Localization and minimal normalization of some basic mixed boundary value problems *

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Abstract. We consider a class of mixed boundary value problems in spaces of Bessel potentials. By localization, an operator L associated with the BVP is related through operator matrix identities to a family of pseudodifferential operators which leads to a Fredholm criterion for L. But particular attention is devoted to the non-Fredholm case where the image of L is not closed. Minimal normalization, which means a certain minimal change of the spaces under consideration such that either the continuous extension of L or the image restriction, respectively, is normally solvable, leads to modified spaces of Bessel potentials. These can be characterized in a physically relevant sense and seen to be closely related to operators with transmission property (domain normalization) or to problems with compatibility conditions for the data (image normalization), respectively.

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1. Introduction to mixed boundary value problems and normalization

We confine our attention to the following model boundary value problem (BVP) based on considerations in [?, p. 186 ff.] and [?]. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\Gamma = \partial\Omega$ divided into two simply connected parts and their common boundary points, i.e. (see Figure 1.1)

$$\Gamma = \Gamma^1 \cup \Gamma^2 \cup \{x^1, x^2\}. \tag{1.1}$$

Let A be a linear differential operator with smooth coefficients in Ω of order $2m$ where $m \in \mathbb{N}_1$ and B^1, B^2 are vectors of linear boundary operators both with smooth coefficients on Γ (extendible to $\overline{\Omega}$) of order $m^1 = (m^1_1, \ldots, m^1_m)$

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and $m^2 = (m^2_1, \ldots, m^2_m)$, respectively, such that $0 \le m^1_j, m^2_j \le 2m-1$. More precisely we have B^k with components

$$b^k_j = b^k_j(x,D) = \sum_{|s|\le m^k_j} b^k_{j,s}(x) T^k_0(D^s\varphi) = T^k_0\left(\sum_{|s|\le m^k_j} b^k_{j,s}(x) D^s\varphi\right), \quad k=1,2 \tag{1.2}$$

where T^k_0 denotes the (usual) trace operator on Γ^k.

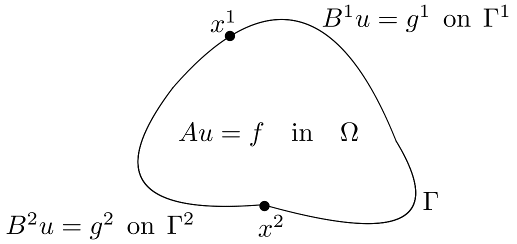

Figure 1.1: Mixed boundary value problem.

We look for all solutions $u \in H^{2m+l}(\Omega)$, $l \ge 0$, such that

$$\begin{aligned} Au(x) &= A(x,D)u(x) = f(x), \qquad x\in\Omega \\ B^k u(x) &= \left(b^k_1(x,D)u(x),\ldots,b^k_m(x,D)u(x)\right) \\ &= \left(g^k_1(x),\ldots,g^k_m(x)\right), \qquad x\in\Gamma^k, \quad k=1,2, \end{aligned} \tag{1.3}$$

where $f \in H^l(\Omega)$, $g^k_j \in H^{2m+l-m^k_j-1/2}(\Gamma^k)$ are (arbitrarily) given, $j=1,\ldots,m$, and refer, for short, to the *mixed BVP* (**??**). It is called *piecewise elliptic*, if B^k $(k=1,2)$ have extensions $\widetilde{B^k}$ to the whole Γ such that (**??**) with B^k replaced by $\widetilde{B^k}$ and with Γ^k, $k=1,2$, replaced by Γ are both elliptic ($k=1$ or 2), i.e., cf. [**?**, p. 187],
I: the principle part of the Fourier symbol of A does not degenerate:

$$A^{pr}(x,\xi) \ne 0, \qquad 0 \ne \xi \in \mathbb{R}^2, \quad x\in\overline{\Omega}; \tag{1.4}$$

II: the Shapiro-Lopatinsky condition is satisfied for A^{pr}, $\widetilde{B^1}^{\,pr}$ and for A^{pr}, $\widetilde{B^2}^{\,pr}$ as well [**?**, §11]. This means that the following two initial value problems have only the trivial solution in $H^{2m}(\mathbb{R}\times\mathbb{R}_+)$

$$\begin{aligned} A^{pr}\left(x_0,\xi,\tfrac{1}{i}\tfrac{d}{dt}\right)v(t) &= 0 \\ \widetilde{B^k}^{\,pr}\left(x_0,\xi,\tfrac{1}{i}\tfrac{d}{dt}\right)v(t) &= 0 \end{aligned} \tag{1.5}$$

for $k=1,2$, $x_0\in\Gamma$, $\xi\in\mathbb{R}\backslash\{0\}$, where (in brief) the operators result from (**??**) by fixing the coefficients, taking the principal parts, applying locally a linear transformation such that x_0 becomes the origin, the x_1-axis is tangential to Γ and Ω lies locally left at zero, and replacing the differential operators $D^{(1,0)}$ by ξ and $D^{(0,1)}$ by $\frac{1}{i}\frac{d}{dt}$, respectively, see [?, §11] for details and for other equivalent formulations.

REMARK 1.1. Smooth transformations do not destroy the Shapiro-Lopatinsky condition. This fact will be important for the localization principle presented in Section **??**.

Later, in Section **??**, we shall discuss modifications and generalizations of the model problem.

EXAMPLES 1.2. 1. The very basic mixed Dirichlet/Neumann problem for the Laplace equation

$$\begin{array}{rcllll} \Delta u &=& 0 & \text{in} & \Omega \\ u &=& g^1 & \text{on} & \Gamma^1 \\ \dfrac{\partial u}{\partial n} &=& g^2 & \text{on} & \Gamma^2 \end{array} \tag{1.6}$$

where $u \in H^1(\Omega)$ is unknown and $g^1 \in H^{1/2}(\Gamma^1)$, $g^2 \in H^{-1/2}(\Gamma^2)$ are given, is well-posed [?]. Normalization is not needed in this space setting.

2. BVPs (as well as transmission problems) for the Helmholtz equation in half-spaces in various cases of basic boundary conditions [?, ?], yield *compatibility conditions* as to be necessary for the solvability or for the Fredholm property in the initial spaces (H^1 formulation) [?, ?, ?, ?]. For instance the mixed impedance BVP for the upper half-plane

$$\begin{array}{rcllll} \left(\Delta + k_0^2\right) u &=& 0 & \text{in} & \Omega = \left\{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\right\} \\ \dfrac{\partial u}{\partial x_2} + iz^1 u &=& g^1 & \text{on} & \Gamma^1 = \mathbb{R}_+ \times \{0\} \\ \dfrac{\partial u}{\partial x_2} + iz^2 u &=& g^2 & \text{on} & \Gamma^2 = \mathbb{R}_- \times \{0\} \end{array} \tag{1.7}$$

where $k_0, z^1, z^2 \in \mathbb{C}$, $\Im m\, k_0 > 0$, $u \in H^1(\Omega)$, $g^k \in H^{-1/2}(\Gamma^k)$ implies that

$$g^1 - Rg^2 \;\in\; r_{\Gamma_1} H^{-1/2}_{\overline{\Gamma_1}}(\Gamma) \;\subsetneq\; H^{-1/2}(\Gamma_1) \tag{1.8}$$

where R is the reflection operator with respect to x_1 and $H^{-1/2}_{\overline{\Gamma_1}}(\Gamma)$ denotes the $H^{-1/2}$ distributions defined on $\Gamma = \partial\Omega = \mathbb{R} \times \{0\}$ and supported on $\overline{\Gamma_1}$, which represents, as restricted on Γ_1, a proper dense subspace of $H^{-1/2}(\Gamma_1) = r_{\Gamma_1} H^{-1/2}(\Gamma)$. In other words: the distributions of $r_{\Gamma_1} H^{-1/2}_{\overline{\Gamma_1}}(\Gamma)$ are extendible by zero to Γ within $H^{-1/2}(\Gamma)$ but the elements of $H^{-1/2}(\Gamma_1)$ are not, in general.

Under this additional compatibility condition, the problem is already well-posed, which can be understood in the following sense: the operator defined by

$$\begin{array}{rclcl} L_0 &:& X_0 = \left\{u \in H^1(\Omega) : (\Delta + k_0^2)u = 0\right\} &\rightarrow& H^{-1/2}(\Gamma^1) \times H^{-1/2}(\Gamma^2) \\ && u &\mapsto& (g^1, g^2) \end{array} \tag{1.9}$$

is not normally solvable. However, restricting the image of L_0 by (**??**) and changing the norm correspondingly, e.g. by using the zero extension operator $\ell^0_{\Gamma_1\to\Gamma} : r_{\Gamma_1} H^{-1/2}_{\Gamma_1}(\Gamma) \to H^{-1/2}(\Gamma)$

$$\|(g_1, g_2)\| = \|g_1\|_{H^{-1/2}(\Gamma_1)} + \left\|\ell^0_{\Gamma_1\to\Gamma}(g^1 - Rg^2)\right\|_{H^{-1/2}(\Gamma)} \tag{1.10}$$

the (image) restricted operator becomes normally solvable and, moreover, bounded invertible (up to some parameters of z^k in a set of $\mathbb{C}$–measure zero [**?**]). We call this the *image normalization* of L_0, cf. [**?**] and Section **??** for the general concept.

