_p(\mathbb{R}^n)$, which are supported in the half space $\operatorname{supp}\varphi \subset \overline{\mathbb{R}}^n_+$ whereas $\mathbb{H}^s_p(\mathbb{R}^n_+)$ denotes the quotient space $\mathbb{H}^s_p(\mathbb{R}^n_+) = \mathbb{H}^s_p(\mathbb{R}^n)/\widetilde{\mathbb{H}}^s_p(\mathbb{R}^n \setminus \mathbb{R}^n_+)$ and can be identified with the space of distributions φ on $\mathbb{R}^n_+$ which admit an extension $\ell\varphi \in \mathbb{H}^s_p(\mathbb{R}^n)$. Therefore $r_{\mathcal{M}}\mathbb{H}^s_p(\mathbb{R}^n) = \mathbb{H}^s_p(\mathbb{R}^n_+)$.

The spaces $\widetilde{\mathbb{B}}^s_{p,q}(\mathbb{R}^n_+)$ and $\mathbb{B}^s_{p,q}(\mathbb{R}^n_+)$ are defined similarly [Tr1, Tr2].

1.2. Symbol classes

If the convolution operator in (1.2) has the bounded extension

$$W^0_a : L_p(\mathbb{R}^n) \longrightarrow L_p(\mathbb{R}^n),$$

then we write $a \in M_p(\mathbb{R}^n)$ and $a(\xi)$ is called a (Fourier) L_p–multiplier. For $\nu \in \mathbb{R}$ let

$$M^{(\nu)}_p(\mathbb{R}^n) = \{\langle\xi\rangle^\nu a(\xi) : a \in M_p(\mathbb{R}^n)\}\ .$$

By using the isomorphism (1.3) and the obvious property

$$W^0_{a_1}W^0_{a_2} = W^0_{a_1a_2}, \quad a_j \in M^{(\nu_j)}_p(\mathbb{R}_n), \quad j = 1, 2, \tag{1.4}$$

we get that the operator

$$W^0_a : \mathbb{H}^s_p(\mathbb{R}^n) \longrightarrow \mathbb{H}^{s-\nu}_p(\mathbb{R}^n)$$

is bounded if and only if $a \in M^{(\nu)}_p(\mathbb{R}^n)$.

Vice versa: if $A : \mathbb{H}^s_p(\mathbb{R}^n) \longrightarrow \mathbb{H}^{s-\nu}_p(\mathbb{R}^n)$ is a bounded operator for all $s \in \mathbb{R}$, is translation invariant $AV_\lambda = V_\lambda A$ where $V_\lambda\varphi(x) := \varphi(x - \lambda)$ for all $\lambda > 0$, then obviously $A : C^\infty_0(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$ is continuous and, due to [Hr1, Theorem 4.2.1], this implies $A = W^0_a$ with $a \in M^{(\nu)}_p(\mathbb{R}^n)$.

The next theorem is a slight modification of the Mikhlin–Hörmander–Lizorkin multiplier theorem. Proofs can be found in [Sr1] and in [Hr1, Theorem 7.9.5].

Theorem 1.1. *If*

$$|\xi^\beta\partial^\beta a(\xi)| \le M\langle\xi\rangle^\nu, \quad \xi \in \mathbb{R}^n\,, \tag{1.5}$$

for some $M > 0$ *and all* $|\beta| \le \left[\dfrac{n}{2}\right] + 1,\ \ \beta \le 1,$ *then* $a \in \bigcap\limits_{1<p<\infty} M^{(\nu)}_p(\mathbb{R}^n)$.

Let $a \in M^{(\nu)}_p(\mathbb{R}^n)$. Then the operators

$$\begin{array}{rcrcl} W_a := r_+\mathbf{a}(D) & : & \widetilde{\mathbb{H}}^s_p(\mathbb{R}^n_+) & \longrightarrow & \mathbb{H}^{s-\nu}_p(\mathbb{R}^n_+), \\ & : & \widetilde{\mathbb{B}}^s_{p,q}(\mathbb{R}^n_+) & \longrightarrow & \mathbb{B}^{s-\nu}_{p,q}(\mathbb{R}^n_+), \end{array}$$

are bounded, where $r_+ := r_{\mathbb{R}^n_+}$ is the restriction operator and

$$D := i\partial := i(\partial_1, \cdots, \partial_n) \quad \partial_j := \frac{\partial}{\partial x_j}$$

is used as argument because it corresponds, by the above definition, to the symbol $\xi := (\xi_1, \cdots, \xi_n)$.

The composition rule (1.4) fails in general for half–space operators (1.6). But if there exists an analytic continuation $a_1(\xi', \xi_n - i\lambda)$ (or $a_2(\xi', \xi_n + i\lambda)$) for $\xi_n \in \mathbb{R}$ and $\lambda \in \mathbb{R}^+$ which belongs to $S'(\mathbb{R}^{n-1} \times \mathbb{C}^-)$ (to $S'(\mathbb{R}^{n-1} \times \mathbb{C}^+)$, respectively), where $\mathbb{C}^\pm = \mathbb{R} \oplus i\mathbb{R}^\pm$, then

$$W_{a_1} W_{a_2} = W_{a_1 a_2}. \tag{1.6}$$

If the symbol $a(x, \xi)$ depends on the variable x, then the corresponding convolution operator (see (1.2))

$$\mathbf{a}(x, D)\varphi(x) = W^0_{a(x,\cdot)}\varphi(x) := \left(\mathcal{F}^{-1}_{\xi \longrightarrow x} a(x, \xi)\mathcal{F}_{y \longrightarrow \xi}\varphi(y)\right)(x) \tag{1.7}$$

with the symbol $a \in C(\mathbb{R}^n, S'(\mathbb{R}^n))$ is called a general **pseudodifferential** operator acting on $\varphi \in S(\mathbb{R}^n)$. Here $C(\omega, \mathcal{B})$ denotes the set of all continuous functions $a : \omega \longrightarrow \mathcal{B}$. Let $M^{(s,s-\nu)}_p(\mathbb{R}^n, \mathbb{R}^n)$ denote the class of symbols $a(x, \xi)$ for which the operator in (1.7) extends to the bounded mapping

$$\mathbf{a}(x, D) : \mathbb{H}^s_p(\mathbb{R}^n) \longrightarrow \mathbb{H}^{s-\nu}_p(\mathbb{R}^n) \tag{1.8}$$

and $M^{(\nu)}_p(\mathbb{R}^n, \mathbb{R}^n) := \bigcup\limits_{s \in \mathbb{R}} M^{(s,s-\nu)}_p(\mathbb{R}^n, \mathbb{R}^n)$.

Lemma 1.2. [DW1, Lemma 1.7] *Let $a, b \in M^{(s,s-\nu)}_p(\mathbb{R}^n \times \mathbb{R}^n)$, $s, \nu \in \mathbb{R}$. If there exist analytic continuations $a(x, \xi', \xi_n + i\lambda)$ and $b(x, \xi', \xi_n - i\lambda)$ for all $x \in \mathbb{R}$, $\xi' \in \mathbb{R}$, $\xi_n \in \mathbb{R}$, $\lambda \in \mathbb{R}^+$ with polynomial growth at ∞ (i. e. $|a|$ and $|b|$ are majorized by $(|\xi'| + |\xi_n| + \lambda)^N$ for some N and all $x \in \mathbb{R}^n$ uniformly), then the operators*

$$\begin{array}{rcccl} \mathbf{a}(x, D) & : & \widetilde{\mathbb{H}}^s_p(\mathbb{R}^n_+) & \longrightarrow & \widetilde{\mathbb{H}}^{s-\nu}_p(\mathbb{R}^n_+)\,, \\ r_+\mathbf{b}(x, D)\ell & : & \mathbb{H}^s_p(\mathbb{R}^n_+) & \longrightarrow & \mathbb{H}^{s-\nu}_p(\mathbb{R}^n_+) \end{array} \tag{1.9}$$

are bounded and

$$\begin{array}{rll} r_+\mathbf{a}(x, D)\varphi = \mathbf{a}(x, D)\varphi\,, & & \varphi \in \widetilde{\mathbb{H}}^s_p(\mathbb{R}^n_+), \\ r_+\mathbf{b}(x, D)\ell r_+\psi = r_+\mathbf{b}(x, D)\psi\,, & & \psi \in \mathbb{H}^s_p(\mathbb{R}^n_+)\,. \end{array} \tag{1.10}$$

Here ℓ is an arbitrary extension of $\varphi \in \mathbb{H}^s_p(\mathbb{R}^n_+)$ with $\ell\varphi \in \mathbb{H}^s_p(\mathbb{R}^n)$. The operator in (1.9) *is independent of the choice of ℓ.*

Theorem 1.3. *Let $\mathbb{N}_0 := \{0, 1, \ldots\}$. If*

$$\int\limits_{\mathbb{R}^n} |\xi^\beta \partial^\alpha_x \partial^\beta_\xi a(x, \xi)| dx \leq M_\alpha \langle\xi\rangle^\nu, \quad \xi \in \mathbb{R}^n \tag{1.11}$$

for some $M_\alpha > 0$ and all $\alpha, \beta \in \mathbb{N}^n_0$, $|\beta| \leq \left[\dfrac{n}{2}\right] + 1$, $\beta \leq 1$, then $a \in \bigcap\limits_{1<p<\infty} M^{(\nu)}_p(\mathbb{R}^n, \mathbb{R}^n)$.

Proof.. The claim follows from the general theorems on multipliers [Sh1, Theorems 4.1 and 5.1] and Theorem 1.1. □

Definition 1.4. For any $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N} := \{1, 2, \ldots\}$, $\nu \in \mathbb{R}$ by $\mathcal{R}^m_{hom,\nu}(\Omega, \mathbb{R}^n)$, $m = 0, 1, ..., \infty$ we denote the class of symbols

$$a(x,\xi) = a_\infty(\xi) + a_0(x,\xi), \tag{1.12}$$

where:

1. both summands are homogeneous of order ν in ξ, that is $a_\infty(\lambda\xi) = \lambda^\nu a_\infty(\xi)$ and $a_0(x,\lambda\xi) = \lambda^\nu a_0(x,\xi)$ for all $\lambda > 0$, $\xi \in \mathbb{R}^n$;
2. there exist constants M_β and $M_{\alpha,\beta}$ such that

$$\begin{gathered} |(\xi')^{\beta'} \partial_\xi^\beta a_\infty(\xi)| \le M_\beta |\xi|^{\nu-\beta_n}, \\ \int_\Omega |(\xi')^{\beta'} \partial_x^\alpha \partial_\xi^\beta a_0(x,\xi)| dx \le M_{\alpha,\beta} |\xi|^{\nu-\beta_n} \end{gathered} \tag{1.13}$$

for all $\alpha, \beta = (\beta', \beta_n) \in \mathbb{N}_0^n$, $|\beta'| \le m$, $\beta_n = 0, 1, ...$ and all $\xi \in \mathbb{R}^n$.

Definition 1.5. By $\mathcal{R}^m_\nu(\Omega, \mathbb{R}^n)$ we denote the class of symbols $a(x,\xi) = a_{pr}(x,\xi) + a^0(x,\xi)$ where $a_{pr} \in \mathcal{R}^m_{hom,\nu}(\Omega, \mathbb{R}^n)$ is known as the **homogeneous principal** symbol and $a^0(x,\xi)$ have the following estimates

$$\int_\Omega |(\xi')^{\beta'} \partial_x^\alpha \partial_\xi^\beta a^0(x,\xi)| dx \le M_{\alpha,\beta} |\xi|^{\nu-\beta_n-1} \tag{1.14}$$

for all $\alpha, \beta = (\beta', \beta_n) \in \mathbb{N}_0^n$, $|\beta'| \le m$, $\beta_n = 0, 1, ...,$ and all $\xi \in \mathbb{R}^n$.

Definition 1.6. By $\mathcal{S}^m_{hom,\nu}(\Omega, \mathbb{R}^n)$, $m = 0, 1, ..., \infty$ we denote a subclass of $\mathcal{R}^m_{hom,\nu}(\Omega, \mathbb{R}^n)$, which consists of symbols (1.12) with stronger estimates than (1.13):

$$\begin{gathered} |\partial_\xi^\beta a_\infty(\xi)| \le M_\beta |\xi|^{\nu-|\beta|}, \\ \int_\Omega |(\partial_x^\alpha \partial_\xi^\beta a_0)(x,\xi)| dx \le M_{\alpha,\beta} |\xi|^{\nu-|\beta|}, \end{gathered} \tag{1.15}$$

for all $\alpha, \beta = (\beta', \beta_n) \in \mathbb{N}_0^n$, $|\beta'| \le m$, $\beta_n = 0, 1, ...$ and all $\xi \in \mathbb{R}^n$.

Definition 1.7. By $\mathcal{S}^m_\nu(\Omega, \mathbb{R}^n)$, $m = 0, 1, ..., \infty$ we denote the class of symbols $a(x,\xi) = a_{pr}(x,\xi) + a^0(x,\xi)$ where $a_{pr} \in \mathcal{S}^m_{hom,\nu}(\Omega, \mathbb{R}^n)$ and $a^0(x,\xi)$ has the following estimates (cf. (1.14))

$$\int_\Omega |\partial_x^\alpha \partial_\xi^\beta a^0(x,\xi)| dx \le M_{\alpha,\beta} |\xi|^{\nu-|\beta|-1} \tag{1.16}$$

for all $\alpha, \beta = (\beta', \beta_n) \in \mathbb{N}_0^n$, $|\beta'| \le m$, $\beta_n = 0, 1, ...,$ and all $\xi \in \mathbb{R}^n$.

We will drop $m=\infty$ and use $\mathcal{R}_{hom,\nu}(\Omega,\mathbb{R}^n)$, $\mathcal{R}_{\nu}(\Omega,\mathbb{R}^n)$, $\mathcal{S}_{\nu}(\Omega,\mathbb{R}^n)$, ... instead of $\mathcal{R}^{\infty}_{hom,\nu}(\Omega,\mathbb{R}^n)$, $\mathcal{R}^{\infty}_{\nu}(\Omega,\mathbb{R}^n)$, $\mathcal{S}^{\infty}_{\nu}(\Omega,\mathbb{R}^n)$, ... when this will not lead to a confusion.

Definition 1.8. We write $a\in\mathcal{S}^m_{cl,\nu}(\Omega,\mathbb{R}^n)$ if $a(x,\xi)$ has the following asymptotic expansion

$$a(x,\xi)\simeq a_0(x,\xi)+a_1(x,\xi)+\cdots \tag{1.17}$$

where a_k is a homogeneous symbol of the class $\mathcal{S}^m_{hom,\nu-k}(\Omega,\mathbb{R}^n)$, $k=0,1,...,N$ and for any natural $N\in\mathbb{N}_0$ the difference

$$\widetilde{a}_{N+1}(x,\xi):=a(x,\xi)-a_0(x,\xi)-\cdots-a_N(x,\xi)$$

has the estimate

$$\int\limits_{\Omega}|\partial^{\alpha}_x\partial^{\beta}_{\xi}\widetilde{a}_{N+1}(x,\xi)|dx\le M_{\alpha,\beta}|\xi|^{\nu-|\beta|-N-1} \tag{1.18}$$

for all $\xi\in\mathbb{R}^n$, $|\xi|\ge 1$ and all $\alpha,\beta\in\mathbb{N}^n_0$, $|\beta'|\le m$.

$a_0(x,\xi)=a_{pr}(x,\xi)$ in (1.17) is known as the **homogeneous principal** symbol of $a(x,D)$.

Finally, we use $\mathcal{S}^{\infty}_{cl,\nu}(\Omega,\mathbb{R}^n):=\bigcap\limits_m\mathcal{S}^m_{cl,\nu}(\Omega,\mathbb{R}^n)$.

It is obvious that

$$\begin{gathered}\mathcal{S}^{\infty}_{hom,\nu}(\Omega,\mathbb{R}^n)\subset\mathcal{S}_{cl,\nu}(\Omega,\mathbb{R}^n)\subset\mathcal{S}^m_{\nu}(\Omega,\mathbb{R}^n)\subset\mathcal{R}^m_{\nu}(\Omega,\mathbb{R}^n),\\ \mathcal{S}_{hom,\nu}(\Omega,\mathbb{R}^n)\subset\mathcal{R}_{hom,\nu}(\Omega,\mathbb{R}^n)\subset\mathcal{R}_{\nu}(\Omega,\mathbb{R}^n)\,.\\ \mathcal{R}^m_{hom,\nu}(\Omega,\mathbb{R}^n)\subset\mathcal{R}^m_{\nu}(\Omega,\mathbb{R}^n),\end{gathered} \tag{1.19}$$

For $a\in\mathcal{R}_{\nu}(\Omega,\mathbb{R}^n)$ we can consider the modified symbol

$$\overset{\circ}{a}(x,\xi):=a(x,\langle\xi'\rangle|\xi'|^{-1}\xi',\xi_n), \tag{1.20}$$

(see [DW1] and [Es1, p.91]) and the truncated symbol

$$\overset{\vee}{a}(x,\xi):=[1-\chi_0(\xi)]a(x,\xi), \tag{1.21}$$

where $\chi_0\in C^{\infty}_0(\mathbb{R})$, $\chi_0(\xi)=0$ for $|\xi|\ge 1$, $\chi_0(\xi)=1$ for $|\xi|\le\dfrac{1}{2}$.

Lemma 1.9. *Let* $a\in\mathcal{S}^m_{\nu}(\Omega,\mathbb{R}^n)$ *and* $m\ge\left[\frac{n}{2}\right]+2$.
Then $\overset{\circ}{a},\overset{\vee}{a}\in\bigcap\limits_{1<p<\infty}M^{(\nu)}_p(\Omega,\mathbb{R}^n)$ *and*

$$a-\overset{\vee}{a}\in\bigcap\limits_{-\infty<\mu\le\nu}\mathcal{R}^m_{\mu}(\Omega,\mathbb{R}^n) \tag{1.22}$$

$$(1-\chi_1)(a-\overset{\circ}{a})\in\mathcal{R}^m_{\nu-1}(\Omega,\mathbb{R}^n), \tag{1.23}$$

where $\chi_1(\xi):=\chi_0\left(\dfrac{1}{2}\xi\right)$.

Proof. Inclusions in the multiplier classes (the first claim) follow from Theorems 1.1 and 1.3 since for both $\overset{\circ}{a}$ and $\overset{\vee}{a}$ inequalities (1.13) and (1.14) hold with $|\xi|$ replaced by $\langle\xi\rangle$.

(1.22) is obvious since $a - \overset{\vee}{a}$ has a compact support.

In the proof of (1.23) we follow [Es1, page 91] and [DW1, Lemma 1.4]: by the mean value theorem

$$a(x,\xi) - \overset{\circ}{a}(x,\xi) = a(x,\xi',\xi_n) - a(x,\xi'+\omega,\xi_n) = \sum_{k=1}^{n-1} (\partial_{\xi_k} a)(x,\xi'+\theta\omega,\xi_n)\omega_k$$

$$\text{with} \quad \omega := \omega(\xi') = (<\xi'> - |\xi'|)|\xi'|^{-1}\xi' = \frac{1}{<\xi'> + |\xi'|}|\xi'|^{-1}\xi'$$

for some $0 < \theta < 1$. Obviously,

$$|\omega| \le 1, \ |(\xi')^{\alpha'} \partial_{\xi'}^{\alpha'} \omega(\xi')| \le M_{\alpha'} < \infty, \quad \text{for all} \quad \alpha' \in \mathbb{N}_0^{n-1}$$

and

$$\frac{1}{2}|\xi| \le |\xi'| + |\xi_n| - 1 \le |\xi' + \theta\omega| + |\xi_n| \le |\xi'| + |\xi_n| + 1 \le \sqrt{2}|\xi| + \frac{1}{2}|\xi| \le 2|\xi|$$

provided $|\xi| \ge 2$; therefore,

$$|(\xi')^{\beta'} \partial_x^\alpha \partial_\xi^\beta [a(x,\xi) - \overset{\circ}{a}(x,\xi)]| = \left| \int_0^1 (\xi')^{\beta'} \partial_x^\alpha \partial_\xi^\beta \partial_\tau a(x,\xi'+\tau\omega,\xi_n) d\tau \right|$$

$$= \left| \sum_{k=1}^{n-1} \sum_{\gamma' \le \beta'} M^0_{\alpha,k,\gamma'} (\xi')^{\beta'} \omega_k(\xi') (\partial_{\xi'}^{\beta'-\gamma'}\omega)(\xi') \int_0^1 (\partial_x^\alpha \partial_{\xi'}^{\gamma'} \partial_{\xi_k} \partial_{\xi_n}^{\beta_n} a)(x,\xi'+\tau\omega(\xi'),\xi_n) d\tau \right|$$

$$\le \sum_{k=1}^{n-1} \sum_{\gamma' \le \beta'} M^1_{\alpha,k,\gamma'} |(\xi' + \theta_{\alpha,k,\gamma'}\omega, \xi_n)|^{\nu-\beta_n-1} \le M_{\alpha,\beta}|\xi|^{\nu-\beta_n-1}, \quad 0 < \theta_{\alpha,k,\gamma'} < 1$$

for $|\xi| \ge 2$ and all $\alpha \in \mathbb{N}_0^n$, $|\beta'| \le \left[\frac{m}{2}\right] + 1$, $\beta_n = 0, 1, \ldots$. □

If $\mathcal{X}_\nu(\mathcal{M}, \mathbb{R}^n)$ is a symbol class, by $\mathcal{TX}_\nu(\mathcal{M}, \mathbb{R}^n)$ we denote those symbols $a \in \mathcal{X}_\nu(\mathcal{M}, \mathbb{R}^n)$ for which $a(x,\xi) = a_0(x, t\omega, \xi_n)$ with $\omega = |\xi'|^{-1}\xi' \in S^{n-2}$, $t = |\xi'| \in \mathbb{R}^+$, $\xi_n \in \mathbb{R}$, the derivatives $\partial_t^k a_0(x, t\omega, \pm 1)$ exist and

$$\begin{gathered} \lim_{t\to 0} \partial_t^k a_0(x,t\omega,-1) = (-1)^k \lim_{t\to 0} \partial_t^k a_0(x,t\omega,1), \\ \text{for all} \quad x \in \overline{\mathcal{M}}, \quad \omega \in S^{n-2}, \quad k = 0, 1, \ldots. \end{gathered} \tag{1.24}$$

Let us note, that for homogeneous $a \in \mathcal{R}^m_{hom,\nu}(\mathcal{M}, \mathbb{R}^n)$ derivatives in (1.24) exist automatically because $a_0(x, t\omega, \pm 1) = t^\nu a_0(x, \omega, \pm t^{-1})$.

(1.24) is known as the transmission property (see [Es1, p.278], [GH1], [Hr1, Sec.18.2], [RS1, Sec.1.1.2]).

The next Lemma is a particular case of [DW1, Lemma 1.8], proved by E. Shargorodsky, and a generalisation of the well–known Lemma on composition of pseudodifferential operators.

Lemma 1.10. *Let* $m \in \mathbb{N}_0$, $\nu_1, \nu_2 \in \mathbb{R}$ *and* $a_j^{(m)} \in \mathcal{R}^m_{\nu_j}(\mathbb{R}^n, \mathbb{R}^n)$, *where* $a_j^{(m)}(x,\xi) := \langle x\rangle^m a_j(x,\xi)$, $j = 1, 2$. *Then*

$$a_1(x,D)a_2(\cdot,D) = (a_1a_2)(x,D) + g(x,D)$$

with $g \in \mathcal{R}^m_{\nu_1+\nu_2-1}(\mathbb{R}^n, \mathbb{R}^n)$.

Note, that if $a_j(x,\xi)$ has a compact support in x–variable, then $a_j \in \mathcal{R}^m_{\nu_j}(\mathbb{R}^n, \mathbb{R}^n)$ implies $a_j^{(m)} \in \mathcal{R}^m_{\nu_j}(\mathbb{R}^n, \mathbb{R}^n)$.

1.3. Anisotropic Bessel potential spaces with weight

Next we define anisotropic Bessel potential spaces with weight, similar to [Es1, Sections 23 and 26].

Let $\mu, s \in \mathbb{R}$, $m \in \mathbb{N}_0$ and $1 < p < \infty$; by $\mathbb{H}^{(\mu,s),m}_p(\mathbb{R}^n)$ we denote the space of functions (of distributions when $\mu < 0$ or $\mu + s < 0$) endowed with the norm

$$\|u\|^{(\mu,s),m}_p = \|u\big|\mathbb{H}^{(\mu,s),m}_p(\mathbb{R}^n)\| := \sum_{k=0}^m \|\langle D'\rangle^\mu \langle D\rangle^{s+k} x_n^k u\big|L_p(\mathbb{R}^n)\|\,, \tag{1.25}$$

$$\xi' = (\xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{n-1}, \quad \xi = (\xi', \xi_n) \in \mathbb{R}^n\,.$$

We shall write $\mathbb{H}^{(\mu,s)}_p(\mathbb{R}^n)$ for $\mathbb{H}^{(\mu,s),0}_p(\mathbb{R}^n)$ and $\mathbb{H}^{s,m}_p(\mathbb{R}^n)$ for $\mathbb{H}^{(0,s),m}_p(\mathbb{R}^n)$, because $\mathbb{H}^{(0,s)}_p(\mathbb{R}^n) = \mathbb{H}^s_p(\mathbb{R}^n)$.

The operator

$$\langle D'\rangle^\nu \langle D\rangle^r : \mathbb{H}^{(\mu,s),m}_p(\mathbb{R}^n) \longrightarrow \mathbb{H}^{(\mu-\nu,s-r),m}_p(\mathbb{R}^n)$$

arranges an isomorphism of spaces for arbitrary $\nu, r \in \mathbb{R}$ and the inverse operator reads $\langle D'\rangle^{-\nu}\langle D\rangle^{-r}$

In fact, the last claim about invertibility follows easily from (1.6). To prove that $\langle D'\rangle^\nu\langle D\rangle^r$ arranges an isomorphism we recall the following equality

$$x_n^k \mathbf{a}(x,D)u(x) = \sum_{l=0}^k \frac{(-i)^l k!}{l!(k-l)!}(\partial^l_{\xi_n}\mathbf{a})(x,D)\left[x_n^{k-l}u(x)\right] \qquad \forall\, u \in \mathbb{S}(\mathbb{R}^n)\,, \tag{1.26}$$

which is verified straightforwardly by applying integration by parts:

$$x_n^k \mathbf{a}(x,D)u(x) \quad = \quad \frac{x_n^k}{(2\pi)^n}\int\limits_{\mathbb{R}^n} e^{-ix\xi}a(x,\xi)\int\limits_{\mathbb{R}^n} e^{i\xi y}u(y)\,dy\,d\xi$$

$$= \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n}\left[(i\partial_{\xi_n})^k e^{-ix\xi}\right]a(x,\xi)\int\limits_{\mathbb{R}^n} e^{i\xi y}u(y)\,dy\,d\xi$$

$$= \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} e^{-ix\xi}(-i\partial_{\xi_n})^k\left[a(x,\xi)\int\limits_{\mathbb{R}^n} e^{i\xi y}u(y)\,dy\right]d\xi$$

$$= \sum_{l=0}^{k}\frac{(-i)^l k!}{l!(k-l)!}(\partial^l_{\xi_n}\mathbf{a})(x,D)x_n^{k-l}u(x)\,.$$

Applying formula (1.26) we proceed as follows

$$\begin{aligned}
&\|\langle D'\rangle^\nu\langle D\rangle^r u \;\big|\; \mathbb{H}_p^{(\mu-\nu,s-r),m}(\mathbb{R}^n)\| \\
&\quad= \sum_{k=0}^{m}\|\langle D'\rangle^{\mu-\nu}\langle D\rangle^{s-r}x_n^k\langle D'\rangle^\nu\langle D\rangle^r u\big|\mathbb{L}_p(\mathbb{R}^n)\| \\
&\quad= \sum_{k=0}^{m}\|\langle D'\rangle^{\mu}\langle D\rangle^{s-r}x_n^k\langle D\rangle^r u\big|\mathbb{L}_p(\mathbb{R}^n)\| \\
&\quad\le \sum_{k=0}^{m}\sum_{l=0}^{k}\frac{k!}{l!(k-l)!}\|g_n^{r,l}(D)\langle D'\rangle^{\mu}\langle D\rangle^{s-r}x_n^{k-l}u\big|\mathbb{L}_p(\mathbb{R}^n)\| \\
&\quad\le M_m\sum_{k=0}^{m}\|\langle D'\rangle^{\mu}\langle D\rangle^{r}x_n^{k}u\,\big|\mathbb{L}_p(\mathbb{R}^n)\,\| = M_m\|u\big|\mathbb{H}_p^{(\mu,s),m}(\mathbb{R}^n)\|
\end{aligned}$$

with $g_n^{r,l}(\xi) := \partial^l_{\xi_n}\langle\xi\rangle^r$, since $\langle\xi\rangle^{r-l}g_n^{r,l}(\xi) = \langle\xi\rangle^{r-l}\partial^l_{\xi_n}\langle\xi\rangle^r$ is an $\mathbb{L}_p$-multiplier due to Theorem 1.1.

The same is true for the inverse $\langle D'\rangle^{-\nu}\langle D\rangle^{-r}$ and the claimed isomorphism is established.

Lemma 1.11. *Let $\nu,\mu,s\in\mathbb{R}$, $m\in\mathbb{N}_0$, $\sigma(\nu,m)=\max\{0,m-\nu\}$ and*

$$\mathbb{H}_p^{(\infty,s),m}(\mathbb{R}^n) := \bigcap_{\mu\in\mathbb{R}}\mathbb{H}_p^{(\mu,s),m}(\mathbb{R}^n)$$

be a Freschet space with a standard metric. The operator

$$\begin{aligned}
\langle D_n\rangle^\nu \quad &: \quad \mathbb{H}_p^{(\mu,s),m}(\mathbb{R}^n) \longrightarrow \mathbb{H}_p^{(\mu-\sigma(\nu,m),s-\nu),m}(\mathbb{R}^n)\,, \\
&: \quad \mathbb{H}_p^{(\infty,s),m}(\mathbb{R}^n) \longrightarrow \mathbb{H}_p^{(\infty,s-\nu),m}(\mathbb{R}^n)\,,
\end{aligned}\tag{1.27}$$

is bounded for both pairs of spaces.

