Singular Integral Equations on Piecewise Smooth Curves in Spaces of Smooth Functions

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To Bernd Silbermann on the occasion of his sixtieth birthday

Abstract. We prove the boundedness of the Cauchy singular integral operator in modified weighted Sobolev $\mathbb{KW}_p^m(\Gamma,\rho)$, Hölder-Zygmund $\mathbb{KZ}_\mu^0(\Gamma,\rho)$, Bessel potential $\mathbb{KH}_p^s(\Gamma,\rho)$ and Besov $\mathbb{KB}_{p,q}^s(\Gamma,\rho)$ spaces under the assumption that the smoothness parameters m,μ,s are large. The underlying contour Γ is piecewise smooth with angular points and even with cusps. We obtain Fredholm criteria and an index formula for singular integral equations with piecewise smooth coefficients and complex conjugation in these spaces provided the underlying contour has angular points but no cusps. The Fredholm property and the index turn out to be independent of the integer parts of the smoothness parameters m,μ,s. The results are applied to an oblique derivative problem (the Poincaré problem) in plane domains with angular points and peaks on the boundary.

Introduction

When considering a Cauchy singular integral equation with complex conjugation

$$A\varphi(t) \equiv a(t)\varphi(t) + \frac{b(t)}{\pi i}\int_\Gamma \frac{\varphi(\tau)d\tau}{\tau - t} + \frac{c(t)}{\pi i}\int_\Gamma \frac{\varphi(\tau)\overline{d\tau}}{\overline{\tau} - \overline{t}} = f(t)\,, \quad t\in\Gamma \tag{0.1}$$

on a piecewise smooth contour Γ (see § 1 below) we are restricted in the choice of the spaces where we can solve equation (0.1). Namely, the operator A in equation (0.1) is not bounded in important spaces of smooth functions: in the usual weighted Sobolev $\mathbb{W}_p^m(\Gamma,\rho)$, Hölder-Zygmund $\mathbb{Z}_\mu(\Gamma,\rho)$, Bessel potential $\mathbb{H}_p^s(\Gamma,\rho)$ and Besov $\mathbb{B}_{p,q}^s(\Gamma,\rho)$ spaces for large values of the smoothness parameters $m = 2,3,\ldots$, $\mu > 1$ and $|s| > 1+1/p$. These spaces cannot be even defined properly (i.e., independently of the choice of a parametrization) if Γ has knots, such as angular points or cusps.

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Even if Γ is sufficiently smooth and the spaces $\mathbb{W}^m_p(\Gamma,\rho)$, $\mathbb{Z}_\mu(\Gamma,\rho)$ etc. can be defined properly, the problem arises again when we take piecewise smooth coefficients $a(t)$, $b(t)$, $c(t)$ with jumps at the knots (for conciseness we relate discontinuity points to the knots of Γ as well).

On the other hand, especially in applications and numerical analysis, it is important to establish additional smoothness properties for the solutions at least outside the knots when the right-hand side f is sufficiently smooth.

We suggest the introduction of weighted spaces $\mathbb{KW}^m_p(\Gamma,\rho)$, $\mathbb{KZ}_\mu(\Gamma,\rho)$, $\mathbb{KH}^s_p(\Gamma,\rho)$, and $\mathbb{KB}^s_{p,q}(\Gamma,\rho)$ with the help of "Fuchs"-derivatives

$$\vartheta(t)\partial_t\varphi(t) := \vartheta(t)\frac{\partial\varphi(t)}{\partial t}\,, \quad \text{where} \quad \vartheta(t) := \prod_{t_j\in\mathcal{T}_\Gamma}(t-t_j) \tag{0.2}$$

and $\mathcal{T}_\Gamma$ is the collection of knots of Γ, instead of the usual derivatives $\partial_t\varphi(t)$ (see Lemmata 1.2, 1.3, and 2.4). It turns out that the operator A in (0.1) with piecewise smooth coefficients $a, b, c \in \mathbb{PC}^{\mathsf{m}}(\Gamma,\mathcal{T}_\Gamma)$ (and even with $a, b, c \in \mathbb{KPC}^{\mathsf{m}}(\Gamma,\mathcal{T}_\Gamma)$; see § 1 for the definitions) is bounded in the modified spaces $\mathbb{KW}^m_p(\Gamma,\rho)$, $\mathbb{KZ}_\mu(\Gamma,\rho)$, $\mathbb{KH}^s_p(\Gamma,\rho)$, and $\mathbb{KB}^s_{p,q}(\Gamma,\rho)$ provided the smoothness parameters m, μ and s are sufficiently large (see Lemmata 1.2, 1.3, 2.4 and Theorem 3.1). Moreover, the operator defined by (0.1) has one and the same kernel and cokernel in the spaces $\mathbb{KW}^{\widetilde{m}}_p(\Gamma,\rho)$, $\mathbb{KZ}_{\widetilde{\mu}}(\Gamma,\rho)$ and $\mathbb{KH}^{\widetilde{s}}_p(\Gamma,\rho)$, $\mathbb{KB}^{\widetilde{s}}_{p,q}(\Gamma,\rho)$ whatever the integer parts of the smoothness parameters $\widetilde{m} = 0,\dots,m$, $0 < \widetilde{\mu} \le \mu$, and $|\widetilde{s}| \le s$ are (see Theorem 3.2 and Remarks 3.4, 3.5).

The results on the Fredholm properties and also those on the boundedness of the operator A in the usual (non-modified) weighted Bessel potential and Besov spaces $\mathbb{H}^s_p(\Gamma,\rho)$ and $\mathbb{B}^s_{p,q}(\Gamma,\rho)$ for small s, $1/p-1 < s < 1/p$, when the multiplication by piecewise continuous functions represents a bounded operator (see Theorem 2.3), are new.

Although the space $\mathbb{KW}^m_p(\Gamma,\rho)$ coincides with $\mathbb{KH}^m_p(\Gamma,\rho)$ for any nonnegative integer m, we formulate the results for the modified Sobolev space $\mathbb{KW}^m_p(\Gamma,\rho)$ because these spaces are more common in applications and the proofs are simpler.

It is well-known that the Bessel potential spaces are as natural in the theory of pseudodifferential operators as the Sobolev spaces are in the theory of partial differential operators. The norm in $\mathbb{H}^s_p(\mathbb{R}^2)$ is especially simple for even $s = 2m, m = 0, 1, 2, \dots$:

$$\|f\,|\,\mathbb{H}^{2m}_p(\mathbb{R}^2)\| = \|(I-\Delta)^m f\big|\mathbb{L}_p(\mathbb{R}^2)\|, \qquad \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

But in the theory of boundary value problems we cannot confine ourselves to the Bessel potential spaces since the traces of functions $\Phi \in \mathbb{H}^s_p(\Omega^\pm)$ on the boundary belong to the Besov spaces $\mathbb{B}^{s-\frac{1}{p}}_{p,p}(\Gamma)$, provided the boundary Γ is sufficiently smooth and $s > 1/p$.

The Besov spaces can be considered as the integral analogue of the Zygmund spaces.

In favor of the Hölder-Zygmund spaces we remark that many important operators, including singular integral operators, are unbounded in the spaces $C^m(\Gamma)$ and in the Hölder spaces $H_m(\Gamma)$ but that they are bounded in $\mathbb{Z}_m(\mathbb{R}^n)$ for every integer $m \in \mathbb{N} := \{1, 2, \dots\}$. The Hölder-Zygmund spaces are the natural extensions of the scale of Hölder spaces to integer values of the smoothness exponent and have an important interpolation property (see § 2).

In § 4 we apply the obtained results to the oblique derivative problem for the Laplacian in domains with piecewise smooth boundary.