Instead of restricting the image of L_0, we can extend the domain. Let us explain this in the special case of $z^1 = z^2 = z$ (which numbers do not effect the compatibility condition (**??**)). The solutions of the Helmholtz equation in $H^1(\Omega)$ can be written as [**?**, **?**]

$$u(x) \;=\; F^{-1}_{\xi\mapsto x_1}\left\{\exp[-x_2 t(\xi)] \cdot \widehat{u_0}(\xi)\right\}, \qquad x_1 \in \mathbb{R}, \quad x_2 > 0 \tag{1.11}$$

where (for the trace u_0 of u) $Fu_0(\xi) = \widehat{u_0}(\xi)$ denotes the one-dimensional Fourier transformation and $t(\xi) = \left(\xi^2 - k_0^2\right)^{1/2}$ with vertical branch cut over infinity. Thinking of smooth data in a dense subspace $\mathcal{S}(\mathbb{R}_+)$ instead of $H^{-1/2}(\Gamma^k)$ we can write the boundary conditions as

$$F^{-1}(iz - t) \cdot Fu_0 \;=\; g = \begin{cases} g^1 & \text{on } \Gamma^1 \\ g^2 & \text{on } \Gamma^2 \end{cases} \tag{1.12}$$

i.e. we understand the boundary value due to the boundary conditions of (**??**) in the sense of

$$H^{-1/2}(\mathbb{R}) \subset H^{-1/2}(\Gamma^1) \oplus H^{-1/2}(\Gamma^2) \tag{1.13}$$

which embedding is dense, proper and continuous. Evidently X_0 is [**?**] isomorphic to $H^{-1/2}(\mathbb{R})$ by the formulas (**??**)–(**??**) (provided $iz - t$ has no zeros on $\mathbb{R}$). If we equip X_0 with the norm induced by the direct sum and take its completion $\overline{X_0}$, we evidently get a well-posed problem associated with the *domain normalization* of L_0, cf. [**?**].

REMARK 1.3. The above examples are treated in the sense of weak solutions ($m = 1$, $l = -1$, $2m + l = 1$) in contrast to the assumptions before ($l \geq 0$). We plan to come back to these questions later.

2. Associated operators

The following operators shall be considered. They are fundamental for the Fredholm property and normalization, respectively, as well.

$$L = \begin{pmatrix} A \\ B^1 \\ B^2 \end{pmatrix} : H^{2m+l}(\Omega) \to \begin{cases} H^l(\Omega) \\ \times \prod_{j=1}^m H^{2m+l-m_j^1-1/2}(\Gamma^1) \\ \times \prod_{j=1}^m H^{2m+l-m_j^2-1/2}(\Gamma^2) \end{cases} \tag{2.1}$$

$$L^{pr}_{x_0} = \begin{pmatrix} A^{pr}_{x_0} \\ B^{1\,pr}_{x_0} \\ B^{2\,pr}_{x_0} \end{pmatrix} \; : \; H^{2m+l}(\Omega) \to \begin{cases} H^l(\Omega) \\ \times \prod_{j=1}^m H^{2m+l-m_j^1-1/2}(\Gamma^1) \\ \times \prod_{j=1}^m H^{2m+l-m_j^2-1/2}(\Gamma^2) \end{cases}$$

where $A^{pr}_{x_0}$ denotes the principal part of A with frozen coefficients in $x_0 \in \overline{\Omega}$ and $B^{k\,pr}_{x_0}$ is defined in the same way if $x_0 \in \Gamma^k \cup \{x^1, x^2\}$; otherwise we put $B^{k\,pr}_{x_0} = 0$, i.e. for all $x_0 \in \Omega \cup \Gamma^{3-k}$. This means that $L^{pr}_{x_0}$ is defined for each $x_0 \in \overline{\Omega}$ as a mapping in the same space setting, but has a different form, namely

$$L^{pr}_{x_0} = \begin{pmatrix} A^{pr}_{x_0} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} A^{pr}_{x_0} \\ B^{1\,pr}_{x_0} \\ 0 \end{pmatrix}, \begin{pmatrix} A^{pr}_{x_0} \\ 0 \\ B^{2\,pr}_{x_0} \end{pmatrix}, \begin{pmatrix} A^{pr}_{x_0} \\ B^{1\,pr}_{x_0} \\ B^{2\,pr}_{x_0} \end{pmatrix} \tag{2.2}$$

for $x_0 \in \Omega$, $x_0 \in \Gamma^1$, $x_0 \in \Gamma^2$ or $x_0 \in \{x^1, x^2\}$, respectively.

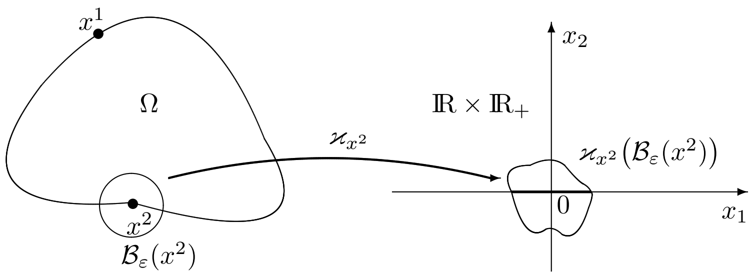

Figure 2.1: Local transformation.

Further, denoting by $\mathcal{B}_\varepsilon(x_0)$ the ε-ball centered at $x_0 \in \overline{\Omega}$, we consider smooth transformations $\varkappa_{x_0} : \mathcal{B}_\varepsilon(x_0) \to \mathbb{R}^2$ of neighborhoods of x_0 into $\mathbb{R}^2$ such that (see Figure 2.1 for the case $x_0 = x^2$)

$$\begin{aligned} &\varkappa_{x_0}(x_0) = 0, && \text{for} \quad x_0 \in \overline{\Omega} \\ &\varkappa_{x_0}\left(\Omega \cap \mathcal{B}_\varepsilon(x_0)\right) \subset \left\{x \in \mathbb{R}^2 : x_2 > 0\right\} && \\ &\varkappa_{x_0}\left(\Gamma \cap \mathcal{B}_\varepsilon(x_0)\right) \subset \left\{x \in \mathbb{R}^2 : x_2 = 0\right\} && \text{for} \quad x_0 \in \Gamma \end{aligned} \tag{2.3}$$

and the transformed operator

$$L^{\langle pr \rangle}_{x_0,0} = \begin{pmatrix} A^{\langle pr \rangle}_{x_0,0} \\ B^{k\,pr}_{x_0,0} \\ B^{(3-k)\,pr}_{x_0,0} \end{pmatrix} : H^{2m+l}(\mathbb{R} \times \mathbb{R}_+) \to \begin{cases} H^l(\mathbb{R} \times \mathbb{R}_+) \\ \times \prod_{j=1}^m H^{2m+l-m_j^k-1/2}(\mathbb{R}_-) \\ \times \prod_{j=1}^m H^{2m+l-m_j^{3-k}-1/2}(\mathbb{R}_+) \end{cases} \tag{2.4}$$

where $A^{pr}_{x_0,0}$ (etc.) are the principal parts of the transformed operators and $A^{\langle pr\rangle}_{x_0,0} = F^{-1}\Phi(\langle\cdot\rangle)\cdot F$ corresponds with the PDO with Fourier symbol of $A^{pr}_{x_0,0}$ replacing the variable $\xi \in \mathbb{R}^2$ by $\left(\frac{1+\xi^2}{\xi^2}\right)^{1/2}\xi$ which makes $A^{\langle pr\rangle}_{x_0,0}$ a bijection provided (**??**) is satisfied, cf. [**?**, Lemma 7.1].

REMARK 2.1. A similar idea, namely the replacement of ξ by $\frac{1+|\xi|}{|\xi|}\xi$, was used in [**?**, §7], which is modified here for computational reasons, see Remark **??** and Corollary **??**. Again $B^{k\,pr}_{x_0,0} = 0$ for $x_0 \in \Omega \cup \Gamma^{3-k}$.

3. Localization

We like to apply a modification of the local principle [**?**, **?**, **?**] in a convenient form. Following those well-known concepts, the above-mentioned operators L (etc.) are called *operators of local type*, if the composition $\varphi \cdot L\psi\cdot$ is compact for any $\varphi, \psi \in C^\infty(\dot{\mathbb{R}}^2)$ with $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \emptyset$. An equivalent condition can be formulated as follows: the commutator $[L, g\cdot] = Lg\cdot - g\cdot L$ is a compact operator for all $g \in C_0^\infty(\dot{\mathbb{R}}^2)$. Here $\dot{\mathbb{R}}^2$ is the one-point compactification of $\mathbb{R}^2$ and $\varphi\cdot$ denotes the multiplication operator generated by the function φ in the corresponding spaces, i.e. restricted to Ω and Γ^k, respectively. This is obviously the case under the assumptions of Section **??**. Note that the multiplication operator $g\cdot$ acts in a scalar way on the right side of L and as a diagonal 3×3 matrix on the left side of L in the sense of the image space of (**??**). In other words

$$g\cdot L = \begin{pmatrix} g_{|\Omega}\cdot & 0 & 0 \\ 0 & g_{|\Gamma^1}\cdot & 0 \\ 0 & 0 & g_{|\Gamma^2}\cdot \end{pmatrix} \circ L \tag{3.1}$$

and we shall work with the short notation on the left side of this identity for convenience (also for smooth functions defined at least on $\overline{\Omega}$).

To define local equivalence (see below) we need localizing classes, which must be covering, to provide proper localization, see [**?**, Chapter 5].

Let $g \in C^\infty(\Omega)$, $0 \le g(x) \le 1$, $g(x) = 1$ for $x \in \mathcal{B}_1(0)$ and $g(x) = 0$ for $x \notin \mathcal{B}_2(0)$. Further we define the family of functions

$$M_{x_0} = \left\{ g_{x_0,\delta} \in C^\infty(\overline{\Omega}) : g_{x_0,\delta}(x) = g\left(\frac{x-x_0}{\delta}\right)\ ,\ 0<\delta<d(x_0) \right\} \text{ for } x_0 \in \overline{\Omega}\,, \tag{3.2}$$

where

$$d(x_0) = \frac{1}{2}\begin{cases} \operatorname{dist}(x_0,\Gamma)\,, & x_0 \in \Omega \\ \operatorname{dist}(x_0,\Gamma^{3-k})\,, & x_0 \in \Gamma^k\,,\quad k=1,2 \\ \operatorname{dist}(x^1,x^2)\,, & x_0 \in \{x^1,x^2\} \end{cases} \tag{3.3}$$

Note, that for $x_0 \in \Omega$ the class M_{x_0} consists of functions which are supported only inside Ω, i.e. supp $g_{x_0,\delta} \cap \Gamma = \emptyset$. For $x_0 \in \Gamma$ we have $g_{x_0,\delta}(x^k) = 0$ if $x_0 \neq x^k$ and $g_{x_0,\delta}(x^k) \neq 0$ if $x_0 = x^k$.