P r o o f. The boundedness for the second pair of (Freschet) spaces is an obvious consequence of the first one. The boundedness result $\mathbb{H}_p^{(\mu,s),0}(\mathbb{R}^n) \to \mathbb{H}_p^{(\mu-\sigma(\nu,0),s-\nu),m}(\mathbb{R}^n)$ (the case $m=0$) follows from Theorem 1.1 because is equivalent that $\langle\xi_n\rangle^\nu\langle\xi'\rangle^{-\sigma(\nu,m)}\langle\xi\rangle^{-\nu}$ is an $\mathbb{L}_p$-multiplier.

Now let $m = 1, 2, \ldots$. we apply (1.26) and proceed as follows:

$$\begin{aligned}
\|\langle D_n\rangle^\nu u \;\;\big|\;\; &\mathbb{H}_p^{(\mu-\sigma(\nu,m),s-\nu),m}(\mathbb{R}^n)\| \\
= \;& \sum_{k=0}^m \|x_n^k \langle D_n\rangle^\nu u \big| \mathbb{H}_p^{(\mu-\sigma(\nu,m),s-\nu+k)}(\mathbb{R}^n)\| \\
\le \;& \sum_{k=0}^m \sum_{l=0}^k M_{k,l}(\nu) \|h_n^{\nu,l}(D_n) x_n^{k-l} u \big| \mathbb{H}_p^{(\mu-\sigma(\nu,m),s-\nu+k)}(\mathbb{R}^n)\| \\
\le \;& \sum_{k=0}^m \sum_{l=0}^k M_{k,l}(\nu) \|\langle D'\rangle^{\mu-\sigma(\nu,m)} \langle D\rangle^{s-\nu+k} h_n^{\nu,l}(D_n) x_n^{k-l} u \big| \mathbb{L}_p(\mathbb{R}^n)\| \\
\le \;& \sum_{k=0}^m \sum_{l=0}^k M_{k,l}(\nu) \|\langle D'\rangle^{-\sigma(\nu,m)} \langle D\rangle^{l-\nu} h_n^{\nu,l}(D_n) \\
& \qquad\qquad \times \left[\langle D\rangle^{s-l+k} x_n^{k-l} u\right] \big| \mathbb{L}_p(\mathbb{R}^n)\| \\
\le \;& M_0(\nu) \sum_{k=0}^m \sum_{l=0}^k \|\langle D\rangle^{s-l+k} x_n^{k-l} u \big| \mathbb{L}_p(\mathbb{R}^n)\| \le M(\nu) \|u | \mathbb{H}_p^{(\mu,s),m}(\mathbb{R}^n)\| ,
\end{aligned}$$

with $h_n^{\nu,l}(\xi_n) := \partial_{\xi_n}^l \langle \xi_n\rangle^\nu$, since

$$\langle \xi'\rangle^{-\sigma(\nu,m)} \langle \xi\rangle^{l-\nu} h_n^{r,l}(\xi_n) = \langle \xi'\rangle^{-\sigma(\nu,m)} \langle \xi\rangle^{l-\nu} \partial_{\xi_n}^l \langle \xi_n\rangle^\nu$$

is an $\mathbb{L}_p$-multiplier due to Theorem 1.1. □

The boundedness property (1.27) is a clear advantages of the anisotropic spaces $\mathbb{H}_p^{(\mu,s),m}(\mathbb{R}^n)$, especially with $\mu = \infty$. The next theorem generalises this property.

Theorem 1.12. *Let $m \in \mathbb{N}_0$, $1 < p < \infty$. If $\partial_{\xi_n}^k a(x,\xi) \in M_p^{(\nu-k)}(\mathbb{R}^n, \mathbb{R}^n)$ and $\partial_x^\alpha \partial_{\xi_n}^k b(x,\xi_n) = 0(|\xi|^{(\nu-k)})$ for all $k = 0, 1, \cdots, m$, $\alpha \in \mathbb{N}^n$, $x \in \mathbb{R}^n)$ $\xi \in \mathbb{R}$, then the operators*

$$\begin{aligned}
\mathbf{a}(x,D) \;&:\; \mathbb{H}_p^{(\mu,s),m}(\mathbb{R}^n) \longrightarrow \mathbb{H}_p^{(\mu,s-\nu),m}(\mathbb{R}^n) , \\
\mathbf{b}(x,D_n) \;&:\; \mathbb{H}_p^{(\mu,s),m}(\mathbb{R}^n) \longrightarrow \mathbb{H}_p^{(\mu-\sigma(\nu,m),s-\nu),m}(\mathbb{R}^n) , \\
\mathbf{b}(x,D_n) \;&:\; \mathbb{H}_p^{(\infty,s),m}(\mathbb{R}^n) \longrightarrow \mathbb{H}_p^{(\infty,s-\nu),m}(\mathbb{R}^n) ,
\end{aligned}$$

where $\sigma(\nu,m) = \max\{0, m-\nu\}$, are bounded for arbitrary $\mu, s \in \mathbb{R}$, $m \in \mathbb{N}_0$ and $1 < p < \infty$.

In particular, if $a \in \mathcal{R}_\nu^\gamma(\mathbb{R}^n, \mathbb{R}^n)$ and $\gamma \ge \left[\dfrac{n}{2}\right] + 1$, then $\mathbf{a}(x,D)$ is bounded for all $m \in \mathbb{N}_0$ and $\mu, s \in \mathbb{R}$.

Proof. Multipliers of the anisotropic Bessel potential spaces without a weight $\mathbb{H}_p^{(\mu,s)}(\mathbb{R}^n)$ coincide with $M_p(\mathbb{R}^n)$ since for arbitrary $g \in M_p^{(\nu)}(\mathbb{R}^n)$ the equality $\langle D'\rangle^\mu \langle D\rangle^s W_g^0 = W_g^0 \langle D'\rangle^\mu \langle D\rangle^s$ holds. Moreover, the boundedness for $m = 0$ can be proved as for corresponding theorems [Sh1, Theorems 4.1, 5.1] if we invoke Theorem 1.1 (cf. Theorem 1.3).

To prove the boundedness in the weighted spaces we apply (1.26) and proceed as follows

$$\begin{aligned}||\mathbf{a}(x,D)u\big|\mathbb{H}_p^{(\mu,s-\nu),m}(\mathbb{R}^n)|| &= \sum_{k=0}^{m}||x_n^k\mathbf{a}(x,D)u\big|\mathbb{H}_p^{(\mu,s+k-\nu)}(\mathbb{R}^n)|| \\ &\le \sum_{k=0}^{m}\sum_{l=0}^{k}\frac{k!}{l!(k-l)!}||(\partial_{\xi_n}^l\mathbf{a})(x,D)x_n^{k-l}u\big|\mathbb{H}_p^{(\mu,s+k-\nu)}(\mathbb{R}^n)|| \\ &\le M_0\sum_{j=0}^{m}\|x_n^j u\big|\mathbb{H}_p^{(\mu,s+j)}(\mathbb{R}^n)\| \le M_0||u\big|\mathbb{H}_p^{(\mu,s),m}(\mathbb{R}^n)||\,.\end{aligned}$$

As for $\mathbf{b}(x,D_n)$: we note that $b_0(x,\xi_n)=b(x,\xi_n)\langle\xi_n\rangle^{-\nu}\in\mathcal{R}_0^\infty(\mathbb{R}^n,\mathbb{R})$ and therefore

$$\begin{aligned}\mathbf{b}(x,D_n)=\mathbf{b}_0(x,D_n)\langle D_n\rangle^\nu:\mathbb{H}_p^{(\mu,s),m}(\mathbb{R}^n) &\longrightarrow \mathbb{H}_p^{(\mu-\sigma(\nu,m),s-\nu),m}(\mathbb{R}^n)\,,\\ \mathbb{H}_p^{(\infty,s),m}(\mathbb{R}^n) &\longrightarrow \mathbb{H}_p^{(\infty,s-\nu),m}(\mathbb{R}^n)\end{aligned}$$

are bounded as it is clear from (1.27) and the proved part of the theorem. □

The spaces $\widetilde{\mathbb{H}}_p^{(\mu,s),m}(\mathbb{R}_+^n)$ and $\mathbb{H}_p^{(\mu,s),m}(\mathbb{R}_+^n)$ are defined similarly to $\widetilde{\mathbb{H}}_p^s(\mathbb{R}_+^n)$ and to $\mathbb{H}_p^s(\mathbb{R}_+^n)$ (see Subsection 1.1).

Theorem 1.13. *If $a\in\mathcal{R}_\nu^\gamma(\mathbb{R}_+^n,\mathbb{R}^n)$, the operators*

$$r_+\overset{\circ}{\mathbf{a}}(x,D),\quad r_+\overset{\vee}{\mathbf{a}}(x,D):\widetilde{\mathbb{H}}_p^{(\mu,s),m}(\mathbb{R}_+^n)\longrightarrow\mathbb{H}_p^{(\mu,s-\nu),m}(\mathbb{R}_+^n)$$

are bounded for all $\gamma\ge\left[\frac{n}{2}\right]+1$, $\mu,s\in\mathbb{R}$, $m\in\mathbb{N}_0$ and $1<p<\infty$.

In particular,

$$\begin{aligned}r_+\langle D'\rangle^\nu\boldsymbol{\lambda}_+^r(D):\widetilde{\mathbb{H}}_p^{(\mu,s),m}(\mathbb{R}_+^n)&\longrightarrow\widetilde{\mathbb{H}}_p^{(\mu-\nu,s-r),m}(\mathbb{R}_+^n),\\ r_+\langle D'\rangle^\nu\boldsymbol{\lambda}_-^r(D)\ell:\mathbb{H}_p^{(\mu,s),m}(\mathbb{R}_+^n)&\longrightarrow\mathbb{H}_p^{(\mu-\nu,s-r),m}(\mathbb{R}_+^n),\\ \lambda_\pm^r(\xi):=(\xi_n\pm i|\xi'|\pm i)^r,&\,\xi=(\xi',\xi_n)\in\mathbb{R}^n\end{aligned}$$

(cf. (1.9), (1.10)*) arrange isomorphisms of the corresponding spaces.*

If $a\in\mathcal{TR}_\nu(\mathbb{R}_+^n,\mathbb{R}^n)$, then the operators with modified and truncated symbols

$$r_+\overset{\circ}{\mathbf{a}}(x,D),\quad r_+\overset{\vee}{\mathbf{a}}(x,D):\mathbb{H}_p^{(\mu,s),m}(\mathbb{R}_+^n)\longrightarrow\mathbb{H}_p^{(\mu,s-\nu),m}(\mathbb{R}_+^n)$$

(cf. ((1.20), (1.21)*) are also bounded.*

Proof. The first claim follows from Lemma 1.9 and Theorem 1.12.

The second claim follows form the first one and Lemma 1.2 (see [SD1], [Sh2, Theorem 1.12]).

The third claim is proved as in [Es1, p.278], [GH1], [Hr1, Sec.18.2], [RS1, Sec.1.1.2]. □

Lemma 1.14. *The multiplication operator by the* HEAVISIDE *function*

$$\theta_+ I = \ell_0 r_+ : \mathbb{H}_p^{(\mu,s),m}(\mathbb{R}^n) \longrightarrow \mathbb{H}_p^{(\mu,s),m}(\mathbb{R}^n)\,, \qquad \theta_+(\xi) := \frac{1}{2}(1+\operatorname{sgn}\xi_n)\,,$$

where ℓ_0 extends a function by 0 to $\mathbb{R}^n$, is bounded provided

$$\frac{1}{p}-1 < s < \frac{1}{p}\,, \qquad 1<p<\infty\,, \quad \mu\in\mathbb{R}\,, \quad m\in\mathbb{N}_0\,.$$

In particular, under the asserted conditions the spaces $\widetilde{\mathbb{H}}_p^{(\mu,s),m}(\mathbb{R}^n_+)$ and $\mathbb{H}_p^{(\mu,s),m}(\mathbb{R}^n_+)$ can be identified: if $\varphi\in\mathbb{H}_p^{(\mu,s),m}(\mathbb{R}^n_+)$, then $\ell_0\varphi\in\widetilde{\mathbb{H}}_p^{(\mu,s),m}(\mathbb{R}^n_+)$.

Proof. The second claim of the lemma is an equivalent reformulation of the first one.

For $\mu=m=0$ the proof of the first claim can be found in [Sr3, St1], [Tr2, §1.8.7] and can be derived from [Du1, Theorem 1.12], because is equivalent to the invertibility of the operator $\boldsymbol{\lambda}^s_-(D)\boldsymbol{\lambda}^{-s}_+(D)=W^0_{g_s}$ with the symbol $g_s(\xi):=(\xi_n-i|\xi'|-i)^s(\xi_n+i|\xi'|+i)^{-s}$ in the LEBESGUE space $L_p(\mathbb{R}^n)$ (cf. Theorem 1.13).

In the case $\mu\neq 0$, $m\neq 0$ we proceed as follows:

$$\|\theta_+u|\mathbb{H}_p^{(\mu,s),m}(\mathbb{R}^n)\| = \sum_{k=0}^m \|\langle D'\rangle^\mu x_n^k\theta_+u|\mathbb{H}_p^{s+k}(\mathbb{R}^n)\|$$
$$\leq M_1\sum_{k=0}^m\sum_{|\alpha|\leq k}\|\partial_x^\alpha x_n^k\langle D'\rangle^\mu\theta_+u|\mathbb{H}_p^s(\mathbb{R}^n)\| \leq M_2\sum_{k=0}^m\sum_{|\alpha|\leq k}\|\theta_+\partial_x^\alpha x_n^k\langle D'\rangle^\mu u|\mathbb{H}_p^s(\mathbb{R}^n)\|$$
$$\leq M_2\sum_{k=0}^m\sum_{|\alpha|\leq k}\|\partial_x^\alpha x_n^k\langle D'\rangle^\mu u|\mathbb{H}_p^s(\mathbb{R}^n)\| \leq M_3\|u|\mathbb{H}^{(\mu,s),m}(\mathbb{R}^n)\|\,.$$
□

1.4. Pseudodifferential operators on manifolds

Let $\mathcal{M}$ be a compact, closed, C^∞–smooth n–dimensional manifold with a smooth boundary $\Gamma:=\partial\mathcal{M}\neq\emptyset$. Then $\mathcal{M}$ can be embedded in some manifold $\mathcal{M}\subset\widetilde{\mathcal{M}}$ of the same smoothness.

Let $\{Y_j\}_{j=1}^\ell$ be a sufficiently refined covering of $\mathcal{M}$. A special local coordinate system (s.l.c.s.) $x^{(j)}$ is defined as in [Es1]: in any chart Y_j which has a non–empty intersection with the boundary Γ the variable $x_n^{(j)}$ is the directed distance to the boundary (and is taken positive for $x\in\mathcal{M}\setminus\Gamma$), whereas the tangential variables $x'_{(j)}=(x_1^{(j)},...,x_{n-1}^{(j)})$ are a coordinate system on Γ.

The spaces $\mathbb{H}_p^s(\mathcal{M})$, $\widetilde{\mathbb{H}}_p^s(\mathcal{M})$, $\mathbb{B}_{p,q}^s(\mathcal{M})$, $\widetilde{\mathbb{B}}_{p,q}^s(\mathcal{M})$, $\mathbb{H}_p^{(\mu,s),m}(\mathcal{M})$ and $\widetilde{\mathbb{H}}_p^{(\mu,s),m}(\mathcal{M})$ can be defined by a partition of unity $\{\psi_j\}_{j=1}^\ell$ subordinated to the covering $\{Y_j\}_{j=1}^\ell$ and local coordinate diffeomorphism

$$\varpi_j : X_j \longrightarrow Y_j, \qquad X_j\subset\mathbb{R}^n_+. \tag{1.28}$$

If $\mathcal{B}^*$ denotes the dual space to the space $\mathcal{B}$ and $\partial\mathcal{M} \neq \emptyset$, then the following relations are valid (see [Tr1]):

$$\left(\widetilde{\mathbb{H}}^s_p(\mathcal{M})\right)^* = \mathbb{H}^{-s}_{p'}(\mathcal{M}), \qquad \left(\widetilde{\mathbb{B}}^s_{p,q}(\mathcal{M})\right)^* = \mathbb{B}^{-s}_{p',q'}(\mathcal{M}), \tag{1.29}$$

provided $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$, $p' = \dfrac{p}{p-1}$, $q' = \dfrac{q}{q-1}$ and

$$\left(\mathbb{H}^s_p(\mathcal{M})\right)^* = \widetilde{\mathbb{H}}^{-s}_{p'}(\mathcal{M}), \qquad \left(\mathbb{B}^s_{p,q}(\mathcal{M})\right)^* = \widetilde{\mathbb{B}}^{-s}_{p',q'}(\mathcal{M}), \tag{1.30}$$

provided $s \geq \dfrac{1}{p}$, $1 < p < \infty$, $1 \leq q \leq \infty$.

If $\mathcal{M}$ is embedded in $\mathbb{R}^l$, $n < l$, then the trace operators

$$\begin{aligned} \gamma_{\mathcal{M}} \quad : \quad \mathbb{H}^s_p(\mathbb{R}^l) &\longrightarrow \mathbb{B}^{s-\frac{l-n}{p}}_{p,p}(\mathcal{M}), \\ : \quad \mathbb{B}^s_{p,q}(\mathbb{R}^l) &\longrightarrow \mathbb{B}^{s-\frac{l-n}{p}}_{p,q}(\mathcal{M}) \end{aligned} \tag{1.31}$$

are correctly defined and bounded, provided

$$1 < p < \infty, \qquad 1 \leq q \leq \infty, \qquad \frac{l-n}{p} < s\,.$$

The next lemma follows from (1.31), as noted in [Gr2, (3.20)] and in [Sh2].

Lemma 1.15. *Let* $0 \leq \dim\mathcal{M} = n < l$, $\varphi \in \mathbb{B}^s_{p,p}(\mathcal{M})$ $(\varphi \in \mathbb{B}^s_{p,q}(\mathcal{M}))$ *and* $1 < p < \infty$ $(1 \leq q \leq \infty)$, $s < 0$.

Then $\varphi \otimes \delta_{\mathcal{M}} \in \mathbb{H}^{s-\frac{l-n}{p'}}_p(\mathbb{R}^l)$ $(\varphi \otimes \delta_{\mathcal{M}} \in \mathbb{B}^{s-\frac{l-n}{p'}}_{p,q}(\mathbb{R}^l))$, *where*

$$\langle \varphi \otimes \delta_{\mathcal{M}}, \psi\rangle := \langle \varphi, \gamma_{\mathcal{M}}\psi\rangle \quad \text{for} \quad \psi \in S(\mathbb{R}^l). \tag{1.32}$$

It is easy to prove that the symbols of the class $\mathcal{R}^m_\nu(\mathcal{M}, \mathbb{R}^l)$ are invariant with respect to the diffeomorphism $(x, \xi) \to (g_0(x, \xi), g_1(x, \xi))$, $g_k \in \mathcal{R}^m_{hom,k}(\mathcal{M}, \mathbb{R}^l)(k = 0, 1)$ (cf. [Sb2, Lemma 1.2]). Therefore the symbol class $\mathcal{R}^m_\nu(T^*\mathcal{M})$ on the cotangent manifold $T^*\mathcal{M}$ is defined correctly (see [Sb2, Subsection 4.3]).

Moreover, the principal symbol $a_{pr}(x, \xi)$ is defined invariantly, is independent of the chart (i.e. of $j = 1, \ldots, \ell$) and $a_{pr} \in \mathcal{R}^m_{hom,\nu}(\mathcal{T}^*\mathcal{M})$.

Definition 1.16. (see [Hr1, Sb2] etc.). An operator

$$\mathbf{A} : \widetilde{\mathbb{H}}^{(\mu,s),m}_p(\mathcal{M}) \longrightarrow \mathbb{H}^{(\mu,s-\nu),m}_p(\mathcal{M}) \tag{1.33}$$

is called pseudodifferential with the symbol $a \in \mathcal{R}^m_\nu(\mathcal{T}^*\mathcal{M})$, if:

(i) $\chi_1\mathbf{A}\chi_2 I \;:\; \mathbb{H}^{(\mu,s),m}_p(\mathcal{M}) \longrightarrow C^\infty(\mathcal{M})$ are continuous for all pairs $\chi_1, \chi_2 \in C^\infty(\mathcal{M})$ with disjoint supports $\text{supp}\chi_1 \bigcap \text{supp } chi_2 = \emptyset$ (i.e. $\chi_1 A\chi_2 I$ has order $-\infty$);

(ii) the transformed operators

$$\varpi_{j,*}\mathbf{A}\varpi_{j,*}^{-1}u = \mathbf{a}^{(j)}(x,D)u, \qquad u \in C_0^\infty(\mathbb{R}^n_+), \qquad j = 1,...,\ell\,,$$

where

$$\varpi_{j,*}u(x) := \begin{cases} \psi_j^0(x)u(\varpi_j(x)), & \text{when} \quad x \in X_j, \\ 0, & \text{when} \quad x \notin X_j, \end{cases}$$

$$\varpi_{j,*}^{-1}\varphi(x) := \begin{cases} \psi_j(t)\varphi(\varpi_j^{-1}(t)), & \text{when} \quad t \in Y_j, \\ 0, & \text{when} \quad t \notin Y_j, \end{cases} \tag{1.34}$$

$$\psi_j^0(x) := \psi_j(\varpi_j(x))\,, \quad x \in X_j \subset \mathbb{R}^n_+, \quad t \in \mathcal{M}$$

and $\{\psi_j\}_{j=1}^{\ell}$ is the partition of unity (see above), are pseudodifferential with the symbols

$$a^{(j)}(\varpi_j(x),\xi) = \psi_j^0(x)a(\varpi_j(t),\xi)\psi_j^0(x)\,.$$

The principal homogeneous symbol is responsible for the FREDHOLM properties and the index of the corresponding operator. Moreover, it is responsible for the exponent of the leading term in the asymptotic expansion of the solution to the pseudodifferential equation $\mathbf{a}(x,D)u = f$, $f \in \mathbb{H}_p^{s-r}(\mathcal{M})$, $u \in \widetilde{\mathbb{H}}_p^s(\mathcal{M})$ in the vicinity of the boundary Γ. But to get further (lower) entries of the asymptotic expansion of the solution, we should involve the full symbols $a(t,\xi)$ (see [Es1, Section 26], [Be1] and Theorem 2.1 below).

If symbol $a(\varpi_j(x),\xi)$ of a pseudodifferential operator $\mathbf{a}(t,D)$ in (1.33) has the transmission property (1.24) ($j = 1,...,\ell$), the operator

$$\mathbf{a}(t,D) : \mathbb{H}_p^{(\mu,s),m}(\mathcal{M}) \longrightarrow \mathbb{H}_p^{(\mu,s-\nu),m}(\mathcal{M})$$

is correctly defined and bounded.

Example 1.17. Let $\mathcal{M} \subset \mathbb{R}^3$ be a 2–dimensional, compact, $C^\infty-$ smooth surface in $\mathbb{R}^3$ with a smooth boundary $\partial\mathcal{M} = \Gamma$,

$$\mathcal{M} = \bigcup_{j=1}^{\ell} Y_j, \quad \varpi_j = (\varpi_{j0},\varpi_{j1},\varpi_{j2}) \ : \ X_j \longrightarrow Y_j, \quad X_j \subset \mathbb{R}^2_+ = \mathbb{R}\times\mathbb{R}^+ \tag{1.35}$$

be a smooth atlas on the surface $\mathcal{M}$ (cf. (1.28)) and $\mu, s \in \mathbb{R}$, $m \in \mathbb{N}_0$, $1 < p < \infty$.

Let $\ell_j(x) = (\ell_{j1}(x),\ell_{j2}(x),\ell_{j3}(x))$, $x \in \mathbb{R}^3$, $j = 1,2$ be two vector fields on $\mathbb{R}^3$ which coincide with linearly independent tangent vectors to the surface $\mathcal{M}$. Restriction of the differential operator

$$\partial_{\ell(x)} := (\partial_{\ell_1(x)},\partial_{\ell_2(x)})\,, \qquad \partial_{\ell_j(x)} := \sum_{k=1}^{3} \ell_{jk}(x)\partial_{x_k}$$

to the surface $\gamma_{\mathcal{M}}\partial_\ell$ is defined correctly. Then a matrix $N \times N$ differential operator of order $\nu \in \mathbb{N}$

$$\mathbf{a}(t, D) := \sum_{|\alpha|=0}^{\nu} c_\alpha(t)\partial^\alpha_{\ell(t)}$$

with $C^\infty(\mathcal{M})$–coefficient is pseudodifferential

$$\mathbf{a}_{\mathcal{M}}(t, D) := r_{\mathcal{M}}\mathbf{a}(t, D) : \widetilde{\mathbb{H}}_p^{(\mu,s),m}(\mathcal{M}) \longrightarrow \mathbb{H}_p^{(\mu,s-\nu),m}(\mathcal{M}) \tag{1.36}$$

and the symbol reads

$$a_{\mathcal{M}}(t, \xi) := a(t, \mathcal{J}^{-1}_{\varpi_j}(\varpi_j^{-1}(t))^\top\xi), \quad a_{\mathcal{M}} \in \mathcal{S}_{cl,\nu}(\mathcal{T}^*\mathcal{M})\,;$$

here $t \in Y_j$ and $\mathcal{J}_{\varpi_j}(x) = \varpi_j'(x) = ||\partial_k\varpi_{jl}(x)||_{3\times2}$, $x = (x_1, x_2) \in X_j$ denotes the Jacoby matrix of transformation (1.35) (with k–th row $(\partial_1\varpi_{jk}, \partial_2\varpi_{jk})$, $k = 0, 1, 2$); $\mathcal{A}^\top$ denotes the transposed matrix to $\mathcal{A}$.

In fact, let

$$\begin{aligned}
&\widetilde{\varpi}_j : \widetilde{X}_j \longrightarrow \widetilde{Y}_j, && \widetilde{X}_j, \widetilde{Y}_j \subset \mathbb{R}^3, \qquad \widetilde{Y}_j \cap \mathcal{M} = Y_j,\\
&\widetilde{X}_j := (-\varepsilon, \varepsilon) \times X_j, && \widetilde{Y}_j := \{\lambda\vec{n}(t) : -\varepsilon < \lambda < \varepsilon\,,\ t \in Y_j\}\,,\\
&\widetilde{\varpi}_j\,|_{X_j} = \widetilde{\varpi}_j(0, x) = \varpi_j(x), && j = 1, 2, \dots, \ell\,,
\end{aligned} \tag{1.37}$$

where $\vec{n}(t)$ is the unit normal at $t \in \mathcal{M}$, be extensions of the diffeomorphisms in (1.35). By $\mathcal{J}_{\widetilde{\varpi}_j}(\widetilde{x}) = \widetilde{\varpi}_j'(\widetilde{x}) = ||\partial_k\widetilde{\varpi}_{jl}(\widetilde{x})||_{3\times3}$ for $\widetilde{x} = (x_0, x_1, x_2) \in \widetilde{X}_j$ we denote the corresponding Jacoby matrix. $\mathcal{J}_{\varpi_j}(x)$ coincides with $\mathcal{J}_{\widetilde{\varpi}_j}(0, x)$ for $x \in X_j$ if we delete the first column, i. e. the entries $(\partial_0\widetilde{\varpi}_{jk})(0, x)$, $k = 0, 1, 2$; therefore $\mathcal{J}_{\widetilde{\varpi}_j}(0, x)(0, y) = \mathcal{J}_{\varpi_j}(x)y$ for $x \in X_j$, $y \in \mathbb{R}^2$. It is known, that

$$\begin{aligned}
&\mathcal{J}_{\widetilde{\varpi}_j}(0, x) = (e_0(x), e_1(x), e_2(x))\,,\\
&e_k = (\partial_k\widetilde{\varpi}_{j1}, \partial_k\widetilde{\varpi}_{j2}, \partial_k\widetilde{\varpi}_{j3})^\top, \quad k = 0, 1, 2
\end{aligned} \tag{1.38}$$

where vector–columns $e_0(x)$, $e_1(x)$ and $e_2(x)$ can be chosen orthogonal on the boundary $x \in X_j \cap \mathbb{R}^n_+$.

The unit vectors $e_1(x)$ and $e_2(x)$ are not usually orthogonal (in contrast to the pairs e_0, e_1 and e_0, e_2).

As a consequence the Jacoby matrix $\mathcal{J}_{\widetilde{\varpi}_j}(\widetilde{x})$ becomes orthogonal on the boundary

$$\det\mathcal{J}_{\widetilde{\varpi}_j}(0, x) = 1\,, \mathcal{J}_{\widetilde{\varpi}_j}(0, x)^\top = [\mathcal{J}_{\widetilde{\varpi}_j}(0, x)]^{-1} \quad \text{for all} \quad x \in X_j \cap \partial\mathbb{R}^n_+\,. \tag{1.39}$$

Since

$$\begin{aligned}
&\widetilde{\varpi}_{j,*}\mathrm{grad}\,_t\widetilde{\varpi}^{-1}_{j,*} = \mathcal{J}^{-1}_{\widetilde{\varpi}_j}(0, t)^\top\mathrm{grad}\,_x\\
&\mathcal{J}^{-1}_{\varpi_j}(t)^\top\xi := \mathcal{J}^{-1}_{\widetilde{\varpi}_j}(0, t)^\top(0, \xi)\,,
\end{aligned} \tag{1.40}$$

we find that the transformed operator $\varpi_{j,*}\mathbf{a}_{\mathcal{M}}(t,D)\varpi_{j,*}^{-1}$ to (1.36) reads as

$$\varpi_{j,*}\mathbf{a}_{\mathcal{M}}(t,D)\varpi_{j,*}^{-1}=\mathbf{a}_{\mathcal{M}}^{(j)}(x,D),\quad a_{\mathcal{M}}^{(j)}(x,\xi):=a^{(j)}(x,\mathcal{J}_{\varpi_j}^{-1}(x)^{\top}\xi)\,,\quad x\in X_j\,,$$

which means that $a_{\mathcal{M}}\in\mathcal{S}_{cl,\nu}(\mathcal{T}^*\mathcal{M})$, as claimed above.