The space $\mathcal{L}^1_{p,\beta}(\Gamma) := \mathbb{KW}^1_p(\Gamma, \mathcal{T}_\Gamma, |t - t_1|^{\beta-1})$ (i.e., the particular case where $m = 1$, $\mathcal{T}_\Gamma = \{t_1\}$ and $\rho(t) := |t - t_1|^{\beta-1}$) was applied in [MS1] to the investigation of boundary integral equations. The anisotropic Bessel potential spaces $\mathbb{H}^{(s,\nu),m}_p(\mathcal{M})$, similar to $\mathbb{KW}^m_p(\Gamma)$, were introduced in [CD1] for the multi-dimensional case in which $\mathcal{M} = \mathbb{R}^n_+$ or $\mathcal{M}$ is a manifold with smooth boundary. In [CD1] the boundedness of a certain class of pseudodifferential operators was proved and a Fredholm criterion for them was established. The spaces $L^{p,m}(\mathbb{R}^+)$ and $X^{p,m}_\rho(\mathbb{R}^+)$, also similar to $\mathbb{KW}^m_p(\mathbb{R}^+, \{0\})$, were used by J. Elschner in a spline approximation method for convolution equations (see [El1] and [PS1, Ch. 5]).

Some results of §§ 1-2 were already announced in [DS2].

1. Weighted Sobolev and Hölder-Zygmund spaces

Let Γ be a piecewise smooth curve that consists of a finite union of smooth arcs which have in common at most endpoints, called knots:

$$\Gamma = \bigcup_{j=1}^{\ell} \Gamma_j\,, \quad \Gamma_j := [t_j, t_{j+1}] = \overset{\smile}{t_j t_{j+1}}, \quad j = 1, \dots, n, \quad t_{n+1} = t_1.$$

Let $\mathcal{T}_\Gamma := \{t_j\}$ denote the collection of all different knots (i.e., all different endpoints of smooth arcs) of Γ. The curve may contain cusps, i.e., the angles between some arcs are allowed to be 0.

The closed arcs Γ_j, $j = 1, \dots, n$, between the knots are sufficiently smooth: any parametrizations

$$\omega_j \ : \ \mathcal{I} := [0, 1] \longrightarrow \Gamma_j\,, \qquad \omega_j(0) = t_j\,, \quad \omega_j(1) = t_{j+1} \tag{1.1}$$

are μ-smooth, $\omega_j \in \mathbb{Z}_\mu(\mathcal{I})$, $\mu \geq 1$, $j = 1, \dots, n$, where $\mathbb{Z}_\mu(\mathcal{I})$ denotes the Hölder-Zygmund space (see below).

Let us suppose, until formula (1.7), that Γ is either a single smooth arc or a single smooth closed contour and that the natural parametrization of Γ with the help of the arc length parameter $0 < s \leq \ell$,

$$t \ : \ [0, \ell] \longrightarrow \Gamma\,, \qquad s \longmapsto t(s)\,, \quad t(0) = t(\ell), \tag{1.2}$$

is μ-smooth, that is, $t(\cdot) \in \mathbb{Z}_\mu([0, \ell])$.

For $0 < \nu \le \mu \le 1$, the Hölder-Zygmund space $\mathbb{Z}_\nu(\Gamma)$ is defined as the space of functions with finite norm

$$\|\psi \mid \mathbb{Z}_\nu(\Gamma)\| := \sup_{t\in\Gamma} |\psi(t)| + \sup_{\substack{0<s\le\ell \\ h>0}} \frac{|\Delta_h^2 \psi(t(s))|}{h^\nu}\,, \tag{1.3}$$

$$\Delta_h \varphi(s) := \varphi(s+h) - \varphi(s)\,, \qquad \Delta_h^2 \varphi(s) := \varphi(s+h) - 2\varphi(s) + \varphi(s-h)\,.$$

Note that for $\nu = 1$ the definition of $\mathbb{Z}_1(\Gamma)$ requires that Γ be smooth, $t(\cdot) \in C^1([0,\ell])$, and that Γ have no angular points: each knot $t_j \in \mathcal{T}_\Gamma$ is an endpoint of a separate arc.

If Γ is μ-smooth and

$$\mu = m + \nu \ge 1\,, \quad 0 < \nu \le 1\,, \quad m \in \mathbb{N}_0 := \{0, 1, \ldots\}\,, \tag{1.4}$$

then angular points on Γ are absent and the Hölder-Zygmund space $\mathbb{Z}_\mu(\Gamma)$ is defined as a collection of all functions $\psi(t)$ which have finite norm

$$\|\psi \mid \mathbb{Z}_\mu(\Gamma)\| := \sum_{k=0}^{m} \sup_{t\in\Gamma} |\psi^{(k)}(t)| + \|\psi^{(m)} \big| \mathbb{Z}_\nu(\Gamma)\|\,, \tag{1.5}$$

$$\varphi^{(k)}(x) := \partial_x^k \varphi(x) = \frac{\partial^k \varphi(x)}{\partial x^k}\,.$$

The Hölder space $H_\mu(\Gamma)$ is defined as a collection of all functions $\psi(t)$ which have finite norm

$$\|\psi \mid H_\mu(\Gamma)\| := \sum_{k=0}^{m} \sup_{t\in\Gamma} |\psi^{(k)}(t)| + \sup_{\substack{0<s\le\ell \\ h>0}} \frac{|\Delta_h \psi^{(m)}(t(s))|}{h^\nu}\,. \tag{1.6}$$

If $\mu = m + \nu \notin \mathbb{N} = \{1, 2, \ldots\}$ is not an integer, $0 < \nu < 1$, the norms in (1.5) and in (1.6) are equivalent and the spaces coincide: $\mathbb{Z}_\mu(\Gamma) = H_\mu(\Gamma)$ for $\mu \notin \mathbb{N}$ (see [St1]). Note that for an integer $m = 1, 2, \ldots$ the spaces $H_m(\Gamma)$ and $\mathbb{Z}_m(\Gamma)$ differ essentially from each other and from the space $C^m(\Gamma)$ of smooth functions with the natural norm

$$\|\psi \mid C^m(\Gamma)\| := \sum_{k=0}^{m} \sup_{t\in\Gamma} |\psi^{(k)}(t)|$$

(see [St1]) and that we have the following proper embedding instead:

$$C^m(\Gamma) \subset H_m(\Gamma) \subset \mathbb{Z}_m(\Gamma)\,. \tag{1.7}$$

For an arbitrary piecewise smooth curve Γ, we denote by $\mathbb{PX}(\Gamma, \mathcal{T}_\Gamma)$ the space of piecewise smooth functions with jump discontinuities at the knots $t_j \in \mathcal{T}_\Gamma$:

$$\mathbb{PX}(\Gamma, \mathcal{T}_\Gamma) := \{g \in \mathbb{X}(\Gamma_j) \ : \ j = 1, \ldots, n\}\,,$$

where $\mathbb{X}(\Gamma_j)$ denotes one of the spaces $C^m(\Gamma_j)$, $H_\mu(\Gamma_j)$, or $\mathbb{Z}_\mu(\Gamma_j)$, $j = 1, ..., n$. For $m = 0$ we use the notation $\mathbb{PC}(\Gamma, \mathcal{T}_\Gamma)$ instead of $\mathbb{PC}^0(\Gamma, \mathcal{T}_\Gamma)$.