We denote by $\mathfrak{M}_{x_0} = \mathfrak{M}_{x_0}(X)$ the class of multiplication operators of the form $G_{x_0,\delta} = g_{x_0,\delta} \cdot = g_{x_0,\delta} I_X$ for all spaces appearing in Section 2 (scalar and vectorial as well, cf. (**??**)). It is obvious that $M = \cup_{x_0 \in \overline{\Omega}} M_{x_0}$ has the characteristics of (a family of functions that generates) a *covering system of localizing classes for mixed BVPs (**??**)*, i.e.

(j) For all $x_0 \in \overline{\Omega}$ and $g \in M_{x_0}$, a multiplication operator $G = g I_X \in \mathfrak{M}_{x_0}(X)$ is bounded but not compact for any admissible space X;

(jj) If $g_1, g_2 \in M_{x_0}$, there exists a function $g \in M_{x_0}$ such that $g_1 g = g_2 g = g$;

(jjj) If $S \subset M$ such that, for every $x_0 \in \overline{\Omega}$ there exists a function $g \in S$ with $g(x_0) \neq 0$, then there exist numbers $N, N_1 \in \mathbb{N}$ such that $g = g_1 + \cdots + g_N$ is invertible in $C^\infty(\overline{\Omega})$ where $g_1 \in M_{x^1}$, $g_2 \in M_{x^2}$, $g_3 + \cdots + g_{N_1} \in M_{\Gamma^1 \cup \Gamma^2} = \cup_{x_0 \in \Gamma^1 \cup \Gamma^2} M_{x_0}$, and $g_{N_1+1} + \cdots + g_N \in M_\Omega = \cup_{x_0 \in \Omega} M_{x_0}$, i.e. $g_3(x) = \cdots = g_N(x) = 0$ for $x \in \{x^1, x^2\}$, $g_{N_1+1}(x) = \cdots = g_N(x) = 0$ for $x \in \Gamma$.

REMARK 3.1. All conditions can be reformulated in terms of multiplication operators on the admissible spaces, e.g.

(jj′) If $G_1, G_2 \in \mathfrak{M}_{x_0}(X)$, there exists another $G \in \mathfrak{M}_{x_0}(X)$ such that $G_1 G = G G_1 = G_2 G = G G_2 = G$.

and (jjj′) can be written by analogy.

If we had dropped the (redundant) zeros in (**??**), the different sizes of multiplication operators in different x_0's do not allow this reformulation of (jjj). In that case, relations resulting from the partition of unity had a more complicated form.

Since we shall have a finite number of local transformations

$$\varkappa_j \ : \ U_j \to V_j \,, \qquad U_j \subset \Omega \,, \qquad V_j \subset \mathbb{R}^2 \,, \qquad \mathrm{j} = 1, 2, \ldots, \mathrm{K}$$

(see Fig. 2.1 and (**??**)), we will use, for convenience, $\varkappa(x)$ which coincides with $\varkappa_j(x)$ for considerations in U_j. Similarly we use $\varkappa^{-1}(z)$, dropping the index j, for the inverse transformation.

The operators L and $L^{pr}_{x_0}$ are said to be *locally equivalent in* $x_0 \in \overline{\Omega}$, if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$||| \left(L - L^{pr}_{x_0} \right) G_{x_0,\delta} ||| < \varepsilon \,, \quad ||| G_{x_0,\delta} \left(L - L^{pr}_{x_0} \right) ||| < \varepsilon \tag{3.4}$$

where $|||\cdot|||$ denotes the norm in the quotient space of bounded linear modulo compact operators. The relation (**??**) is obviously true for the operators introduced before, cf. [**?**]. Moreover, we have the *local quasi-equivalence* of $L^{pr}_{x_0}$ in $x_0 \in \Gamma$ and $L^{\langle pr \rangle}_{x_0,0}$ as well as $L^{pr}_{x_0,0}$ in 0 (we will apply both concepts later):

$$\begin{aligned} ||| G_{x_0,\delta} \left(L^{pr}_{x_0} - T_\varkappa L^{\langle pr \rangle}_{x_0,0} T_{\varkappa^{-1}} \right) ||| = ||| \left(L^{pr}_{x_0} - T_\varkappa L^{\langle pr \rangle}_{x_0,0} T_{\varkappa^{-1}} \right) G_{x_0,\delta} ||| < \varepsilon \,, \\ ||| G_{x_0,\delta} \left(L^{pr}_{x_0} - T_\varkappa L^{pr}_{x_0,0} T_{\varkappa^{-1}} \right) ||| = ||| \left(L^{pr}_{x_0} - T_\varkappa L^{pr}_{x_0,0} T_{\varkappa^{-1}} \right) G_{x_0,\delta} ||| < \varepsilon \,, \end{aligned} \tag{3.5}$$

where we define the operators $T_\varkappa$ in the following way.

If $x_0 \in U_j$, $\varkappa = \varkappa_j$ and $\varphi \in X = H^{2m+l}(\Omega)$, let

$$\mathcal{T}_\varkappa \varphi(x) = \begin{cases} \varphi(\varkappa_j(x))\,, & x \in \Omega \cap U_j \\ 0\,, & x \in \Omega \setminus U_j \end{cases} \tag{3.6}$$

If X is the image space of L, we modify $\mathcal{T}_\varkappa$ in the sense of (??) and $\mathcal{T}_\varkappa^{-1}$ is defined similarly in the corresponding space settings.

Further we call the above operators *locally Fredholm* in x_0 if there exist *local left* and *right regularizers* $R^l_{x_0}$, $R^r_{x_0}$such that

$$\begin{aligned} &|||\left(R^l_{x_0} L - I\right) G_{x_0,\delta}||| = 0 \\ &|||G_{x_0,\delta}\left(LR^r_{x_0} - I\right)||| = 0 \end{aligned} \tag{3.7}$$

for suitable $G_{x_0,\delta} \in \mathfrak{M}_{x_0}$.

On the plane $\mathbb{R}^2$ we consider the "pull-back" localizing classes

$$\mathfrak{M}'_{z_0} = \{G'_{z_0,\delta}(z) = G_{x_0,\delta}(\varkappa^{-1}(z)) \;:\; G_{x_0,\delta} \in \mathfrak{M}_{x_0}\}\,, \quad z_0 = \varkappa(x_0) \in \mathbb{R}^2 \tag{3.8}$$

and define locally equivalent and locally Fredholm operators by analogy.

THEOREM 3.2. *Let L be the operator (??) associated to a mixed BVP (??). The following assertions are equivalent:*

(i) L is Fredholm;

(ii) L is locally Fredholm in x_0, for all $x_0 \in \overline{\Omega}$;

(iii) $L^{\langle pr\rangle}_{x_0,0}$ is locally Fredholm in $\varkappa(x_0)$, for all $x_0 \in \overline{\Omega}$.

Proof. In brief: The equivalence of the conditions (i) and (ii) follows from a general localization principle described in [?] – modified for the present settings. Namely, if (i) L is Fredholm, there exist (global) regularizers R^l and R^r acting in spaces inversely to (??) such that $R^l L - I$ and $LR^r - I$ are compact in the corresponding spaces. This implies (??) for any $g \in C^\infty(\dot{\mathbb{R}}^2)$, i.e. condition (ii). Conversely, if (ii) is satisfied, since $\{\mathfrak{M}_{x_0}\}_{x_0 \in \overline{\Omega}}$ is a covering system of localizing classes in all Bessel potential spaces, from the collection $\{G_{x_0,\delta}\}_{x_0 \in \overline{\Omega}, \delta > 0}$ of operators $G_{x_0,\delta} \in \mathfrak{M}_{x_0}$, which provide localization in (??), we can select a finite collection $\{\{G_{x_j,\delta_j}\}_{j=1,\dots,N}$, such that $\sum_{j=1,\dots,N} G_{x_j,\delta_j}$ is invertible. Therefore,

$$R^l = \sum_{j=1}^{N} R^l_{x_j} G_{x_j,\delta_j}$$

is a left regularizer of L because

$$\begin{aligned} R^l L - I \;&=\; \sum_{j=1}^{N} R^l_{x_j} G_{x_j,\delta_j} L - I \\ &=\; \sum_{j=1}^{N} \left(R^l_{x_j} L G_{x_j,\delta_j} - G_{x_j,\delta_j}\right) + \text{compact} \end{aligned}$$

is compact. Here we used that $G_{x_j,\delta_j} L = L G_{x_j,\delta_j}$ up to lower order terms which are compact according to the well-known embedding theorems in spaces of Bessel potentials. The same argument holds for the right global and local regularizers.

The equivalence of (ii) and (iii) is an obvious consequence of the fact that operators which are of local type and locally quasi-equivalent in $x_0 \in \overline{\Omega}$ and $z_0 = \varkappa(x_0) \in \mathbb{R}^2$, like L and $L_{x_0}^{\langle pr \rangle}$, are simultaneously locally Fredholm in x_0 and in $z_0 = \varkappa(x_0)$ or not, respectively.

REMARK 3.3. The same reasoning implies the equivalence of the local Fredholm property of the operators L_{x_0}, $L_{x_0}^{pr}$, $L_{x_0}^{\langle pr \rangle}$ and $L_{x_0,0}^{pr}$, respectively, in each $x_0 \in \overline{\Omega}$. But these conditions are less important for our purposes.

THEOREM 3.4. *If the BVP (**??**) is piecewise elliptic, the associated operator L is Fredholm if and only if $L_{x_0,0}^{\langle pr \rangle}$ is Fredholm for $x_0 \in \{x^1, x^2\}$.*

Proof. First, for the operator $L_{x_0,0}^{\langle pr \rangle}$, the Fredholm property and the local Fredholm property in 0 amount to the same. This is a consequence of the fact that $L_{x_0,0}^{\langle pr \rangle}$ is locally Fredholm anyway in all other points of $\mathbb{R} \times \mathbb{R}_+$ and at infinity due to the assumption of piecewise ellipticity [**?**].