Example 1.18.(see similar in [Ag1, DNS1, DNS2]). Let $\mathcal{M}$ be as in Example 1 (cf. (1.35)) $-\infty<\nu\leq -1$ and

$$\begin{aligned}&a(\xi)=\mathcal{F}k(\xi)\simeq a_\nu(\xi)+a_{\nu-1}(\xi)+\cdots+a_{\nu-k}(\xi)+\cdots\,,\\&a_{\nu-k}(\lambda\xi)=\lambda^{\nu-k}a_{\nu-k}(\xi)\quad \xi\in\mathbb{R}^3,\quad \lambda>0\end{aligned}$$

be a classical $N\times N$ matrix–symbol $a\in\mathcal{S}_{cl,\nu}(\mathbb{R}^3)$.

If $\nu\neq -1$ the trace

$$\begin{aligned}\mathbf{a}_{\mathcal{M}}(t,D)\varphi(t)=\gamma_{\mathcal{M}}\mathbf{a}(D)(\varphi\otimes\delta_{\mathcal{M}}(t)=\int\limits_{\mathbb{R}^3}k(t-y)(\varphi\otimes\delta_{\mathcal{M}})(y)dy\\=\int\limits_{\mathcal{M}}k(t-\tau)\varphi(\tau)d_\tau\mathcal{M}\,,\quad t\in\mathcal{M}\end{aligned}\tag{1.41}$$

(see (1.32)) is a pseudodifferential operator

$$\mathbf{a}_{\mathcal{M}}(t,D):\widetilde{\mathbb{H}}_p^{(\mu,s),m}(\mathcal{M})\to\mathbb{H}_p^{(\mu,s-\nu-1),m}(\mathcal{M})\,.$$

This operator has a classical symbol

$$a_{\mathcal{M}}(t,\xi')\simeq\sum_{k=0}^{\infty}a_{\mathcal{M},\nu+1-k}(t,\xi')\,,\quad a_{\mathcal{M},\nu+1-k}\in\mathcal{S}^{\infty}_{hom,\nu+1-k}(\mathcal{T}^*\mathcal{M})\,,\quad \xi'\in\mathbb{R}^2\,,$$

$$\begin{aligned}a_{\mathcal{M},\nu+1-k}\,(\,\varpi_j(x),\xi')=\sum_{m=0}^{k}\sum_{\substack{|\beta|+|\gamma|-|\alpha|=k-m\\2\alpha\leq\beta}}\frac{(-i)^{|\alpha+\beta+\gamma|}b_{\alpha,\beta}(x)\partial_x^\gamma\mathcal{G}_{\varpi_j}(x)}{2\pi\det\mathcal{J}_{\widetilde{\varpi}_j}(0,x)\gamma!}\\\times(-\xi')^\alpha\int\limits_{-\infty}^{\infty}\partial_{\xi'}^{\beta+\gamma}a_{\nu-m}\left(\mathcal{J}_{\widetilde{\varpi}_j}^{-1}(0,x)^{\top}(\xi',\lambda)\right)d\lambda,\end{aligned}\tag{1.42}$$

where

$$\mathcal{G}_{\varpi_j}:=(\det\|(\partial_k\varpi_j,\partial_l\varpi_j)\|_{2\times 2})^{\frac{1}{2}}\quad\text{ with }\quad \partial_k\varpi_j:=(\partial_k\varpi_{j1},\partial_k\varpi_{j2},\partial_k\varpi_{j3})^{\top}$$

denotes the square root of the GRAM determinant of the vector–function $\varpi_j=(\varpi_{j1},\varpi_{j2},\varpi_{j3})^{\top}$ for $j=1,2,\ldots,N$, $b_{0,\beta}(x)=1$ and coefficients $b_{\alpha,\beta}(x)$ for $|\alpha|>0$ are found from the following equality

$$\frac{1}{\alpha!}\left[\sum_{|\delta|=2}^{m}\frac{(-1)^{|\delta|+1}}{\delta!}\partial^\delta\varpi_j(x)\tau^\delta\right]^\alpha=\sum_{|\beta|=2|\alpha|}^{m+2}b_{\alpha,\beta}(x)\tau^\beta+\sum_{|\beta|=m+3}^{m|\alpha|}g_{\alpha,\beta}^{(m)}(x)\tau^\beta\,,\quad \alpha\in\mathbb{N}^n\,.$$

In particular, the homogeneous principal symbol reads

$$(1.43)\qquad \begin{aligned} a_{\mathcal{M},pr}(\varpi_j(x),\xi) &:= a_{\mathcal{M},\nu+1}(\varpi_j(x),\xi') \\ &= \frac{\mathcal{G}_{\varpi_j}(x)}{2\pi\det\mathcal{J}_{\widetilde{\varpi}_j}(0,x)}\int\limits_{-\infty}^{\infty} a_\nu\left(\mathcal{J}^{-1}_{\widetilde{\varpi}_j}(0,x)^\top(\xi',\lambda)\right)d\lambda\,,\quad x\in X_j\,. \end{aligned}$$

If $\nu=-1$ we can not write (1.41) but

$$(1.44)\quad \mathbf{a}_\mathcal{M}(t,D)\varphi(t)=\gamma_\mathcal{M}\mathbf{a}(D)(\varphi\otimes\delta_\mathcal{M})(t)=c_0(t)\varphi(t)+\int\limits_\mathcal{M} k_0(t,t-\tau)\varphi(\tau)d_\tau\mathcal{M}\,.$$

is a pseudodifferential operator of order zero $\mathbf{a}_\mathcal{M}(t,D):\widetilde{\mathbb{H}}^{(\mu,s),m}_p(\mathcal{M})\to\mathbb{H}^{(\mu,s),m}_p(\mathcal{M})$ i.e. is a singular integral operator; the integral in (1.44) is understood in the CAUCHY principal value sense and (see [Es1, (3.26)])

$$c_0(t)=\frac{\Gamma\left(\frac{n-1}{2}\right)}{2\pi^{\frac{n-1}{2}}}\int\limits_{|\omega|=1} a_{\mathcal{M},pr}(t,\omega)d_\omega S\,,\quad k_0(t,\tau)=\mathcal{F}^{-1}_{\xi\to\tau}[a_{\mathcal{M},pr}(t,\xi)-c_0(t)]\,,\quad t,\tau\in\mathcal{M}\,.$$

In fact, it is known that

$$\int\limits_\mathcal{M} g(\tau)d_\tau\mathcal{M}=\sum_{j=1}^N\int\limits_{\mathbb{R}^{n-1}}\psi^0_j(y)\mathcal{G}_{\varpi_j}(y)g(\varpi_j(y))dy$$

(see [Sc2, §IV.10.38], [Sl1, §3.6]). Therefore

$$\begin{aligned} \mathbf{a}^{(j)}_\mathcal{M}(x,D)\varphi(x) &= \varpi_{j,*}\mathbf{a}_\mathcal{M}(t,D)\varpi^{-1}_{j,*}\varphi(x)=c_0(\varpi_j(x))(\psi^0_j(x))^2\varphi(x) \\ &\quad+\psi^0_j(x)\int\limits_{\mathbb{R}^{n-1}}\psi^0_j(y)\mathcal{G}_{\varpi_j}(y)k(\varpi_j(x)-\varpi_j(y))\varphi(y)dy \end{aligned}$$

$$=c_0(\varpi_j(x))(\psi^0_j(x))^2\varphi(x)+\sum_{m=0}^\infty\psi^0_j(x)\int\limits_{\mathbb{R}^{n-1}}\psi^0_j(y)\mathcal{G}_{\varpi_j}(y)k_{\nu-m}(\varpi_j(x)-\varpi_j(y))\varphi(y)dy\,,$$

where

$$\mathcal{F}k_{\nu-m}=a_{\nu-m}\,,\quad k_{\nu-m}(\lambda t)=\lambda^{n-\nu+m}k_{\nu-m}(t)\,,\quad \lambda>0,\;\; t\in\mathbb{R}^n\,.$$

By the TAYLOR formula we get the asymptotic expansion

$$(1.45)\qquad \varpi_j(x)-\varpi_j(y)=\mathcal{J}_{\varpi_j}(x)(x-y)+\sum_{|\delta|=2}^\infty\frac{(-1)^{|\delta|+1}}{\delta!}\partial^\delta\varpi_j(x)(x-y)^\delta\,.$$

Applying the TAYLOR formula again with the help of (1.45) we get

$$k_{\nu-m}(\varpi_j(x)-\varpi_j(y))\;=\;\sum_{|\alpha|=0}^\infty\frac{1}{\alpha!}\partial^\alpha k_{\nu-m}(\mathcal{J}_{\varpi_j}(x)(x-y))$$

$$\times\left[\sum_{|\delta|=2}^{\infty}\frac{(-1)^{|\delta|+1}}{\delta!}\partial^{\delta}\varpi_j(x)(x-y)^{\delta}\right]^{\alpha}$$

$$(1.46)\qquad = \sum_{|\alpha|=0}^{\infty}\sum_{|\beta|=2|\alpha|}^{\infty} b_{\alpha,\beta}(x)\partial^{\alpha}k_{\nu-m}(\mathcal{J}_{\varpi_j}(x)(x-y))(x-y)^{\beta}\,,$$

where $b_{0,\beta}=1$ and other coefficients $b_{\alpha,\beta}(x)$, $|\alpha|>0$ are defined above.

Applying the Taylor formula once more we get

$$(1.47)\qquad \mathcal{G}_{\varpi_j}(y)=\mathcal{G}_{\varpi_j}(x)+\sum_{|\gamma|=1}^{\infty}\frac{(-1)^{|\gamma|}}{\gamma!}\partial^{\gamma}\mathcal{G}_{\varpi_j}(x)(x-y)^{\gamma}\,.$$

Applying formulae (1.45)–(1.47) we proceed as follows

$$a_{\mathcal{M},\nu+1-k}(\varpi_j(x),\xi')=\sum_{m=0}^{k}\sum_{2\alpha\leq\beta|\beta|+|\gamma|-|\alpha|=k-m}\frac{b_{\alpha,\beta}(x)(-\partial_x)^{\gamma}\mathcal{G}_{\varpi_j}(x)}{\gamma!}$$
$$\times\mathcal{F}_{z\to\xi'}\left[z^{\beta+\gamma}\partial_z^{\alpha}k_{\nu-m}\left(\mathcal{J}_{\varpi_j}(x)z\right)\right]$$
$$=\sum_{m=0}^{k}\sum_{\substack{|\beta|+|\gamma|-|\alpha|=k-m\\2\alpha\leq\beta}}\frac{(-i)^{|\alpha+\beta+\gamma|}b_{\alpha,\beta}(x)(-\partial_x)^{\gamma}\mathcal{G}_{\varpi_j}(x)}{2\pi\det\mathcal{J}_{\widetilde{\varpi}_j}(0,x)\gamma!}$$
$$(1.48)\qquad \times(-\xi')^{\alpha}\int_{-\infty}^{\infty}\partial_{\xi'}^{\beta+\gamma}a_{\nu-m}\left(\mathcal{J}_{\widetilde{\varpi}_j}^{-1}(0,x)^{\top}(\xi',\lambda)\right)d\lambda\ ,$$

because

$$\mathcal{F}_{z\to\xi'}\ \left[\ z^{\beta+\gamma}\partial_z^{\alpha}k_{\nu-m}\left(\mathcal{J}_{\varpi_j}(x)z\right)\ \right]$$
$$=\int_{\mathbb{R}^{n-1}}e^{i\xi' z}z^{\beta+\gamma}\partial_z^{\alpha}\left[\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{-i\mathcal{J}_{\varpi_j}(x)z\eta}a_{\nu-m}(\eta)d\eta\right]$$
$$=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^{n-1}}z^{\beta+\gamma}\left[(-\partial_z)^{\alpha}e^{i\xi' z}\right]\int_{\mathbb{R}^n}e^{-i\mathcal{J}_{\widetilde{\varpi}_j}(0,x)(0,z)\eta}a_{\nu-m}(\eta)d\eta$$
$$=\frac{1}{(2\pi)^n\det\mathcal{J}_{\widetilde{\varpi}_j}(0,x)}(-i\partial_{\xi'})^{\beta+\gamma}\int_{\mathbb{R}^{n-1}}(-\xi')^{\alpha}e^{i\xi' z}\int_{\mathbb{R}^n}e^{-iz\eta}a_{\nu-m}(\mathcal{J}_{\widetilde{\varpi}_j}^{-1}(0,x)^{\top}\eta)d\eta$$
$$=\frac{(-i)^{|\beta+\gamma|}}{2\pi\det\mathcal{J}_{\widetilde{\varpi}_j}(0,x)}\partial_{\xi'}^{\beta+\gamma}\mathcal{F}_{z\to\xi'}\mathcal{F}_{\eta'\to z}^{-1}\left[\int_{-\infty}^{\infty}a_{\nu-m}(\mathcal{J}_{\widetilde{\varpi}_j}^{-1}(0,x)^{\top}(\eta',\lambda))d\lambda\right]$$
$$=\frac{(-i)^{|\beta+\gamma|}}{2\pi\det\mathcal{J}_{\widetilde{\varpi}_j}(0,x)}\partial_{\xi'}^{\beta+\gamma}\int_{-\infty}^{\infty}a_{\nu-m}(\mathcal{J}_{\widetilde{\varpi}_j}^{-1}(0,x)^{\top}(\xi',\lambda))d\lambda\,.$$

aaa

1.5. Solvability results

Let $\mathcal{M}$ be a smooth manifold with a smooth boundary and consider $N \times N$ system of pseudodifferential equations

$$r_{\mathcal{M}}\mathbf{a}(x,D)u = v, \tag{1.49}$$

$$a \in \mathcal{R}^{\gamma}_{\nu}(\mathcal{T}^*\mathcal{M}), \qquad u \in \widetilde{\mathbb{H}}^{(\mu,s),m}_p(\mathcal{M}), \quad v \in \mathbb{H}^{(\mu,s-\nu),m}_p(\mathcal{M}),$$

$$m,\gamma \in \mathbb{N}_0, \qquad \gamma \geq \left[\frac{n}{2}\right] + 2, \quad \mu, s, \nu \in \mathbb{R}, \quad 1 < p < \infty.$$

We suppose the homogeneous principal symbol $a_{pr}(x,\xi)$ is **elliptic**

$$\inf_{\substack{x\in\overline{\mathcal{M}}\\ |w|=1}} |\det a_{pr}(x,\omega)| > 0 \tag{1.50}$$

and consider the matrix

$$a_0 = a_0(x') := [a_{pr}(x',+1)]^{-1}a_{pr}(x',-1), \tag{1.51}$$

$$a_{pr}(x',\pm 1) := a_{pr}(x',0,\dots,0,\pm 1), \quad x' \in \partial\mathcal{M}.$$

Let $\lambda_1(x'),\dots,\lambda_\ell(x')$ be all eigenvalues of the matrix $a_0(x')$ with the Riesz indices $m_1(x'),\cdots,m_\ell(x')$, respectively (i.e. $\lambda_j(x')$ defines $m_j(x')$ linearly independent associated vectors for $a_0(x')$; see [Ga1]) and

$$\delta_j = \delta_j(x') := \frac{1}{2\pi i}\log\lambda_j(x'), \ \frac{1}{p} - 1 < s - \operatorname{Re}\delta_j - \frac{\nu}{2} \leq \frac{1}{p}, \quad j = 1,\dots,\ell. \tag{1.52}$$

Theorem 1.19. *Let the homogeneous principal symbol $a_{pr} \in \mathcal{R}^{\gamma}_{hom,\nu}(\mathcal{T}^*\mathcal{M})$ of equation* (1.49) *be elliptic (see* (1.50)*). Then*

$$a_{pr}(x',\xi) = [a^-_{pr}(x',\xi)]^{-1}\Xi_{a_{pr}}(x',\xi)a^+_{pr}(x',\xi),$$

$$a^{\pm}_{pr}(x',\xi) = (\xi_n \pm i|\xi'|)^{\pm\frac{\nu}{2}}g_{\pm}(x',\xi), \tag{1.53}$$

$$\Xi_{a_{pr}}(x',\xi) = \left(\frac{\xi_n - i|\xi'|}{\xi_n + i|\xi'|}\right)^{-\Delta(x')+\sigma(x')} B^0_{a_{pr}}\left(\frac{1}{2\pi i}\log\frac{\xi_n - i|\xi'|}{\xi_n + i|\xi'|}\right).$$

Here:

(i) the functions $g^{\pm 1}_-(x',\xi',\xi_n - it)$ and $g^{\pm 1}_+(x',\zeta',\zeta_n + it)$ have uniformly bounded analytic continuation for $t > 0$, are homogeneous of order 0 $g_{\pm}(x',\lambda\xi) = g_{\pm}(x',\xi)$ $(\lambda > 0)$ and the estimates

$$|\partial^k_{\xi_n} g^{\pm 1}_+(x',\xi)| \leq M|\xi|^{-k}, \qquad |\partial^k_{\xi_n} g^{\pm 1}_-(x',\xi)| \leq M|\xi|^{-k} \tag{1.54}$$

hold for all $k = 0,1,\dots$, $\xi \in \mathbb{R}^n$, and $x \in \partial\mathcal{M}$;

(ii) *numbers* δ_j *are defined in* (1.52), *the vector* $\Delta := (\delta_1, \dots, \delta_\ell)$ *has length* N *(each* δ_j *occurs, according to its algebraic multiplicity,* m_j *times),* $\sigma = (\sigma_1, \dots, \sigma_N) \in \mathbb{N}_0^N$ *are integers (known as the partial indices of* $a_{pr}(x,\xi)$*) and*

$$h^{\Delta+\sigma} := \mathrm{diag}\,\{h^{\delta_1+\sigma_1}, \dots, h^{\delta_\ell+\sigma_N}\} \quad \text{for} \quad h \in \mathbb{C}\ ; \tag{1.55}$$

(iii) $a_{pr}(x, 0, \pm 1)$ *are positive definite or* $a_0(x')$ *is a normal matrix, commuting with the transposed*

$$a_0(x')[a_0(x')]^\top = [a_0(x')]^\top a_0(x')\,, \tag{1.56}$$

then $B_{a_{pr}}(t) = I$*; the* $N \times N$ *matrix* $B_{a_{pr}}(t) = \|b_{jk}(t)\|_{N\times N}$ *is polynomial, upper triangular* ($b_{jk}(t) = 0$ *for* $j > k$) *with identities on the main diagonal* ($b_{jj}(t) \equiv 1$). $B_{a_{pr}}(t)$ *commutes with the diagonal matrix* ζ^δ*:* $B^0_{a_{pr}}(t)\zeta^\delta = \zeta^\delta B^0_{a_{pr}}(t)$.

If equation (1.49) *is Fredholm for some* $p \in (1,\infty)$, $\mu, s \in \mathbb{R}$ *and* $m \in \mathbb{N}_0$*, then all partial indices vanish* $\sigma_1(x') = \dots = \sigma_N(x') = 0$ *for* $x' \in \partial\mathcal{M}$ *and* $[a^-_{pr}]^{\pm1}$, $[a^+_{pr}]^{\mp1} \in \mathcal{R}^{\gamma-1}_{hom,\pm\frac{\nu}{2}}(\mathcal{T}^*\mathcal{M})$.

We postpone the proof of this theorem until Subsection 1.8. Here we formulate an important corollary which will be proved also later in the same Subsection 1.9.

Lemma 1.20. *If the matrix* $a_0(x')$ *(see* (1.51), (1.79)*) is normal, then it is simple* $\ell = N$ *(i.e. each eigenvalue* $\lambda_j(x')$ *has algebraic multiplicity 1) and* $a_0(x')$ *is unitarily similar with the diagonal one*

$$\begin{aligned} &B_{a_{pr}}(\xi) \equiv I\,, \quad a_0(x') = \mathcal{K}(x')\mathrm{diag}\,\{\lambda_1(x'), \dots, \lambda_N(x')\}\,\mathcal{K}^*(x'),\\ &\mathcal{K} \in C^\infty(\partial\mathcal{M}), \quad \det\mathcal{K}(x') \neq 0, \quad \mathcal{K}^{-1}(x') = \mathcal{K}^*(x')\,. \end{aligned} \tag{1.57}$$

If the principal symbol $a_{pr} \in \mathcal{R}^\gamma_{hom,\nu}(\mathcal{T}^*\mathcal{M})$ *is* **strongly elliptic** *on* $\partial\mathcal{M}$*, that is there exists a constant* $M > 0$ *such that the inequality*

$$\mathrm{Re}\,(a_{pr}(x',\xi)\eta, \eta) \geq M|\xi|^\nu|\eta|^2 \tag{1.58}$$

holds for all $x' \in \partial M$, $\xi \in \mathbb{R}^n$ *and all* $\eta \in \mathbb{C}^n$*. Then it admits factorization* (1.53) *with*

$$\sigma_1(x') = \dots = \sigma_N(x') \equiv 0 \quad \text{and} \quad [a^-_{pr}]^{\mp1}, [a^+_{pr}]^{\pm1} \in \mathcal{R}^{\gamma-1,0}_{hom,\pm\frac{\nu}{2}}(\mathcal{T}^*\mathcal{M})\,. \tag{1.59}$$

Moreover, if $a_{pr} \in \mathcal{R}^\gamma_{hom,\nu}(\mathcal{T}^*\mathcal{M})$ *is* **positive definite** *on* $\partial\mathcal{M}$*, that is*

$$(a_{pr}(x',\xi)\eta, \eta) \geq M|\xi|^\nu|\eta|^2 \quad \text{for all} \quad x' \in \partial\mathcal{M}, \quad \xi \in \mathbb{R}^n \quad \text{and} \quad \eta \in \mathbb{C}^n$$

Precise description of $B^0_{a_{pr}}(t)$, connected with the Jordan normal form of a_0, see in Subsection 1.7, (1.79)–(1.80).

The same assertion holds if we replace condition (1.58) by the following less restrictive one:

$$|(a_{pr}(x',\xi)\eta, \eta)| \geq M|\xi|^\nu|\eta|^2$$

(noted by I.Nadiradze).

with some constant $M > 0$, then $a_0(x')$ in (1.51) *is simple and*

$$\sigma_1(x') = \cdots = \sigma_N(x') \equiv 0, \qquad \operatorname{Re}\delta_1(x') = \cdots = \operatorname{Re}\delta_\ell(x') \equiv 0\,. \tag{1.60}$$

Now we can prove the following

Theorem 1.21. *Equation* (1.49) *is* FREDHOLM *if and only if:*

(i) The homogeneous principal symbol $a_{pr}(x,\xi)$ is elliptic (*cf.* (1.50))*;*

(ii) all partial indices of factorization (1.53) *are trivial on $\partial\mathcal{M}$:*

$$\sigma_1'(x) = \cdots \sigma_N(x') = 0 \quad \text{for all} \quad x' \in \partial\mathcal{M};$$

(iii) (1.52) *holds with the strong inequality* $\operatorname{Re}\delta_j(x') \neq s - \dfrac{1}{p} - \dfrac{\nu}{2}$ *for all $j = 1, \ldots, \ell$ and all $x' \in \partial\mathcal{M}$.*

If equation (1.49) *is* FREDHOLM*, it has one and the same kernel* Ker $r_{\mathcal{M}}\mathbf{a}(x,D)$ *and the same index* $\operatorname{Ind} r_{\mathcal{M}}\mathbf{a}(x,D)$ *in all spaces* $\widetilde{\mathbb{H}}_p^{(\mu,s),m}(\mathcal{M}) \to \mathbb{H}_p^{(\mu,s-\nu),m}(\mathcal{M})$ *which meet the conditions (i)–(ii) and are independent of $m \in \mathbb{N}_0$.*

Proof. First we replace $\mathbf{a}(x,D)$ by the operator $\overset{\vee}{\mathbf{a}}_{pr}(x,D)$ with the truncated symbol (cf. (1.21)). Since the manifold $\mathcal{M}$ is compact and the difference is smoothing operator of order $-\infty$ (cf Lemma 1.15), the difference is compact operator in the spaces $\widetilde{\mathbb{H}}_p^{(\mu,s),m}(\mathcal{M}) \to \mathbb{H}_p^{(\mu,s-\nu),m}(\mathcal{M})$ and has no influence on the FREDHOLM properties.

Now we apply the "quasilocalization" technique (see [Si1]), which means "freezing coefficients" and transforming the operator from the manifold to $\mathbb{R}^n_+$. For the BESSEL potential spaces this approach is described in details e.g. in [Du1, § 3.2°] and we suppose the reader is familiar with the quasiequivalence and local invertibility of operators. We remind that quasiequivalent operators are locally invertible only simultaneously (see [Du1, § 3.2°]) and if the operator $\mathbf{a}(x,D)$ in (1.49) is locally invertible for all $x \in \overline{\mathcal{M}}$, it is FREDHOLM (see [Du1, § 3.2 °] and [GK1]).

We find easily that operators

$$\overset{\vee}{\mathbf{a}}\,(x,D)\ :\ \widetilde{\mathbb{H}}_p^{(\mu,s),m}(\mathcal{M}) \to \mathbb{H}_p^{(\mu,s-\nu),m}(\mathcal{M})$$

and

$$\begin{aligned} &\overset{\vee}{\mathbf{a}}_{pr}\,(x_0,D) : \mathbb{H}_p^{(\mu,s),m}(\mathbb{R}^n) \longrightarrow \mathbb{H}_p^{(\mu,s),m}(\mathbb{R}^n) \quad \text{for} \quad x_0 \in \mathcal{M}\backslash\partial\mathcal{M}\,, \\ &r_+\,\overset{\vee}{\mathbf{a}}_{pr}\,(x_0',D) : \widetilde{\mathbb{H}}_p^{(\mu,s),m}(\mathbb{R}^n_+) \longrightarrow \mathbb{H}_p^{(\mu,s),m}(\mathbb{R}^n_+) \quad \text{for} \quad x_0' \in \partial\mathcal{M} \end{aligned} \tag{1.61}$$

are quasi-equivalent. Therefore equation (1.49) is FREDHOLM if and only if the operators $\overset{\vee}{\mathbf{a}}_{pr}\,(x_0,D)$ and $r_+\,\overset{\vee}{\mathbf{a}}_{pr}\,(x_0',D)$ in (1.61) are locally invertible at the respective points $t_0 = \varpi_j(x_0)$ in the respective spaces.

Numbers δ_j in (1.52) and ν_j in [DSW1, (A.32)] are related as follows: $\delta_j = -i\nu_j$.

A condition for local invertibility of the convolution operator $W^0_{\overset{\vee}{a}_{pr}(x_0,\cdot)} = \overset{\vee}{\mathbf{a}}_{pr}(x_0,D)$ is well–known and coincides with the ellipticity condition at an inner point $x_0 \in \mathcal{M}\backslash\partial\mathcal{M}$ (see e.g. [Du1, § 4]). We leave the details to the reader and proceed to the case $x_0 \in \partial\mathcal{M}$ which is more complicated.

If we apply the "lifting" Bessel potential operators $\langle D'\rangle^{\mu}\lambda_+^{-s}(D)$ and $\langle D'\rangle^{-\mu}\lambda_-^{s-\nu}(D)$ (see Theorem 1.13) and recall that

$$\langle D'\rangle^{\pm\mu} r_+ \overset{\vee}{\mathbf{a}}_{pr}(x_0,D) = r_+ \overset{\vee}{\mathbf{a}}_{pr}(x_0,D)\langle D'\rangle^{\pm\mu}$$

for a fixed $x_0 \in \partial\mathcal{M}$ and $\lambda^{\pm\mu}(\xi')$ independent of ξ_n, we get: the second operator in (1.61) is locally invertible iff

$$r_+\widetilde{\mathbf{a}}_{pr}(x_0,D) := r_+\boldsymbol{\lambda}_-^{s-\nu}(D)\, \overset{\vee}{\mathbf{a}}_{pr}(x_0,D)\boldsymbol{\lambda}_+^{-s}(D) \tag{1.62}$$

is locally invertible at 0 as an operator in the space $\mathbb{H}_p^{0,m}(\mathbb{R}^n_+)$ for all $x_0 \in \partial\mathcal{M}$. We remind, that $\widetilde{\mathbb{H}}_p^{0,m}(\mathbb{R}^n_+) = \mathbb{H}_p^{0,m}(\mathbb{R}^n_+)$ (see Lemma 1.14).

The operators (1.62) and

$$r_+\widetilde{\mathbf{a}}^{\infty}_{pr}(x_0,D) : \mathbb{H}_p^{0,m}(\mathbb{R}^n_+) \longrightarrow \mathbb{H}_p^{0,m}(\mathbb{R}^n_+) \tag{1.63}$$

are locally equivalent (see [DS1, § 3.1]); here

$$\widetilde{a}^{\infty}_{pr}(x_0,\xi) := \lim_{t\to\infty}\widetilde{a}_{pr}(x_0,t\xi) = \left(\frac{\xi_n - i|\xi'|}{\xi_n + i|\xi'|}\right)^s (\xi_n - i|\xi'|)^{-\nu} a_{pr}(x_0,\xi) \tag{1.64}$$

is the radial limits of $\widetilde{a}_{pr}(x_0,\xi)$ at ∞.

If we introduce an equivalent norm in $\mathbb{H}_p^{0,m}(\mathbb{R}^n_+)$:

$$\|u|\mathbb{H}_p^{0,m}(\mathbb{R}^n_+)\| := \sum_{k=0}^{m}\sum_{|\alpha|=k}\|\partial^\alpha x_n^k u|L_p(\mathbb{R}^n)\|$$

(cf. (1.25)) we find that the dilation operator

$$\mathbf{V}_\tau u(x) := \tau^{-\frac{1}{p}}u(\tau x), \quad \tau > 0, \quad k \in \mathbb{R}^n_+,$$

is an isomorphism in $\mathbb{H}_p^{0,m}(\mathbb{R}^n_+)$. Therefore we can apply [Du1, Lemma 3.6] and find out that the local invertibility of operator (1.63) at 0 coincides with the (global) invertibility, because $V_\tau r_+\widetilde{\mathbf{a}}^{\infty}_{pr}(x_0,D) = r_+\widetilde{\mathbf{a}}^{\infty}_{pr}(x_0,D)V_\tau$ due homogeneity of the symbol $\widetilde{a}^{\infty}_{pr}(x_0,\tau\xi) = \widetilde{a}^{\infty}_{pr}(x_0,\xi)$ $(\tau > 0)$.