The Sobolev space $\mathbb{W}_p^m(\mathcal{I})$ on the unit interval is defined as

$$\mathbb{W}_p^m(\mathcal{I}) := \left\{\varphi \in \mathbb{L}_p(\mathcal{I}) \ : \ \partial^k\varphi \in \mathbb{L}_p(\mathcal{I}), \ \ k = 0, \dots, m\right\}$$

and is endowed with the norm

$$\|\varphi \mid \mathbb{W}_p^m(\mathcal{I})\| := \left(\sum_{k=0}^m \|\partial_x^k\varphi \mid \mathbb{L}_p(\mathcal{I})\|^p\right)^{\frac{1}{p}} = \left(\sum_{k=0}^m \int_0^1 |\partial_x^k\varphi(x)|^p dx\right)^{\frac{1}{p}} .$$

With the help of the parametrization (1.1) we define the Sobolev space $\mathbb{W}_p^m(\Gamma_j)$ on the smooth arcs Γ_j for $1 < p < \infty$ and $m \leq \mu$ as the space of all functions φ for which $\varphi(\omega_j(x))$ is in $\mathbb{W}_p^m(\mathcal{I})$. For the entire piecewise smooth curve Γ the space $\mathbb{W}_p^m(\Gamma)$ can be defined only for $m = 0, 1$. In fact, for any parametrization

$$\omega \ : \ \mathcal{R} := [0, R] \longrightarrow \Gamma \tag{1.8}$$

of the entire curve Γ (cf. (1.2)) the derivative $(\partial_t\varphi)(\omega(x)) = [\omega'(x)]^{-1}\partial_x\varphi(\omega(x))$ involves a piecewise continuous factor $[\omega']^{-1} \in \mathbb{PC}(\mathcal{R}, \mathcal{T}_\mathcal{R})$, where

$$\mathcal{T}_\mathcal{R} := \{x_j \in \mathcal{R} \ : \ \omega(x_j) = t_j \in \mathcal{T}_\Gamma\}$$

is the set of all "knots" of $\mathcal{R}$. Therefore, the second derivative $(\partial_t^2\varphi)(\omega(x))$ is not defined properly because the second derivative $\partial_x^2\omega(x)$ of the parametrization, which participates as a factor, may contain delta functions:

$$\partial_x^2\omega(x) = \omega_0^{(2)}(x) - \sum_{j=1}^n [\omega'(x_j+0) - \omega'(x_j-0)]\delta_j(x) \, , \tag{1.9}$$

$$\omega_0^{(2)} \in \mathbb{PC}^{m-2}(\Gamma, \mathcal{T}_\Gamma), \qquad \omega_0^{(2)}(x_j \pm 0) = \omega^{(2)}(x_j \pm 0) = (\partial_x^2\omega)(x_j \pm 0) \, ,$$
$$\langle\delta_j, \psi\rangle := \psi(x_j) \, , \qquad \psi \in \mathbb{C}(\mathcal{R}) \, , \qquad \forall\, \omega(x_j) = t_j \in \mathcal{T}_\Gamma \, .$$

To prove (1.9) we represent $\omega'(x)$ in the form

$$\omega'(x) = \omega_0'(x) - \sum_{j=1}^n [\omega'(x_j+0) - \omega'(x_j-0)]\chi_+(x - x_j) \, , \tag{1.10}$$

where $\omega_0'(x)$ is continuous $\omega_0' \in \mathbb{C}(\mathcal{R}) \cap \mathbb{PC}^{m-1}(\mathcal{R}, \mathcal{T}_\mathcal{R})$, $\partial\omega_0'(x_j \pm 0) = \omega'(x_j \pm 0)$ and $\chi_+(x)$ is the Heaviside function: $\chi_+(x) = 0$ for $x < 0$, $\chi_+(x) = 1$ for $x > 0$. It is easy to ascertain that the functions $\omega'(x)$ and $\omega_0'(x)$ differ by a piecewise constant function with jumps at x_j, $\omega(x_j) = t_j \in \mathcal{T}_\Gamma$ and, therefore, their derivatives coincide:

$$(\partial_x\omega')(x) = (\partial_x\omega_0')(x) = \omega^{(2)}(x) \quad \forall\, x \neq x_1, \dots, x_n$$

and even

$$(\partial_x\omega')(x_j \pm 0) = (\partial_x\omega_0')(x_j \pm 0) = \omega^{(2)}(x_j \pm 0) \quad \forall\, j = 1, \dots, n \, .$$

Since $\chi_+' = \delta$ in the sense of distributions, from (1.10) we derive (1.9).

By the same reason a multiplication operator

$$gI \ : \ \mathbb{W}_p^m(\Gamma) \longrightarrow \mathbb{W}_p^m(\Gamma) \, , \qquad g \in \mathbb{PC}^m(\Gamma, \mathcal{T}_\Gamma) \tag{1.11}$$

is bounded only for $m = 0$ (i.e., in the Lebesgue space $\mathbb{L}_p(\Gamma)$ only).

In order to treat finally singular integral equations for spaces of smooth functions in an efficient way, we need the boundedness (j) of differential operators, (jj) of multiplication operators (both were discussed before), and (jjj) of the Cauchy singular integral operator (see Theorem 3.1 below).

To guarantee all three listed space properties, we suggest to consider a special Sobolev space $\mathbb{KW}^m_p(\Gamma,\rho)$ with a power weight

$$\rho(t) := \prod_{j=1}^{n} (t-t_j)^{\alpha_j}\,, \qquad \alpha_j \in \mathbb{C}\,, \quad 1 < p < \infty \tag{1.12}$$

defined as follows:

$$\mathbb{KW}^m_p(\Gamma,\rho) := \left\{\varphi \in \mathbb{L}_p(\Gamma,\rho) \ : \ \partial^k\varphi \in \mathbb{L}_p(\Gamma,\rho^{(k)}), \ \ k = 0,\dots,m\right\}\,,$$

$$\rho^{(k)}(t) := \prod_{j=1}^{n} (t-t_j)^{\alpha_j+k}\,. \tag{1.13}$$

The space is endowed with a natural norm,

$$\|\varphi \mid \mathbb{KW}^m_p(\Gamma,\rho)\| := \left(\sum_{k=0}^{m} \|\partial^k_t\varphi \mid \mathbb{L}_p(\Gamma,\rho^{(k)})\|^p\right)^{\frac{1}{p}}$$

$$:= \left(\sum_{k=0}^{m} \int_\Gamma |\rho^{(k)}(t)\partial^k_t\varphi(t)|^p|dt|\right)^{\frac{1}{p}}\,, \tag{1.14}$$

which makes it a Banach space. It can be verified straightforwardly that the derivatives

$$\vartheta^k(t)\partial^k_t\varphi(t) \quad \text{and} \quad \partial^k_t\vartheta^k(t)\varphi(t)$$

(see (0.2) for $\vartheta(t)$) exist in the usual sense and that the following norms are equivalent to the original norm in (1.14):

$$\|\varphi \mid \mathbb{KW}^m_p(\Gamma,\rho)\|_1 := \left(\sum_{k=0}^{m} \|(\vartheta\partial_t)^k\varphi \mid \mathbb{L}_p(\Gamma,\rho)\|^p\right)^{\frac{1}{p}}\,,$$

$$\|\varphi \mid \mathbb{KW}^m_p(\Gamma,\rho)\|_2 := \left(\sum_{k=0}^{m} \|\partial^k_t\vartheta^k\varphi \mid \mathbb{L}_p(\Gamma,\rho)\|^p\right)^{\frac{1}{p}}\,. \tag{1.15}$$

Let $\mu = m + \nu$ be as in (1.4), $\vartheta(t)$ be as in (0.2), and

$$\mathbb{KPZ}_\mu(\Gamma,\mathcal{T}_\Gamma) := \left\{g \in \mathbb{PZ}_\nu(\Gamma) \ : \ \vartheta^k\partial^k g \in \mathbb{PZ}_\nu(\Gamma,\mathcal{T}_\Gamma) \quad \forall j = 1,\dots,m\right\}\,. \tag{1.16}$$

Let us prove that $\mathbb{PZ}_\mu(\Gamma,\mathcal{T}_\Gamma) \subset \mathbb{KPZ}_\mu(\Gamma,\mathcal{T}_\Gamma)$. In fact, from the definition of the δ-function and the above considerations it is clear that

$$(t-t_j)\delta_j(t) = 0\,, \quad j = 1,\dots,n \tag{1.17}$$

(see (1.9)). Therefore, $\lim_{t\to t_j}\vartheta(t)g'(t)=0$. Thus, dealing with "Fuchs" derivatives of $g\in\mathbb{KW}^m_p(\Gamma,\rho)$, we can ignore the δ-functions and take $\vartheta^k\partial^k_t g\in\mathbb{PZ}_{m-k}(\Gamma,\mathcal{T}_\Gamma)\subset\mathbb{PZ}_\nu(\Gamma,\mathcal{T}_\Gamma)$ for all $k=1,\dots,m$.