Considering Theorem **??** we have to prove the "if" part, i.e., assuming that the BVP (**??**) is piecewise elliptic and that $L_{x_0,0}^{\langle pr \rangle}$ is Fredholm for $x_0 \in \{x^1, x^2\}$, we like to construct a two-sided regularizer of L.

First we think of local regularizers in $x_0 \in \Omega$ and $x_0 \in \Gamma^1$. The BVP

$$\begin{aligned} Au = f \quad &\in \ H^l(\Omega) = Y_0 \\ \widetilde{B^1} u = g \ &\in \ \textstyle\prod_{j=1}^m H^{2m+l-m_j^1-1/2}(\Gamma) = Y_1 \end{aligned} \tag{3.9}$$

is elliptic for $u \in H^{2m+l}(\Omega) = X$. Thus

$$\widetilde{L^1} = \begin{pmatrix} A \\ \widetilde{B^1} \end{pmatrix} : X \to Y_0 \times Y_1 \tag{3.10}$$

has a regularizer $\widetilde{R^1} = \left(\widetilde{R_{11}}, \widetilde{R_{12}}\right) : Y_0 \times Y_1 \to X$. Putting

$$R^1 = \left(\widetilde{R_{11}}, \widetilde{R_{12}} E_{\Gamma_1 \to \Gamma}, 0\right) : Y_0 \times Y_{1|\Gamma^1} \times Y_{2|\Gamma^2} \to X \tag{3.11}$$

where $E_{\Gamma_1 \to \Gamma} : Y_{1|\Gamma^1} \to Y_1$ is a continuous extension operator (of local type), with obvious space identifications, we obtain a local regularizer $R_{x_0} = R^1$ of L for all $x_0 \in \Omega \cup \Gamma^1$.

By analogy we have local regularizers for $x_0 \in \Gamma^2$. Together with the second assumption and Theorem **??**, we get a family of local regularizers for all $x_0 \in \overline{\Omega}$ and can construct a global regularizer as in the proof of Theorem **??**.

4. Reduction to semi-homogeneous problems

In the present section we study the localized operator $L^{\langle pr \rangle}_{x_0,0}$ for $x_0 \in \{x^1, x^2\}$ in Theorem **??** and (**??**) in detail.

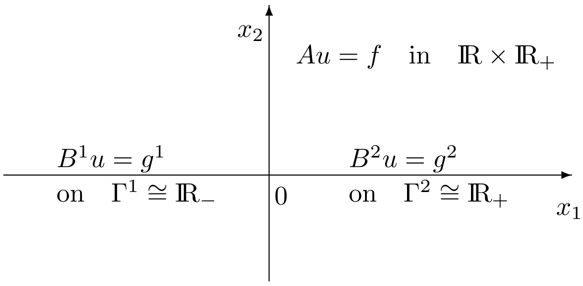

Figure 4.1: Canonical mixed BVP.

It is the operator associated with the *canonical mixed BVP* for the upper half-plane where now, abbreviating the previous notation, the operators A and B^k have constant coefficients. Figure 4.1 corresponds with the quasi-localization in $x_0 = x^1$ in Figure 1.1 and the roles of B^1 and B^2 are exchanged, if we consider $x_0 = x^2$.

We also consider the two BVPs where the mixed boundary condition is replaced either by $B^1_{\mathbb{R}} u = g$ on the full line $\Gamma = \{(x_1, x_2) : x_2 = 0\} = \Gamma^1 \cup \{(0,0)\} \cup \Gamma^2 \cong \mathbb{R}$ or by $B^2_{\mathbb{R}} u = g$ in the corresponding spaces, i.e. associated with the operators (in short notation)

$$L^k_{\mathbb{R}} = \begin{pmatrix} A \\ B^k_{\mathbb{R}} \end{pmatrix} : H^{2m+l}(\mathbb{R} \times \mathbb{R}_+) \to \begin{cases} H^l(\mathbb{R} \times \mathbb{R}_+) \\ \times \prod_{j=1}^m H^{2m+l-m^k_j-1/2}(\mathbb{R}). \end{cases} \tag{4.1}$$

Both are elliptic (see Section **??**) and admit therefore the following results.

PROPOSITION 4.1. *The elements of* $\ker A$ *are represented by*

$$\begin{aligned} u(x_1, x_2) = F^{-1}_{\xi \mapsto x_1} \Big\{ \Big(\widehat{u_{11}}(\xi) + x_2 \widehat{u_{12}}(\xi) + \cdots + x_2^{\mu_1 - 1} \widehat{u_{1\mu_1}}(\xi) \Big) \exp[-t_1(\xi) x_2] \\ + \cdots + \Big(\widehat{u_{\kappa 1}}(\xi) + x_2 \widehat{u_{\kappa 2}}(\xi) + \cdots + x_2^{\mu_\kappa - 1} \widehat{u_{\kappa \mu_\kappa}}(\xi) \Big) \exp[-t_\kappa(\xi) x_2] \Big\} \end{aligned} \tag{4.2}$$

where κ is the number of square roots $t_1, \ldots, t_\kappa$ with positive real part of the characteristic polynomial $A(i\xi, z)$ with respect to z, the numbers $\mu_1, \ldots, \mu_\kappa$ are the corresponding multiplicities, $\sum_{\lambda=1}^{\kappa} \mu_\lambda = m$, and

$$u_{\lambda,\nu} \in H^{2m+l-\nu+1/2}, \qquad \nu = 1, \ldots, \mu_\lambda, \quad \lambda = 1, \ldots, \kappa. \tag{4.3}$$

Proof. Making use of the characterization of elliptic PDOs [**?**, p. 150], the representation formula (**??**) is a consequence of partial Fourier transformation with respect to the first variable and solution of a homogeneous ODE by an exponential ansatz. (**??**) results from estimates in spaces of Bessel potentials, see [**?**] for the case of second order PDEs.

PROPOSITION 4.2. *The operators*

$$B^k_{\mathbb{R}} : \ker A \mapsto Y^k_1 = \prod_{j=1}^{m} H^{2m+l-m^k_j-1/2}(\mathbb{R}) \tag{4.4}$$

and $L^k_{\mathbb{R}}$ *in (??) are homeomorphisms.*

Proof. $\ker A$ as a closed subspace of H^{2m+l} is toplinear isomorphic to the topological product space of the data (??), which are in one-to-one (homeomorphic) correspondence with the data space Y^k_1 by means of linear algebra and elementary estimates in Bessel potential spaces. Together with the Fredholm property of L^k [?] and elementary space decomposition, this implies the statement.

These results allow the following interpretations.

COROLLARY 4.3. *In short, the operators* $L^k_{\mathbb{R}}$ *can be written as*

$$L^k_{\mathbb{R}} = \begin{pmatrix} A \\ B^k_{\mathbb{R}} \end{pmatrix} : X \to \begin{cases} Y_0 \\ \times Y^k_1 \end{cases} = Y^k \tag{4.5}$$

where $B^k_{\mathbb{R}}$ *is a retraction [?], i.e. right invertible by an extension (or co-retraction) operator* E^k.

The data (product) space Y^0_1 *given by (??) is the image of a particular boundary operator* $B^0_{\mathbb{R}}$ *of an elliptic BVP due to* $L^0 = (A, B^0_{\mathbb{R}})^T$.

The spaces Y^k_1, $k = 0, 1, 2$, *are isomorphically connected by translation invariant operators*

$$\begin{array}{ccccc} Y^1_1 & \longleftarrow & Y^0_1 & \longrightarrow & Y^2_1 \\ & B_- & & B_+ & \end{array} \tag{4.6}$$

where the Fourier symbols $\Psi_\pm$ *of* $B_\pm = F^{-1}\Psi_\pm \cdot F$ *are rational* $m \times m$ *matrix functions of the variables* ξ, $t_1(\xi), \ldots, t_\kappa(\xi)$.

Proof. This is a consequence of the well-posedness of the two elliptic BVPs in combination with Proposition ?? and Proposition ??.

Coming to the question of reducing BVPs to semi-homogeneous BVPs, where $Au = 0$ instead of $Au = f$, we study bounded linear operators in Banach spaces of the form

$$L = \begin{pmatrix} A \\ B \end{pmatrix} : X \to Y = \begin{pmatrix} Y_0 \\ Y_1 \end{pmatrix} \tag{4.7}$$

and like to relate L to

$$L^0 = B_{|X_0} : X_0 \to Y_1 \tag{4.8}$$

where $X_0 = \ker A$. The question is, if the two operators are *toplinear equivalent after extension* (in brief: equivalent after extension) i.e. [?, ?, ?] if there exist Banach spaces Z_1, Z_2 and homeomorphisms E_1, E_2 such that

$$\begin{pmatrix} L^0 & 0 \\ 0 & I_{Z_1} \end{pmatrix} = E_1 \begin{pmatrix} L & 0 \\ 0 & I_{Z_2} \end{pmatrix} E_2 \tag{4.9}$$

which yields that the two operators have similar properties (invertibility etc.) and various explicit results (representation of generalized inverses and normalization, e.g.) can be obtained from corresponding results for the other (simpler) one. Here we have:

LEMMA 4.4. *Assuming (??) where A is right invertible, i.e. surjective and $X_0 = \ker A$ is complemented, then the operators (??) and (??) are toplinear equivalent after extension.*

Proof. It is sufficient to observe that

$$\begin{aligned} \begin{pmatrix} L^0 & 0 \\ 0 & I_{Y_0} \end{pmatrix} &= \begin{pmatrix} L^0 & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \widetilde{R} \end{pmatrix} \\ &: \begin{pmatrix} Y_1 \\ Y_0 \end{pmatrix} \leftarrow \begin{pmatrix} X_0 \\ X_1 \end{pmatrix} \leftarrow \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} \end{aligned} \tag{4.10}$$

provided $A\widetilde{R} = I_{Y_0}$, $X_1 = \operatorname{im} R = RAX$, $\widetilde{R} = \operatorname{Rst} R : Y_0 \to X_1$ (i.e. image restricted) and the first factor on the right is equivalent to L (coincides up to homeomorphisms).