Further localisation with respect to $\omega \in S^{n-1}$ (see e.g. [Du1, § 1.4^0], [DS1, Theorem 3.20], [Sh2, Lemma 1.20]) leads to the following result: ellipticity $\det a_{pr}(x_0,\omega) \neq 0$, $\omega \in S^{n-1}$ is necessary condition for the invertibility of operator (1.63).

Since $a_{pr}(x_0,\xi)$ is elliptic, from (1.53)–(1.54) and (1.64) we find

$$\begin{aligned}\widetilde{a}^{\infty}_{pr}(x_0,\xi) &= g_-^{-1}(x_0,\xi)\left(\frac{\xi_n - i|\xi'|}{\xi_n + i|\xi'|}\right)^{s-\Delta(x_0)-\nu/2+\sigma(x_0)}\\ &\times B^0_{a_{pr}}\left(\frac{1}{2\pi}\log\frac{\xi_n - i|\xi'|}{\xi_n + i|\xi'|}\right)g_+(x_0,\xi),\end{aligned} \tag{1.65}$$

where under the sum $s-\Delta(x_0)-\dfrac{\nu}{2}+\sigma(x_0)$ of vectors Δ,σ (see (1.54)) and the scalar $s-\dfrac{\nu}{2}$ is meant $(s-\delta_1-\dfrac{\nu}{2}+\sigma_1,\dots,s-\delta_\ell-\dfrac{\nu}{2}+\sigma_N)$.

Due to estimates (1.54) $g_-^{\pm1}(x_0,\cdot)$, $g_+^{\pm1}(x_0,\cdot)\in M_p^{(0)}(\mathbb{R}^n)$ (see Theorem 1.1); therefore the convolution operators $W_{g_-^{-1}(x_0,\cdot)}=r_+\mathbf{g}_-^{-1}(x_0,D)$ and $W_{g_+(x_0,\cdot)}=r_+\mathbf{g}_+(x_0,D)$ are invertible in $\mathbb{H}_p^{0,m}(\mathbb{R}_+^n)$ (see (1.6) and Lemma 1.2) with the inverses $r_+\mathbf{g}_-(x_0,D)$ and $r_+\mathbf{g}_+^{-1}(x_0,D)$, respectively.

The convolution operator $r_+[\boldsymbol{\zeta}(D)]^{s-\Delta-\frac{\nu}{2}+\sigma}\mathbf{B}_{a_{pr}}(D)$ with the symbols

$$\zeta(\xi):=\frac{\xi_n-i|\xi'|}{\xi_n+i|\xi'|},\quad B_{a_{pr}}(\xi)=B^0_{a_{pr}}\left(\frac{1}{2\pi}\frac{\xi_n-i|\xi'|}{\xi_n+i|\xi'|}\right),\quad \xi\in\mathbb{R}^n$$

is bounded in the space $\mathbb{H}_p^{0,m}(\mathbb{R}_+^n)$ due to Theorems 1.1, 1.19.iii and to Lemma 1.14.

From (1.65) and (1.6) we find

$$(1.66)\qquad r_+\widetilde{\mathbf{a}}^\infty_{pr}(x_0,D)=r_+\mathbf{g}_-^{-1}(x_0,D)r_+[\boldsymbol{\zeta}(D)]^{s-\Delta-\frac{\nu}{2}+\sigma}\mathbf{B}_{a_{pr}}(D)r_+\mathbf{g}_+(x_0,D)$$

and since $r_+\mathbf{g}_-^{-1}(x_0,D)$, $r_+\mathbf{g}_+(x_0,D)$ are invertible, invertibility of (1.66) in $\mathbb{H}_p^{0,m}(\mathbb{R}_+^n)$ is equivalent with the invertibility of

$$(1.67)\qquad r_+[\boldsymbol{\zeta}(D)]^{s-\Delta-\frac{\nu}{2}+\sigma}\mathbf{B}_{a_{pr}}(D):\mathbb{H}_p^{0,m}(\mathbb{R}_+^n)\longrightarrow\mathbb{H}_p^{0,m}(\mathbb{R}_+^n).$$

The operator $\mathbf{A}:=r_+[\boldsymbol{\zeta}(D)]^{s-\delta-\frac{\nu}{2}+\sigma}\mathbf{B}_{a_{pr}}(D)$ is upper triangular with scalar operators $r_+[\boldsymbol{\zeta}(D)]^{s-\delta_j-\frac{\nu}{2}+\sigma_j}$, $j=1,\dots,N$ on the main diagonal (we remind that $\mathbf{B}_{a_{pr}}(D)$ has identities on the main diagonal). Invertibility of these scalar operators ensure invertibility of $\mathbf{A}$.

First let us suppose conditions (ii) of the theorem fulfilled. Due to (1.6) the operator

$$(1.68)\qquad r_+[\boldsymbol{\zeta}(D)]^{s-\delta_j-\frac{\nu}{2}}=r_+(D_n-i|D'|)^{s-\delta_j-\frac{\nu}{2}}r_+(D_n+i|D'|)^{-s+\delta_j+\frac{\nu}{2}}$$

$$r_+[\boldsymbol{\zeta}(D)]^{s-\delta_j-\frac{\nu}{2}}:\mathbb{H}_p^{0,m}(\mathbb{R}_+^n)\longrightarrow\mathbb{H}_p^{0,m}(\mathbb{R}_+^n)$$

is formally invertible and the operator

$$(1.69)\qquad (r_+[\boldsymbol{\zeta}(D)]^{s-\delta_j-\frac{\nu}{2}})^{-1}:=r_+(D_n+i|D'|)^{s-\delta_j-\frac{\nu}{2}}\chi_+(D_n-i|D'|)^{-s+\delta_j+\frac{\nu}{2}}$$

is the formal inverse to (1.68). It remains to prove that (1.69) is bounded in $\mathbb{H}_p^{0,m}(\mathbb{R}_+^n)$ for $j=1,\dots,N$.

Due to Theorem 1.13 invertibility is equivalent to the boundedness of the operators

$$\theta_+I:\mathbb{H}^{s-\delta_j-\frac{\nu}{2},m}(\mathbb{R}^n)\longrightarrow\mathbb{H}^{s-\delta_j-\frac{\nu}{2},m}(\mathbb{R}^n)\quad\text{for}\quad j=1,\dots,N,$$

which follow from Lemma 1.14 due to conditions (1.52) provided $\dfrac{1}{p}\neq s-\operatorname{Re}\delta_j(x')-\dfrac{\nu}{2}$ (cf. condition (iii) of Theorem 1.21).

The explicit inverse to $r_+\mathbf{B}_{a_{pr}}(D)$ see in Remark 1.28.

By a standard arguments it can be proved that if the integer $\sigma_j \neq 0$ in (1.68), then this operator would have either infinite dimensional kernel (provided $\sigma_j < 0$) or infinite dimensional co–kernel (provided $\sigma_j > 0$), which is incompatible with the Fredholm criteria we look for (see [Du1, § 4.4]).

The last step is to prove that the conditions $s - \operatorname{Re}\delta_j(x') - \dfrac{\nu}{2} \neq \dfrac{1}{p}$ $(j = 1, \ldots, N)$ are necessary.

For this we note, that if $s - \operatorname{Re}\delta_j(x') - \dfrac{\nu}{2} = \dfrac{1}{p}$ for at least one $1 \leq j \leq N$, then the operators $\mathbf{A}_{\pm\varepsilon} := r_+[\boldsymbol{\zeta}(D)]^{s-\delta_j-\frac{\nu}{2}\pm\varepsilon}$ for sufficiently small $\varepsilon > 0$ is close (by norm) to $\mathbf{A}_0 := r_+[\boldsymbol{\zeta}(D)]^{s-\delta_j-\frac{\nu}{2}}$ and has different partial indices: 0 for $\mathbf{A}_{-\varepsilon}$ and $+1$ for $\mathbf{A}_{+\varepsilon}$. If we assume $\mathbf{A}_0$ is FREDHOLM, the same holds for $\mathbf{A}_{\pm\varepsilon}$, which is a contradiction, because $\mathbf{A}_{-\varepsilon}$ is Fredholm as proved above, whereas $\mathbf{A}_{+\varepsilon}$ is not.

The last claim about the kernel and the index follows from [DNS2, Lemma 19] since $\mathbf{a}(x, D)$ has one and the same regulariser in all spaces where it is FREDHOLM. □

1.6. Hölder spaces

Let $0 < \nu \leq 1$ and $\mathcal{H}^\nu(\mathbb{R})$ denote the space of HÖLDER continuous functions on $\mathbb{R}$ endowed with the norm

$$\|\varphi\big|\mathcal{H}^\nu(\mathbb{R})\| := \sup_{t\in\mathbb{R}} |\varphi(t)| + \sup_{\substack{t_1,t_2\in\mathbb{R}\\ t_1\neq t_2}} \frac{|\varphi(t_2) - \varphi(t_1)|}{\left|\dfrac{t_2}{t_2 + i} - \dfrac{t_1}{t_1 + i}\right|^\nu} .$$

Norm can also be represented in two following forms:

$$\begin{aligned} \|\varphi\big|\mathcal{H}^\nu(\mathbb{R})\| &= \sup_{t\in\mathbb{R}} |\varphi(t)| + \sup_{\substack{t_1,t_2\in\mathbb{R}\\ t_1\neq t_2}} \frac{|\varphi(t_2) - \varphi(t_1)|}{\left|\dfrac{1}{t_2 + i} - \dfrac{1}{t_1 + i}\right|^\nu} \\ &= \sup_{t\in\mathbb{R}} |\varphi(t)| + 2 \sup_{\substack{t_1,t_2\in\mathbb{R}\\ t_1\neq t_2}} \frac{|\varphi(t_2) - \varphi(t_1)|}{\left|\dfrac{t_2 - i}{t_2 + i} - \dfrac{t_1 - i}{t_1 + i}\right|^\nu} . \end{aligned} \tag{1.70}$$

The space $\mathcal{H}^\nu(\mathbb{R})$ differs from $C^\nu(\mathbb{R})$ (see § 1.1) since $\mathbb{R}$ is not compact; for a compact curve Γ the spaces $\mathcal{H}^\nu(\Gamma)$ and $C^\nu(\Gamma)$ are isomorphic.

The spaces $\mathcal{H}^\nu(\mathbb{R})$ and $\mathcal{H}^\nu(\Gamma_0) = C^\nu(\Gamma_0)$, where $\Gamma_0 = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle, are isomorphic:

$$\varpi_* \ : \ \mathcal{H}^\nu(\mathbb{R}) \longrightarrow \mathcal{H}^\nu(\Gamma_0) , \quad \varpi_*\varphi(z) := \varphi\left(i\frac{1+z}{1-z}\right) , \quad z \in \Gamma_0 . \tag{1.71}$$

The inverse isomorphism reads

$$\varpi_*^{-1}\psi(t) := \psi\left(\frac{t-i}{t+i}\right) , \quad t \in \mathbb{R} .$$

In fact,

$$
\begin{aligned}
&\|\varpi_*\varphi\big|\mathcal{H}^\nu(\Gamma_0)\| \\
&\quad = \sup_{z\in\Gamma_0}\left|\varphi\left(i\frac{1+z}{1-z}\right)\right| + \sup_{\substack{z_1,z_2\in\Gamma_0\\ z_1\neq z_2}} \frac{\left|\varphi\left(i\frac{1+z_2}{1-z_2}\right)-\varphi\left(i\frac{1+z_1}{1-z_1}\right)\right|}{|z_2-z_1|^\nu} \\
&\quad = \sup_{t\in\mathbb{R}}|\varphi(t)| + \sup_{\substack{t_1,t_2\in\mathbb{R}\\ t_1\neq t_2}} \frac{|\varphi(t_2)-\varphi(t_1)|}{\left|\dfrac{t_2-i}{t_2+i}-\dfrac{t_1-i}{t_1+i}\right|^\nu}
\end{aligned}
$$

and, due to (1.70),

$$
\|\varphi\big|\mathcal{H}^\nu(\mathbb{R})\| \le \|\varpi_*\varphi\big|\mathcal{H}^\nu(\Gamma_0)\| \le 2\|\varphi\big|\mathcal{H}^\nu(\mathbb{R})\|\,.
$$

Lemma 1.22. *The* HILBERT *transform*

$$
H_{\mathbb{R}}\varphi(t) := \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\varphi(\tau)d\tau}{\tau-t} \tag{1.72}
$$

is bounded in $\mathcal{H}^\nu(\mathbb{R})$ *and in* $\mathcal{H}^\nu_0(\mathbb{R}) := \{\varphi\in\mathcal{H}^\nu(\mathbb{R}) : \varphi(\infty)=0\}$ *provided* $0<\nu<1$.

Proof. The CAUCHY singular integral operator

$$
S_{\Gamma_0}\psi(z) := \frac{1}{\pi i}\int_{\Gamma_0}\frac{\psi(\zeta)d\zeta}{\zeta-z}
$$

is bounded in $\mathcal{H}^\nu(\Gamma_0)$ for $0<\nu<1$ by the PRIVALOV'S theorem (see [GK1, MP1, Mu1]). Under the isomorphism (1.71)) the transformed operator $\varpi_*^{-1}S_{\Gamma_0}\varpi_*$ acquires the form

$$
\begin{aligned}
\varpi_*^{-1}S_{\Gamma_0}\varpi_*\varphi(t) &= \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\varphi(\tau)}{\dfrac{\tau-i}{\tau+i}-\dfrac{t-i}{t+i}}\,\frac{2i d\tau}{(\tau+i)^2} = \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{t+i}{\tau+i}\,\frac{\varphi(\tau)d\tau}{\tau-t} \\
&= H_{\mathbb{R}}\varphi(t) - K_1\varphi\,, \qquad K_1\varphi := \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\varphi(\tau)d\tau}{\tau+i}\,.
\end{aligned}
$$

Since one–dimensional operator K_1 is bounded in $\mathcal{H}^\nu(\mathbb{R})\to\mathbb{C}\subset\mathcal{H}^\nu(\mathbb{R})$, the operator $H_{\mathbb{R}}$ is bounded in $\mathcal{H}^\nu(\mathbb{R})$.

Boundedness in the space $\mathcal{H}^\nu_0(\mathbb{R})$ is a consequence of the equalities:

$$
\mathcal{H}^\nu(\mathbb{R}) = \{\mathrm{const}\}\dot{+}\mathcal{H}^\nu_0(\mathbb{R})\,, \quad H_{\mathbb{R}}c = 0 \ \text{ for } \ c=\mathrm{const}\,. \qquad \square
$$

For a positive $\mu>0$, $\mu=m+\nu$, $m\in\mathbb{N}$, $0<\nu\le 1$ we consider the following BANACH algebra

$$
\mathcal{H}^\mu(\mathbb{R}) := \{\varphi\in C^m(\mathbb{R}) : (t+i)^k\partial_t^k\varphi\in\mathcal{H}^\nu(\mathbb{R}), k=0,1,\dots,m\}\,,
$$

endowed with the norm

$$\|\varphi\big|\mathcal{H}^{\mu}(\mathbb{R})\| := \sum_{k=0}^{m} \|(t+i)^k \partial_t^k \varphi\big|\mathcal{H}^{\nu}(\mathbb{R})\| \,.$$

If $\varphi \in \mathcal{H}^{\mu}(\mathbb{R})$ by sending in (1.70) $t_1 \to 0$ and setting $t_2 = t \in \mathbb{R}$ we get

$$\partial_t^k[\varphi(t) - \varphi(\infty)] = \mathcal{O}\left(|t+i|^{-\nu-k}\right)\,, \quad k = 0, 1, \dots, [\mu] = m\,. \tag{1.73}$$

The next lemma states a certain inverse estimates to (1.73).

Lemma 1.23. *Let* $0 < \nu \leq 1$, $\varphi \in C^m(\mathbb{R})$ *and*

$$C_{k,\nu} := \sup_t \left||t+i|^{k+\nu}\partial_t^k[\varphi(t) - \varphi(\infty)]\right| < \infty \quad \textit{for} \quad k = 0, 1, ..., m\,.$$

Then $\varphi \in \mathcal{H}^{m-1+\nu}(\mathbb{R})$ *and* $\|\varphi\big|\mathcal{H}^{m-1+\nu}(\mathbb{R})\| \leq M \sum\limits_{k=0}^{m} C_{k,\nu}$, *where* M *=const is independent of* φ.

If $\varphi \in \mathcal{H}_0^{m+\nu}(\mathbb{R})$ *and*

$$\partial_t^k b(t) = \mathcal{O}\left(|t+i|^{-k}\right) \quad \textit{for} \quad k = 0, 1, ...m\,,$$

then $b\varphi \in \mathcal{H}_0^{m+\nu}(\mathbb{R})$.

Proof. To prove the first part of the lemma by the definition of $\mathcal{H}^{m-1+\nu}(\mathbb{R})$ we need to check that $\varphi_k(t) := (t+i)^k\partial_t^k\varphi(t)$ belong to $\mathcal{H}^{\nu}(\mathbb{R})$ for $k = 0, \dots, m-1$ and the norms can be estimated with constants $C_0, \dots, C_m$.

For the proof we need the following inequalities from [Mu1, § 5])

$$\begin{aligned} &(x+y)^{\sigma} \leq (x^{\sigma} + y^{\sigma}) \leq 2^{1-\sigma}(x+y)^{\sigma}\,, \quad 0 < \sigma \leq 1\,, \\ &|x^{\sigma} - y^{\sigma}| \leq |x-y|^{\sigma}\,, \quad x \neq y\,, \quad x, y \in [0, \infty)\,, \end{aligned} \tag{1.74}$$

which are easy to check directly.

Let $t_1, t_2 \in [0, \infty)$; applying (1.74), we proceed as follows

$$\begin{aligned} |\varphi_k(t_2) - \varphi_k(t_1)| &= \left|\int_{t_1}^{t_2} \partial_\tau\varphi_k(\tau)d\tau\right| \leq \int_{t_1}^{t_2} \Big|k(\tau+i)^{k-1}\partial_\tau^k\varphi(\tau) \\ &\quad + (\tau+i)^k\partial_\tau^{k+1}\varphi(\tau)\Big|d\tau \leq (kC_{k,\nu} + C_{k+1,\nu})\int_{t_1}^{t_2} |\tau+i|^{-1-\nu}d\tau \\ &\leq (kC_{k,\nu} + C_{k+1,\nu})\int_{t_1}^{t_2} (\tau^2+1)^{-\frac{1+\nu}{2}}d\tau \leq 2^{-\frac{\nu+1}{2}}(kC_{k,\nu} + C_{k+1,\nu})\int_{t_1}^{t_2} (\tau+1)^{-1-\nu}d\tau \end{aligned}$$

As an example of the function $b(t)$ can be taken $(t+i)^{i\mu}$, $\mu \in \mathbb{R}$. A similar assertion is proved in [Mu1, Chapt.1, § 6] for the functions on a smooth curve when $m = 1$.

$$\leq C'_{k,\nu}\left|(t_2+1)^{-\nu}-(t_1+1)^{-\nu}\right| = C'_{k,\nu}\frac{[(t_2+1)^{\nu}-(t_1+1)^{\nu}]}{[(t_2^2+1)(t_1^2+1)]^{\frac{\nu}{2}}}$$
$$\leq C'_{k,\nu}\frac{|t_2-t_1|^{\nu}}{[|t_2+i||t_1+i|]^{\nu}} = C'_{k,\nu}\left|\frac{t_2}{t_2+i}-\frac{t_1}{t_1+i}\right|^{\nu},$$

where

$$C'_{k,\nu} := \frac{2^{-\frac{\nu+1}{2}}(kC_{k,\nu}+C_{k+1,\nu})}{\nu}.$$

Similar inequality holds if $t_1, t_2 \in (-\infty, 0]$.

Next we have to consider the cases when t_1 and t_2 have different signs. Without loss of generality we can assume $t_2 > 0$ and $t_1 < 0$; since $\varphi(t)$ is continuous at $t = 0$ and at $t = \infty$ (which means $\lim_{t\to-\infty}\varphi(t) = \lim_{t\to+\infty}\varphi(t) = \varphi(\infty)$), applying (1.74), we find:

$$|\varphi_k(t_2)-\varphi_k(t_1)| = |\varphi_k(t_2)-\varphi_k(t_0)| + |\varphi_k(t_0)-\varphi_k(t_1)|$$
$$\leq C'_{k,\nu}\left|\frac{t_2}{t_2+i}-t'_0\right|^{\nu} + C'_{k,\nu}\left|t'_0-\frac{t_1}{t_1+i}\right|^{\nu}$$
$$\leq 2^{1-\nu}C'_{k,\nu}\left(\left|\frac{t_2}{t_2+i}-t'_0\right| + \left|t'_0-\frac{t_1}{t_1+i}\right|\right)^{\nu},$$

where

$$t'_0 := \lim_{t\to t_0}\frac{t}{t+i} = \begin{cases} 0 & \text{for} \quad t_0 = 0\,, \\ 1 & \text{for} \quad t_0 = \infty\,, \end{cases} \qquad t_0 := \begin{cases} 0 & \text{for} \quad t_2 - t_1 \leq 2\,, \\ \infty & \text{for} \quad t_2 - t_1 > 2\,. \end{cases}$$

Let $t_2 - t_1 \leq 2$, then $t'_0 = 0$ and

$$|\varphi_k(t_2)-\varphi_k(t_1)| \leq 2^{1-\nu}C'_{k,\nu}\left(\frac{t_2}{|t_2+i|}-\frac{t_1}{|t_1+i|}\right)^{\nu}$$
$$\leq 2^{1-\nu}C'_{k,\nu}\left(\frac{t_2(t_1^2+1)^{\frac12}-t_1(t_2^2+1)^{\frac12}}{|t_2+i||t_1+i|}\right)^{\nu}$$
$$\leq 2^{1-\nu}C'_{k,\nu}\left(\frac{t_2(-t_1+1)-t_1(t_2+1)}{|t_2+i||t_1+i|}\right)^{\nu}$$
$$\leq 2^{1-\nu}C'_{k,\nu}\left(\frac{t_2-t_1-2t_2t_1}{|t_2+i||t_1+i|}\right)^{\nu} \leq 2^{1-\nu}C'_{k,\nu}\left(\frac{t_2-t_1+(t_2-t_1)^2}{|t_2+i||t_1+i|}\right)^{\nu}$$
$$\leq 2^{1-\nu}C'_{k,\nu}\left(\frac{3(t_2-t_1)}{|t_2+i||t_1+i|}\right)^{\nu} \leq 2^{1-\nu}3^{\nu}C'_{k,\nu}\left|\frac{t_2}{t_2+i}-\frac{t_1}{t_1+i}\right|^{\nu}.$$

If $2 < t_2 - t_1$ then $t'_0 = 1$ and

$$|\varphi_k(t_2)-\varphi_k(t_1)| \leq C'_{k,\nu}\left(|t_2+i|^{-\frac{\nu}{2}}+|t_1+i|^{-\frac{\nu}{2}}\right)$$
$$\leq C'_{k,\nu}\left(\frac{(t_1^2+1)^{\frac{\nu}{2}}+(t_2^2+1)^{\frac{\nu}{2}}}{|t_2+i|^{\nu}|t_1+i|^{\nu}}\right) \leq C'_{k,\nu}\left(\frac{(t_2+1)^{\nu}+(-t_1+1)^{\nu}}{|t_2+i|^{\nu}|t_1+i|^{\nu}}\right)$$
$$\leq 2^{1-\nu}C'_{k,\nu}\left(\frac{t_2-t_1+2}{|t_2+i||t_1+i|}\right)^{\nu} \leq 2^{1-\nu}C'_{k,\nu}\left(\frac{2(t_2-t_1)}{|t_2+i||t_1+i|}\right)^{\nu}$$
$$\leq 2C'_{k,\nu}\left|\frac{t_2}{t_2+i}-\frac{t_1}{t_1+i}\right|^{\nu}.$$

The proved inequalities can be summarised as follows

$$|\varphi_k(t_2)-\varphi_k(t_1)| \le \frac{12}{\nu}(kC_{k,\nu}+C_{k+1,\nu})\left|\frac{t_2}{t_2+i}-\frac{t_1}{t_1+i}\right|^{\nu}. \tag{1.75}$$

Thus, $\varphi \in \mathcal{H}^{m-1+\nu}(\mathbb{R})$ and $\|\varphi\big|\mathcal{H}^{m-1+\nu}(\mathbb{R})\| \le M\sum\limits_{k=0}^{m} C_{k,\nu}$.

Let us prove the second assertion. We have to prove that

$$\psi_k \in \mathcal{H}^{\nu}(\mathbb{R}), \quad \psi_k(t) := (t+i)^k\partial^k[b(t)\varphi(t)] \quad \text{for} \quad k=0,1,\dots,m\,.$$

Since

$$\psi_k(t) = \sum_{j=0}^{k} b_j(t)\varphi_{k-j}(t)\,, \quad b_j(t) := (t+i)^j\partial^j b(t)\,, \quad \varphi_j(t) := (t+i)^j\partial^j\varphi(t)\,,$$

the claim has to be proved only for $m=1$: $\varphi_{k-j}\in\mathcal{H}^{\nu}_0(\mathbb{R})$ and $\partial^l_t b_j(t) = 0\left(|t+i|^{-l}\right)$ for $l=0,1$ imply $b_j\varphi_{k-j}\in\mathcal{H}^{\nu}_0(\mathbb{R})$ and, finally $\psi_k\in\mathcal{H}^{\nu}_0(\mathbb{R})$.

Due to isomorphism (1.71) it suffices to prove $\mathrm{ae}_*(b\varphi) = \mathrm{ae}_* b\,\mathrm{ae}_*\varphi \in \mathcal{H}^{\nu}_0(\Gamma_0)$. Since

$$\begin{aligned}
&\mathrm{ae}_*\varphi\in\mathcal{H}^{\nu}_0(\Gamma_0)\,, \quad \mathrm{ae}_*\varphi(1)=\varphi(\infty)=0\,, \quad \mathrm{ae}_* b(\zeta)=\mathcal{O}(1) \;\text{ as }\; \zeta\to 1\,,\\
&\partial_\zeta \mathrm{ae}_* b(\zeta) = \partial_\zeta b\left(i\frac{1+\zeta}{1-\zeta}\right) = (\partial_t b)\left(i\frac{1+\zeta}{1-\zeta}\right)\left[i\frac{1}{1-\zeta}+i\frac{1+\zeta}{(1-\zeta)^2}\right]\\
&\qquad = \mathcal{O}\left(\left|i\frac{1+\zeta}{1-\zeta}+i\right|^{-1}\left|\frac{2i}{(1-\zeta)^2}\right|\right) = \mathcal{O}\left(|1-\zeta|^{-1}\right) \;\text{ as }\; \zeta\to 1\,,
\end{aligned}$$

conditions of the assertion in [Mu1, § 6.1^0] are fulfilled and the inclusion $\mathrm{ae}_*(b\varphi) = \mathrm{ae}_* b\,\mathrm{ae}_*\varphi\in\mathcal{H}^{\nu}_0(\mathbb{R})$ follows. □

Corollary 1.24. *If $0<\mu_1\le\mu_2$, the embedding $\mathcal{H}^{\mu_2}(\mathbb{R})\subset\mathcal{H}^{\mu_1}(\mathbb{R})$ is continuous.*

P r o o f. The claim follows from the foregoing Lemma 1.23 and inequality (1.82). □

Rational functions

$$r_\ell(t) = \sum_{|k|\le\ell} c_k\left(\frac{t-i}{t+i}\right)^k, \quad t\in\mathbb{R}\,, \quad c_k\in\mathbb{C} \tag{1.76}$$

belong to all $\mathcal{H}^{\mu}(\mathbb{R})$ (see Lemma 1.23) and let $\overset{\circ}{\mathcal{H}}{}^{\mu}(\mathbb{R})$ denote the sub-algebra of $\mathcal{H}^{\mu}(\mathbb{R})$ obtained by closing the algebra of rational functions (1.83). The algebra $\overset{\circ}{\mathcal{H}}{}^{\mu}(\mathbb{R})$ is rationally dense by the definition in [BG1] (see also [CG1]).

In [Ta1, § 1.3.4] the sub-algebra $\overset{\circ}{\mathcal{H}}{}^{\mu}(\mathbb{R})$ is characterised for $0<\mu<1$ (the same holds for all non–integer $\mu\in\mathbb{R}^+\setminus\mathbb{N}_0$) as follows: $\varphi\in\overset{\circ}{\mathcal{H}}{}^{\mu}(\mathbb{R})$ iff

$$\lim_{\varepsilon\to 0}\;\sup_{\substack{|t'-t|<\varepsilon\\ t'\ne t}} \frac{|\varphi(t')-\varphi(t)|}{\left|\dfrac{t'}{t'+i}-\dfrac{t}{t+i}\right|^{\nu}} = 0$$

uniformly for all $t \in \mathbb{R} \cup \{\infty\}$.

Lemma 1.25. *If* $0 < \mu = m + \nu < \mu' = m' + \nu'$, $m, m' \in \mathbb{N}_0$, $0 < \nu, \nu' < 1$, *then the embedding* $\mathcal{H}^{\mu'}(\mathbb{R}) \subset \overset{\circ}{\mathcal{H}}{}^{\mu}(\mathbb{R})$ *is continuous and dense.*

If $\varphi \in \overset{\circ}{\mathcal{H}}{}^{\mu}_0(\mathbb{R})$ *and*

$$\partial_t^k b(t) = \mathcal{O}\left(|t+i|^{-k}\right) \quad \textit{for} \quad k = 0, 1, ...m\,,$$

then $b\varphi \in \overset{\circ}{\mathcal{H}}{}^{\mu}_0(\mathbb{R})$.