Moreover, $\mathbb{KPZ}_\mu(\Gamma,\mathcal{T}_\Gamma)$ is a Banach algebra. In fact, for arbitrary $g,h\in\mathbb{KPZ}_\mu(\Gamma,\mathcal{T}_\Gamma)$ we have

$$\vartheta^k\partial^k_t(gh)=\sum_{j=0}^{k}\begin{pmatrix} j\\ k\end{pmatrix}\vartheta^{k-j}(\partial^{k-j}_t g)\vartheta^j(\partial^j_t h)\in\mathbb{PZ}_\mu(\Gamma,\mathcal{T}_\Gamma) \tag{1.18}$$

for all $k=0,1,\dots,m$, which implies that $g\cdot h\in\mathbb{KPZ}_\mu(\Gamma,\mathcal{T}_\Gamma)$.

The space $\mathbb{KPZ}_\mu(\Gamma,\mathcal{T}_\Gamma)$ is essentially larger than $\mathbb{PZ}_\mu(\Gamma,\mathcal{T}_\Gamma)$: the first contains, e.g., the functions $g_1(t)+(t-t_j)^\gamma g_2(t)$ with a complex γ such that $\operatorname{Re}\gamma>0$ and $g_1,g_2\in\mathbb{PZ}_\mu(\Gamma,\mathcal{T}_\Gamma)$, which are absent in the second one.

Lemma 1.1. *The space $\mathbb{KPZ}_\mu(\Gamma,\mathcal{T}_\Gamma)$, defined by* (1.16)*, and the space $\mathbb{KPC}^m(\Gamma,\mathcal{T}_\Gamma)$, defined similarly, are Banach algebras and the embeddings*

$$\mathbb{PZ}_\mu(\Gamma,\mathcal{T}_\Gamma)\subset\mathbb{KPZ}_\mu(\Gamma,\mathcal{T}_\Gamma)\,,\quad \mathbb{PC}^m(\Gamma,\mathcal{T}_\Gamma)\subset\mathbb{KPC}^m(\Gamma,\mathcal{T}_\Gamma)$$

are proper.

As usual, for a negative $m=-1,-2,\dots$ the space $\mathbb{KW}^m_p(\Gamma,\rho)$ is defined as the dual space to $\mathbb{KW}^{-m}_{p'}(\Gamma,\rho^{-1})$, where $p':=p/(p-1)$.

Lemma 1.2. *Let Γ be piecewise μ-smooth, $m=0,\pm1,\pm2,\dots$, and $|m|<\mu$.*

The space $\mathbb{KW}^m_p(\Gamma,\rho)$ is defined correctly and is independent of the choice of parametrizations $\omega_j:\mathcal{I}\to\Gamma_j$, $j=1,\dots,n$ of the arcs Γ_j (see (1.1)*).*

The multiplication operator gI is bounded in $\mathbb{KW}^m_p(\Gamma,\rho)$ for arbitrary $g\in\mathbb{KPC}^{|m|}(\Gamma,\mathcal{T}_\Gamma)$.

Proof. We have to consider the case $m=0,1,\dots$ only. For a negative $m=-1,-2\dots$ both assertions follow by duality. In fact, it suffices to prove that the dual space is correctly defined and that the dual operator to gI is $\overline{g}I$.

The equality

$$\vartheta^k\partial^k_t(g\varphi)=\sum_{j=0}^{k}\begin{pmatrix} j\\ k\end{pmatrix}\vartheta^{k-j}(\partial^{k-j}_t g)\vartheta^j(\partial^j_t\varphi)\in\mathbb{W}^{m-k}_p(\Gamma,\rho)\,,\qquad k=0,1,\dots,m\,,$$

which is similar to (1.18), immediately implies that the multiplication operator gI is bounded in the space $\mathbb{KW}^m_p(\Gamma,\mathcal{T}_\Gamma)$.

¿From the equalities

$$\begin{aligned}&\vartheta\partial_x\varphi(\omega)=\vartheta(\partial_t\varphi)(\omega)\partial_x\omega\,,\\ &\vartheta^2\partial^2_x\varphi(\omega)=\vartheta^2(\partial^2_t\varphi)(\omega)(\partial_x\omega)^2+\vartheta(\partial_t\varphi)(\omega)\vartheta\partial^2_x\omega\end{aligned} \tag{1.19}$$

and similar formulas for higher derivatives and from the boundedness of the multiplication operators proved in the first part of the lemma it follows that the transformation operator

$$\omega_* \ : \ \mathbb{KW}^m_p(\Gamma,\rho) \longrightarrow \mathbb{KW}^m_p(\mathcal{I},\rho_0)\,, \qquad \omega_*\varphi(x) := \varphi(\omega(x))\,, \tag{1.20}$$
$$\omega(1-0) = \omega(0+0) = t_1\,, \ \ \omega(x_j) = t_j\,, \ \ j = 2,\dots,n\,,$$
$$\rho_0(x) := x^{\alpha_1}(x-1)^{\alpha_1}\prod_{j=2}^{n}(x-x_j)^{\alpha_j}$$

is a homeomorphism. Therefore the space $\mathbb{KW}^m_p(\Gamma,\rho)$ is independent of the choice of the parametrization of Γ. □

Let us consider the weighted Hölder-Zygmund space

$$\mathbb{Z}^0_\mu(\Gamma,\rho) := \{\varphi_0 := \rho\varphi \in \mathbb{Z}_\mu(\Gamma) \ : \ \varphi_0(t_j) = 0\,, \ \ k = 0,\dots,m\}\,, \quad 0 < \mu \le 1\,,$$

which is endowed with a natural norm (cf. (1.3)):

$$\|\varphi \mid \mathbb{Z}^0_\mu(\Gamma,\rho)\| = \|\rho\varphi \mid \mathbb{Z}_\mu(\Gamma)\|\,. \tag{1.21}$$

For $0 < \mu < 1$, the weighted Hölder-Zygmund space $\mathbb{Z}^0_\mu(\Gamma,\rho)$ coincides with the weighted Hölder space $H^0_\mu(\Gamma,\rho)$ considered in [Du1, Du2, Du3]. As for the spaces $H^0_1(\Gamma,\rho)$ and $\mathbb{Z}^0_1(\Gamma,\rho)$, they are essentially different (see [St1]).

To give a straightforward definition of the Hölder-Zygmund spaces $\mathbb{Z}^0_\mu(\Gamma,\rho)$ for $\mu = m+\nu \ge 1$ we need a μ-smooth contour Γ. For a piecewise smooth Γ we suggest the following modification of the Hölder-Zygmund space without weight:

$$\mathbb{KZ}_\mu(\Gamma) := \{\varphi \in \mathbb{Z}_\nu(\Gamma) \ : \ \vartheta^k\partial^k\varphi \in \mathbb{Z}_\nu(\Gamma)\,, \ \ k = 1,\dots,m\}\,,$$

provided $0 < \nu < 1$ (i.e., $\mu \notin \mathbb{N}$). For a weighted space we set

$$\mathbb{KZ}^0_\mu(\Gamma,\rho) := \{\rho\varphi \in \mathbb{Z}_\nu(\Gamma) \ : \ \varphi_k := \rho^{(k)}\partial^k\varphi \in \mathbb{Z}_\nu(\Gamma), \tag{1.22}$$
$$\varphi_k(t_j) = 0\,, \ \ k = 0,\dots,m\,, \ j = 0,\dots,n\}\,.$$

We endow these spaces with the following natural norms (cf. (1.3) and (1.21)):

$$\|\varphi \mid \mathbb{KZ}_\mu(\Gamma)\| := \sum_{k=0}^{m-1}\sup_{t\in\Gamma}|\vartheta^k(t)\partial^k_t\varphi(t)| + \|\vartheta^m\partial^m\varphi \mid \mathbb{Z}_\nu(\Gamma)\|\,,$$
$$\|\varphi \mid \mathbb{KZ}^0_\mu(\Gamma,\rho)\| := \sum_{k=0}^{m-1}\sup_{t\in\Gamma}|\varphi_k(t)| + \|\varphi_m \mid \mathbb{Z}_\nu(\Gamma)\|\,. \tag{1.23}$$

Equivalent norms can be written down as in (1.15).