REMARK 4.5. For this step, only properties of A (not of B) are important. Here a right inverse is obtained by continuous extension of $H^l(\mathbb{R}^2_+)$ to $H^l(\mathbb{R}^2)$, inversion of the PDO A (with constant coefficients) on $\mathbb{R}^2$, and restriction to the half-space.

Evidently the same arguments hold when B stands for a “mixed boundary operator”.

PROPOSITION 4.6. *Consider the operator in Theorem ?? defined by (??) and now abbreviated by*

$$L_M = \begin{pmatrix} A \\ B^1 \\ B^2 \end{pmatrix} : X \to \begin{pmatrix} Y_0 \\ r_-Y_1^1 \\ r_+Y_1^2 \end{pmatrix}. \tag{4.11}$$

This operator is toplinear equivalent after extension to each of the following:

$$L_M^0 = \begin{pmatrix} B^1 \\ B^2 \end{pmatrix}_{|X_0} : X_0 = \ker A \to \begin{pmatrix} r_-Y_1^1 \\ r_+Y_1^2 \end{pmatrix} \tag{4.12}$$

$$L_0^0 = B^2{}_{|X_0^0} : X_0^0 = X_0 \cap \ker B^1 \to r_+Y_1^2. \tag{4.13}$$

Proof. The first equivalence relation is clear from the foregoing consideration. A right inverse of the differential operator A with constant coefficients, see (**??**) and (**??**), is obtained by

$$R = rF^{-1}\Phi_A^{-1} \cdot F\ell$$

where $\ell : H^l(\mathbb{R}\times\mathbb{R}_+) \to H^l(\mathbb{R}^2)$ denotes a continuous extension operator, Φ_A is the Fourier symbol of A, and $r : H^{2m+l}(\mathbb{R}^2) \to H^{2m+l}(\mathbb{R}\times\mathbb{R}_+)$ stands for the restriction to the half-plane.

The second relation follows similarly by Lemma **??**. B^1 is right invertible, since a right inverse is obtained by continuous extension from $r_-Y_1^1$ to Y_1^1, inversion of the operator $L^{1,0}$ associated to the (non-mixed, semi-homogeneous) BVP due to (A, B^1) in $\mathbb{R}^2_+$ by Corollary **??**, and restriction to the half-line $\mathbb{R}_+$.

5. The Fredholm property

Now we focus on the two operators $L^{\langle pr\rangle}_{x_0,0}$, $x_0 \in \{x^1, x^2\}$, in (**??**) which are both directly associated to the mixed canonical BVP described in Section **??**.

Let W_{x^k} denote the two Wiener-Hopf operators of the form

$$W_{x^k} = r_+B_+B_-^{-1} : \prod_{j=1}^{m} H_+^{2m+l-m_j^k-1/2} \to H^{2m+l-m_j^{3-k}-1/2}(\mathbb{R}_+) \tag{5.1}$$

where $k = 1, 2$ and $B_\pm$ are the operators of (**??**) resulting from the operator $L^{\langle pr\rangle}_{x^k,0}$ of (**??**), $k = 1, 2$ (or the corresponding mixed canonical BVP described in Figure 3.1 for $k = 1$).

PROPOSITION 5.1. *The operator L^0_{0,x^k} defined by (**??**) in the case of $L_M = L^{\langle pr\rangle}_{x^k,0}$, is equivalent to W_{x^k} (in the sense of toplinear [**?**] equivalent or isomorphic [**?**] or just equivalent [**?**] operators in Banach spaces).*

Proof. Let us consider $k = 1$, the situation corresponding to Figure 4.1.

The elements of $X_0 = \ker A$ can be represented in the form (**??**) due to Proposition **??**, in particular the elements of $X_0^0 = X_0 \cap \ker B^1$ can be written in this form. Proposition **??** implies that the restriction of the operator $L^1_{\mathbb{R}}$ given by (**??**) on the space X_0^0, acting into its image, i.e.

$$\operatorname{Rst} L^1_{\mathbb{R}} = \begin{pmatrix} A \\ B^1_{\mathbb{R}} \end{pmatrix} : X_0^0 \to \begin{cases} \{0\} \\ \times\prod_{j=1}^m H_+^{2m+l-m_j^1-1/2} \end{cases} \tag{5.2}$$

is a homeomorphism according to Proposition **??** and to the fact that X_0^0 is a complemented subspace of $H^{2m+l}(\mathbb{R}^2_+)$. With the corresponding restrictions of the boundary operators

$$\begin{array}{lllll} \operatorname{Rst} B^1_{\mathbb{R}} & : & X_0^0 & \to & H_+^r, & r = (2m+l-m_j^1-1/2)_{j=1,\ldots,m} \\ \operatorname{Rst} B^2 & : & X_0^0 & \to & H^s(\mathbb{R}_+), & s = (2m+l-m_j^2-1/2)_{j=1,\ldots,m} \end{array} \tag{5.3}$$

we have the composition of toplinear mappings

$$W_{x^1} = \operatorname{Rst} B^2 \circ \mathcal{P}^{-1} \circ \mathcal{P} \circ \left(\operatorname{Rst} B^1_{\mathbb{R}}\right)^{-1} : H^r_+ \to X^0_0 \to Y^0_1 \to X^0_0 \to H^s(\mathbb{R}_+) \tag{5.4}$$

where $\mathcal{P}$ is the Poisson operator defined by (**??**) and we put $B_+ = \operatorname{Rst} B^2 \circ \mathcal{P}^{-1}$, $B_- = \operatorname{Rst} B^1_{\mathbb{R}} \circ \mathcal{P}^{-1}$ in accordance with (**??**).

THEOREM 5.2. *Let L be the operator (**??**) associated with the mixed BVP (**??**). Then L is a Fredholm operator if and only if the two operators W_{x^k} of (**??**) are Fredholm.*

Proof. This result connects Theorem **??** with Proposition **??** via Proposition **??**.

COROLLARY 5.3. *The Fourier symbols Φ^k of*

$$W_{x^k} = r_+ F^{-1} \Phi^k \cdot F : H^{r^k}_+ \to H^{s^k}(\mathbb{R}_+) \tag{5.5}$$

are regular, i.e. $F^{-1}\Phi^k \cdot F : H^{r^k} \to H^{s^k}$ are bijective, and they are $m \times m$ matrices of functions from $\mathcal{R}\left(\xi, t_1(\xi), \ldots, t_\kappa(\xi)\right)$, i.e. they are rational in these variables.

Proof. Let us assume firstly that $x^1 = (0,0)$ and $\partial\Omega$ is locally like Figure 4.1 near the origin. The principle parts of the Fourier symbols $\Psi^k_{\mathbb{R}}$ of $\operatorname{Rst} B^1_{\mathbb{R}} \circ \mathcal{P}^{-1}$ and $\operatorname{Rst} B^2 \circ \mathcal{P}^{-1}$ are computed from (**??**) and (**??**) as

$$\left(\Psi^k_{\mathbb{R}}\right)_{(\lambda,\nu),j} = \sum_{|\sigma| = m^k_j} b^k_{j,\sigma}(0)\,(i\xi)^{\sigma_1}\, D^{\sigma_2}_{x_2}(x_2^\nu \exp(-t_\lambda x_2))_{|x_2=0} \tag{5.6}$$

showing the dependence of the Fourier transformed principal parts of the data in terms of the ansatz functions $\widehat{u_{\lambda,\nu}}$ by a multiplication operator.

Thus the orders of the mappings result from the increase orders of the symbols, bijectivity from the one-to-one correspondence in appropriate Banach spaces, see Proposition **??** and Corollary **??**, and the algebraic form is obvious from (**??**) and (**??**). The localization does not effect the above properties and the case $k = 2$ holds by analogy.

COROLLARY 5.4. *The two operators W_{x^k} are (toplinear) equivalent to the* lifted operators

$$W_{x^k,0} = r_+ F^{-1} \Phi^k_0 \cdot F : \left(L^2_+\right)^m \to L^2(\mathbb{R}_+)^m \tag{5.7}$$

where

$$\begin{aligned} \Phi^k_0 &= \lambda^{s^k}_- \Phi^k \lambda^{-r^k}_+ \\ \lambda^s_\pm(\xi) &= \operatorname{diag}\left((\xi \pm i)^{s_1}, \ldots, (\xi \pm i)^{s_m}\right) \end{aligned} \tag{5.8}$$

for $s = (s_1, \ldots, s_m) \in \mathbb{R}^m$.

Proof. See e.g. [?, ?, ?].

REMARK 5.5. Instead of $\lambda_{\pm}(\xi) = \xi \pm i$ one can apply $\xi \pm k_0$ where $\Im m k_0 > 0$ or different functions of this type in different places of the matrices (??) in order to reduce the "algebraic complexity" of Φ_0^k [?]. For instance, if $t_1 = t_2 = \cdots = t_\kappa$ in (??) (A^{pr} being a power of the Laplacian, e.g.) and $l \in \mathbb{N}_0$ (see the original BVP in Section ??), then

$$\Phi_0^k(\xi) \in \mathcal{R}\left(\xi, t_1(\xi)\right)^{m\times m}, \qquad t_1(\xi) = \left(\xi^2+1\right)^{1/2} \tag{5.9}$$

taking the usual branch due to a vertical cut between $\pm i$ over infinity. But, in any case, we have the following result:

$$\left(\frac{\xi - i}{\xi + i}\right)^{-l} \Phi_0^k(\xi) \in \mathcal{R}\left(\xi, t_1(\xi), \ldots, t_\kappa(\xi)\right)^{m\times m}, \qquad k = 1, 2. \tag{5.10}$$

COROLLARY 5.6. *The Fourier symbols of the two lifted Wiener-Hopf operators $W_{x^k,0}$ due to the two locally quasi-equivalent operators $L^{\langle pr\rangle}_{x^k,0}$ of Theorem ?? are regular elements of the algebra of Hölder continuous $m\times m$ matrix functions defined on the two-point compactification of $\mathbb{R}$:*

$$\Phi_0^k \in \mathcal{G}C^{\mu}(\ddot{\mathbb{R}})^{m\times m}, \qquad \mu \in]0,1[\tag{5.11}$$

with a possible jump at infinity.

Proof. This is an elementary consequence of the form of Φ_0^k discussed before, cf. [?].