Proof. It is known that $C^{\nu'}(\Gamma_0) \subset \overset{\circ}{C}{}^{\nu}(\Gamma_0)$ provided $0 < \nu < \nu'$, where $\overset{\circ}{C}{}^{\nu}(\Gamma_0)$ is obtained by closing rational functions $\sum\limits_{|k| \le \ell} c_k \zeta^k$ in $C^{\nu}(\Gamma_0)$ (see [Mu1], [Ta1, § 1.3.4]. The claimed inclusion $\mathcal{H}^{\nu'}(\mathbb{R}) \subset \overset{\circ}{\mathcal{H}}{}^{\nu}(\mathbb{R})$ (for the case $m = m' = 0$) follows automatically if isomorphism (1.71) is applied, because the rational functions on the axes $r_\ell(t)$ (see (1.76)) and on the unit circle are related via the isomorphism:

$$(\mathrm{ae}_* r_\ell)(\zeta) = \sum_{|k| \le \ell} c_k \zeta^k\,.$$

Now let $m = 1, 2, \ldots$, $\varphi \in \mathcal{H}^{\mu'}(\mathbb{R})$ and $\varphi_m(t) = (t+i)^m \partial_t^m \varphi(t)$. By the definition of the space $\varphi_m \in \mathcal{H}^{\nu'}(\mathbb{R})$ and as we already noted for arbitrary $\varepsilon > 0$ there exists a rational function

$$r_{m,\varepsilon}(t) = \sum_{|j| \le N} c_{m,j,\varepsilon} \left(\frac{t-i}{t+i}\right)^j$$

(cf. (1.76)) such that

$$\|\varphi_m - r_{m,\varepsilon} \big| \mathcal{H}^{\nu}(\mathbb{R})\| < \varepsilon\,. \tag{1.77}$$

We can assume that $r_{m,\varepsilon}(\infty) = \varphi_m(\infty)$ since otherwise we can take $\widetilde{r}_{m,\varepsilon}(t) := \varphi_m(\infty) - [r_{m,\varepsilon}(t) - r_{m,\varepsilon}(\infty)]$ and find

$$\begin{aligned}
&\widetilde{r}_{m,\varepsilon}(\infty) = \varphi(\infty)\,,\\
&\|\varphi_m - \widetilde{r}_{m,\varepsilon} \big| \mathcal{H}^{\nu}(\mathbb{R})\| \le \|\varphi_m - r_{m,\varepsilon} \big| \mathcal{H}^{\nu}(\mathbb{R})\| + |\varphi_m(\infty) - r_{m,\varepsilon}(\infty)| < 2\varepsilon\,.
\end{aligned}$$

Since $\mu > 1$ due to Corollary 1.24 $\varphi \in \mathcal{H}^1(\mathbb{R})$, $\varphi(\infty)$ exists and is finite; then

$$\begin{aligned}
r_\varepsilon(t) = r_{m,\varepsilon}(\infty) + \int_{-\infty}^{t} d\tau_{m-1} \int_{-\infty}^{\tau_{m-1}} \cdots \int_{-\infty}^{\tau_2} d\tau_1 \int_{-\infty}^{\tau_1} \frac{r^0_{m,\varepsilon}(\tau) d\tau}{(\tau+i)^m}\\
= r_{m,\varepsilon}(\infty) + \int_{-\infty}^{t} d\tau_{m-1} \int_{-\infty}^{\tau_{m-1}} \cdots \int_{-\infty}^{\tau_2} \frac{r^0_{m,\varepsilon}(\tau) d\tau}{(\tau+i)^m} \int_{\tau}^{\tau_2} d\tau_1\\
= r_{m,\varepsilon}(\infty) + \int_{-\infty}^{t} d\tau_{m-1} \int_{-\infty}^{\tau_{m-1}} \cdots \int_{-\infty}^{\tau_2} (\tau_2 - \tau) \frac{r^0_{m,\varepsilon}(\tau) d\tau}{(\tau+i)^m}
\end{aligned}$$

$$= \cdots = r_{m,\varepsilon}(\infty) + \int\limits_{-\infty}^{t} \frac{(\tau-t)^{m-1}}{(m-1)!} \frac{r^0_{m,\varepsilon}(\tau)d\tau}{(\tau+i)^m}\,,$$

where

$$r^0_{m,\varepsilon}(t) := r_{m,\varepsilon}(t) - r_{m,\varepsilon}(\infty)\,.$$

Easy to verify that r_ε is a rational function of the form (1.76).

Similarly,

$$\varphi(t) = \varphi(\infty) + \int\limits_{-\infty}^{t} d\tau_{m-1} \int\limits_{-\infty}^{\tau_{m-1}} \cdots \int\limits_{-\infty}^{\tau_2} d\tau_1 \int\limits_{-\infty}^{\tau_1} \frac{\varphi^0(\tau)d\tau}{(\tau+i)^m}$$

$$= \varphi(\infty) + \int\limits_{-\infty}^{t} \frac{(\tau-t)^{m-1}}{(m-1)!} \frac{\varphi^0(\tau)d\tau}{(\tau+i)^m}\,, \quad \varphi^0(t) := \varphi(t) - \varphi(\infty)\,.$$

Since $\varphi(\infty) = r_{m,\varepsilon}(\infty)$ we proceed as follows:

$$\left|(t+i)^{k+\nu}\partial_t^k\left[\varphi(t) - r_\varepsilon(t)\right]\right| = \left|(t+i)^{k+\nu}\int\limits_{-\infty}^{t} d\tau_{m-k-1} \cdots \int\limits_{-\infty}^{\tau_1} \frac{\varphi^0(\tau) - r^0_{m,\varepsilon}(\tau)}{(\tau+i)^m} d\tau\right|$$

$$\leq |t+i|^{k+\nu} \sup_\tau \left|\varphi^0(\tau) - r^0_{m,\varepsilon}(\tau)\right|^\delta \int\limits_{-\infty}^{t} d\tau_{m-k-1} \cdots \int\limits_{-\infty}^{\tau_1} \frac{\left|\varphi^0(\tau) - r^0_{m,\varepsilon}(\tau)\right|^{1-\delta}}{(\tau+i)^m} d\tau$$

$$\leq M_1|t+i|^{k+\nu} \sup_\tau |\varphi(\tau) - r_{m,\varepsilon}(\tau)|^\delta \int\limits_{-\infty}^{t} d\tau_{m-k-1} \cdots \int\limits_{-\infty}^{\tau_1} |\tau+i|^{(\delta-1)\nu'-m} d\tau$$

$$\leq M_2\varepsilon^\delta |t+i|^{k+\nu} \int\limits_{-\infty}^{t} d\tau_{m-k-1} \cdots \int\limits_{-\infty}^{\tau_1} |\tau+i|^{-\nu-m} d\tau \leq M_2\varepsilon^\delta \tag{1.78}$$

for $k = 0, 1, \ldots, m-1$, where

$$0 < \delta := 1 - \frac{\nu}{\nu'} < 1 \quad \text{and} \quad (1-\delta)\nu' = \nu\,.$$

Recalling the norm estimate from Lemma 1.23 and applying inequalities (1.77) and (1.78) we find

$$\|\varphi - r_\varepsilon\big|\mathcal{H}^\mu(\mathbb{R})\| \leq M \sum_{k=0}^{m-1} \sup_t ||t+i|^{k+\nu}\partial^k[vf(t) - r_\varepsilon(t)]| + M\|\varphi_m - r_{m,\varepsilon}\big|\mathcal{H}^\nu(\mathbb{R})\|$$

$$\leq M[mM_2\varepsilon^\delta + \varepsilon]$$

and the convergence $r_\varepsilon \to \varphi$ in $\mathcal{H}^\mu(\mathbb{R})$ as $\delta \to 0$ is proved.

The second claim follows from the proved one and the second claim of Lemma 1.23. □

Lemma 1.26. *If $\mu \in \mathbb{R}^+ \setminus \mathbb{N}_0$ ($\mu = m + \nu$, $m \in \mathbb{N}_0$, $0 < \nu < 1$), then $\mathcal{H}^\mu(\mathbb{R})$ and $\overset{\circ}{\mathcal{H}}{}^\mu(\mathbb{R})$ are decomposable Banach algebras, i.e. the* HILBERT *transform $H_{\mathbb{R}}$ is bounded in these algebras.*

Proof. Boundedness in $\overset{\circ}{\mathcal{H}}{}^\mu(\mathbb{R})$ is a consequence of boundedness in $\overset{\circ}{\mathcal{H}}{}^{\mu'}(\mathbb{R})$ with $0 < \mu < \mu'$ (see Lemma 1.25). Applying integration by parts we get the following

$$(t+i)^k \partial_t^k H_{\mathbb{R}}\varphi = H_{\mathbb{R}}(t+i)^k \partial_t^k \varphi \,.$$

Applying Lemma 1.22 we proceed as follows:

$$\begin{aligned}
\|H_{\mathbb{R}}\varphi \big| \mathcal{H}^\mu(\mathbb{R})\| &= \sum_{k=1}^m \|(t+i)^k \partial_t^k H_{\mathbb{R}}\varphi \big| \mathcal{H}^\nu(\mathbb{R})\| \\
&= \sum_{k=1}^m \|H_{\mathbb{R}}(t+i)^k \partial_t^k \varphi \big| \mathcal{H}^\nu(\mathbb{R})\| \leq \|H_{\mathbb{R}}\| \sum_{k=1}^m \|(t+i)^k \partial_t^k \varphi \big| \mathcal{H}^\nu(\mathbb{R})\| \\
&= \|H_{\mathbb{R}}\| \|\varphi \big| \mathcal{H}^\mu(\mathbb{R})\| \,.
\end{aligned}$$ □

It is possible to define the algebra $\mathcal{H}^\mu(\mathbb{R})$ with the help of the ZYGMUND spaces $\mathbb{Z}^\mu(\mathbb{R})$

$$\mathbb{Z}^\mu(\mathbb{R}) := \{\varphi \in \mathbb{C}^m(\mathbb{R}) : (t+i)^k \partial_t^k \varphi \in \mathbb{Z}^\nu(\mathbb{R}), k = 0, 1, \ldots, m\}\,, \quad \mu = m + \nu\,,$$

and endow it with the norm

$$\|\varphi \big| \mathbb{Z}^\mu(\mathbb{R})\| := \sum_{k=0}^m \|(t+i)^k \partial_t^k \varphi \big| \mathbb{Z}^\nu(\mathbb{R})\| \,.$$

Then spaces $\mathbb{Z}^\mu(\mathbb{R})$ and $\mathcal{H}^\mu(\mathbb{R})$ are the same for non integer $\mu \in \mathbb{R}^+ \setminus \mathbb{N}_0$ and boundedness of the HILBERT transform $H_{\mathbb{R}}$ in $\mathbb{Z}^\mu(\mathbb{R})$ holds even for an integer $\mu = 1, 2, \ldots$, which is not the case for $\mathcal{H}^m(\mathbb{R})$.

1.7. Factorization of symbols

Let $\mathcal{M}$, $a_{pr}(x', \xi)$, $a_0 = a_0(x')$ be as in Subsection 1.5, $\lambda_1(x'), \ldots, \lambda_\ell(x')$ be the eigenvalues of $a_0(x')$ (see (1.51)) and $m_1, \ldots, m_\ell$ be their algebraic multiplicities, i.e. the lengths of the corresponding chains of associated vectors $\sum\limits_{j=1}^{\ell} m_j = N$. Then $a_0(x')$ has the following decomposition

$$\begin{aligned}
&a_0(x') = \mathcal{K}_0(x')\mathcal{J}_{a_{pr}}(x')\mathcal{K}_0^{-1}(x') = \mathcal{K}(x')\Lambda_{a_{pr}} B^0_{a_{pr}}(1)\mathcal{K}^{-1}(x')\,, \\
&\det\mathcal{K}_0(x') \neq 0, \quad \det\mathcal{K}(x') \neq 0, \quad x' \in \partial\mathcal{M}
\end{aligned} \tag{1.79}$$

(see (1.57)), where the matrices $B^0_{a_{pr}}$ and $\mathcal{J}_{a_{pr}}(x')$ are quasi–diagonal

$$\begin{aligned}
\mathcal{J}_{a_0}(x') &:= \Lambda_{a_{pr}}(x') + H_{a_{pr}} = \mathrm{diag}\,\{\lambda_1(x')I_{m_1} + H_{m_1}, \ldots, \lambda_\ell(x')I_{m_\ell} + H_{m_\ell}\} \\
B^0_{a_{pr}}(t) &:= \mathrm{diag}\,\{B_{m_1}(t), \ldots, B_{m_\ell}(t)\}\,, \quad t \in \mathbb{C}\,, \\
B_m(z) &:= \exp(zH_m), \qquad z \in \mathbb{C}\,, \\
\Lambda_{a_{pr}}(x') &:= \mathrm{diag}\,\{\lambda_1(x')I_{m_1}, \ldots, \lambda_\ell(x')I_{m_\ell}\}\,, \\
H_{a_{pr}} &:= \mathrm{diag}\,\{H_{m_1}, \ldots, H_{m_\ell}\}\,;
\end{aligned}$$

I_m is the identity and H_m is the nilpotent matrix $H_m^m = 0$:

$$I_m := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{m\times m}, \quad H_m := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{m\times m}.$$

The first representation in (1.79) is known as the normal Jordan form and $\lambda I_m + H_m$ is the Jordan cell of the dimension m

$$\lambda I_m + H_m = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}_{m\times m}$$

Since $B_m(z) = \exp(zH_m)$, $z \in \mathbb{C}$ and H_m is nilpotent, the exponent has a finite expansion

$$\begin{aligned}
B_m(z) &:= \exp(zH_m) := I + \sum_{k=1}^{m-1} \frac{z}{k} H_m^k \\
&= \begin{pmatrix} 1 & \dfrac{z}{1!} & \dfrac{z^2}{2!} & \cdots & \dfrac{z^{m-2}}{(m-2)!} & \dfrac{z^{m-1}}{(m-1)!} \\ 0 & 1 & \dfrac{z}{1!} & \cdots & \dfrac{z^{m-3}}{(m-3)!} & \dfrac{z^{m-2}}{(m-2)!} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & \dfrac{z}{1!} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{m\times m}, \qquad z \in \mathbb{C}\,.
\end{aligned}$$

The sets $\left\{B^0_{a_{pr}}(z)\right\}_{z\in\mathbb{C}}$ and $\{B_m(z)\}_{z\in\mathbb{C}}$ are one parameter groups (see [Ar1, §§ 14–23]) of matrix–operators and have the following properties:

$$\begin{aligned}
&B^0_{a_{pr}}(z_1 + z_2) = B^0_{a_{pr}}(z_1)B^0_{a_{pr}}(z_2)\,, \\
&B^0_{a_{pr}}(0) = I_N, \qquad B^0_{a_{pr}}(-z) = \{B^0_{a_{pr}}(z)\}^{-1}\,, \\
&\{B^0_{a_{pr}}(z)\}^\gamma := \exp(z\gamma H_{a_{pr}}) = B^0_{a_{pr}}(\gamma z)\,, \qquad z, \gamma \in \mathbb{C}\,.
\end{aligned} \tag{1.80}$$

According to the definition e.g. in [Ga1, § V.1]

$$b(x') = \frac{1}{2\pi i}\log a_0(x') := \frac{1}{(2\pi)^2}\int_\Gamma [a_0(x') - zI]^{-1}\log\, z\, dz,$$

where I is the identity matrix, Γ is a closed contour, circumventing all eigenvalues $\lambda_1(x'),\ldots,\lambda_\ell(x')$ of $a_0(x')$ and leaving outside the negative real half–axes Re $z \leq 0$. We assume $\log z := \log|z| + i\mathrm{Arg} z, \quad -\pi < \mathrm{Arg} z < \pi$.

Here is the "pure algebraic" definition of the above defined logarithm:

$$b(x') := \frac{1}{2\pi i}\log a_0(x') := \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}a_0^k(x') = \frac{1}{2\pi i}\mathcal{K}(x')\log\left[\Lambda_{a_{pr}}B^0_{a_{pr}}(1)\right]\mathcal{K}^{-1}(x')$$

$$(1.81)\qquad = \mathcal{K}(x')\left\{\Delta(x') + \frac{1}{2\pi i}H_{a_{pr}}\right\}\mathcal{K}^{-1}(x'), \quad \delta_j(x') := \frac{1}{2\pi i}\log\lambda_j(x'),$$

$$\Delta(x') := \frac{1}{2\pi i}\log\Lambda_{a_{pr}(x')} = \mathrm{diag}\,\{\underbrace{\delta_1(x'),\ldots,\delta_1(x')}_{m_1\ \text{–times}},\ldots,\underbrace{\delta_\ell(x'),\ldots,\delta_\ell(x')}_{m_\ell\text{–times}}\}.$$

Introducing the notation

$$(1.82)\qquad B_\pm(t) := B^0_{a_{pr}}\left(\frac{1}{2\pi i}\log(t\pm i)\right),$$

where the branch of the logarithm is fixed in the complex plane cut along the ray $\{z\in\mathbb{C} : \arg z = \gamma_0\}$, we find (cf. (1.80))

$$B^0_{a_{pr}}\left(\frac{1}{2\pi i}\log\frac{t-i}{t+i}\right) = B_-(t)B_+^{-1}(t)$$

$$(1.83)\qquad = \begin{cases} [B^0_{a_{pr}}(1)]^{-1} + \mathcal{O}\left(|t-i|^{-1}\right) & \text{if}\quad t\to-\infty, \\ I_N + \mathcal{O}\left(|t+i|^{-1}\right) & \text{if}\quad t\to+\infty. \end{cases}$$

1.8. Proof of Theorem 1.18

Since

$$a_{pr}(x',\xi) = |\xi|^\nu a_{pr}(x',|\xi|^{-1}\xi) = (\xi_n - i|\xi'|)^{\frac{\nu}{2}}(\xi_n + i|\xi'|)^{\frac{\nu}{2}}a^0_{pr}(x',\xi),$$

$$(1.84)\qquad a^0_{pr}(x',\xi) := a_{pr}(x',|\xi|^{-1}\xi), \quad a^0_{pr}(x',\lambda\xi) = a^0_{pr}(x',\xi), \quad \lambda > 0$$

we can suppose that $\nu = 0$ and $a_{pr}(x',\lambda\xi) = a_{pr}(x',\xi)$ itself is homogeneous of order 0.

Let

$$(1.85)\qquad a^*(\omega,t) = (t-i)^\Delta B_-(t)a_1(\omega,t)B_+^{-1}(t)(t+i)^{-\Delta},$$

$$\omega = |\xi'|^{-1}\xi' \in \mathcal{S}^{n-1} := \{\omega\in\mathbb{R}^{n-1} : |\omega| = 1\}, \quad t = |\xi'|^{-1}\xi_n \in \mathbb{R},$$

$$a_1(\omega,t) = \mathcal{K}^{-1}a^{-1}_{pr}(0,+1)a_{pr}(\omega,t)\mathcal{K}$$

(we have dropped the variable $x' \in \partial\mathcal{M}$ for simplicity). Next we prove that

$$(1.86)\quad \omega^{\alpha'}\partial_\omega^{\alpha'}\partial_t^k[a^*(\omega,t)-I_N]_{jq} = O[|t+i|^{\mathrm{Re}\,[\delta_j-\delta_q]+\varepsilon-k-1}) = O[|t+i|^{\delta_0+\varepsilon-k-1})$$

for all $k\in\mathbb{N}_0$, $\alpha'\in\mathbb{N}_0^{n-1}$, $|\alpha'|\le\gamma-1$, small $\varepsilon>0$ and δ_0 defined by the relations (see (1.52))

$$(1.87)\qquad \delta_0 := \max_{j,q=1,\dots,N}\{\mathrm{Re}\,(\delta_j-\delta_q)\} < \delta_0+3\varepsilon<1\,.$$

First let us prove that

$$(1.88)\qquad \omega^{\alpha'}\partial_t^k\partial_\omega^{\alpha'}[a_1(\omega,t)-a_1(0,\pm1)] = \mathcal{O}(|t+i|^{-k-1})\,,$$

for $|\alpha'|\le\gamma-1\,,\ \ k\in\mathbb{N}_0$, where

$$a_1(0,-1) = \lim_{\xi_n\to-\infty} a_1(\xi',\xi_n) = \mathcal{K}^{-1}a_{pr}^{-1}(0,+1)a_{pr}(0,-1)\mathcal{K} = \Lambda_{pr}B_{pr}^0(1)$$

$$(1.89)\qquad a_1(0,+1) = \lim_{\xi_n\to+\infty} a_1(\xi',\xi_n) = I_N$$

(see (1.51), (1.79), (1.85)).

$a_1(\omega,t)$ is homogeneous of order 0, and $a_1(\cdot,\pm1)\in C^\gamma(S^{n-2})$; TAYLOR expansion at $t=\pm\infty$ provides

$$\begin{aligned} a_1(\omega,t)-a_1(0,\pm1) &= a_1(|t|^{-1}\omega,\pm1)-a_1(0,\pm1)\\ &= \sum_{j=1}^{n-1}\omega_j(\partial_{\omega_j}a_1)(0,\pm1)|t|^{-1}+\mathcal{O}(|t|^{-2}) \quad \text{as} \quad t\to\pm\infty. \end{aligned}$$

For $0<|\alpha'|\le\gamma-1$ estimate (1.88) follows if we differentiate the foregoing identity.

The function $a^*(\omega,t)$ from (1.85) can be rewritten as follows (cf. (1.89))

$$(1.90)\qquad \begin{aligned} & a^*(\omega,t) = a_2^\pm(\omega,t)+a_3^\pm(\omega,t)\,,\\ & a_2^\pm(\omega,t) = (t-i)^\Delta B_-(t)[a_1(\omega,t)-a_1(0,\pm1)]B_+^{-1}(t)(t+i)^{-\Delta}\,,\\ & a_3^+(\omega,t) = (t-i)^\Delta B_-(t)B_+^{-1}(t)(t+i)^{-\Delta}\\ &\qquad = \left(\frac{t-i}{t+i}\right)^\Delta B^0_{a_{pr}}\left(\frac{1}{2\pi i}\log\frac{t-i}{t+i}\right)\,,\\ & a_3^-(\omega,t) = (t-i)^\Delta B_-(t)\Lambda_{a_{pr}}B^0_{a_{pr}}(1)B_+^{-1}(t)(t+i)^{-\Delta}\,. \end{aligned}$$

If we apply (1.88), we get estimates for $a_2^\pm(\omega,t)$

$$(1.91)\qquad \begin{aligned} & \omega^{\alpha'}\partial_\omega^{\alpha'}\partial_t^k[a_2^\pm(\omega,t)]_{j,q} = \mathcal{O}(|t+i|^{\mathrm{Re}\,(\delta_j-\delta_q)+\varepsilon-k-1})\\ &\qquad = \mathcal{O}(|t+i|^{\delta_0+\varepsilon-k-1}),\quad k+|\alpha'|\le\gamma\,,\quad \text{as}\quad |t|\to\infty\,, \end{aligned}$$

where $\varepsilon>0$ and δ_0 are defined in (1.87).

To prove a similar estimate for $a_3^\pm(\omega,t)$ we note that according to the definition of function $(t\pm i)^{\pm\Delta}$ (see (1.52), (1.55))

$$(1.92)\quad (t-i)^{\pm\Delta}(t+i)^{\mp\Delta} = \left(\frac{t-i}{t+i}\right)^{\pm\Delta} = \begin{cases} I_N+\mathcal{O}(|t+i|^{-1}) & \text{as} \quad t\to+\infty,\\ \Lambda_{a_{pr}}^{\mp1}+\mathcal{O}(|t+i|^{-1}) & \text{as} \quad t\to-\infty. \end{cases}$$

Applying (1.83), (1.88), (1.89), (1.90), (1.92) we proceed as follows

$$\omega^{\alpha'}\partial_\omega^{\alpha'}\partial_t^k\left[a_3^+(\omega,t)-I_N\right]=\omega^{\alpha'}\partial_\omega^{\alpha'}\partial_t^k\left[(t-i)^\Delta B_-(t)B_+^{-1}(t)(t+i)^{-\Delta}-I_N\right]$$
$$=\omega^{\alpha'}\partial_\omega^{\alpha'}\partial_t^k\left[\left(\frac{t-i}{t+i}\right)^\Delta B^0_{a_{pr}}\left(\frac{1}{2\pi i}\log\frac{t-i}{t+i}\right)-I_N\right]$$
$$=\mathcal{O}(|t+i|^{-k-1})\quad\text{as}\quad t\to+\infty \tag{1.93}$$
$$\omega^{\alpha'}\partial_\omega^{\alpha'}\partial_t^k\left[a_3^-(\omega,t)-I_N\right]$$
$$=\omega^{\alpha'}\partial_\omega^{\alpha'}\partial_t^k\left[(t-i)^\Delta B_-(t)\Lambda_{a_{pr}}B^0_{a_{pr}}(1)B_+^{-1}(t)(t+i)^{-\Delta}-I_N\right]$$
$$=\omega^{\alpha'}\partial_\omega^{\alpha'}\partial_t^k\left[\Lambda_{a_{pr}}B^0_{a_{pr}}(1)\left(\frac{t-i}{t+i}\right)^\Delta B^0_{a_{pr}}\left(\frac{1}{2\pi i}\log\frac{t-i}{t+i}\right)-I_N\right]$$
$$=\omega^{\alpha'}\partial_\omega^{\alpha'}\partial_t^k\left[\Lambda_{a_{pr}}B^0_{a_{pr}}(1)\Lambda^{-1}_{a_{pr}}B^0_{a_{pr}}(-1)-I_N\right]+\mathcal{O}(|t+i|^{-k-1})$$
$$=\mathcal{O}(|t+i|^{-k-1})\quad\text{as}\quad t\to-\infty\,, \tag{1.94}$$

because the diagonal matrices $\Lambda_{a_{pr}}$, $(t\pm i)^{\pm\Delta}$ commute with the block–diagonal matrices $B_\pm(t)$, $B^0_{a_{pr}}(1)$ (the diagonal matrices are constant inside the blocks of the block–diagonal ones) and $B^0_{a_{pr}}(z_1), B^0_{a_{pr}}(z_2)$ commute as well (see (1.80)).

From (1.90), (1.91), (1.93), (1.94) we get (1.86) and, by virtue of Lemma 1.23, $a^*\in\mathcal{H}^{m-\delta_0-\varepsilon}(\mathbb{R})\subset\overset{\circ}{\mathcal{H}}{}^{m-\delta_0-2\varepsilon}(\mathbb{R})$ for all $m=1,2,\ldots$.

The elliptic matrix–function a^* in the decomposable and rationally dense algebra $\overset{\circ}{\mathcal{H}}{}^{m-\delta_0-2\varepsilon}(\mathbb{R})$ (see § 2) admits a factorization

$$a^*(t)=[a^*_-(t)]^{-1}\left(\frac{t-i}{t+i}\right)^\sigma a^*_+(t), \tag{1.95}$$
$$\sigma=(\sigma_1,\ldots,\sigma_N)\in\mathbb{Z}^N,\quad \mathbb{Z}=\{0,\pm1,\ldots\}$$

with factors $[a^*_-(t)]^{\pm1}$, $[a^*_+(t)]^{\pm1}$ which belong to $\overset{\circ}{\mathcal{H}}{}^{m-\delta_0-2\varepsilon}(\mathbb{R})$ and have uniformly bounded analytic continuations into the half–planes $\operatorname{Im}t<0$ and $\operatorname{Im}t>0$, respectively (see [BG1, CG1]).

Since the limits $a^*_\pm(\infty)$ and $a^*(\infty)=I_N$ exist (see (1.73), from (1.95) and (1.86) we find

$$[a^*_-(\infty)]^{-1}a^*_+(\infty)=a^*(\infty)=I_N$$

and, without loss of generality, we can suppose $a^*_\pm(\infty)=I_N$; then (see (1.73))

$$\partial_t^k[a^*_\pm(t)-I_N]=\mathcal{O}\left(|t+i|^{\delta_0+2\varepsilon-k-1}\right)\quad\text{as}\quad t\to\infty\,. \tag{1.96}$$

From (1.85) and (1.95), inserting $\omega=|\xi'|^{-1}\xi'$ and $t=|\xi'|^{-1}\xi_n$ we find the components of the factorization (1.53) (we remind that $\nu=0$ and, therefore, $g_\pm(x',\xi)=a^\pm_{pr}(x',\xi)$ in (1.53)):

$$a^\pm_{pr}(x',\xi)=a^0_\pm(x',|\xi'|^{-1}\xi',|\xi'|^{-1}\xi_n),$$
$$a^0_\pm(x',\omega,t)=(t\pm i)^{-\Delta}B^{-1}_\pm(x',t)a^*_\pm(x',\omega,t)B_\pm(x',t)(t\pm i)^\Delta\widetilde{a}_\pm(x')$$

$$(1.97)\quad = \widetilde{a}_\pm(x') + (t\pm i)^{-\Delta} B_\pm^{-1}(x',t)[a^*_\pm(x',\omega,t) - I_N] B_\pm(x',t)(t\pm i)^{\Delta}\widetilde{a}_\pm(x')\,,$$
$$\widetilde{a}_+(x') := \mathcal{K}^{-1}(x')\,, \quad \widetilde{a}_-(x') := \mathcal{K}^{-1}(x') a_{pr}^{-1}(x',+1)\,,$$
$$B_{a_{pr}}(\xi) := B^0_{a_{pr}}\left(\frac{1}{2\pi i}\log\frac{\xi_n - i|\xi'|}{\xi_n + i|\xi'|}\right).$$

The theorem will be proved if we can prove the estimates

$$\omega^{\beta'}\partial_\omega^{\beta'}\partial_t^k\left([a^0_+(\omega,t)]^{\pm1}_{j,q} - [\widetilde{a}_+]^{\pm1}_{j,q}\right) = \mathcal{O}(|t+i|^{\theta-k-1})\,,$$
$$(1.98)\quad \omega^{\beta'}\partial_\omega^{\beta'}\partial_t^k\left([a^0_-(\omega,t)]^{\pm1}_{j,q} - [\widetilde{a}_-]^{\pm1}_{j,q}\right) = \mathcal{O}(|t+i|^{\theta-k-1})$$

for all $j,q = 1,...,N$, $k = 0,1,...$ and some $0<\theta<1$; concerning β': $|\beta'| = 0$ if $\mathbf{a}(x,D)$ **is not** FREDHOLM, $|\beta'|\le\gamma-1$ if $\mathbf{a}(x,D)$ **is** FREDHOLM.