Lemma 1.3. *The spaces* $\mathbb{KZ}_\mu(\Gamma)$ *for* $\mu \notin \mathbb{N}$ *and* $\mathbb{KZ}^0_\mu(\Gamma,\rho)$ *for arbitrary* $\mu > 0$ *are correctly defined and are independent of the choice of the parametrizations* $\omega_j \ : \ \mathcal{I} \to \Gamma_j$, $j = 1,\dots,n$ *of the curve* Γ *(see* (1.1)*).*

The multiplication operator gI *is bounded in the space* $\mathbb{KZ}^0_\mu(\Gamma,\rho)$ *for arbitrary* $g \in \mathbb{KPZ}_\mu(\Gamma,\mathcal{T}_\Gamma)$.

Proof. The proof follows word for word the proof of the preceding Lemma 1.2 with obvious modifications. □

2. Weighted Bessel potential and Besov spaces

It is possible to define new spaces by interpolation. Without going into the details of interpolation theory (we refer the reader to [Tr1] for that) let us note that interpolation assigns to a pair of Banach spaces $\mathbb{X}_0$, $\mathbb{X}_1$ embedded in a bigger Banach space, $\mathbb{X}_0, \mathbb{X}_1 \subset \mathbb{X}$ (an interpolation pair), a new space $\mathbb{X}_\vartheta := [\mathbb{X}_0, \mathbb{X}_1]_\vartheta$, $0 \le \vartheta \le 1$ (an interpolated space), with a comfortable interpolation property, stated in the next lemma. This lemma summarizes results on interpolation of operators by different methods exposed, e.g., in [Tr1, §§ 1.10.1, 1.16.4].

Lemma 2.1. (Interpolation Property). *If the operator*

$$\begin{aligned} A \ &: \ \mathbb{X}_0 \longrightarrow \mathbb{Y}_0 \\ &: \ \mathbb{X}_1 \longrightarrow \mathbb{Y}_1 \end{aligned}$$

is bounded in both pairs, then A is bounded between pairs of interpolated spaces

$$A : \mathbb{X}_\vartheta := [\mathbb{X}_0, \mathbb{X}_1]_\vartheta \longrightarrow \mathbb{Y}_\vartheta := [\mathbb{Y}_0, \mathbb{Y}_1]_\vartheta$$

for all $0 \le \vartheta \le 1$ and, for some positive constant C_ϑ,

$$\|A\big|\mathbb{X}_\vartheta \to \mathbb{Y}_\vartheta\| \le C_\vartheta \|A\big|\mathbb{X}_0 \to \mathbb{Y}_0\|^{1-\vartheta} \|A\big|\mathbb{X}_1 \to \mathbb{Y}_1\|^\vartheta .$$

Moreover, if $\mathbb{Y}_0 = \mathbb{Y}_1$ or $\mathbb{X}_0 = \mathbb{X}_1$ and the operator $A \ : \ \mathbb{X}_k \longrightarrow \mathbb{Y}_k$ is compact for $k = 0$ or for $k = 1$, then $A \ : \ \mathbb{X}_\vartheta \longrightarrow \mathbb{Y}_\vartheta$ is compact for all[1] $0 < \vartheta < 1$.

The Bessel potential space $\mathbb{H}^s_p(\Gamma)$, $s \ge 0$, $1 \le p \le \infty$, where

$$s = m + \vartheta\,, \qquad m \in \mathbb{N}\,, \quad 0 < \vartheta \le 1\,, \tag{2.1}$$

can be defined as the result of complex interpolation of Sobolev spaces (cf. [Tr1, § 1.9, § 2.3]),

$$\mathbb{H}^s_p(\Gamma) = (\mathbb{W}^m_p(\Gamma), \mathbb{W}^{m+1}_p(\Gamma))_\vartheta\,, \tag{2.2}$$

while the Besov space $\mathbb{B}^s_{p,q}(\Gamma)$ is the result of real interpolation (cf. [Tr1, § 1.3, § 2.3]),

$$\mathbb{B}^s_{p,q}(\Gamma) = (\mathbb{W}^m_p(\Gamma), \mathbb{W}^{m+1}_p(\Gamma))_{\vartheta,q}\,, \quad 1 \le q \le \infty\,. \tag{2.3}$$

Similar definitions are valid for the spaces $\mathbb{H}^s_p(\Omega^\pm)$, $\mathbb{B}^s_{p,q}(\Omega^\pm)$.

We note that the spaces can also be defined rigorously if Γ is only s-smooth. Therefore, for piecewise smooth Γ we can take $s \le 1$ (or, even, $s \le 1 + 1/p$).

For the definition of equivalent norms in Bessel potential and Besov spaces we need some standard definitions and notations.

$\mathcal{S}(\mathbb{R}^n)$ denotes the space of rapidly decreasing smooth functions and $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions, i.e., the space of continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$.

---

[1] The first result on interpolation of compact operators was obtained, to our knowledge, by M. Krasnosel'skij [Kr1] in 1960. This reference is missing in H. Triebel's fundamental monograph [Tr1, § 1.16.4]; he only cites papers devoted to the subject since 1964.

The direct and the inverse Fourier transforms $\mathcal{F}$ and $\mathcal{F}^{-1}$ are defined as follows:

$$\begin{aligned}\mathcal{F}\varphi(\xi) &:= \int_{\mathbb{R}^n} e^{i\xi x}\varphi(x)\,dx, \qquad \xi\in\mathbb{R}^n,\\ \mathcal{F}^{-1}\psi(x) &= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{-ix\xi}\psi(\xi)\,d\xi, \qquad x\in\mathbb{R}^n\,.\end{aligned} \tag{2.4}$$

For the Euclidean space $\mathbb{R}^n$, the Bessel potential space $\mathbb{H}^s_p(\mathbb{R}^n)$ $(s\in\mathbb{R},\ 1\le p\le\infty)$ is the subset of $\mathcal{S}'(\mathbb{R}^n)$ $(s\in\mathbb{R},\ 1<p<\infty)$ consisting of the elements f with finite norm

$$\|f\big|\mathbb{H}^s_p(\mathbb{R}^n)\| := \|\mathcal{F}^{-1}\langle\xi\rangle^s\mathcal{F}f\,|\,\mathbb{L}_p(\mathbb{R}^n)\|,$$

where

$$\langle x\rangle = \left(1+|x|^2\right)^{\frac{1}{2}}\,, \quad x\in\mathbb{R}^n\,. \tag{2.5}$$

The Besov space $\mathbb{B}^s_{p,q}(\mathbb{R}^n)$ $(s=m+\vartheta>0,\ 1\le p,q\le\infty)$ is equipped with the norm

$$\|f\big|\mathbb{B}^s_{p,q}(\mathbb{R}^n)\| := \|f\big|\mathbb{W}^m_p(\mathbb{R}^n)\| + \sum_{|\alpha|=m}\left(\int_{\mathbb{R}^n}\frac{\|\Delta^2_h\partial^\alpha f\big|L_p(\mathbb{R}^n)\|^q}{h^{\vartheta q}}\frac{dh}{h^n}\right)^{1/q}$$

for $1\le q<\infty$ and with the obvious "esssup" modification (instead of the integration) when $q=\infty$.

For domains $\Omega^\pm\subset\mathbb{R}^2$, the spaces $\mathbb{H}^s_p(\Omega^\pm)$ and $\mathbb{B}^s_{p,q}(\Omega^\pm)$ are defined as the restrictions of $\mathbb{H}^s_p(\mathbb{R}^2)$ and $\mathbb{B}^s_{p,q}(\mathbb{R}^2)$ to $\Omega^\pm$, while $\widetilde{\mathbb{H}}^s_p(\Omega^\pm)$ and $\widetilde{\mathbb{B}}^s_{p,q}(\Omega^\pm)$ are subspaces of the corresponding spaces on $\mathbb{R}^2$ and consist of functions which are supported in $\overline{\Omega^\pm}$.