THEOREM 5.7. *Let the mixed BVP (??) be piecewise elliptic. Then it is Fredholm if and only if the two (for short)* assigned Wiener-Hopf operators *$W_{x^k,0}$ from (??) have symbols which satisfy*

$$\det\left(\lambda\Phi_0^k(-\infty) + (1-\lambda)\Phi_0^k(+\infty)\right) \neq 0, \qquad \lambda \in [0,1], \quad k = 1, 2. \tag{5.12}$$

If this condition is violated, the operator L associated with the mixed boundary value problem, is not normally solvable: $\operatorname{im} L$ *is not closed but the* defect numbers *are finite:*

$$\begin{aligned} \alpha(L) &= \dim\ker L &< \infty \\ \beta(L) &= \operatorname{codim}\overline{\operatorname{im} L} &< \infty. \end{aligned} \tag{5.13}$$

Proof. The question of the BVP (??) to be Fredholm was already reduced before to the corresponding question for the operators (??) with symbols (??). But the Fredholm criterion is known for Wiener-Hopf (matrix) operators in L^2 spaces with piecewise continuous symbols [?]. Considering L in dependence of the smoothness parameter $l \geq 0$, say $L(l)$ in (**??**), we have evidently the monotony property

$$\left.\begin{array}{rcl}\alpha\,(L(l_1)) & \leq & \alpha\,(L(l_2)) \\ \beta\,(L(l_1)) & \geq & \beta\,(L(l_2))\end{array}\right\} \qquad \text{for} \qquad l_1 > l_2 \tag{5.14}$$

and the set of non-Fredholm points of $L(l)$ on the l-semi-axis is locally finite due to (**??**), i.e.

$$\#S_{NF}\,([a,b]) = \#\,\{l \in [a,b] : L(l) \text{ is not Fredholm}\} < \infty \tag{5.15}$$

for any $0 \leq a < b < \infty$. This yields (**??**).

6. Normalization – the basic idea

The normalization problem for a single bounded linear operator $T : X_0 \to Y_0$, which is not normally solvable in given Banach spaces X_0, Y_0 can be formulated as to find another pair of Banach spaces X_1, Y_1 such that T maps $X_0 \cap X_1$ into Y_1, $X_0 \cap X_1$ is dense in X_1 and the operator T restricted on $X_0 \cap X_1$, as an operator into Y_1, has a continuous extension on X_1, in brief

$$\overline{T} = \operatorname{Ext} T_{|X_0 \cap X_1} : X_1 \to Y_1 \tag{6.1}$$

and such that $\overline{T}$ is normally solvable. Then we say that the pair $(X_1, Y_1) \in \mathcal{N}(T)$ *solves the normalization problem for* T. See [**?**, **?**, **?**, **?**] for details, modifications and generalizations of the concept.

For mixed BVPs (**??**) we get the following result immediately from Theorem **??**.

COROLLARY 6.1. *Let $L(l)$ denote the operator associated to the mixed BVP (**??**) and $L(l_0)$ be not normally solvable for some $l_0 \geq 0$. Then there exists an $\epsilon > 0$ such that $L(l) : X(l) \to Y(l)$, abbreviating the spaces in (**??**), is Fredholm for $l \in I_\epsilon =]l_0 - \epsilon, l_0 + \epsilon[\cap [0, \infty[\setminus \{l_0\}$, i.e. $(X(l), Y(l)) \in \mathcal{N}\,(L(l_0))$. Further ϵ can be chosen such that both defect numbers $\alpha(L(l))$ and $\beta(L(l))$ have only one jump within I_ϵ, namely at l_0.*

As we saw in the second item of Examples **??**, and for various other reasons [**?**, **?**, **?**], it may be interesting, not to change the topologies of both spaces, X_0 and Y_0, simultaneously. This leads to the question whether a normalization problem is solvable under the additional assumptions

$$X_1 = X_0, \quad Y_1 \subset Y_0 \qquad \text{or} \qquad X_1 \supset X_0, \quad Y_1 = Y_0 \tag{6.2}$$

respectively. We call these *minimal normalization problems* [**?**, **?**] and denote the corresponding normalized operators by

$$\overset{<}{T} = \operatorname{Rst} T : X_0 \to Y_1 \qquad \text{and} \qquad \overset{>}{T} = \operatorname{Ext} T : X_1 \to Y_0 \tag{6.3}$$

provided $(X_0, Y_1) \in \mathcal{N}(T)$ or $(X_1, Y_0) \in \mathcal{N}(T)$ holds, respectively. The first one is referred to as *image normalization*, the second as *domain normalization*.

PROPOSITION 6.2. *If a Banach space operator $T \in \mathcal{L}(X_0, Y_0)$ is not normally solvable, but $\ker T$ and $\overline{\operatorname{im} T}$ are complemented, then the two minimal normalization problems are uniquely solvable up to isomorphy of the spaces in question.*

Proof. For the image normalization decompose

$$\begin{aligned} X_0 &= \ker T \ \oplus \ \widetilde{X_0} \\ Y_0 &= \overline{\operatorname{im} T} \ \oplus \ \widetilde{Y_0}. \end{aligned} \tag{6.4}$$

Since $\operatorname{Rst} T : \widetilde{X_0} \to \operatorname{im} T$ is bijective (but not bounded invertible with respect to the given norms of X_0 and Y_0), we choose

$$Y_1 \ = \ \operatorname{im} T \ \oplus \ \widetilde{Y_0} \tag{6.5}$$

with the norm, for $y = z + \widetilde{y}$ due to this decomposition,

$$\|y\|_{Y_1} \ = \ \left\| (\operatorname{Rst} T)^{-1} z \right\|_{X_0} + \|\widetilde{y}\|_{Y_0}. \tag{6.6}$$

The desired properties of Y_1 are evident. Uniqueness follows from considering two analogous decompositions of normalized space pairs and the isomorphy of finite dimensional spaces with the same dimension and their complements, respectively. The case of domain normalization is left as an exercise.

REMARK 6.3. The assumptions of Proposition **??** are satisfied for Hilbert space operators (every closed subspace is complemented) or in the case where the defect numbers (**??**) of T are finite.

In order to solve the two concrete minimal normalization problems for mixed BVPs, we now start with normalizing the assigned Wiener-Hopf operators (**??**) in the scalar case $m = 1$ corresponding with a second order PDE in (**??**). So we study (dropping the dependence on x^k) operators of the form

$$W_0 = r_+ F^{-1} \Phi_0 \cdot F : L^2_+ \to L^2(\mathbb{R}_+) \tag{6.7}$$

where

$$\Phi_0 \in \mathcal{G}C^\mu(\ddot{\mathbb{R}}), \qquad \mu \in]0, 1[. \tag{6.8}$$

It is convenient to study simultaneously the restrictions for $s > 0$ and continuous extensions for $s < 0$ of W_0, briefly denoted by

$$W_s = r_+ F^{-1} \Phi_0 \cdot F : H^s_+ \to H^s(\mathbb{R}_+). \tag{6.9}$$

LEMMA 6.4 ([**?**, **?**]). *Putting "the complex winding number"*

$$w = w(\Phi_0) = \frac{1}{2\pi i} \int_{\mathbb{R}} d \log \Phi_0 = \sigma + i\tau \tag{6.10}$$

$$\sigma = \frac{1}{2\pi} \int_{\mathbb{R}} d \arg \Phi_0, \qquad \tau = \frac{1}{2\pi} \log \left| \frac{\Phi_0(-\infty)}{\Phi_0(+\infty)} \right|$$

we can write any $\Phi_0 \in \mathcal{G}C^\mu(\ddot{\mathbb{R}})$ *in the form*

$$\Phi_0 = \left(\frac{\lambda_-}{\lambda_+}\right)^w \Psi \tag{6.11}$$

where

$$\begin{aligned} &\Psi \in \mathcal{G}C^\mu(\dot{\mathbb{R}}) \\ &\operatorname{ind}\Psi = \frac{1}{2\pi}\int_{\mathbb{R}} d\arg\Psi = 0 \\ &\Psi(+\infty) = \Psi(-\infty) = \Phi(+\infty). \end{aligned} \tag{6.12}$$

Let us recall that the spaces of Bessel potentials [**?**] were introduced by [**?**] and [**?**] as convolutions of L^p-integrable functions with modified Bessel functions of the third kind for $s > 0$. With the help of the Fourier transformation the spaces H^s can be written, more rigorously, as images of the so-called [**?**, **?**, **?**] *Bessel potential operators*

$$\Lambda^{-s} = F^{-1}\lambda^{-s} \cdot F : L^2 \to H^s, \qquad s \in \mathbb{R} \tag{6.13}$$

where $\lambda(\xi) = \left(\xi^2 + 1\right)^{1/2}$ and, moreover,

$$\Lambda^w_\pm = F^{-1}\lambda^w_\pm \cdot F : H^s \to H^{s-\Re e w} \tag{6.14}$$

where $\lambda_\pm(\xi) = \xi \pm i$, $s \in \mathbb{R}$, $w \in \mathbb{C}$ and

$$\begin{array}{lcll} \operatorname{Rst}\Lambda^w_+ & : & H^s & \to \ H^{s-\Re e w} \\ r_+\Lambda^w_-\ell^{(s)} & : & H^s(\mathbb{R}_+) & \to \ H^{s-\Re e w}(\mathbb{R}_+) \end{array} \tag{6.15}$$

are isomorphisms, $\ell^{(s)}\varphi$ denoting any extension of $\varphi \in H^s(\mathbb{R}_+)$ to $\ell^{(s)}\varphi \in H^s$, see [**?**]. So it is natural to consider the following modified spaces as images of combined Bessel potential operators which are not normally solvable in the above-mentioned spaces $H^s_\pm$ and $H^s(\mathbb{R}_\pm)$, cf. [**?**, Definition 2.1].