In fact, from (1.97) we find

$$\xi_j\partial_{\xi_j}\partial^{\beta_n}_{\xi_n}a_+(\xi) = -\sum_{\substack{k\le n-1\\ k\ne j}}(\partial_{\omega_k}\partial_t^{\beta_n}a^0_+)(|\xi'|^{-1}\xi', |\xi'|^{-1}\xi_n)\frac{\xi_j^2\xi_k}{|\xi'|^{\beta_n+3}}$$
$$+(\partial_{\omega_j}\partial_t^{\beta_n}a^0_+)(|\xi'|^{-1}\xi', |\xi'|^{-1}\xi_n)\frac{\xi_j}{|\xi'|^{\beta_n+1}} - (\partial_t^{\beta_n+1}a^0_+)(|\xi'|^{-1}\xi', |\xi'|^{-1}\xi_n)\frac{\xi_j^2\xi_n}{|\xi'|^{\beta_n+3}}$$

and, due to (1.98),

$$|\xi_j\partial_{\xi_j}\partial^{\beta_n}_{\xi_n}a_+(\xi)| \le M_1\left(\frac{\left|i+|\xi'|^{-1}\xi_n\right|^{-\beta_n}}{|\xi'|^{\beta_n}} + \frac{\left|i+|\xi'|^{-1}\xi_n\right|^{-\beta_n-1}|\xi_n|}{|\xi'|^{\beta_n+1}}\right)2M_1|\xi|^{-\beta_n}.$$

By similar estimates

$$(1.99)\quad \left|(\xi')^{\beta'}\partial^\beta_\xi a_+(\xi)\right| \le M_2\xi|^{-\beta_n}, \quad \beta_n = 0,1,..., \quad |\beta'|\le\gamma-1,$$

where again $|\beta'| = 0$ if $\mathbf{a}(x,D)$ **is not** FREDHOLM, $|\beta'|\le\gamma-1$ if $\mathbf{a}(x,D)$ **is** FREDHOLM.

From (1.99) we get estimates (1.54) (if $\mathbf{a}(x,D)$ **is not** FREDHOLM) and the inclusions $[a^-_{pr}]^{\pm1}$, $[a^+_{pr}]^{\mp1}\in\mathcal{R}^{\gamma-1}_{hom,0}(\mathcal{T}^*\mathcal{M})$ (if $\mathbf{a}(x,D)$ **is** FREDHOLM; we remind that we are treating the case $\nu = 0$).

First we will prove estimates (1.98) for $\beta' = 0$, i.e. when $\mathbf{a}(x,D)$ **is not** FREDHOLM. Consider, for definiteness, $a^0_+(t) = a^0_+(\omega,t)$. Other estimates are similar.

A typical entry of the matrix a^0_+ is

$$(1.100)\quad (a^0_+)_{jq}(\omega,t) - \widetilde{a}_+ = (t+i)^{\delta_q-\delta_j}\sum_{l\le q}c_{jql}\left[(a^*_+)_{jl}(\omega,t) - \delta_{jl}\right][\ln(t+i)]^{m_{lq}}\,,$$

where $m_{qq} = 0$, $\delta_l = \delta_j$, δ_{jl} is the KRONEKER's symbol.

Invoking (1.73) we find

$$\partial_t^k\left([a^0_+(\omega,t)]_{jq} - [\widetilde{a}_+]_{jq}\right) = \partial_t^k\Big((t+i)^{\delta_q-\delta_j}\sum_{l\le q}c_{jql}(\omega)[(a^*_+)_{jl}(\omega,t) - \delta_{jl}][\ln(t+i)]^{m_{lq}}\Big)$$
$$= \begin{cases} O\left((t+i)^{\mathrm{Re}\,(\delta_q-\delta_j)+\delta_0+3\varepsilon-k-1}\right) & \text{if}\quad \mathrm{Re}\,\delta_q > \mathrm{Re}\,\delta_j\,,\\ \mathcal{O}\left((t+i)^{\delta_0+3\varepsilon-k-1}\right) & \text{if}\quad \mathrm{Re}\,\delta_q \le \mathrm{Re}\,\delta_j\,.\end{cases}$$

From (1.95) we have

$$a_+^* - a_-^* = \left[I_N - \left(\frac{t-i}{t+i}\right)^\sigma \right] a_+^* + [a_-^* - I_N][a^* - I_N] + a^* - I_N \tag{1.101}$$

and applying (1.86), (1.96) we obtain

$$\begin{aligned}\partial_t^k [a_+^* - a_-^*]_{jl}(t) = \mathcal{O}(|t+i|^{-k-1}) + \partial_t^k \sum_{r=1}^N \{[a_-^*]_{jr} - \delta_{jr}\}[a^* - I_N]_{rl} \\ + \mathcal{O}\left(|t+i|^{\mathrm{Re}\,(\delta_j - \delta_l) + \varepsilon - k - 1}\right) = \sum_{r=1}^N \mathcal{O}\left(|t+i|^{\mathrm{Re}\,(\delta_r - \delta_l) + \varepsilon - 1 - k}\right) \\ = \mathcal{O}\left(|t+i|^{\delta_l^+ + \varepsilon - 1 - k}\right), \quad k = 0, 1, \dots,\end{aligned}$$

where $\delta_j^+ := \max_q \{\mathrm{Re}\,[\delta_q - \delta_j]\} = \mathrm{Re}\,[\delta_{j_*} - \delta_j]$ for a certain $1 \le j \le n$ (note, that we have inserted $\partial_t^l \{[a_-^*]_{jr}(\omega, t) - \delta_{jl}\} = \mathcal{O}(|t+i|^{\delta_0 + 2\varepsilon - l - 1}) = \mathcal{O}(|t+i|^{-l})$; cf. (1.96)). Invoking Lemma 1.23 we conclude $[a_+^* - a_-^*]_{jl} \in \overset{\circ}{\mathcal{H}}{}^{m - \delta_l^+ - 2\varepsilon}(\mathbb{R})$ for all $m = 1, 2, \dots$.

The projections $P_{\mathbb{R}}^\pm = \frac{1}{2}(I \pm H_{\mathbb{R}})$ eliminate functions, analytic in the half planes $\mp \mathrm{Im}\, t < 0$ (see [CG1, GK1, LS1]); hence

$$(a_\pm^*)_{jl} = \pm P_{\mathbb{R}}^\pm [a_+^* - a_-^*]_{jl} \in \overset{\circ}{\mathcal{H}}{}^{m - \delta_l^+ - 2\varepsilon}(\mathbb{R})$$

and therefore (see (1.73) and cf. (1.96))

$$\partial_t^k \left[(a_\pm^*)_{jl}(\omega, t) - I_N\right]_{jl} = \mathcal{O}\left(|t+i|^{\delta_l^+ + 2\varepsilon - k - 1}\right), \quad k = 0, 1, \dots .$$

Inserting the obtained asymptotic for $\left[(a_-^*)_{jl}(\omega, t) - I_N\right]_{jl}$ into (1.101) and again invoking (1.95) we get more precise asymptotic

$$\begin{aligned}&\partial_t^k [a_+^* - a_-^*]_{jl}(t) = \mathcal{O}\left(|t+i|^{-k-1}\right) \\ &\quad + \sum_{r=1}^N \mathcal{O}\left(|t+i|^{\delta_r^+ + 2\varepsilon - 1 + \mathrm{Re}\,(\delta_r - \delta_l) + \varepsilon - k - 1}\right) + \mathcal{O}\left(|t+i|^{\mathrm{Re}\,(\delta_j - \delta_l) + \varepsilon - k - 1}\right) \\ &\quad = \sum_{r=1}^N \mathcal{O}\left(|t+i|^{\mathrm{Re}\,(\delta_{r_*} - \delta_l) + 3\varepsilon - k - 2}\right) + \mathcal{O}\left(|t+i|^{\mathrm{Re}\,(\delta_j - \delta_l) + \varepsilon - k - 1}\right) \\ &\quad = \mathcal{O}\left(|t+i|^{\mathrm{Re}\,(\delta_j - \delta_l) + \varepsilon - k - 1}\right), \quad k = 0, 1, \dots,\end{aligned}$$

where $\delta_{r_*} := \delta_r + \delta_r^+$.

Thus, $[a_+^* - a_-^*]_{jl} \in \overset{\circ}{\mathcal{H}}{}^{m - \mathrm{Re}\,(\delta_j - \delta_l) - 2\varepsilon}(\mathbb{R})$ for all $m = 1, 2, \dots$ and we conclude, as above, $(a_\pm^*)_{jl} = \pm P_{\mathbb{R}}^\pm [a_+^* - a_-^*]_{jl} \in \overset{\circ}{\mathcal{H}}{}^{m - \mathrm{Re}\,(\delta_j - \delta_l) - 2\varepsilon}(\mathbb{R})$. The latter yields (cf. (1.96))

$$\partial_t^k \left[(a_\pm^*)_{jl}(\omega, t) - I_N\right]_{jl} = \mathcal{O}\left(|t+i|^{\mathrm{Re}\,(\delta_j - \delta_l) + 2\varepsilon - k - 1}\right), \quad k = 0, 1, \dots .$$

By virtue of (1.100)

$$\partial_t^k [a_+^0]_{jq}(t) = \mathcal{O}\left(|t+i|^{\mathrm{Re}\,(\delta_q-\delta_j)+\mathrm{Re}\,(\delta_j-\delta_l)+3\varepsilon-k-1}\right) = \mathcal{O}(|t+i|^{\theta-k-1})$$

for all $k=0,1,\dots$ since $\delta_l=\delta_j$ (see (1.97)) and $\mathrm{Re}\,(\delta_q-\delta_j)+3\varepsilon=\theta<1$ (see (1.87)).

Now we will prove estimates (1.98) for $\beta'\neq 0$, i.e. when $\mathbf{a}(x,D)$ **is** FREDHOLM.

If equation (1.49) is FREDHOLM for some p,μ,s and m; then the partial indices of factorization (1.95) vanish

$$\sigma_1(x') = ... = \sigma_N(x') = 0 \quad \text{for all} \quad x'\in\partial\mathcal{M} \tag{1.102}$$

and components $a^*_\pm(x',\omega,t)$ of the factorization (1.95) depend on $x'\in\partial\mathcal{M}$ and on $\omega\in S^{n-2}$. Conditions (1.102) ensure that $[a_\pm(x',\omega,t)]^{\pm1}$ satisfy the same condition with respect to the variables $x'\in\partial\mathcal{M}$ and $\omega\in S^{n-2}$ as $a^*(x',\omega,t)$, i.e. as $[a_{pr}(x',\omega,t)$ (see [Sb1]) and the inclusions $[a^-_{pr}]^{\mp1}$, $[a^+_{pr}]^{\pm1}\in\mathcal{R}^{\gamma-1}_{hom,0}(\mathcal{T}^*\mathcal{M})$ follow.

1.9. Proof of Lemma 1.18

For the first claim of the Lemma we quote [La1, Theorem 2.10.2].

The pseudodifferential equation (1.49) with the strongly elliptic symbol $a^*_{pr}(x,\xi)$ is FREDHOLM (see e.g. [DS1, Theorem 3.26], [DW1, Theorem 1.7] etc. for the case $\mu=m=0$; the case $\mu\neq0$, $m\neq0$ is similar). Therefore the partial indices vanish (see (1.102) and (1.60)) and imply the inclusions $[a^-_{pr}]^{\pm1}$, $[a^+_{pr}]^{\mp1}\in\mathcal{R}^{\gamma-1,0}_{\pm\frac{\nu}{2}}(\mathcal{T}^*\mathcal{M})$ as in the foregoing theorem.

The remaining assertions are proved in [DSW1, Lemma A.6] as follows.

Since the matrices $a_{pr}(\omega,\pm1)$ are positive definite, there exist the square roots $[a_{pr}(\omega,+1)]^{\pm\frac12}$ and the matrix

$$\begin{aligned} a^1_{pr}(\omega) &:= [a_{pr}(\omega,+1)]^{\frac12}\, a^0_{pr}(\omega)\, [a_{pr}(\omega,+1)]^{-\frac12} \\ &= [a_{pr}(\omega,+1)]^{-\frac12}\, a_{pr}(\omega,-1)\, [a_{pr}(\omega,+1)]^{-\frac12}\,, \end{aligned}$$

due to similarity, has the common eigenvalues, the common eigenvectors and the common JORDAN chains of associated vectors with $a^0_{pr}(\omega)$. On the other hand $a^1_{pr}(\omega)$ is self adjoint, i.e. is normal and. has no associated vectors as noted above; moreover, $\mathcal{K}\in C^\infty(\partial\mathcal{M})$. Let $\eta(\omega),\dots,\eta_N(\omega)\in\mathbb{C}^N$ be eigenvectors corresponding to the eigenvalues $\lambda_1(\omega),\dots,\lambda_N(\omega)$; then

$$a^0_{pr}(\omega)\eta_j(\omega)=\lambda_j\eta_j(\omega),\quad j=1,\dots,N$$

and we get

$$\lambda_j(\omega)=\frac{\left(a^0_{pr}(y',0,+1)\eta_j(\omega),\eta_j(\omega)\right)}{\left(a^0_{pr}(y',0,-1)\eta_j(\omega),\eta_j(\omega)\right)}>0$$

because of the positive definiteness of $a_{pr}(\omega,\pm1)$. This implies (1.60). □

Remark 1.27. Let an elliptic symbol $a_{pr}(x,\xi)$ have restricted smoothness $a_{pr} \in \mathcal{R}^{\varpi,\gamma,m}_{hom,\nu}(\mathcal{T}^*\mathcal{M})$, which reads: $a_{pr}(x,\xi)$ is positive homogeneous of order ν in ξ and

$$
\begin{aligned}
&|(\xi')^{\beta'}\partial_x^\alpha\partial_\xi^\beta a_0(x,\xi)| \le M_{\alpha,\beta}|\xi|^{\nu-\beta_n}, \quad \xi \in \mathbb{R}^n\,,\\
&\text{for all } |\alpha| \le \varpi \quad |\beta'| \le \gamma\,, \quad \gamma \ge 1\,, \quad k = 1,\dots,m\,, \quad m \ge 2\,.
\end{aligned} \tag{1.103}
$$

Then for the components of factorization (1.53)–(1.54) we get the inclusions $[a^-_{pr}]^{\mp 1}$, $[a^+_{pr}]^{\pm 1} \in \mathcal{R}^{\varpi,\gamma-1,m-2}_{hom,\pm\frac{\nu}{2}}(\mathcal{T}^*\mathcal{M})$.

Remark 1.28. The convolution operator $r_+\mathbf{B}_{a_{pr}}(D)$ (see (1.54)) (1.66)) is bounded in both spaces $\widetilde{H}^{(\mu,s),m}_p(\mathbb{R}^n_+)$ and $H^{(\mu,s),m}_p(\mathbb{R}^n_+)$ for all $\mu, s \in \mathbb{R}$, $1 < p < \infty$ and $m = 0, 1, \ldots$ due to Theorems 1.3, 1.19.iii and to Lemma 1.14, because

$$
B_{a_{pr}}(\xi) := B^0_{a_{pr}}\left(\frac{1}{2\pi i}\log\frac{\xi_n - i|\xi'|}{\xi_n + i|\xi'|}\right) = B^0_{a_{pr}}\left(-\frac{1}{2\pi i}\log\frac{\xi_n + i|\xi'|}{\xi_n - i|\xi'|}\right) = B^{-1}_{a_{pr}}(\xi', -\xi_n)
$$

(cf. (1.97)) has analytic extensions in both complex half–spaces (see Lemma 1.2).

The inverse to this operator can be written explicitly based on the factorization

$$
B_{a_{pr}}(\xi) = B^{-1}_-(\xi)B_+(\xi)\,, \quad B_\pm(\xi) := B^0_{a_{pr}}\left(-\frac{1}{2\pi i}\log(\xi_n \pm i|\xi'|)\right)
$$

(see (1.83)), on the properties (1.80) and on (1.6); namely

$$
\left[r_+\mathbf{B}_{a_{pr}}(D)\right]^{-1} = \mathbf{B}^{-1}_+(D)\mathbf{B}_-(D). \tag{1.104}
$$

In fact, the only property which needs to be verified is the boundedness of the inverse operator $\left[r_+\mathbf{B}_{a_{pr}}(D)\right]^{-1}$ in the spaces $\widetilde{H}^{(\mu,s),m}_p(\mathbb{R}^n_+)$ and $H^{(\mu,s),m}_p(\mathbb{R}^n)_+)$, which follows since it is a formal inverse (see (1.6)) and $r_+\mathbf{B}_{a_{pr}}(D)$ is invertible, as proved in Theorem 1.21 (we remind that $\mathbf{B}_{a_{pr}}(D)$ is upper triangular with identities on the main diagonal).

Theorem 1.29. *Let $\mathcal{M}^+ := \partial\mathcal{M} \times \mathbb{R}^+$, $a, b \in C^\infty(\mathcal{M})$ and*

$$
a^\infty(x', \xi_n) = a(x')\langle\xi_n\rangle^\nu \left(\frac{\xi_n - i}{\xi_n + i}\right)^{-\Delta} B^0_{a_{pr}}\left(\frac{1}{2\pi i}\log\frac{\xi_n - i}{\xi_n + i}\right) b(x')
$$

(see (1.79)–(1.80) for $B^0_{a_{pr}}$).

Then the corresponding pseudodifferential operator

$$
\mathbf{a}^\infty(x', D_n) : \widetilde{\mathbb{H}}^{(\infty,s),m}_p(\mathcal{M}^+) \longrightarrow \mathbb{H}^{(\infty,s-\nu),m}_p(\mathcal{M}^+) \tag{1.105}
$$

with the symbol $a^\infty(x', \xi_n)$ is bounded.

The equation

$$
\mathbf{a}^\infty(x', D_n)u = v, \qquad v \in \mathbb{H}^{(\infty,s-\nu),m}_p(\mathcal{M}^+) \tag{1.106}
$$

has a unique solution $u \in \widetilde{\mathbb{H}}_p^{(\infty,s),m}(\mathcal{M}^+)$ *for all* $m \in \mathbb{N}_0$ *provided* a, b *are elliptic (non–degenerate) matrices and the conditions*

$$\frac{1}{p} - 1 < s - \operatorname{Re}\delta_j - \frac{\nu}{2} < \frac{1}{p}\,, \qquad j = 1, \cdots, \ell$$

hold. The solution reads

$$u = b^{-1}(x')(D_n + i)^{-\Delta-\frac{\nu}{2}}\mathbf{B}_+(D_n)\theta_+(D_n - i)^{\Delta-\frac{\nu}{2}}\mathbf{B}_-^{-1}(D_n)a^{-1}(x')v\,.$$

Proof. The proof follows word in word the proof of Theorem 1.21, but we will expose a simpler version of the proof.

The variable $x' \in \partial\mathcal{M}$ can be localised and the localised operator

$$\mathbf{a}^\infty(x_0', D_n) : \widetilde{\mathbb{H}}_p^{s,m}(\mathbb{R}^+) \longrightarrow \mathbb{H}_p^{s-\nu,m}(\mathbb{R}^+) \tag{1.107}$$

is a one–dimensional PsDO ($\mathbf{a}^\infty(x', D_n)$ in (1.105) is invertible iff $\mathbf{a}^\infty(x_0', D_n)$ in (1.107) are invertible for all $x_0' \in \partial\mathcal{M}$; (see [Du1, Sec. 3.2]).

Further we apply the lifting (see [DS1, Sec. 3.1] and Theorem 1.13 above): $\mathbf{a}^\infty(x_0', D_n)$ in (1.107) is invertible iff the lifted convolution operator

$$W_{a^\infty_{\nu,s}(x_0',\cdot)} = \mathbf{a}^\infty_{\nu,s}(x_0', D_n) : \mathbb{H}_p^{0,m}(\mathbb{R}^+) \longrightarrow \mathbb{H}_p^{0,m}(\mathbb{R}^+)$$

with the symbol

$$\begin{aligned} a^\infty_{\nu,s}(x_0', \xi_n) &:= (\xi_n - i)^{s-\nu}a^\infty(x_0', \xi_n)(\xi_n + i)^{-s} \\ &= a(x')\left(\frac{\xi_n - i}{\xi_n + i}\right)^{s-\Delta-\frac{\nu}{2}} B^0_{a_{pr}}\left(\frac{1}{2\pi i}\log\frac{\xi_n - i}{\xi_n + i}\right)b(x') \end{aligned}$$

is invertible.

Invertibility conditions of the convolution operator $W_{a^\infty_{\nu,s}(x_0',\cdot)}$ in the space $L_p(\mathbb{R}^+) = \mathbb{H}_p^{0,0}(\mathbb{R}^+)$ are known (see [Du2]) and coincide with the conditions of the theorem. As for the spaces $\mathbb{H}_p^{0,m}(\mathbb{R}^+)$ with $m \neq 0$, the invertibility condition is independent of $m = 0, 1, 2, \cdots$ since the inverse in $L_p(\mathbb{R}^+)$ is bounded in all spaces $\mathbb{H}_p^{0,m}(\mathbb{R}^+)$ and therefore represents the inverse there. $\square$

2. Asymptotic

Throughout this Section we assume

$$a \in \mathcal{S}^{\gamma}_{cl,\nu}(\mathcal{T}^*\mathcal{M}) \quad \text{and} \quad \gamma \in \mathbb{N}_0 \quad \gamma \geq \left[\frac{n}{2}\right] + 2\,. \quad \nu \in \mathbb{R}\,.$$

Further we suppose that $a(x,\xi)$ has an **elliptic** homogeneous principal part, which reads

$$\det a_{pr}(x;\xi) \neq 0, \quad x \in \overline{\mathcal{M}}, \quad \xi \in \mathbb{R}^n \setminus \{0\}$$

(cf. (1.50)). The notations $a_0(x')$, $\mathcal{K}(x')$, $\mathcal{J}_{a_0(x')}$, $\lambda_1(x')$, $\Lambda_{a_{pr}}(x')$, $\delta_j(x')$, $\Delta(x')$, $B_{a_{pr}}$, B_m, $B_\pm(t)$, $H_{a_{pr}}$, H_m, from Subsections 1.5 and 1.7 (cf. (1.50)–(1.52), (1.55), (1.79)–(1.81)) will be used without further references.

2.1. Formulation of results

Let $\mathcal{M}$ be a compact n–dimensional C^∞–smooth manifold with the C^∞–smooth boundary $\partial\mathcal{M}$ and consider a $N \times N$ system of pseudodifferential equations

$$\mathbf{a}(x;D)u = v, \qquad u \in \widetilde{\mathbb{H}}^{(\infty,s),\varpi}_p(\mathcal{M})\,, \tag{2.1}$$

$$v \in \mathbb{H}^{(\infty,s-\nu),\varpi}_p(\mathcal{M})\,, \quad s \in \mathbb{R}\,, \quad 1 < p < \infty\,, \quad \varpi = 0,1,\dots\,.$$

Let us introduce a special local coordinate system (s.l.c.s.) $(\ x',\ x_{n,+}) \in \mathcal{M}^+ := \partial\mathcal{M} \times \mathbb{R}^+$ on $\mathcal{M}$ in the neighbourhood of $\partial\mathcal{M}$, where $x' \in \partial\mathcal{M}$, while $x_{n,+}$ measures the distance to the boundary $\partial\mathcal{M}$.

The main purpose of the present section is to prove the following.

Theorem 2.1. *Let equation* (2.1) *have a unique solution* $u \in \widetilde{\mathbb{H}}^{(\infty,s),\varpi}_p(\mathcal{M})$ *for each given* $v \in \mathbb{H}^{(\infty,s-\nu),\varpi}_p(\mathcal{M})$. *Then*

$$\frac{1}{p} - 1 < s - \frac{\nu}{2} - \operatorname{Re}\delta_j(x') < \frac{1}{p} \quad \text{for all} \quad j = 1,\cdots,\ell\,.$$

Let further $\frac{\nu}{2} + \operatorname{Re}\delta_j(x') > -1$, $M \in \mathbb{N}_0$, $M \leq \varpi$, $\gamma \geq \left[\frac{n}{2}\right] + M + 4$, $\mathcal{K}, \delta_1, \dots, \delta_\ell \in C^\infty(\partial\mathcal{M})$ *and* $v \in \mathbb{H}^{(\infty,s-\nu+M+1),\varpi}_p(\mathcal{M})$. *Then the solution has the following asymptotic expansion*

$$\begin{aligned} u(x',x_{n,+}) &= \mathcal{K}(x')\ x_{n,+}^{\frac{\nu}{2}+\Delta(x')} B^0_{a_{pr}}\left(-\tfrac{1}{2\pi i}\log x_{n,+}\right)\mathcal{K}^{-1}(x')\Bigg[c_0(x') \\ &+ \sum_{k=1}^{M} x^k_{n,+} \sum_{j=0}^{(2m_0-1)k} c_{kj}(x')\log^j x_{n,+}\Bigg]\chi_0(x_{n,+}) + \widetilde{u}_{M+1}(x',\ x_{n,+}) \end{aligned} \tag{2.2}$$

The inclusions $\mathcal{K}, \delta_1, \dots, \delta_\ell \in C^\infty(\partial\mathcal{M})$ can be guaranteed if either $\mathcal{K}, \delta_1 \in C^\infty(\partial\mathcal{M})$ or $\delta_1, \dots, \delta_\ell \in C^\infty(\partial\mathcal{M})$ and dimensions of JORDAN blocks in the block–diagonal matrix $B^0_{a_{pr}}(1)$ are stable $m_j = \text{const}\,,\ j = 1,\dots,\ell$.

with $\widetilde{u}_{M+1} \in \widetilde{\mathbb{H}}_p^{(\infty,s+M+1),\varpi}(\mathcal{M})$ *and a suitable cut–off function* $\chi_0 \in C_0^\infty(\mathbb{R}^+)$, $\chi_0(x_{n,+}) = 1$ *for small* $x_{n,+} > 0$ *(cf.* (1.55), (1.81) *for notations). Here* $B^0_{a_{pr}}(t)$ *is the block–diagonal matrix defined in* (1.81) *by the homogeneous principal symbol of equation* (2.1) *and* $m_0 = \max\{m_1, \dots, m_\ell\}$ *denotes the size of its maximal block. N–vectors* $c_{00} := c_0$ *and* c_{kj} *belong to* $C^\infty(\partial\mathcal{M})$, c_0 *is defined by the homogeneous principal symbol, while others* $c_{01}, c_{10}, \dots$ *are influenced by the full symbol of equation* (2.1).

Furthermore, for arbitrary $\mu, m, k \in \mathbb{N}_0$ *the a priori estimates*

$$\begin{aligned} C_0 \sum_{k=0}^{M}\sum_{j=0}^{k} \|c_{kj}|C^M(\partial\mathcal{M})\| + C_0\|\widetilde{u}_{M+1}|\widetilde{\mathbb{H}}_p^{(\infty,s+M+1),\varpi}\mathcal{M}\| \\ \le \|u|\widetilde{\mathbb{H}}_p^{(\infty,s),\varpi}(\mathcal{M})\| \le C_1\|v|\mathbb{H}_p^{(\infty,s-\nu+M+1),\varpi}(\mathcal{M})\|\,, \\ \|v|\mathbb{H}_p^{(\infty,s-\nu),\varpi}(\mathcal{M})\| \le C_2\|u|\widetilde{\mathbb{H}}_p^{(\infty,s),\varpi}(\mathcal{M})\| \end{aligned} \tag{2.3}$$

hold with some constants C_0, C_1, C_2 *which are independent of* v.

If chains of associated vectors are trivial $B^0_{a_{pr}} = I$ *(e.g. if* $a_{pr}(x', 0, \pm 1)$ *are positive definite or the matrix* $a_0(x')$ *in* (1.51) *is normal; see* (1.56)) *for all* $x' \in \partial\mathcal{M}$, *then* $m_0 = 1$, *logarithmic terms vanish from the leading term of the asymptotic expansion and it acquires the form*

$$\begin{aligned} u(x', x_{n,+}) &= \mathcal{K}(x')x_{n,+}^{\frac{\nu}{2}+\Delta(x')}\mathcal{K}^{-1}(x') \\ &\times\Big[c_0(x') + \sum_{k=1}^{M} x_{n,+}^k \sum_{j=0}^{k} c_{kj}(x')\ \log^j x_{n,+}\Big]\chi_0(x_{n,+}) + \widetilde{u}_{M+1}(x', x_{n,+})\,. \end{aligned}$$

If $\lambda_1 = \ldots = \lambda_\ell = \lambda$ *are all equal, expansion takes simplest form*

$$u(x', x_{n,+}) = \sum_{k=0}^{M}\sum_{j=0}^{k} c_{kj}(x')x_{n,+}^{\frac{\nu}{2}+\Delta(x')+k} \log^j x_{n,+}\, \chi_0(x_{n,+}) + \widetilde{u}_{M+1}(x', x_{n,+})\,. \tag{2.4}$$

As we noted in the Introduction asymptotic (2.2) was derived in [Es1] and [Be1] for $p = 2$; but even for the case $p = 2$ asymptotic (2.2) is more precise.

Remark 2.2.. The obtained estimate for the exponents of logarithmic terms $(2m_0 - 1)k$ in (2.2) is rough. In the model case of the half–space we have estimate $m_0 k$ (see [DW1] and cf. Lemma 2.6 below).

As it was noted in [Be1] and as it is clear from the proof of Theorem 2.1 in § 2.3 (the case $M = 0$) even if $\mathcal{K} \notin C^\infty(\partial\mathcal{M})$ the leading part of asymptotic (2.2) is the same:

$$\begin{aligned} u(x', x_{n,+}) = \mathcal{K}(x')\ x_{n,+}^{\frac{\nu}{2}+\Delta(x')} B^0_{a_{pr}}\left(-\frac{1}{2\pi i}\log x_{n,+}\right)\mathcal{K}^{-1}(x')c_0(x')\chi_0(x_{n,+}) \\ +\mathcal{K}(x')x_{n,+}^{\frac{\nu}{2}+\Delta(x')+1}\widetilde{u}_1(x',\ x_{n,+}) \end{aligned} \tag{2.5}$$

with "almost bounded" $|x^\varepsilon_{n,+}\widetilde{u}_1(x', x_{n,+})| \le C < \infty$ for arbitrary $\varepsilon > 0$ but we can not claim any more that the "stress intensity factor" is smooth $c_0 \notin C^\infty(\partial\mathcal{M})$.

2.2. Auxiliary propositions

Let us remind that we use the notation from Subsections 1.5, 1.7.