The space $\mathbb{W}^s_p(\Omega^\pm)=\mathbb{B}^s_{p,p}(\Omega^\pm)$ is also known as the Sobolev-Slobodetskij space.

Similarly we define the spaces $\mathbb{H}^s_p(\mathcal{I})$, $\mathbb{B}^s_{p,q}(\mathcal{I})$ (as the restrictions of $\mathbb{H}^s_p(\mathbb{R})$, $\mathbb{B}^s_{p,q}(\mathbb{R})$ to the interval $\mathcal{I}\subset\mathbb{R}$) and $\widetilde{\mathbb{H}}^s_p(\mathcal{I})$, $\widetilde{\mathbb{B}}^s_{p,q}(\mathcal{I})$ (as the subspaces of $\mathbb{H}^s_p(\mathbb{R})$, $\mathbb{B}^s_{p,q}(\mathbb{R})$ of functions which are supported in the closed interval $\overline{\mathcal{I}}=[0,1]$).

Using the parametrizations (1.1), (1.2) we define the spaces $\mathbb{H}^s_p(\Gamma_j),\ldots,$ $\widetilde{\mathbb{B}}^s_{p,q}(\Gamma_j)$ in a standard way.

The weighted spaces $\mathbb{H}^s_p(\Gamma,\rho)$ and $\mathbb{B}^s_{p,q}(\Gamma,\rho)$ consist of all functions $\varphi(t)$ for which $\rho\varphi\in\mathbb{H}^s_p(\Gamma)$ and $\rho\varphi\in\mathbb{B}^s_{p,q}(\Gamma)$, respectively.

Since the spaces $\mathbb{H}^s_p(\Gamma,\rho)$ and $\widetilde{\mathbb{H}}^{-s}_{p'}(\Gamma,\rho^{-1})$ $(s\in\mathbb{R},\ p':=p/(p-1))$ are dual (adjoint), see [Tr1], it is natural to define the spaces $\mathbb{B}^s_{p,q}(\Gamma,\rho)$ and $\widetilde{\mathbb{B}}^s_{p,q}(\Gamma,\rho)$ for a negative $s<0$ as the dual spaces for $\widetilde{\mathbb{B}}^{-s}_{p',q'}(\Gamma,\rho^{-1})$ and $\mathbb{B}^{-s}_{p',q}(\Gamma,\rho^{-1})$, respectively.

The notation $\mathbb{H}^s_p$, ignoring the weight function and the domain of definition, will be used if the subsequent proposition is valid for any weight and any of the domains $\mathbb{R}^n$, $\Omega^\pm$, Γ_j and Γ. Moreover, when writing $\mathbb{X}(\Omega^\pm,\rho)$ we mean any of the spaces $\mathbb{H}^s_p(\Omega^\pm,\rho)$, $\mathbb{B}^s_{p,q}(\Omega^\pm,\rho)$, $\mathbb{W}^s_p(\Omega^\pm,\rho)$, $\mathbb{Z}_s(\Omega^\pm,\rho)$ while $\widetilde{\mathbb{X}}(\Omega^\pm,\rho)$ stands for the "tilde-spaces".

The Besov space $\mathbb{B}^s_{\infty,\infty} = \mathbb{W}^s_\infty$ and the Hölder-Zygmund space $\mathbb{Z}_s$ are isomorphic and coincide.

The equality $\mathbb{H}^s_2 = \mathbb{B}^s_{2,2}$ holds for all $s \in \mathbb{R}$ and, in particular, $\mathbb{H}^s_2(\mathbb{R}^n) = \mathbb{W}^s_2(\mathbb{R}^n)$ for all $s \geq 0$.

For a non-negative integer $m \in \mathbb{N}_0$ the Bessel potential space $\mathbb{H}^m_p$ and the Sobolev space $\mathbb{W}^m_p$ can be identified.

Theorem 2.2. (Interpolation Theorem; [Tr1, §§ 2.4, 3.3]). *Let*

$$s_0, s_1 \in \mathbb{R}, \quad 0 < \vartheta < 1, \quad 1 \leq p_0, p_1, \nu, q_0, q_1 \leq \infty,$$
$$\frac{1}{p} = \frac{1-\vartheta}{p_0} + \frac{\vartheta}{p_1}, \quad \frac{1}{q} = \frac{1-\vartheta}{q_0} + \frac{\vartheta}{q_1}, \quad s = (1-\vartheta)s_0 + \vartheta s_1.$$

For the real $(\cdot,\cdot)_{\vartheta,p}$, complex $(\cdot,\cdot)_\vartheta$, and modified complex $[\cdot,\cdot]_\vartheta$ interpolation functors we have the following:

i. $(\mathbb{H}^s_{p_0}, \mathbb{H}^s_{p_1})_{\vartheta,p} = \mathbb{H}^s_p$ *provided* $1 < p_0,\ p_1 < \infty$*;*

ii. $[\mathbb{H}^{s_0}_{p_0}, \mathbb{H}^{s_1}_{p_1}]_\vartheta = (\mathbb{H}^{s_0}_{p_0}, \mathbb{H}^{s_1}_{p_1})_\vartheta = \mathbb{H}^s_p$ *provided* $1 < p_0,\ p_1 < \infty$*;*

iii. $(\mathbb{H}^{s_0}_r, \mathbb{H}^{s_1}_r)_{\vartheta,\nu} = \mathbb{B}^s_{r,\nu}$ *provided* $s_0 \neq s_1,\ 1 < r < \infty$*;*

iv. $(\mathbb{B}^{s_0}_{r,q_0}, \mathbb{B}^{s_1}_{r,q_1})_{\vartheta,\nu} = \mathbb{B}^s_{r,\nu}$ *provided* $s_0 \neq s_1,\ 1 \leq r \leq \infty$*;*

v. $(\mathbb{B}^{s_0}_{p_0,q_0}, \mathbb{B}^{s_1}_{p_1,q_1})_\vartheta = \mathbb{B}^s_{p,q}$*; if in addition,* $1 < p_0,\ p_1 < \infty$ *and either* $q_0 \neq \infty$ *or* $q_1 \neq \infty$*, then* $[\mathbb{B}^{s_0}_{p_0,q_0}, \mathbb{B}^{s_1}_{p_1,q_1}]_\vartheta = \mathbb{B}^s_{p,q}$*;*

vi. $(\mathbb{Z}_{t_0}, \mathbb{Z}_{t_1})_{\vartheta,\infty} = (\mathbb{Z}_{t_0}, \mathbb{Z}_{t_1})_\vartheta = \mathbb{Z}_t$ *provided* $t_0, t_1 > 0,\ 0 < \vartheta < 1,\ t = (1-\vartheta)t_0 + \vartheta t_1$*.*

The next theorem addresses the boundedness of a multiplication operator and of the Cauchy singular integral operator, which are of a special importance for the theory of the equations in study (see §§ 3-4).