DEFINITION 6.5. For every $w \in \mathbb{C}$, let

$$\begin{aligned} \overset{<}{H}{}^w(\mathbb{R}_+) &= r_+\Lambda_-^{-w-1/2}H_+^{-1/2} = r_+\Lambda_-^{-w-1/2}\Lambda_+^{1/2}L^2_+ \\ &\subset H^{\Re e w}(\mathbb{R}_+) \end{aligned} \tag{6.16}$$

equipped with the norm induced by $H_+^{-1/2}$, namely with

$$\|\psi\|_{\overset{<}{H}{}^w(\mathbb{R}_+)} = \|\ell^0\varphi\|_{H^{-1/2}} \tag{6.17}$$

where $\varphi = r_+\Lambda_-^{w+1/2}\ell^{(\Re e w)}\psi$ and $\ell^{(\Re e w)}\psi$ denotes any extension from $\mathbb{R}_+$ to $\mathbb{R}$ within the space $H^{\Re e w}$ (note that the extension of φ by zero within the space $H^{-1/2}$ is possible due to the given form of φ). Further, we define for $w \in \mathbb{C}$,

$$\overset{>}{H}{}^w_+ = \operatorname{clos}\left\{\psi \in H_+^{\Re e w} : \|\psi\|_{\overset{>}{H}{}^w_+} = \|r_+\Lambda_+^{w-1/2}\psi\|_{H^{1/2}(\mathbb{R}_+)}\right\}. \tag{6.18}$$

It is known [**?**, Corollary 2.2, Corollary 2.4] **(i)** that the embeddings

$$\overset{<}{H}^{w}(\mathbb{R}_+) \subset H^{\Re e w}(\mathbb{R}_+), \qquad \overset{>}{H}{}^{w}_{+} \supset H^{\Re e w}_{+} \tag{6.19}$$

are proper, dense and continuous for any $w \in \mathbb{C}$, **(ii)** that

$$\overset{<}{H}^{-k-1/2}(\mathbb{R}_+) = r_+ H_+^{-k-1/2}, \qquad r_+ \overset{>}{H}{}^{k+1/2}_{+} = H_0^{k+1/2}(\mathbb{R}_+) \tag{6.20}$$

for $k \in \mathbb{N}_0$ (where the last space is the closure of $\mathcal{D}(\mathbb{R}_+) = C_0^\infty(\mathbb{R}_+)$ in $H^{k+1/2}(\mathbb{R}_+)$), and **(iii)** that the following operators are homeomorphisms for any $w_1, w_2 \in \mathbb{C}$:

$$\begin{aligned} \mathrm{Rst}\, r_+ \Lambda_-^{w_1-w_2}\, \ell^{(\Re e w_1)} &: \overset{<}{H}^{w_1}(\mathbb{R}_+) \to \overset{<}{H}^{w_2}(\mathbb{R}_+) \\ \mathrm{Ext}\, \Lambda_+^{w_1-w_2}{}_{|H_+^{\Re e w_1}} &: \overset{>}{H}{}^{w_1}_{+} \to \overset{>}{H}{}^{w_2}_{+} . \end{aligned} \tag{6.21}$$

Finally, we need the basic result about normalization from [**?**, Theorem 2.5]:

THEOREM 6.6. *Let $\Phi_0 \in \mathcal{G}C^\mu(\ddot{\mathbb{R}})$, $\mu \in]0,1[$ and $w = \sigma + i\tau$ be defined by (**??**). Then W_s given by (**??**) is not normally solvable if and only if*

$$\kappa = s + \sigma + \frac{1}{2} \in \mathbb{Z}. \tag{6.22}$$

In this case

$$\left(H_+^s,\, \overset{<}{H}^{s-i\tau}(\mathbb{R}_+)\right), \left(\overset{>}{H}^{s-i\tau}(\mathbb{R}_+),\, H^s(\mathbb{R}_+)\right) \in \mathcal{N}(W_s)\,. \tag{6.23}$$

Moreover, each of the normalized operators is one-sided invertible with index

$$\mathrm{Ind}\, \overset{<}{W}_s = -\kappa, \qquad \mathrm{Ind}\, \overset{>}{W}_s = -\kappa + 1 \tag{6.24}$$

respectively, and one-sided inverses are explicitly obtained by factorization of the Fourier symbols of $W_{s\pm\epsilon}$, $\epsilon \in]0,1[$, and extension or restriction of the corresponding one-sided inverses of $W_{s\pm\epsilon}$, respectively (see [**?**] for details).

Now, the basic idea is, to normalize the original mixed BVP "consequently" provided it is piecewise elliptic and not Fredholm.

REMARK 6.7. In the system's case, there are corresponding solutions of the minimal normalization problem based upon a representation of the (multiplicative) jump of $\Phi_0 \in \mathcal{G}C^\mu(\ddot{\mathbb{R}})^{m\times m}$ at infinity in normal Jordan form

$$\Phi_0^{-1}(+\infty)\Phi_0(-\infty) = M^{-1} J_{\Phi_0} M \tag{6.25}$$

where $M \in \mathcal{G}\,\mathbb{C}^{m\times m}$ and the quasidiagonal matrix

$$J_{\Phi_0} = \mathrm{diag}\left(J_1, \ldots, J_{\widetilde{m}}\right) \tag{6.26}$$

has Jordan blocks of size $l_j \times l_j$ in the diagonal given by

$$J_j = \begin{pmatrix} \mu_j & 1 & 0 & \cdots & 0 \\ 0 & \mu_j & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \mu_j \end{pmatrix} \tag{6.27}$$

which yields

$$\Phi_0(\xi) = \Phi_0(-\infty) M^{-1} \left(\operatorname{diag} \left(\left(\frac{\lambda_-(\xi)}{\lambda_+(\xi)} \right)^{\widetilde{\omega_j}} \right) J_{\Phi_0} + \Psi_0(\xi) \right) M \tag{6.28}$$

where the elements of $\Psi_0(\xi)$ are decreasing of order $|\xi|^{-\mu}$ at infinity, see [**?**, Section 6].

For the sake of shortness we shall not work out the system's case here in detail but like to point out that the corresponding results for the scalar (or diagonal) case can be obtained by analogy including explicit factorization, which is a different and subtle problem in general, already in the case $m = 2$, see [**?**].

REMARK 6.8. As shown in [**?**] there is a direct connection between an image normalization of an elliptic Wiener-Hopf equation in L_p space with a piecewise-continuous symbol (having a jump at infinity) which is not Fredholm (the Widom-Gohberg-Krupnik arc crosses the origin) and the corresponding image normalization of a related Wiener-Hopf equation with continuous symbol which has a zero of order 1 at a finite point. The normalized space is defined in both cases by integral operators of Cesaro type (for details see [**?**, §3.1–3.3]). These results can be modified for the present setting of Bessel potential spaces in order to obtain useful characterizations of the normalized spaces.

7. Minimal normalization in the scalar case

Here we consider the case $m = 1$, which is easily extended to the case $m > 1$ when we have diagonal (or triangular, at least) Wiener-Hopf operators such that we can normalize component-wise.

Let us focus on the image normalization of a mixed BVP (**??**) that is piecewise elliptic but not Fredholm, and work it out, step by step, starting with the assigned Wiener-Hopf operators $W_{x^k,0}$. At least one of them violates the Fredholm condition (**??**), say, only if $k = 1$, for simplicity.

As seen before, its symbol

$$\Phi_{x^1,0} \in \mathcal{GR}\left(\xi, \sqrt{\frac{\xi - i}{\xi + i}} \right), \tag{7.1}$$

because $\Phi_{x^1} \in \mathcal{GR}\left(\xi, \sqrt{\xi^2+1}\right)$ and W_{x^1} acts between spaces of order $r^1 = 2m+l-m^1-1/2$ and $s^1 = 2m+l-m^2-1/2$, i.e. of order $l \pm 1/2$ (since $m=1$, $m^k \in \{0,1\}$). Thus, $\Phi_{x^1,0} \in \mathcal{G}\, C^\mu(\overset{\bullet\bullet}{\mathbb{R}})$ and we can apply Theorem **??** with $s=0$ and $\sigma = 1/2 \mod \mathbb{Z}$. Therefore $W_{x^1,0} : L^2_+ \to L^2(\mathbb{R}_+)$ is normalized by

$$\overset{<}{W}_{x^1,0} \ : \ L^2_+ \to \overset{<}{H}{}^{-i\tau}(\mathbb{R}_+) \tag{7.2}$$

where τ results from the modulus jump of $\Phi_{x^1,0}$ at infinity (the integer part of the winding number does not matter and the fractional real part is guilty for non-Fredholmness), see (**??**) and (**??**).

According to Corollary **??** and property (**??**) of the Bessel potential operators, the unlifted operator $W_{x^1} : H^{r^1}_+ \to H^{s^1}(\mathbb{R}_+)$ is normalized by

$$\overset{<}{W}_{x^1} = r_+\Lambda_-^{-s^1}\,\ell\,\overset{<}{W}_{x^1,0}\,\Lambda_+^{r^1} : H^{r^1}_+ \to r_+\Lambda_-^{-s^1+i\tau-1/2}\Lambda_+^{1/2}L^2_+ = \overset{<}{H}{}^{w}\,(\mathbb{R}_+) \tag{7.3}$$

where $w = s^1 - i\tau$.