Lemma 2.3. *(cf. [Be1, (1.32)]). Let $a \in S^{\gamma}_{cl,\nu}(\mathcal{T}^*\mathcal{M})$ and*

$$\begin{aligned} a^{\infty}(x', \xi_n) &:= \langle\xi_n\rangle^{\nu}\ a_{pr}(x', +1) b_-^{-1}(x', \xi_n) b_+(x', \xi_n)\,, \\ b_{\pm}(x', \xi_n) &:= (\xi_n \pm i)^{b(x')}, \quad b(x') := \frac{1}{2\pi i}\log a_0(x')\,. \end{aligned} \tag{2.6}$$

(cf. (1.81)*). Then*

$$\lim_{\xi_n \to \pm\infty} \langle\xi_n\rangle^{-\nu} a^{\infty}(x'; \xi_n) = a_{pr}(x'; \pm 1)\,, \tag{2.7}$$

$$\partial^{\alpha'}_{x'}\partial^{\beta_n}_{\xi_n}\left[a_{pr}(x'; 0, \xi_n) - a^{\infty}(x'; \xi_n)\right] = \mathcal{O}\left(|\xi_n|^{\nu-\beta_n-1}\right) \quad \text{as} \quad |\xi_n| \to \infty \tag{2.8}$$

for all $\alpha' \in \mathbb{N}_0^{n-1}$, $\beta_n \in \mathbb{N}_0$ (we remind, that $a_{pr}(x'; \pm 1) := a_{pr}(x'; 0, \pm 1)$).

Proof. The equalities

$$\left(\frac{t+i}{t-i}\right)^{\mu} = \begin{cases} 1 + \mathcal{O}(\langle t\rangle^{-1}) & \text{as} \quad t \to +\infty, \\ \exp(2\pi\mu i) + \mathcal{O}(|t|^{-1}) & \text{as} \quad t \to -\infty \end{cases}$$

(cf. (1.92)) and (1.51), (2.6), yield:

$$\begin{aligned} a^{\infty}(x'; \xi_n) &= \langle\xi_n\rangle^{\nu} a_{pr}(x'; +1)\left(\frac{\xi_n + i}{\xi_n - i}\right)^{\frac{1}{2\pi i}\log\ a_0(x')} = \langle\xi_n\rangle^{\nu} a_{pr}(x'; +1) a_0(x') + \mathcal{O}\left(\langle\xi_n\rangle^{\nu-1}\right) \\ &= \langle\xi_n\rangle^{\nu} a_{pr}(x'; -1) + \mathcal{O}\left(\langle\xi_n\rangle^{\nu-1}\right) \quad \text{as} \quad \xi_n \to -\infty\,, \\ a^{\infty}(x'; t) &= \langle\xi_n\rangle^{\nu} a_{pr}(x'; +1) + \mathcal{O}\left(\langle\xi_n\rangle^{\nu-1}\right) \quad \text{as} \quad \xi_n \to +\infty \end{aligned}$$

and (2.7) is proved.

To prove (2.8) we apply the Taylor expansion to $a^{\infty}(x'; \xi_n)$ at $\xi_n \to \pm\infty$ separately:

$$\begin{aligned} a^{\infty}(x'; \xi_n) &:= a_{pr}(x'; +1)|\xi_n|^{\nu}\langle\xi_n^{-1}\rangle^{\nu}\left(\frac{1 + i\xi_n^{-1}}{1 - i\xi_n^{-1}}\right)^{\frac{1}{2\pi i}\log\ a_0(x')} \\ &= \sum_{j=0}^{M} a_j^{\infty}(x'; \operatorname{sgn}\xi_n)\xi_n^{\nu-j} + \widetilde{a}^{\infty}_{M+1}(x'; \xi_n)\,, \end{aligned} \tag{2.9}$$

where

$$a_j^{\infty}(x'; \pm 1) = \frac{1}{j!} a_{pr}(x'; +1) \lim_{t \to \pm 0} \partial_t^j \left[\langle t\rangle^{\nu}\left(\frac{1+it}{1-it}\right)_0^{\frac{1}{2\pi i}\log\ a_0(x')}\right]$$

and the function $g_{\sigma}(t) := \left(\dfrac{1+it}{1-it}\right)_0^{\sigma}$ is continuous on $\mathbb{R} \setminus \{0\}$ except $t = 0$ but including infinity $g_{\sigma}(-\infty) = g_{\sigma}(+\infty)$; the branch of the analytic function $g_{\sigma}(z)$ on

Estimates (2.8) for $\alpha' = 0$, $\beta = 0$ are obtained in [Es1, (26.7)] (cf. also [Be1, (1.32)]). As proved in [Sr1, Sect. 4] derivatives $\partial^{\alpha}_x b(x, \xi)$ up to order $|\alpha| \le n$ must be estimated to get boundedness of the corresponding operator $\mathbf{b}(x, D)$.

the complex plane $\mathbb{C} \setminus [-i, i]$ cut along the interval $[-i, i]$ on the imaginary axis, are fixed as follows

$$g_\sigma(\pm\infty) = e^{\pi\sigma i}\,, \quad g_\sigma(-0) = 1\,, \quad g_\sigma(+0) = e^{2\pi\sigma i}\,.$$

Obviously,

$$a_0^\infty(x'; \pm 1) = a_{pr}(x'; \pm 1)$$

(see (2.7)) and, therefore, $a_0^\infty(x'; \xi_n) = a_{pr}(x'; 0, \xi_n)$. (2.8) is a consequence of (2.9) if we take $M = 0$. □

Lemma 2.4. *Let* $M \in \mathbb{N}_0$, $s \in \mathbb{R}^+$, $\varphi \in \mathbb{H}_p^{(\infty, s+M+1), \varpi}(\mathbb{R}^n)$ *and*

$$\frac{1}{p} - 1 < s < \frac{1}{p}\,, \quad \theta_+ := \frac{1}{2}(1 + \operatorname{sgn} x_n), \quad x_{n,+} := \theta_+ x_n. \tag{2.10}$$

Then

$$\begin{aligned} \theta_+\varphi(x) &= \sum_{k=0}^{M} \frac{(-i)^k}{k!} x_{n,+}^k e^{-x_{n,+}}((D_n + i)^k \varphi(x', 0) + \widetilde{\varphi}_{M+1}(x)\,, \\ \widetilde{\varphi}_{M+1}(x) &:= (D_n + i)^{-M-1}\theta_+(D_n + i)^{M+1}\varphi(x) \end{aligned} \tag{2.11}$$

and $\widetilde{\varphi}_{M+1} \in \widetilde{\mathbb{H}}_p^{(\infty, s+M+1), \varpi}(\mathbb{R}^n_+)$.

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $\overset{\vee}{\varphi}_n (x', t_n) = (\mathcal{F}^{-1}_{x_n \to t_n}\varphi)(x', t_n)$. Then

$$\begin{aligned} \frac{1}{2}(I - S_{\mathbb{R}}) \overset{\vee}{\varphi}_n (x', t_n) &= \frac{1}{2}\left[\overset{\vee}{\varphi}_n (x', t_n) - \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\overset{\vee}{\varphi}_n (x', \tau_n) d\tau_n}{\tau_n - t_n}\right] \\ &= \sum_{k=0}^{M} \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{(\tau_n - i)^k}{(t_n - i)^{k+1}} \overset{\vee}{\varphi}_n (x', \tau_n) d\tau_n \\ &+ \frac{1}{2}\left[\overset{\vee}{\varphi}_n (x', t_n) - \frac{(t_n - i)^{-M-1}}{\pi i}\int_{-\infty}^{\infty} \frac{(\tau_n - i)^{M+1} \overset{\vee}{\varphi}_n (x', \tau_n) d\tau_n}{\tau_n - t_n}\right] \\ &= \sum_{k=0}^{M} \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{(\tau_n - i)^k}{(t_n - i)^{k+1}} \overset{\vee}{\varphi} (t', \tau_n) d\tau_n \\ &+ \frac{1}{2}(t_n - i)^{-M-1}(I - S_{\mathbb{R}})(\tau_n - i)^{M+1} \overset{\vee}{\varphi}_n (x', t_n); \end{aligned}$$

if we apply $\mathcal{F}_n := \mathcal{F}_{t_n \to x_n}$, we obtain (2.11). In fact,

$$\begin{aligned} &\mathcal{F}_n \frac{1}{2}(I - S_{\mathbb{R}})\mathcal{F}_n^{-1}\varphi = \theta_+\varphi \\ &\mathcal{F}_n(t_n - i)^{\pm k}\mathcal{F}_n^{-1}\varphi = (-1)^k \mathcal{F}_n^{-1}(t_n + i)^{\pm k}\mathcal{F}_n\varphi = (-1)^k(D_n + i)^{\pm k}\varphi, \\ &\int_{-\infty}^{\infty} (\tau_n - i)^k \overset{\vee}{\varphi}_n (x', \tau_n) d\tau_n = (-1)^k((D_n + i)^k\varphi)(x', 0), \end{aligned}$$

$$\mathcal{F}_n(t_n-i)^{-k-1}=\int\limits_{-\infty}^{\infty}\frac{e^{ix_nt_n}dt_n}{(t_n-i)^{k+1}}=\frac{1}{k!}(ix_n)^k\int\limits_{-\infty}^{\infty}\frac{e^{ix_nt_n}dt_n}{t_n-i}=2\pi i\frac{i^k}{k!}x^k_{n,+}e^{-x_{n,+}}.$$

For $\varphi\in\mathbb{H}_p^{(\infty,s+M+1),\varpi}(\mathbb{R}^n)$ we get the following inclusions successively: $(D_n+i)^{M+1}\varphi\in\mathbb{H}_p^{(\infty,s),\varpi}(\mathbb{R}^n)$ (see (1.27)) $\Longrightarrow \psi_0=\theta_+(D_n+i)^{M+1}\varphi\in\mathbb{H}_p^{(\infty,s),\varpi}(\mathbb{R}^n_+)=\widetilde{\mathbb{H}}_p^{(\infty,s),\varpi}(\mathbb{R}^n_+)$ (see Lemma 1.14), $\Longrightarrow (D_n+i)^{-M-1}\psi_0\in\widetilde{\mathbb{H}}^{(\infty,s+m+1),\varpi}(\mathbb{R}^n_+)$ (see Theorem 1.13).

For $\varphi\in\mathbb{H}_p^{(\infty,s+M+1)}(\mathbb{R}^n)$ the claim (2.11) follows since $C_0^\infty(\mathbb{R}^n)$ is dense in $\mathbb{H}_p^{(\infty,s+M+1)}(\mathbb{R}^n)$. □

Remark 2.5. Let us note that if conditions of Lemma 2.4 holds, then

$$\theta_-\varphi(x)=\sum_{k=0}^M\frac{(-i)^k}{k!}x^k_{n,-}e^{x_{n,-}}((D_n-i)^k\varphi)(x',0)+\widetilde{\varphi}_{M+1}(x)$$
$$\widetilde{\varphi}_{M+1}:=(D_n-i)^{-M-1}\theta_-(D_n-i)^{M+1}\varphi\in\widetilde{\mathbb{H}}_p^{(\infty,s+M+1),\varpi}(\mathbb{R}^n_+).$$

Note that if we change D_n+i (D_n-i) in (2.11) (in the previous identity) by D_n-i (by D_n+i, respectively), we can not claim any more $\widetilde{\varphi}_{M+1}\in\widetilde{\mathbb{H}}_p^{(\infty,s+M+1,\varpi)}(\mathbb{R}^n_+)$, since from $\psi_0\in\widetilde{\mathbb{H}}_p^{(\infty,s),\varpi}(\mathbb{R}^n_\pm)$ does not follow $(D_n\pm i)^{-M-1}\psi_0\in\widetilde{\mathbb{H}}_p^{(\infty,s+M+1),\varpi}(\mathbb{R}^n_\pm)$.

Lemma 2.6. *Let $\mathcal{M}^+=\partial\mathcal{M}\times\mathbb{R}^+$, $s,\nu\in\mathbb{R}$, $\varpi\in\mathbb{N}_0$, $1<p<\infty$, $a\in\mathcal{S}^\gamma_{cl,\nu}(\mathcal{T}^*\mathcal{M})$ and conditions of Theorem* 1.21 *be fulfilled.*

If $a^\infty(x',\xi_n)$ is defined by (2.6) *the equation*

$$r_+\mathbf{a}^\infty(x',D_n)u=f,\quad f\in\mathbb{H}_p^{(\infty,s-\nu),\varpi}(\mathcal{M}^+)\tag{2.12}$$

has a unique solution in $\mathbb{H}_p^{(\infty,s-\nu)),\varpi}(\mathcal{M}^+)$, represented by the formulae

$$\begin{aligned}u&=\mathcal{K}(x')\mathbf{a}_+^{-1}(D_n)\theta_+\mathbf{a}_-(D_n)a^0f,\\ a^0(x')&:=\mathcal{K}^{-1}(x')[a_{pr}(x',+1)]^{-1},\\ \mathbf{a}_+^{-1}(D_n)&:=\mathbf{B}_+^{-1}(D_n)(D_n+i)^{-\frac{\nu}{2}-\Delta(x')},\\ \mathbf{a}_-(D_n)&:=\mathbf{B}_-(D_n)(D_n-i)^{-\frac{\nu}{2}+\Delta(x')}.\end{aligned}\tag{2.13}$$

where $B_\pm(\xi_n)$ are defined in (1.82).

For arbitrary $M\in\mathbb{N}_0$, $M\le\varpi$, $f\in\mathbb{H}_p^{(\infty,s-\nu+M+1)),\varpi}(\mathcal{M}^+)$ this solution has the following asymptotic expansion

$$\begin{aligned}u(x',x_{n,+})=&\sum_{k=0}^M\mathcal{K}(x')x_{n,+}^{\frac{\nu}{2}+\Delta(x')+k}e^{-x_{n,+}}B^0_{a_{pr}}\left(-\frac{1}{2\pi i}\log x_{n,+}\right)\mathcal{K}^{-1}(x')c_k(x')\\ &+\widetilde{u}_{M+1}(x',x_{n.+}),\qquad \widetilde{u}_{M+1}\in\widetilde{\mathbb{H}}_{p,comp}^{(\infty,s+M+1),\varpi}(\mathbb{R}^n_+)\end{aligned}\tag{2.14}$$

with

$$c_k(x'):=\mathcal{K}(x')\mathbf{B}^0_{a_{pr}}\left(\frac{1}{2\pi i}\partial_\mu\right)\frac{e^{\frac{\pi\mu}{2}i}}{\Gamma(-\mu)}((D_n+i)^k\mathbf{a}_-(D_n)a^0(x')f)(x',0),$$
$$\mu:=-\frac{\nu}{2}-\Delta(x')-k-1,\quad c_k\in C^\infty(\partial\mathcal{M}),\quad k=1,\dots,M$$

and $g(\mu) := \operatorname{diag}\{g(\mu_1), \cdots, g(\mu_N)\}$.

Proof. From (1.92) we find

$$\left(\frac{\lambda+i}{\lambda-i}\right)^{\frac{1}{2\pi i}\log\ a_0(x')} = e^{-\frac{1}{2\pi i}\log\frac{\lambda-i}{\lambda+i}\log\ a_0(x')} = \mathcal{K}(x')\left(\frac{\lambda-i}{\lambda+i}\right)^{-\Delta(x')} e^{-\frac{1}{2\pi i}\log\frac{\lambda-i}{\lambda+i}H_{a_{pr}}}\mathcal{K}^{-1}(x')$$
$$= \mathcal{K}(x')\left(\frac{\lambda-i}{\lambda+i}\right)^{-\Delta(x')} B^0_{a_{pr}}\left(-\frac{1}{2\pi i}\log\frac{\lambda-i}{\lambda+i}\right)\mathcal{K}^{-1}(x')\,.$$

Now the first assertions about solvability of equation (2.12) and solution formulae (2.13) follow from Theorem 1.21 (cf. also Theorem 1.29).

Let us prove (2.14).

Applying (2.11) to (2.13) we proceed as follows:

$$\text{(2.15)}\qquad u(x',x_{n,+}) = \sum_{k=0}^{M}\mathcal{K}(x')\mathbf{a}_+^{-1}(x',D_n)v_k(x_{n,+})c_k^1(x') + \widetilde{u}_{M+1}(x',x_{n,+}),$$
$$v_k(x_{n,+}) := x_{n,+}^k e^{-x_{n,+}},\quad c_k^1(x') = \frac{(-i)^k}{k!}(D_n+i)^k\mathbf{a}_-(x',D_n)a^0(x')f(x',0),$$
$$\widetilde{u}_{M+1} = (D_n+i)^{-M-1}[\mathbf{a}^\infty(x',D_n)]^{-1}(D_n+i)^{M+1}f.$$

Since $(D_n+i)^{M+1}f \in \mathbb{H}_p^{(\infty,s-\nu),\varpi}(\mathcal{M}^+)$ and due to Theorems 1.12, 1.13 and (1.27) the operators

$$\begin{aligned}[]
[\mathbf{a}^\infty(x',D_n)]^{-1} &: \mathbb{H}_p^{(\infty,s-\nu),\varpi}(\mathcal{M}^+) \to \widetilde{\mathbb{H}}_p^{(\infty,s),\varpi}(\mathcal{M}^+)\\
(D_n+i)^{-M-1} &: \widetilde{\mathbb{H}}_p^{(\infty,s),\varpi}(\mathcal{M}^+) \to \widetilde{\mathbb{H}}_p^{(\infty,s+M+1),\varpi}(\mathcal{M}^+)
\end{aligned}$$

are bounded, we get $\widetilde{u}_{M+1} \in \widetilde{\mathbb{H}}_p^{(\infty,s+M+1),\varpi}(\mathcal{M}^+)$.

Due to conditions (iii) of Theorem 1.21 from the Sobolev embedding theorem there follows

$$\text{(2.16)}\quad c_k^1 = \frac{(-i)^k}{k!}(D_n+i)^k(D_n-i)^{-\frac{\nu}{2}-\Delta}\mathbf{B}_-^{-1}(D_n)a^0 f\,,$$
$$c_k^1 \in \mathbb{H}_p^{(\infty,s+M+1-k-\frac{\nu}{2}-\Delta),\varpi}(\mathcal{M}^+) \subset C^{M-k}(\mathbb{R}^+, C^\infty(\partial\mathcal{M}))\,,\quad k=0,1,\dots,M\,,$$

because $s - \dfrac{\nu}{2} - \operatorname{Re}\Delta + 1 > \dfrac{1}{p}$. Under the space $\mathbb{H}_p^{(\infty,\mu),\varpi}(\mathcal{M}^+)$ with a vector $\mu = (\mu_1, \cdots, \mu_N)$ is meant the vector space.

We proceed as follows

$$u(x',x_{n,+}) := \sum_{k=0}^{M}\mathcal{K}(x')\mathbf{a}_+^{-1}(x',D_n)v_k(x_{n,+})c_k^1(x') + \widetilde{u}_{M+1}(x',x_{n,+})$$
$$= \sum_{k=0}^{M}\mathcal{K}(x')(D_n+i)^{-\frac{\nu}{2}-\Delta}\mathbf{B}^0_{a_{pr}}\left(\frac{1}{2\pi i}\log(D_n+i)\right)x_{n,+}^k e^{-x_{n,+}}c_k^1(x')$$

$$+\widetilde{u}_{M+1}(x',x_{n,+}) = \sum_{k=0}^{M}\mathcal{K}(x')\mathcal{F}^{-1}_{\xi_n\to x_n}\Bigg[(\xi_n+i)^{-\frac{\nu}{2}-\Delta}B^0_{a_{pr}}\left(\frac{1}{2\pi i}\log(\xi_n+i)\right)$$

$$\times\mathcal{F}_{x_n\to\xi_n}(x^k_{n,+}e^{-x_{n,+}})\Bigg]c^1_k(x')+\widetilde{u}_{M+1}(x',x_{n,+})$$

$$=\sum_{k=0}^{M}\mathcal{K}(x')\mathcal{F}^{-1}_{\xi_n\to x_n}\left[(\xi_n+i)^{-\frac{\nu}{2}-\Delta-k-1}B^0_{a_{pr}}\left(\frac{1}{2\pi i}\log(\xi_n+i)\right)\right]c^2_k(x')$$

$$+\widetilde{u}_{M+1}(x',x_{n,+})\,,\qquad c^2_k(x') := i^{k+1}k!\;c^1_k(x')\in C^\infty(\partial\mathcal{M}) \tag{2.17}$$

because

$$\mathcal{F}_{x_n\to\xi_n}(x^k_{n,+}e^{-x_{n,+}}) = i^{k+1}k!(\xi_n+i)^{-k-1}\,.$$

By differentiating the formula

$$\mathcal{F}^{-1}_{\lambda\to t}[(\lambda+i\tau)^\mu] = \frac{e^{\frac{\pi\mu}{2}i}}{\Gamma(-\mu)}t_+^{-\mu-1}e^{-\tau t_+}\,,\quad \tau>0 \tag{2.18}$$

with $\operatorname{Re}\mu<0$ (see [Es1, (2.36)]) in μ we find that

$$\begin{aligned}\mathcal{F}^{-1}_{\lambda\to t}\quad[(\lambda+i\tau)^\mu\log^m(\lambda+i\tau)] &= t_+^{-1}e^{-\tau t_+}\partial^m_\mu t_+^{-\mu}\frac{e^{\frac{\pi\mu}{2}i}}{\Gamma(-\mu)}\\ &= t_+^{-\mu-1}e^{-\tau t_+}(-\log t_+ + \partial_\mu)^m\frac{e^{\frac{\pi\mu}{2}i}}{\Gamma(-\mu)}.\end{aligned} \tag{2.19}$$

Applying this formula to (2.17) we get the following

$$\begin{aligned}u(x',x_{n,+}) &= \sum_{k=0}^{M}\mathcal{K}(x')x_{n,+}^{\frac{\nu}{2}+\Delta+k}e^{-x_{n,+}}\\
&\times\exp\left[\frac{1}{2\pi i}\left(-\log x_{n,+}+\partial_\mu\right)H_{a_{pr}}\right]\frac{e^{\frac{\pi\mu}{2}i}}{\Gamma(-\mu)}c^2_k(x')+\widetilde{u}_{M+1}(x',x_{n,+})\\
&=\sum_{k=0}^{M}\mathcal{K}(x')x_{n,+}^{\frac{\nu}{2}+\Delta+k}e^{-x_{n,+}}B^0_{a_{pr}}\left(-\frac{1}{2\pi i}\log x_{n,+}\right)\mathcal{K}^{-1}(x')c_k(x')\\
&+\widetilde{u}_{M+1}(x',x_{n,+}),\qquad \mu := -\frac{\nu}{2}-\Delta-k-1,\\
&c_k(x') = \mathcal{K}(x')(-2\pi i)^{-\mu}B^0_{a_{pr}}\left(\frac{1}{2\pi i}\partial_\mu\right)\frac{e^{\frac{\pi\mu}{2}i}}{\Gamma(-\mu)}c^1_k(x')\,,\end{aligned}$$

because $B^0_{a_{pr}}(t) = e^{tH_{a_{pr}}}$, $\quad t\in\mathbb{C}$ (cf. (1.80)). □

Remark 2.7. Inserting the expansion of $e^{-x_{n,+}}$ into (2.14) and rearranging the sums, the formula acquires the form

$$\begin{aligned}u(x',x_{n,+}) &= \sum_{k=0}^{M}\mathcal{K}(x')x_{n,+}^{\frac{\nu}{2}+\Delta(x')+k}B^0_{a_{pr}}\left(-\frac{1}{2\pi i}\log x_{n,+}\right)\mathcal{K}^{-1}(x')\widetilde{c}_k(x')\\
&+\widetilde{u}_{M+1}(x',x_{n.+})\,,\quad \widetilde{c}_k(x') := \sum_{\ell=0}^{k}\frac{(-1)^{k-\ell}}{(k-\ell)!}c_\ell(x')\,,\end{aligned} \tag{2.20}$$

where $\widetilde{u}_{M+1} \in \widetilde{\mathbb{H}}^{(\infty,s+M+1),\varpi}_{p,comp}(\mathbb{R}^n_+)$ and $c_\ell(x')$ is the same as in (2.14).

Lemma 2.8. *For a given constant* $\gamma \in \mathbb{C}$ *and given functions* $\{a_k(x', \operatorname{sgn} t)\}_0^m$, $a_k(\cdot, \pm 1) \in C^\infty(\partial\mathcal{M}))$, $k = 0, 1, \ldots, m$, $t \in \mathbb{R}$ *the following representation holds*

$$\begin{aligned}
\sum_{k=0}^{m} a_k(x, \operatorname{sgn} t)|t|^\gamma \log^k |t| &= \sum_{k=0}^{m+\sigma(\gamma)} b_k(x')(t - i0)^\gamma \log^k (t - i0) \\
&\quad + \sum_{k=\sigma(\gamma)}^{m+\sigma(\gamma)} c_k(x')(t + i0)^\gamma \log^k (t + i0)\,, \qquad (2.21)\\
x' \in \partial\mathcal{M}, \quad t \in \mathbb{R}, \quad &\sigma(\gamma) := \begin{cases} 0, & \text{if } \gamma \notin \mathbb{Z}, \\ 1, & \text{if } \gamma \in \mathbb{Z} \end{cases},
\end{aligned}$$

where b_k, $c_k \in C^\infty(\partial\mathcal{M})$, $k = 0, \ldots, m + \sigma(\gamma)$.

Representation (2.21) *is unique.*

Proof.(cf. similar assertions in [Es1, Remark 10.3] and [Be1, p.438]). Since

$$(t \pm i0)^\gamma = \begin{cases} |t|^\gamma, & \text{for } t > 0, \\ e^{\pm\pi\gamma i}|t|^\gamma, & \text{for } t < 0, \end{cases} \qquad (2.22)$$

assuming (2.21) we find that

$$\begin{aligned}
\sum_{k=0}^{m} a_k(x', +1)|t|^\gamma \log^k |t| &= \sum_{k=0}^{m+\sigma(\gamma)} b_k(x')|t|^\gamma \log^k |t| + \sum_{k=\sigma(\gamma)}^{m+\sigma(\gamma)} c_k(x')|t|^\gamma \log^k |t|\,, \\
\sum_{k=0}^{m} a_k(x', -1)|t|^\gamma \log^k |t| &= \sum_{k=0}^{m+\sigma(\gamma)} b_k(x')e^{-\pi\gamma i}|t|^\gamma (\log |t| - \pi i)^k + \\
&\quad + \sum_{k=\sigma(\gamma)}^{m+\sigma(\gamma)} c_k(x')e^{\pi\gamma i}|t|^\gamma (\log |t| - \pi i)^k
\end{aligned}$$

if $t > 0$ and if $t < 0$, respectively.

Equating coefficients of $\log^k |t|$ we get

$$\begin{cases}
a_k(x', -1) = \displaystyle\sum_{j=k}^{m+\sigma(\gamma)} \begin{pmatrix} j \\ k \end{pmatrix} (-\pi i)^{j-k} e^{-\pi\gamma i} b_j(x') + \\
\qquad + \displaystyle\sum_{j=k}^{m+\sigma(\gamma)} \begin{pmatrix} j \\ k \end{pmatrix} (\pi i)^{j-k} e^{\pi\gamma i} c_j(x'), \\
c_k(x') = a_k(x', +1) - b_k(x')\,, \quad k = 0, \ldots, m + \sigma(\gamma)\,, \\
a_{m+1}(x', \pm 1) = 0\,, \quad c_0(x') = 0 \quad \text{if} \quad \gamma \in \mathbb{Z}\,.
\end{cases}$$

The system can be rewritten as follows:

$$\begin{cases} \sum_{j=k}^{m+\sigma(\gamma)} \begin{pmatrix} j \\ k \end{pmatrix} (\pi i)^{j-k}[e^{-\pi\gamma i}(-1)^{j-k} - e^{\pi\gamma i}]b_j(x') = \widetilde{a}_k(x'), \\ c_k(x') = a_k(x', +1) - b_k(x')\,, \quad k = 0, \ldots, m+\sigma(\gamma)\,, \\ a_{m+1}(x', \pm 1) = 0\,, \quad c_0(x') = 0 \quad \text{if} \quad \gamma \in \mathbb{Z}\,, \end{cases}$$

where

$$\widetilde{a}_\ell(x') = a_\ell(x', -1) - \sum_{j=\ell}^{m} \begin{pmatrix} j \\ \ell \end{pmatrix} (\pi i)^{j-\ell} e^{\pi\gamma i} a_\ell(x', +1), \quad \ell = 1, \ldots, m,$$
$$\widetilde{a}_{m+1}(x') = 0$$

are known C^∞–functions.

For $\gamma \notin \mathbb{Z}$ the matrix of the system is $(m+1)\times(m+1)$ upper triangular matrix with the entries $-2i\sin\pi\gamma \neq 0$ on the principal diagonal $j = k$; therefore the system has a unique solution which is a vector–function with C^∞–smooth entries.

For $\gamma \in \mathbb{Z}$ the matrix of the system is $(m+2)\times(m+2)$ upper triangular matrix, but the principal diagonal $j = k$ vanishes; therefore the principal becomes the diagonal $k = j+1$ which has the entries $2\left[e^{-\pi\gamma i}(-1) - e^{\pi\gamma i}\right] = -4\cos\pi\gamma = 4(-1)^{\gamma+1}$. Since unknowns are exactly $m+1$ again, the system has a unique solution which is a vector–function with C^∞–smooth entries. □

Lemma 2.9. *Let $b \in S^\infty_\nu(\mathcal{T}^*\mathcal{M})$ have a compact support in the variable ξ_n:*

$$b(x, \xi', \xi_n) = 0 \quad \text{if} \quad |\xi_n| \geq M, \quad \text{for all} \quad x \in \mathcal{M},\ \xi' \in \mathbb{R}^{n-1}. \tag{2.23}$$

Then $\mathbf{b}(x, D)$ is a smoothing operator

$$\mathbf{b}(x, D) : \widetilde{\mathbb{H}}_p^{(\infty,s),\varpi}(\mathcal{M}) \to C^\infty(\mathcal{M}) \quad \text{for all} \quad s \in \mathbb{R}. \tag{2.24}$$

Proof. In a local coordinate system

$$\partial_x^\alpha \mathbf{b}(x, D)u = \sum_{0\leq\gamma\leq\alpha} c_\gamma \mathbf{b}^{(\gamma)}(x, D)\partial_x^{\alpha'-\gamma'}u,$$
$$b^{(\gamma)}(x, \xi) := (-i\xi_n)^{\alpha_n-\gamma_n}(\partial_x^\gamma b)(x, \xi)$$

and, obviously, $b^{(\gamma)} \in S^\infty_\nu(\mathcal{T}^*\mathcal{M})$. Therefore $\partial_x^\alpha \mathbf{b}(x, D)u \in \mathbb{H}^{(\infty,s-\nu)),\varpi}(\mathcal{M})$ for all $\alpha \in \mathbb{N}_0^n$ and this means $\mathbf{b}(x, D)u \in C^\infty(\mathcal{M})$. □

2.3. Proof of Theorem 2.1

Solvability conditions of equation (2.1) follow from Theorem 1.21 and we skip over to the proof of (2.2)–(2.4).