Theorem 2.3.

i. *If* Γ *is sufficiently smooth and if*

$$s \in \mathbb{R}, \quad 1 < p < \infty, \quad \text{and} \quad \mu \begin{cases} = |s| & \text{for} \quad s = 0, \pm 1, \dots, \\ > |s| & \text{for} \quad s \neq 0, \pm 1, \dots, \end{cases} \tag{2.6}$$

then the multiplication operator gI *with* $g \in H_\mu(\Gamma)$ *is bounded in the space* $\mathbb{H}^s_p(\Gamma, \rho)$*.*

ii. *If* Γ *is sufficiently smooth and if*

$$s \in \mathbb{R}, \quad 1 \leq p, q \leq \infty, \quad \text{and} \quad \mu > |s|, \tag{2.7}$$

then the multiplication operator gI *with* $g \in H_\mu(\Gamma)$ *is bounded in the space* $\mathbb{B}^s_{p,q}(\Gamma, \rho)$*.*

iii. *The spaces* $\mathbb{H}^\nu_p(\mathbb{R}^+, x^\alpha)$*,* $\widetilde{\mathbb{H}}^\nu_p(\mathbb{R}^+, x^\alpha)$ *and the spaces* $\mathbb{B}^\nu_{p,q}(\mathbb{R}^+, x^\alpha)$*,* $\widetilde{\mathbb{B}}^\nu_{p,q}(\mathbb{R}^+, x^\alpha)$ *are pairwise isomorphic and can be identified (extending the restricted* *functions by* 0) *provided*

$$\frac{1}{p} - 1 < \nu < \frac{1}{p}\,, \qquad 1 < p < \infty\,. \tag{2.8}$$

iv. *If* Γ *is piecewise smooth and* $\mu > |\nu|$, *then the multiplication operator* gI *with* $g \in \mathbb{PZ}_\mu(\Gamma, \mathcal{T}_\Gamma)$ *is bounded in the spaces* $\mathbb{H}^\nu_p(\Gamma,\rho)$ *and* $\mathbb{B}^\nu_{p,q}(\Gamma,\rho)$ *provided condition* (2.8) *holds.*

Proof. For the proofs of assertion i we refer to [Tr1, § 3] (see also [MSh1]). Assertion ii follows from assertion i and Theorem 2.2.iii. Assertion iii is a slight modification of the corresponding assertion from [Tr1, § 2.10] (see also [MSh1]). Namely, it suffices to prove the equivalent boundedness of the multiplication operator $\chi_+ I$ by a characteristic function of the half-line in $\mathbb{H}^\nu_p(\mathbb{R}, x^\alpha)$ and in $\mathbb{B}^\nu_{p,q}(\mathbb{R}, x^\alpha)$. By the definition of the weighted spaces, this is equivalent to the boundedness of the operator $x^{-\alpha}\chi_+ x^\alpha I = \chi_+ I$ in the unweighted spaces $\mathbb{H}^\nu_p(\mathbb{R})$ and $\mathbb{B}^\nu_{p,q}(\mathbb{R})$, which is proved in [Tr1, § 2.10].

To prove assertion iv we recall the definition of the weighted Bessel potential space on a piecewise smooth contour. Due to assertion iii, under condition (2.8) the space $\mathbb{H}^\nu_p(\Gamma,\rho)$ can be identified with the space

$$\widetilde{\mathbb{H}}^\nu_p(\Gamma,\rho) := \{\varphi \,:\, \rho\varphi \in \widetilde{\mathbb{H}}^\nu_p(\Gamma_j) \ \ \forall\, j = 1,\ldots,n\}\,. \tag{2.9}$$

On the other hand, any function $g \in \mathbb{PZ}_\mu(\Gamma, \mathcal{T}_\Gamma)$ can be represented as a finite sum

$$g(t) = \sum_{j=1}^n g_j(t)\chi_j(t)\,, \qquad g_j \in \mathbb{Z}_\mu(\Gamma)\,, \quad j = 1,\ldots,n\,,$$

where $\chi_j(t)$ is the characteristic function of the smooth arc $\Gamma_j \subset \Gamma$, and due to assertion i it suffices to consider only the case $g(t) = \chi_j(t)$.

We are left to prove that $\chi_j I$ is bounded in $\widetilde{\mathbb{H}}^\nu_p(\Gamma,\rho)$ or, as in assertion iii, in $\widetilde{\mathbb{H}}^\nu_p(\Gamma)$, which is a simple exercise.

For the space $\mathbb{B}^\nu_{p,q}(\Gamma,\rho)$ the result can be proved similarly or obtained from the proved assertion by interpolation. □

¿From Theorem 2.3.iii we easily find that if Γ is piecewise smooth, then the spaces $\mathbb{H}^s_p(\Gamma,\rho)$ and $\mathbb{B}^s_{p,q}(\Gamma,\rho)$ can be defined rigorously, provided conditions (2.8) hold. As we have already noted above, the definition cannot be extended to large $|s|$. We must impose restrictions on s to ensure the boundedness of a multiplication operator by a piecewise smooth function and of the Cauchy singular integral operator (see Theorems 2.3 and 3.1).

Therefore we suggest to consider the following spaces: if

$$s = m + \nu\,, \qquad m \in \mathbb{N}_0 \tag{2.10}$$

and (2.8) holds, we define

$$\mathbb{KH}^s_p(\Gamma,\rho) := \left\{\varphi \in \mathbb{H}^\nu_p(\Gamma,\rho) \,:\, \vartheta^k\partial^k\varphi \in \mathbb{H}^\nu_p(\Gamma,\rho), \ \ k = 0,\ldots,m\right\}\,, \tag{2.11}$$

$$\mathbb{KB}^s_{p,q}(\Gamma,\rho) := \left\{\varphi \in \mathbb{B}^\nu_{p,q}(\Gamma,\rho) \,:\, \vartheta^k\partial^k\varphi \in \mathbb{B}^\nu_{p,q}(\Gamma,\rho), \ \ k = 0,\ldots,m\right\} \tag{2.12}$$

(see (0.2) for $\vartheta(t)$). The spaces are endowed with natural norms:

$$\|\varphi \mid \mathbb{KH}^s_p(\Gamma,\rho)\| := \left(\sum_{k=0}^{m} \|\vartheta^k \partial_t^k \varphi \mid \mathbb{H}^\nu_p(\Gamma,\rho)\|^p\right)^{\frac{1}{p}},$$

$$\|\varphi \mid \mathbb{KB}^s_{p,q}(\Gamma,\rho)\| := \left(\sum_{k=0}^{m} \|\vartheta^k \partial_t^k \varphi \mid \mathbb{B}^\nu_{p,q}(\Gamma,\rho)\|^p\right)^{\frac{1}{p}},$$

which make them be Banach spaces. It can be verified straightforwardly that equivalent norms can be defined analogously as in (1.15).

For a negative $s < 0$, the spaces $\mathbb{KH}^s_p(\Gamma,\rho)$ and $\mathbb{KB}^s_{p,q}(\Gamma,\rho)$ are defined as the dual spaces to $\mathbb{KH}^{-s}_{p'}(\Gamma,\rho^{-1})$ and $\mathbb{KB}^{-s}_{p',q'}(\Gamma,\rho^{-1})$, respectively, where $p' := p/(p-1)$, $q' := q/(q-1)$.

Lemma 2.4. *Let Γ be piecewise μ-smooth and $|s| < \mu$.*

The spaces $\mathbb{KH}^s_p(\Gamma,\rho)$ and $\mathbb{KB}^s_{p,q}(\Gamma,\rho)$ are defined correctly and are independent of the choice of the parametrizations $\omega_j : \mathcal{I} \to \Gamma_j$, $j = 1,\dots,n$ of the curve Γ (see (1.1)).

The multiplication operator gI is bounded in $\mathbb{KH}^s_p(\Gamma,\rho)$ and in $\mathbb{KB}^s_{p,q}(\Gamma,\rho)$ for arbitrary $g \in \mathbb{KPZ}_\mu(\Gamma,\mathcal{T}_\Gamma)$.