Now we modify the boundary operator B^2 into

$$\overset{<}{B}{}^2 \colon H^{2m+l}(\Omega) \to \overset{<}{H}{}^{s^1}\left(\Gamma^2, x^1\right) \tag{7.4}$$

imposing the local behavior of $\overset{<}{H}{}^{w}(\mathbb{R}_+)$ near zero by transformation in the image of B^2 near x^1; more precisely

$$\overset{<}{H}{}^{s^1}\left(\Gamma^2, x^1\right) = \left\{\varphi \in H^{s^1}\left(\Gamma^2\right) : \ell^0\left((\omega\varphi)\circ\varkappa_{x^1}^{-1}\right) \in \overset{<}{H}{}^{w}(\mathbb{R}_+)\right\} \tag{7.5}$$

where $\omega \in C^\infty\left(\Gamma^2\right)$, $0 \le \omega \le 1$, $\omega = 1$ in $\mathcal{B}_\epsilon(x^1)\cap\Gamma^2$, $\omega = 0$ in $\mathcal{B}_\epsilon(x^2)\cap\Gamma^2$, $\varkappa_{x^1}$ is the transformation from (**??**) and ℓ^0 denotes zero extension to $\mathbb{R}_+$ which represents a continuous operator according to the smooth cut-off by ω. The induced norm is given by

$$\|\varphi\|_{\overset{<}{H}{}^{s^1}(\Gamma^2,x^1)} = \left\|\ell^0\left((\omega\varphi)\circ\varkappa_{x^1}^{-1}\right)\right\|_{\overset{<}{H}{}^{w}(\mathbb{R}_+)} + \left\|\ell^0\left(((1-\omega)\varphi)\circ\varkappa_{x^1}^{-1}\right)\right\|_{H^{s^1}(\mathbb{R}_+)}. \tag{7.6}$$

Finally, the image of L has to be adapted, in order to end up with a normalized operator due to (**??**). Starting again with localization of

$$\overset{<}{L} = \begin{pmatrix} A \\ B^1 \\ \overset{<}{B}{}^2 \end{pmatrix} : H^{2m+l}(\Omega) \to \begin{cases} H^l(\Omega) \\ \times H^{r^1}\left(\Gamma^1\right) \\ \times \overset{<}{H}{}^{s^1}\left(\Gamma^2, x^1\right) \end{cases} \tag{7.7}$$

we obtain similar results due to local behavior, continuous embedding and compactness criteria, ending up with the fact that $\overset{<}{L}$ is Fredholm in the described situation. The modification for the case where $W_{x^2,0}$ violates (**??**) is obvious.

Domain normalization runs analogously. Various details follow directly from the fact that the normalized spaces are images of Bessel potential operators, e.g. the local type property [**?**, Prop. 1.3] of the normalized operators and compactness of lower order terms etc. So we proved the following result (up to analogous conclusions).

THEOREM 7.1. *Consider the mixed BVP (**??**) that is piecewise elliptic, see (**??**), (**??**), suppose $m = 1$ (i.e. the PDE is of second order) and $l \geq 0$. Let $W_{x^k,0}(l)$ denote the two assigned (scalar) Wiener-Hopf operators from (**??**) and $w^k(l) = \sigma^k(l) + i\tau^k(l)$ the numbers defined in (**??**) due to the Fourier symbols $\Phi_0^k(l)$ of $W_{x^k,0}(l)$. Then the BVP (**??**), i.e. the associated operator L in (**??**), is Fredholm for all $l \geq 0$ up to a set*

$$S_{NF}(\mathbb{R}_+) = \big(l^1 + \mathbb{N}_1\big) \cup \big(l^2 + \mathbb{N}_1\big) \tag{7.8}$$

where $l^k \in [0, 1[$. More precisely

$$\begin{aligned} w^k(l) &= w^k(0) + l \\ l^k &= -\big(\sigma^k(0) + \tfrac{1}{2}\big) \qquad \operatorname{mod} \mathbb{Z} \end{aligned} \tag{7.9}$$

*for $k = 1, 2$. For the critical orders $l \in S_{NF}(\mathbb{R}_+)$ (where the BVP is not Fredholm), the image of L is not closed, but the defect numbers are finite, and L can be normalized as follows: If $l^1 \neq l^2$ and $l \in \big(l^1 + \mathbb{N}_1\big)$, the image normalized operator (**??**) of L is Fredholm, and, for $l \in l^2 + \mathbb{N}_1$, an obvious normalization of B^1 helps. If $l^1 = l^2$ and $l \in S_{NF}(\mathbb{R}_+)$ the image normalization is given by simultaneous image normalization of B^1 and B^2. Completely analogous results hold for the domain normalization (based upon Theorem **??**).*

EXAMPLE 7.2. The full scale of normalized spaces $\overset{<}{H}{}^w(\mathbb{R}_+)$ (scalar case) appears already for the following class of mixed canonical problems given in Figure 7.1, where a, b and c are complex constants.

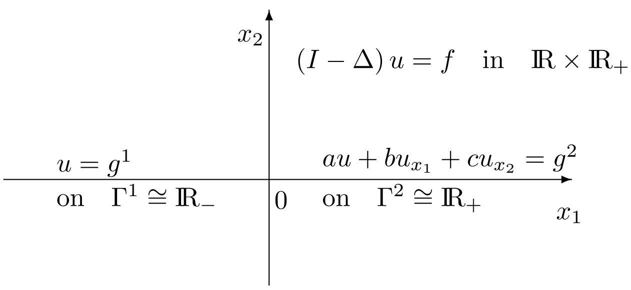

Figure 7.1: Mixed BVP with an oblique derivative boundary condition.

Consider only the semi-homogeneous problem due to $f = 0$, $g^1 = 0$ (cf. Section **??**). The representation formula for $u \in \ker(I - \Delta)$ in the upper half-plane

$$u(x) = F^{-1}_{\xi \mapsto x_1} \widehat{u_0}(\xi) \exp[-x_2 t(\xi)] \tag{7.10}$$

gives us the Wiener-Hopf operator (see Section **??**)

$$W = r_+ F^{-1} \, (a - ib\xi - ct) \cdot F : H_+^{2+l-1/2} \to H^{2+l-3/2}(\mathbb{R}_+) \qquad (7.11)$$

and the lifted operator $W_0 : L_+^2 \to L^2(\mathbb{R}_+)$ with symbol

$$\Phi_0 = \left(\frac{\xi - i}{\xi + i}\right)^{l+1/2} \frac{a - ib\xi - ct}{\xi + i} \qquad (7.12)$$

and the condition for (piecewise) ellipticity:

$$a - ib\xi - ct(\xi) \neq 0, \qquad \xi \in \mathbb{R}. \qquad (7.13)$$

We compute the complex winding number

$$w = \frac{1}{2\pi i} \int_{\mathbb{R}} d \log \Phi_0 = \sigma + i\tau \qquad (7.14)$$

and find a non-Fredholm operator for $\sigma = \frac{1}{2} \mod \mathbb{Z}$, i.e., thinking of $\Phi_0(+\infty) = -c - ib$, $\Phi_0(-\infty) = (c - ib) \exp[-2\pi i(l + \frac{1}{2})]$, for

$$l + \arg \frac{-c - ib}{c - ib} \in \mathbb{Z}. \qquad (7.15)$$

This condition has plenty of realizations, e.g.:
Case 1. Let $l = 0$ and $b, c \in \mathbb{R}$, then $c = 0$ suffices, i.e. the principle part of the oblique derivative is tangential (and not really oblique);
Case 2. Let $l = 0$, $b, c \in \mathbb{C}$ and $(-c - ib)/(c - ib) = p > 0$; this is satisfied for $c = ib(p+1)/(p-1)$, i.e. an imaginary ratio of the coefficients b and c; and so on for other $l \notin \mathbb{N}_0$.

8. Concluding remarks

The case $m > 1$ can be treated with the idea of Remark **??** and yields, in general, rather complicated formulas. Then, the compatibility conditions for the data that result from image normalization, cf. (**??**), are not necessarily "local". For instance, in the canonical problem treated in Section **??**, thinking of the Bi-Laplacian for A and higher order boundary operators, one can meet conditions which combine data after application of convolution type operators on the boundary.

Various generalizations are possible: for three - or n-dimensional configurations, systems of PDE (as resulting from Maxwell's and Lamé's equations), pseudodifferential equations, weak formulations ($l < 0$), less smooth boundaries and other spaces of Besov-Triebel-Lizorkin type [**?**, **?**].

Beside of the interpretation of image normalization in terms of compatibility conditions, one can understand the domain normalization in some cases as imposing a transmission property [?]. A simple case is an operator in (??) of the form

$$W = r_+ B : H_+^{l+1/2} \to H^{l+1/2}(\mathbb{R}_+), \qquad l \in \mathbb{N}_0 \tag{8.1}$$

where $B = I + c \cdot \Lambda_+^{-\epsilon}$ with $\epsilon > 0$ which maps $H_+^{l+1/2}$ onto $r_+ H_+^{l+1/2} \subset H^{l+1/2}(\mathbb{R}_+)$ and is not Fredholm.

In the case $l = 0$ the embedding $r_+ H_+^{1/2} \subset H^{1/2}(\mathbb{R}_+)$ is proper and dense [?]. Thus $\overset{>}{W}$ can be identified with (is equivalent to)

$$\overset{>}{W} \ell^0 = \operatorname{Ext} W \ell^0 : H^{1/2}(\mathbb{R}_+) \to H^{1/2}(\mathbb{R}_+) \tag{8.2}$$

which obviously has the transmission property [?].

If $l = 1, 2, 3, \ldots$, the closure of $r_+ H_+^{l+1/2}$ in $H^{l+1/2}(\mathbb{R}_+)$ has codimension l [?], thus the operator

$$\widetilde{W} = r_+ B \ell^{l+1/2} : H^{l+1/2}(\mathbb{R}_+) \to H^{l+1/2}(\mathbb{R}_+) \tag{8.3}$$

with arbitrary extension $\ell^{l+1/2}$ into $H^{l+1/2}$, which has the transmission property, coincides with $\overset{>}{W} \ell$ (defined on that closure) up to an operator of characteristic l, i.e. is equivalent after extension by finite dimensional operators.

Further interesting studies may be based on the characterization of the normalizing conditions (compatibility or transmission property, respectively) in terms of integral conditions and resulting conclusions for the asymptotic behavior of solutions. For instance, due to (??), the elements $\psi \in \overset{>}{H}{}_+^{l+1/2}$, $l \in \mathbb{N}_0$, are characterized by

$$\left\| r_+ \Lambda_+^l \psi \right\|_{H^{1/2}(\mathbb{R}_+)} < \infty \tag{8.4}$$

which is equivalent to $\psi \in \ell^0 H^{1/2}(\mathbb{R}_+)$ with

$$\begin{aligned} &D^j \psi(0) = 0, && j = 0, \ldots, l-1 \\ &\int_0^\infty \tfrac{1}{x} \left| D^l \psi(x) \right|^2 dx < \infty, && \end{aligned} \tag{8.5}$$

see [?, p. 11].

Some of these questions will be treated in a forthcoming paper.

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