We will apply iteration, starting with the case $M = 0$. Some formulae, which will be used repeatedly, will be derived for general M.

Since the assertion is local we can suppose $\mathcal{M}$ is the half–space, but functions are compactly supported. Then equation (2.1) can be written in the following equivalent form

$$r_+\mathbf{a}^\infty(x', D_n)u = v_0^1\,, \tag{2.25}$$

where $v_0^1 := v - \widetilde{\mathbf{a}}_1(x, D)u - [\mathbf{a}_0(x, D) - \mathbf{a}^\infty(x, D)]u$ and we applied the expansion

$$a = \sum_{j=0}^{M} a_j + \widetilde{a}_{M+1}, \quad a_j \in \mathcal{S}^\gamma_{hom,\nu-j}(\mathcal{T}^*\mathcal{M})\,, \quad \widetilde{a}_{M+1} \in \mathcal{S}^\gamma_{\nu-M-1}(\mathcal{T}^*\mathcal{M}) \tag{2.26}$$

of the classical symbol for $M = 0$.

Homogeneous symbols and kernels of corresponding PsDOs with negative order have singularities at 0 and multiplying them by a function $\chi_0^n \in C^\infty\mathbb{R})$, where $\chi_0^n(\xi_n) = 0$ for $|\xi_n| < 1$ and $\chi_0^n(\xi_n) = 1$ for $|\xi_n| > 2$ we cut-off the singularity. Due to Lemma 2.9 the perturbation operator is smoothing $[I - \boldsymbol{\chi}_0^n(D_n)]\psi \in C^\infty(\mathcal{M})$ for arbitrary $\psi \in \mathbb{H}^{(\infty,\mu),\varpi}_p[\mathbb{R}^n)$ and we can ignore it. Although we will not write cut–off function, we suppose its presence and can forget about singularities of symbols at $\xi_n = 0$.

Applying the Taylor formula at $x_{n,+} = 0$ and at $\xi' = 0$, invoking Lemma 2.3, we find

$$\begin{aligned}
&a(x', x_{n,+}; \xi) - a^\infty(x', x_{n,+}; \xi) = [a_0(x', x_{n,+}; \xi', \xi_n) - a_0(x', 0; \xi', \xi_n)]\\
&\qquad+[a_0(x', 0; \xi', \xi_n) - a_0(x', 0; 0, \xi_n)] + [a_0(x', 0; 0, \xi_n) - a^\infty(x'; \xi_n)]\\
&(2.27)\;+\widetilde{a}_1^{\nu-1}(x; \xi) = \widetilde{a}_2^{\nu-1}(x'; \xi) + \widetilde{a}_3^{\nu-1}(x'; \xi_n)\,,\\
&\qquad\widetilde{a}_1^{\nu-1}(x; \xi) = x_{n,+}(\partial_{x_n} a_0)(x', \theta_0 x_{n,+}; \xi)\,, \quad \widetilde{a}_2^{\nu-1}(x'; \xi) = \sum_{j=1}^{n-1} \xi_j(\partial_{\xi_j} a_0)(x'; \theta_j\xi)\,,\\
&\qquad\widetilde{a}_3^{\nu-1}(x'; \xi_n) = a_0(x'; 0, \xi_n) - a^\infty(x'; \xi_n)\,,
\end{aligned}$$

where $0 < \theta_j < 1$. The inclusion $\widetilde{\mathbf{a}}_1^{(\nu-1)}(x; D)u \in \mathbb{H}^{(\infty,s-\nu+1),\varpi}_p(\mathcal{M})$ follows since $\partial_{x_n} a_0 \in \mathcal{S}^{\gamma-1}_\nu(\mathcal{T}^*\mathcal{M})$ and the factor $x_{n,+}$ increases the smoothness of the product by 1 (see the definition of the weighted spaces $\mathbb{H}^{(\infty,s-\nu+1),\varpi}_p(\mathcal{M})$ in § 1.3). The inclusion $\widetilde{\mathbf{a}}_2^{(\nu-1)}(x'; D)u \in \mathbb{H}^{(\infty,s-\nu+1),\varpi}_p(\mathcal{M})$ follows since $\partial_{\xi_j} a_0 \in \mathcal{S}^\gamma_{\nu-1}(\mathcal{T}^*\mathcal{M})$ and the operator $D_j = i\partial_{x_j}$, which corresponds to the symbol ξ_j, is bounded in the space $\mathbb{H}^{(\infty,s-\nu),\varpi}_p(\mathcal{M})$ (does not changes it) provided $j = 1, \dots, n-1$ (see § 1.3). The inclusion $\widetilde{\mathbf{a}}_3^{(\nu-1)}(x'; D_n)u \in \mathbb{H}^{(\infty,s-\nu+1),\varpi}_p(\mathcal{M})$ follows due to Lemma 2.3 (see (2.8)) and to Theorem 1.12.

Thus, $v_0^1 \in \mathbb{H}^{(\infty,s-\nu+1),\varpi}_p(\mathcal{M})$ (see (2.25)).

By invoking Lemma 2.6 and Remark 2.7 we derive expansion (2.2) for $M = 0$:

$$u = u_0 + \widetilde{u}_1, \qquad \widetilde{u}_1 \in \widetilde{\mathbb{H}}^{(\infty,s+1),\varpi}_p(\mathcal{M})\,, \tag{2.28}$$

$$u_0(x', x_{n,+}) = \mathcal{K}(x')x_{n,+}^{\frac{\nu}{2}+\Delta(x')} B^0_{a_{pr}}\Big(-\tfrac{1}{2\pi i}\log x_{n,+}\Big)\mathcal{K}^{-1}(x')c_0(x')\chi_0(x_n)\,, \quad c_0 \in C^\infty(\partial\mathcal{M}).$$

Now let $M \geq 1$ and suppose we have proved

$$(2.29)\quad \begin{aligned} u = \sum_{k=0}^{M-1} u_k + \widetilde{u}_M\,, \quad u_k(x', x_{n,+}) := \mathcal{K}(x') x_{n,+}^{\frac{\nu}{2}+\Delta(x')+k} B^0_{a_{pr}}\left(-\tfrac{1}{2\pi i}\log x_{n,+}\right) \\ \times\mathcal{K}^{-1}(x') \sum_{j=0}^{\sigma(k)} \log^j x_{n,+} c_{kj}(x') e^{-x_{n,+}}\,, \quad c_{kj} \in C^\infty(\partial\mathcal{M})\,, \quad c_{00}(x') = c_0(x')\,, \\ \widetilde{u}_M \in \widetilde{\mathbb{H}}_p^{(\infty, s+M), \varpi}(\mathcal{M})\,. \end{aligned}$$

Next we shall prove that $u_k \in \widetilde{\mathbb{H}}_p^{(\infty,s+k),\varpi}(\mathcal{M}^+)$ for $k = 0, 1, \ldots$. In fact, for this it obviously suffices to prove that

$$(2.30)\qquad v_k \in \widetilde{\mathbb{H}}_p^{s+k}(\mathbb{R}^+) \quad \text{where} \quad v_k(t) := e^{-t} t^{\frac{\nu}{2}+\Delta+k} \log^m t, \quad t \in \mathbb{R}^+$$

for $k = 0, 1 \ldots$. Since

$$(D_n + i)^{s+k} \;:\; \widetilde{H}_p^{s+k}(\mathbb{R}^+) \longrightarrow L_p(\mathbb{R}^+)$$

is an isomorphism, we have to prove the inclusion $\widetilde{v}_0 := (D_n + i)^{s+k} v_k \in L_p(\mathbb{R}^+)$.

Let us recall the formulae

$$(2.31)\quad \mathcal{F}_{t\to\lambda}[t_+^\mu \log^m t_+ e^{-\tau t_+}] = (\tau - i\lambda)^{-\mu-1}[-\log(\tau - i\lambda) + \partial_\mu]^m \Gamma(\mu+1)\,,$$

which is the inverse formula to (2.19) and follows from the formula

$$\mathcal{F}_{t\to\lambda}[t_+^\mu e^{-\tau t_+}] = \Gamma(\mu+1)(\tau - i\lambda)^{-\mu-1}\,, \quad \operatorname{Re}\mu > -1\,, \quad \tau > 0$$

(see [Es1, (2.36)] and cf. (2.18)) by differentiating in μ.

Invoking (2.31) and applying (2.19), we proceed as follows

$$\begin{aligned} v_0(t) &= (D_n + i)^{s+k} v_k(t) = \mathcal{F}_{\lambda\to t}\left\{(\lambda + i)^{s+k} \mathcal{F}_{y\to\lambda}[v_k(y)]\right\} \\ &= \mathcal{F}_{\lambda\to t}\left[(\lambda+i)^{s-\frac{\nu}{2}-\Delta(x')-1} \sum_{j=0}^m c_j \log^j(\lambda + i)\right] = t_+^{-s+\frac{\nu}{2}+\Delta(x')} \sum_{j=0}^m d_j \log^j(t_+ + i) \end{aligned}$$

and the inclusion $v_0 \in L_p(\mathbb{R}^+)$ follows since $-s + \dfrac{\nu}{2} + \operatorname{Re}\Delta(x') > -\dfrac{1}{p}$.

Locally equation (2.1) can be represented as follows

$$(2.32)\quad \begin{aligned} r_+ \mathbf{a}^\infty(x'; D_n) u = v^1_{M+1} - \sum_{k=0}^{M-1} \sum_{j=1}^{M-k} r_+ \mathbf{a}_j(x; D) u_k \\ - \sum_{k=0}^{M-1} r_+[\mathbf{a}_0(x; D) - \mathbf{a}^\infty(x'; D_n)] u_k\,, \end{aligned}$$

where

$$\begin{aligned} v^1_{M+1} = v - r_+ \widetilde{\mathbf{a}}_{M+1}(x; D) u - \sum_{k=0}^{M-1} \sum_{j=M-k+1}^{M-1} r_+ \mathbf{a}_j(x; D) u_k \\ - r_+[\mathbf{a}_0(x; D) - \mathbf{a}^\infty(x'; D_n)] \widetilde{u}_M\,. \end{aligned}$$

It is almost obvious that $v^1_{M+1} \in \mathbb{H}^{(\infty,s-\nu+M+1),\varpi}_p(\mathcal{M})$ because the operator in the square brackets has order $\nu-1$ (cf. (2.27)).

The TAYLOR formula, applied at $x_{n,+}=0$, and then at $|\xi_n|^{-1}\xi'=0$, gives:

$$a_j(x',x_{n,+};\xi',\xi_n) = \sum_{m=0}^{M-k-j} \frac{x^m_{n,+}}{m!}(\partial^m_{x_n} a_j)(x',0;|\xi_n|^{-1}\xi',\operatorname{sgn}\xi_n)|\xi_n|^{\nu-j}$$
$$+x^{M-k-j+1}_{n,+} a^{(1)}_{jk}(x',x_{n,+};\xi) = \sum_{m=0}^{M-k-j} \frac{x^m_{n,+}}{m!} \sum_{\ell=0}^{M-k-j-m} |\xi_n|^{\nu-j-\ell}$$
$$\times \sum_{|\gamma'|=\ell} \frac{(\xi')^{\gamma'}}{(\gamma')!}(\partial^m_{x_n}\partial^{\gamma'}_{\xi'} a_j)(x',0;0,\operatorname{sgn}\xi_n) + a^{(2)}_{jk}(x',x_{n,+};\xi)\,, \tag{2.33}$$

where

$$a^{(2)}_{jk}(x;\xi) = x^{M-k-j-1}_{n,+} a^{(1)}_{jk}(x',x_{n,+};\xi) + \sum_{m=0}^{M-k-j} x^m_{n,+} a^{(2)}_{jkm}(x';\xi),$$
$$a^{(1)}_{jk} \in \mathcal{S}^{\gamma-k-j}_{\nu-j}(\mathcal{T}^*\mathcal{M})\,, \quad a^{(2)}_{jkm} \in \mathcal{S}^{\gamma-M}_{\nu-M+m+k-1}(\mathcal{T}^*\mathcal{M}) \tag{2.34}$$

and, obviously, $\mathbf{a}^{(2)}_{jk}(x;D)u_k \in \mathbb{H}^{(\infty,s-\nu+M+1),\varpi}_p(\mathcal{M}^+)$.

Similarly (cf. (2.6)),

$$a^\infty(x';\xi_n) = |\xi_n|^\nu \langle \xi_n^{-1}\rangle^\nu a_{pr}(x';+1)\left(\frac{1+i\xi_n^{-1}}{1-i\xi_n^{-1}}\right)^{b(x')} = \sum_{\ell=0}^{M-k} |\xi_n|^{\nu-\ell} a_\ell(x';\xi_n)$$
$$+\widetilde{a}^\infty_{M-k+1}(x';\xi_n), \qquad \widetilde{\mathbf{a}}^\infty_{M-k+1}(x';D_n)u_k \in \mathbb{H}^{(\infty,s-\nu+M+1),\varpi}_p(\mathcal{M}^+)\,,$$
$$a_\ell(x';\xi_n) := \frac{a_{pr}(x';+1)}{\ell!}\left[\partial^\ell_t \langle t\rangle^\nu \left(\frac{1+it}{1-it}\right)^{b(x')}\right]_{t=0} (\operatorname{sgn}\xi_n)^\ell\,. \tag{2.35}$$

As we see the coefficients of expansions (2.33) with $j=0$ and of (2.35) coincide (cf. (2.7)). Therefore (2.32)–(2.35) yield

$$r_+\mathbf{a}^\infty(x';D_n)u = \sum_{k=0}^{M-1}\sum_{j=0}^{M-k}\sum_{m=0}^{M-k-j}\sum_{\substack{\ell=0\\ j+m+\ell>0}}^{M-k-j-m} x^m_{n,+}\mathbf{a}^0_{jm\ell}(x';D)u_k + v^2_{M+1}\,, \tag{2.36}$$

$$a^0_{jm\ell}(x';\xi) := \sum_{|\gamma'|=\ell}\left[a_{jm\gamma'}(x';\operatorname{sgn}\xi_n) - \delta_{j+m,0}\delta_{\gamma_1,\ell}a_\ell(x';\operatorname{sgn}\xi_n)\right](\xi')^{\gamma'}|\xi_n|^{\nu-j-\ell},$$

$$\gamma' = (\gamma_1,\ldots,\gamma_{n-1})\,, \quad v^2_{M+1} \in \mathbb{H}^{(\infty,s-\nu+M+1),\varpi}_p(\mathcal{M}^+)\,, \quad a_{jm\gamma'}(\cdot;\pm1) \in C^\infty(\partial\mathcal{M})\,,$$

where $\delta_{j\ell}$ is KRONEKER's delta. By inverting operator $\mathbf{a}^\infty(x';D_n)$ (see Lemma 2.6) we find

$$u = \sum_{k=0}^{M-1}\sum_{\ell=0}^{M-k}\sum_{m=0}^{M-k-j}\sum_{\substack{j=0\\ \ell+m+j>0}}^{M-k-\ell} \mathcal{K}(x')[\mathbf{a}_+(x';D_n)]^{-1}\theta_+\mathbf{a}_-(x';D_n)a^0(x')x^m_{n,+}$$
$$\times\mathbf{a}^0_{jm\ell}(x',D)u_k + r_+[(\mathbf{a}^\infty(x';D_n)]^{-1}v^2_{M+1}\,, \tag{2.37}$$

where $v^3_{M+1} \in \widetilde{\mathbb{H}}_p^{(\infty,s+M+1),\varpi}(\mathcal{M}^+)$.

Applying (2.31) we find the following (see (2.29) for u_k):

$$\begin{aligned}
&\mathcal{F}_{y_n\to\xi_n}[u_k(x',y_{n,+})]\\
&\quad= \mathcal{K}(x')\mathcal{F}_{y_n\to\xi_n}\left[y_{n,+}^{\frac{\nu}{2}+\Delta(x')+k}\exp\left(-\frac{1}{2\pi i}H_{a_{pr}}\log\, y_{n,+}\right)\right.\\
&\quad\left.\sum_{q=0}^{\sigma(k)}\log^q y_{n,+}\mathcal{K}^{-1}(x')c_{kq}(x')\right]\\
&\quad= \mathcal{K}(x')(0-i\xi_n)^{-\frac{\nu}{2}-\Delta(x')-k-1}\exp\left(\left[-\frac{1}{2\pi i}\log(0-i\xi_n)+\frac{1}{2\pi i}\partial_\mu\right]H_{a_{pr}}\right)\\
&\quad\times\sum_{q=0}^{\sigma(k)}\left[\frac{1}{2\pi i}\log(\xi_n+i0)+\frac{1}{4}+\frac{1}{2\pi i}\partial_\mu\right]^q\Gamma\left(\frac{\nu}{2}+\Delta(x')+k+1\right)c_{kq}(x')\\
&\quad= \mathcal{K}(x')(\xi_n+i0)^{-\frac{\nu}{2}-\Delta(x')-k-1}B^0_{a_{pr}}\left(-\frac{1}{2\pi i}\log(\xi_n+i0)\right)\\
&\quad\times\sum_{q=0}^{\sigma(k)}\log^q(\xi_n+i0)c^1_{kq}(x')\,, \qquad (2.38)
\end{aligned}$$

since $B^0_{a_{pr}}(t)=e^{tH_{a_{pr}}}$, $t\in\mathbb{C}$ (cf. (1.80)).

Invoking (2.38) and expansion of $a^0_{jm\ell}$ from (2.36), Inserting $(\xi_n+i0)^\sigma=\theta_+(\xi_n)|\xi_n|^\sigma+e^{\pi\sigma i}\theta_-(\xi_n)|\xi_n|^\sigma$ (see (2.22)), we obtain

$$\begin{aligned}
&u_{jm\ell k}(x',x_{n,+}) := a^0(x')x^m_{n,+}\mathbf{a}^0_{jml}(x';D)u_k(x',x_{n,+})\\
&= \sum_{|\gamma'|=\ell}a^0(x')x^m_{n,+}\mathcal{F}^{-1}_{\xi_n\to x_{n,+}}\left\{a^1_{jm\gamma'}(x';\operatorname{sgn}\xi_n)|\xi_n|^{\nu-j-\ell}(i\partial_{x'})^{\gamma'}\mathcal{F}_{y_n\to\xi_n}[u_k(x',y_{n,+})]\right\}\\
&(2.39)\qquad = \sum_{q=0}^{\sigma(k)}\sum_{|\gamma'|=\ell}a^0(x')\mathcal{F}^{-1}_{\xi_n\to x_{n,+}}\left\{(i\partial_{\xi_n})^m a^2_{jm\gamma'}(x';\operatorname{sgn}\xi_{n,+})\right.\\
&\left.\times(i\partial_{x'})^{\gamma'}|\xi_n|^{\frac{\nu}{2}-\Delta(x')-k-j-\ell-1}B^0_{a_{pr}}\left(\frac{1}{2\pi i}\log|\xi_n|\right)\log^q|\xi_n|c_{kq}(x')\right\}\\
&= \sum_{q=0}^{\sigma(k)+\ell+m_0-1}\mathcal{F}^{-1}_{\xi_n\to x_{n,+}}\left\{a^3_{jm}(x';\operatorname{sgn}\xi_n)|\xi_n|^{\frac{\nu}{2}-\Delta(x')-k-j-\ell-m-1}\log^q|\xi_n|\right\}c^0_{kq}(x')\\
&= \sum_{q=0}^{\sigma(k)+\ell+m_0-1}\mathcal{F}^{-1}_{\xi_n\to x_{n,+}}\left\{|\xi_n|^{\frac{\nu}{2}-\Delta(x')-k-j-\ell-m-1}\log^q|\xi_n|a^4_{jmq}(x';\operatorname{sgn}\xi_n)\right\},
\end{aligned}$$

where $a^0_{jm\ell}(x';\theta)$ is defined in (2.36) and $a^1_{jm\gamma'}(x';\theta)$, $a^2_{jm}(x';\theta)$, $a^3_{jm}(x';\theta)$, $a^4_{jm}(x';\theta)$ are similar (we remind that symbols are cut off at $\xi_n=0$). The powers of logarithmic terms increased by m_0-1 due to the factor $B^0_{a_{pr}}\left(\frac{1}{2\pi i}\log|\xi_n|\right)$ and by $|\gamma'|=\ell\le M-k$ due to the differentiation $\partial^{\gamma'}_{x'}|\xi_n|^{\Delta(x')}$. We proceed as follows

$$u^1_{jm\ell k}(x',x_{n,+}) := r_+\mathbf{a}_-(x';D)u_{jm\ell k}(x,x_{n,+})$$

$$= r_+\mathcal{F}^{-1}_{\xi_n\to x_{n,+}}\left\{B_-(\xi_n)(\xi_n-i)^{-\frac{\nu}{2}+\Delta(x')}\mathcal{F}_{y_{n,+}\to\xi_n}[u_{jm\ell k}(x',y_{n,+})]\right\}$$

$$= \sum_{q=0}^{\sigma(k)+\ell+m_0-1} r_+\mathcal{F}^{-1}_{\xi_n\to x_{n,+}}\left\{B^0_{a_{pr}}\left(\frac{1}{2\pi i}\log(\xi_n-i)\right)(\xi_n-i)^{-\frac{\nu}{2}+\Delta(x')}\right.$$
$$\left.\times|\xi_n|^{\frac{\nu}{2}-\Delta(x')-j-k-\ell-m-1}\log^q|\xi_n|a^4_{jm}(x',\operatorname{sgn}\xi_n)\right\}. \tag{2.40}$$

As in (2.38), (2.39) we replace functions $(\xi_n-i)^\sigma$ and $\log^q(\xi_n-i)$ by the Taylor sums of $|\xi_n|^{\sigma-k}$ and $\log^{q-j}|\xi_n|$, respectively and apply Lemma 2.8, that might increase the powers of logarithmic terms by 1. The factor $B^0_{a_{pr}}\left(\frac{1}{2\pi i}\log(\xi_n-i)\right)$ increases powers of logarithmic terms by m_0-1.

Ignoring summands with the argument ξ_n-i0 (they are deleted after the Fourier transform by r_+) and applying (2.19) with $\tau\to 0$ we get

$$ru^1_{jm\ell k}(x',x_{n,+}) = \sum_{q=0}^{\sigma(k)+2m_0+M-k-1} r_+\mathcal{F}^{-1}_{\xi_n\to x_{n,+}}\left\{(\xi_n+i0)^{-j-k-\ell-m-1}\log^q(\xi_n+i0)\right\}c^1_{kjmq\ell}(x')$$
$$+v^3_{jm\ell k,M+1} = \sum_{q=0}^{\sigma(k+1)} x^{j+k+\ell+m}_{n,+}\log^q x_{n,+}\, c^2_{kjmq\ell}(x') + v^3_{jm\ell k,M+1}\,, \tag{2.41}$$

where $c^1_{kjmq}, c^2_{kjmq}\in C^\infty(\partial\mathcal{M})$ and, due to the Taylor expansion, the remainder $v^3_{jm\ell k,M+1}\in\widetilde{\mathbb{H}}^{(\infty,s-\frac{\nu}{2}+M+1),\varpi}_p(\mathcal{M}^+)$.

From (2.37)–(2.41) we get

$$u = \sum_{k=0}^{M-1}\sum_{j=1}^{M-k}\sum_{m=0}^{M-k-j}\sum_{\substack{\ell=0\\ j+m+\ell>0}}^{M-k-j-m}\mathcal{K}(x')[\mathbf{a}_+(x';D_n)]^{-1}u^1_{jm\ell k}$$
$$+r_+[a^\infty(x';D_n)]^{-1}v^2_{M+1}+v^4_{M+1}\,. \tag{2.42}$$

Similarly to (2.37)–(2.41) we find:

$$\mathcal{K}(x')[\mathbf{a}_+(x',D_n)]^{-1}u^1_{jm\ell k} = \mathcal{K}(x')r_+\mathcal{F}^{-1}_{\xi_n\to x_{n,+}}\left\{(\xi_n+i0)^{\frac{\nu}{2}+\Delta(x')+j+k+\ell+m}\right.$$
$$\left.\times B^0_{a_{pr}}\left(\frac{1}{2\pi i}\log(\xi_n+i0)\right)\sum_{q=0}^{\sigma(k+1)}c^3_{kjmq\ell}(x')\log^q(\xi_n+i0)\right\}$$
$$+v^5_{jm\ell k,M+1}\,,\qquad c^3_{kjmq\ell}\in C^\infty(\partial\mathcal{M})\,,$$

where $v^5_{jm\ell k,M+1}\in\widetilde{\mathbb{H}}^{(\infty,s+M+1),\varpi}_p(\mathcal{M}^+)$. Degrees of logarithmic terms in the last formulae does not increase because all symbols have analytic extensions already, depending either on the argument ξ_n+i0 or on ξ_n+i, and we does not need to apply Lemma 2.8; to factors with the arguments ξ_n+i we should apply the Taylor expansion which leaves behind sufficiently smooth remainder $v^5_{jm\ell k,M+1}$.

From (2.44), (2.45), by applying (2.19) with $\tau \to 0$ we derive

$$\mathcal{K}(x')[\mathbf{a}_+(x', D_n)]^{-1} u^1_{jm\ell k} = \mathcal{K}(x') r_+ x_{n,+}^{\frac{\nu}{2}+\Delta(x')+j+k+\ell+m} B^0_{a_{pr}}\left(-\frac{1}{2\pi i}\log x_{n,+}\right)$$

$$\times \sum_{q=0}^{\sigma(k+1)} c^3_{kjmq\ell}(x') \log^q x_{n,+} + v^5_{jm\ell k, M+1}\,, \quad c^3_{kjmq\ell} \in C^\infty(\partial\mathcal{M})\,, \tag{2.43}$$

which, together with (2.42), gives all summands of (2.2) but the leading term, if we replace summations with respect to j, ℓ, m, k by one sum with respect to k and, respectively, replace $j+k+\ell+m$ by k. We have to estimate exponents of logarithmic terms as well.

For exponents of logarithmic terms we get the following estimate:

$$\begin{gathered}\sigma(k+1) \le \sigma(k) + 2m_0 + M - k - 1 \\ (\sigma(k+1) \le \sigma(k) + 2m_0 - 1 \quad \text{provided} \quad \delta(x') \equiv \text{const})\,.\end{gathered} \tag{2.44}$$

Let us prove, based on (2.44), that

$$\sigma(k) \le (2m_0 - 1)k + m_0 - 1 \quad k = 0, 1, \dots\,, \tag{2.45}$$

which implies $\sigma(k) = k$ for $m_0 = 1$ (see (2.4)). To prove (2.45) we set $M = k$ (this gives possibility to find k–th summand of the asymptotic knowing previous summands). Then from (2.44) there follows

$$\sigma(k) \le \sigma(k-1) + 2m_0 - 1 = (2m_0 - 1)k + m_0 - 1\,, \quad k = 1, 2, \dots$$

since, as we already know, $\sigma(0) = m_0 - 1$ (cf. (2.28) and (2.29); let us note that $\sigma(0) = 0$ does not mean that the first term of asymptotic expansion does not contain logarithms–all terms have the factor $B^0_{a_{pr}}\left(-\frac{1}{2\pi i}\log x_{n,+}\right)$).

Thus, due to (2.43) and (2.44) all summands in (2.42), which contain $u^1_{jm\ell k}$, have appropriate asymptotic (cf. (2.2)). These entries does not generate the leading term of asymptotic in (2.2) because $j + m + \ell \ge 1$. The leading term is generated by application of Lemma 2.6 to the summand $r_+[(a^\infty(x'; D_n)]^{-1} v^2_{M+1}$.

Since order of PsDO ord $\mathbf{a}_+(x', D) < -\frac{\nu}{2} + \varepsilon$, where ε is due to the logarithmic terms, and the symbol is analytic, due to Lemma 1.2 the summands $\mathcal{K}(x')[\mathbf{a}_+(x', D_n)]^{-1} u^1_{jm\ell k}$, and the remainder terms $v^3_{jm\ell k, M+1}$, v^4_{M+1}, $v^5_{jm\ell k, M+1}$ all belong to $\widetilde{\mathbb{H}}_p^{(\infty, s+M+1), \varpi}(\mathcal{M}^+)$; therefore they can be included in the remainder $\widetilde{u}_{M+1}$ of asymptotic (2.2).

Concerning the a priori estimates (2.3): the last two inequalities

$$\|u|\widetilde{\mathbb{H}}_p^{(\mu,s),\varpi}(\mathcal{M})\| \le C_1 \|v|\mathbb{H}_p^{(\mu,s-\nu),\varpi}(\mathcal{M})\| \le C_2 \|u|\widetilde{\mathbb{H}}_p^{(\mu,s),\varpi}(\mathcal{M})\|$$

asserting the equivalence of the right–hand side and of the solution, follow due to the boundedness and invertibility of the operator

$$r_{\mathcal{M}} \mathbf{a}(x, D) : \widetilde{\mathbb{H}}_p^{(\infty,s),\varpi}(\mathcal{M}) \longrightarrow \mathbb{H}_p^{(\infty,s-\nu),\varpi}(\mathcal{M})\,.$$

As for the remaining inequalities, they follow since the norms of $c_0(x')$, $c_{kj}(x')$ and of $\widetilde{u}_{M+1}(x', x_{n,+})$ are estimated by norms of the right–hand side $v(x)$ and of the solution $u(x', x_{n,+})$.

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