Proof. The proof is similar to the proof of Lemma 1.2. □

3. Singular integral equations in weighted spaces

If Γ is an m-smooth closed contour and $m \in \mathbb{N}$, then the Cauchy singular integral operator

$$S_\Gamma\varphi(t) := \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)d\tau}{\tau - t}, \qquad t \in \Gamma \tag{3.1}$$

is bounded in the spaces $\mathbb{H}^s_p(\Gamma)$, $\mathbb{B}^s_{p,q}(\Gamma)$, and $\mathbb{Z}_s(\Gamma)$ provided

$$1 < p,q < \infty, \quad s \in \mathbb{R}, \quad s \le m. \tag{3.2}$$

In fact, let us recall that S_Γ is bounded in the Hölder spaces $H_\nu(\Gamma) = \mathbb{Z}_\nu(\Gamma)$ for $0 < \nu < 1$ (the Privalov Theorem; see [Du1, Du2, Mu1, St1]) and in the Lebesgue spaces $L_p(\Gamma)$ (the Riesz Theorem; see [GK1, Kh1, St1]). Since

$$(\partial^j S_\Gamma\varphi)(t) = (S_\Gamma\partial^j\varphi)(t) \qquad \forall\, j \in \mathbb{N}, \tag{3.3}$$

we get

$$\begin{aligned}\|S_\Gamma\varphi\big|\mathbb{X}^{k+\nu}(\Gamma)\| &= \sum_{j=0}^{k}\|\partial^j S_\Gamma\varphi\big|\mathbb{X}^\nu(\Gamma)\| = \sum_{j=0}^{k}\|S_\Gamma\partial^j\varphi\big|\mathbb{X}^\nu(\Gamma)\| \\ &\le C\sum_{j=0}^{k}\|\partial^j\varphi\big|\mathbb{X}^\nu(\Gamma)\| = C\|\varphi\big|\mathbb{X}^{k+\nu}(\Gamma)\|, \quad k \in \mathbb{N}.\end{aligned}$$

This implies the boundedness in the Sobolev spaces $\mathbb{X}^k(\Gamma) := \mathbb{W}^k_p(\Gamma) = \mathbb{H}^k_p(\Gamma)$ for all integers $k = 0, \dots, m$ and in the Hölder-Zygmund spaces $\mathbb{Z}_\mu(\Gamma)$ for all non-integers $\mu = k + \nu$, $0 < \nu < 1$. Due to the Interpolation Theorems 2.2.i, 2.2.iii and 2.2.vi, S_Γ is bounded in the spaces $\mathbb{H}^s_p(\Gamma)$, $\mathbb{B}^s_{p,q}(\Gamma)$, and $\mathbb{Z}_\mu(\Gamma)$ if conditions (3.2) hold.

If Γ is piecewise smooth, the boundedness result (3.1)-(3.2) does not hold any more, especially for the weighted spaces. Instead we can prove the following.

Theorem 3.1. *The Cauchy singular integral operator with a weight,*

$$S_{\Gamma,w}\varphi(t) := \frac{1}{\pi i}\int_\Gamma \frac{w(t)}{w(\tau)}\frac{\varphi(\tau)d\tau}{\tau - t}, \qquad w(t) := \prod_{j=1}^n (t - t_j)^{\beta_j}, \tag{3.4}$$

is bounded in the following spaces:

i. *in the modified weighted Sobolev space $\mathbb{KW}^m_p(\Gamma,\rho)$, $m \in \mathbb{N}_0$ provided*

$$-\frac{1}{p} < \alpha_j + \beta_j < 1 - \frac{1}{p}, \quad j = 1, \dots, n, \quad 1 < p < \infty; \tag{3.5}$$

ii. *in the modified weighted Hölder-Zygmund space $\mathbb{KZ}^0_\mu(\Gamma,\rho)$ provided*

$$\begin{aligned} &\mu = m + \nu, \quad m \in \mathbb{N}_0, \quad 0 < \nu \le 1, \\ &0 < \alpha_j + \beta_j - \nu < 1, \quad j = 1, \dots, n; \end{aligned} \tag{3.6}$$

iii. *in the modified weighted Bessel potential $\mathbb{KH}^s_p(\Gamma,\rho)$ and Besov $\mathbb{KB}^s_{p,q}(\Gamma,\rho)$ spaces, with $s = m + \nu$ for any integer m, provided that* (2.8) *holds and*

$$-\frac{1}{p} < \alpha_j + \beta_j - \nu, \; \alpha_j + \beta_j < 1 - \frac{1}{p}, \quad j = 1, \dots, n, \quad 1 < p, q < \infty. \tag{3.7}$$

Proof. First let us prove the assertions for $m = 0$: the operator

$$S_{\Gamma,w} \; : \; \mathbb{X}(\Gamma,\rho) \longrightarrow \mathbb{X}(\Gamma,\rho) \tag{3.8}$$

is bounded provided

$$\begin{aligned} \mathbb{X}(\Gamma,\rho) &= \mathbb{L}_p(\Gamma,\rho) && \text{and (3.5) holds}, \\ \mathbb{X}(\Gamma,\rho) &= \mathbb{Z}_\nu(\Gamma,\rho) && \text{and (3.6) holds}, \\ \mathbb{X}(\Gamma,\rho) &= \mathbb{H}^\nu_p(\Gamma,\rho) && \text{and (3.7) holds}, \\ \mathbb{X}(\Gamma,\rho) &= \mathbb{B}^\nu_{p,q}(\Gamma,\rho) && \text{and (3.7) holds}. \end{aligned} \tag{3.9}$$

In fact, for the weighted Lebesgue space $\mathbb{X}(\Gamma,\rho) = \mathbb{L}_p(\Gamma,\rho)$ the boundedness result (3.8) is well known (and first proved in [Kh1], see also [GK1] and [BK1]).

For the weighted Hölder space $\mathbb{X}(\Gamma,\rho) = H_\nu(\Gamma,\rho)$, the boundedness result (3.8) is proved in [Du1, Du2] (see also [GK1]) under the constraint $0 < \nu < 1$. In the case $\nu = 1$, the boundedness result can be proved similarly using the boundedness of S_Γ in $\mathbb{Z}_1(\Gamma)$ (see (3.1)-(3.2)).

The operator $S_{\Gamma,w}$ is bounded in the space $\mathbb{H}^\nu_p(\Gamma,\rho) = \widetilde{\mathbb{H}}^\nu_p(\Gamma,\rho)$ if and only if the operators $\chi_j S_{\Gamma,w\rho}\chi_k$ are bounded in $\mathbb{H}^\nu_p(\Gamma)$, where $\chi_j(t)$ is the characteristic function of the smooth arc $\Gamma_j \subset \Gamma$, $j = 1, \dots, n$. By rectification of the smooth arcs we easily derive the required boundedness property from the boundedness of the operators

$$S^{(\alpha)}_{\mathbb{R}^+}\varphi(x) := \frac{1}{\pi i}\int_0^\infty \left(\frac{x}{y}\right)^\alpha \frac{\varphi(y)dy}{y-x}\,, \qquad N^{(\alpha)}_{\mathbb{R}^+,\gamma}\varphi(x) := \frac{1}{\pi i}\int_0^\infty \left(\frac{x}{y}\right)^\alpha \frac{\varphi(y)dy}{y-e^{i\gamma}x}\,,$$

$$S^{(\alpha)}_{\mathbb{R}^+},\ N^{(\alpha)}_{\mathbb{R}^+,\gamma}\ :\ \widetilde{\mathbb{H}}^\nu_p(\mathbb{R}^+) = \mathbb{H}^\nu_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^\nu_p(\mathbb{R}^+)\,, \tag{3.10}$$

$$\frac{1}{p}-1<\nu<\frac{1}{p}\,, \quad -\frac{1}{p}<\alpha-\nu,\ \alpha<1-\frac{1}{p}\,, \quad 0<|\gamma|<\pi\,.$$

To prove the boundedness of the operators in (3.10) we will first consider the operator

$$A := -i\sin\pi\alpha\cos\pi\alpha + \sin^2\pi\alpha\, S_{\mathbb{R}^+}\ :\ \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)\,. \tag{3.11}$$

This operator represents a Mellin convolution:

$$A = \mathfrak{M}^0_{\mathcal{A}_p}\,, \qquad \mathcal{A}_p(\xi) := -i\sin\pi\alpha\cos\pi\alpha + \sin^2\pi\alpha\ \coth\pi\left(\frac{i}{p}+\xi\right)\,, \tag{3.12}$$

where

$$\mathfrak{M}^0_a\varphi(x) := \mathfrak{M}^{-1}_{\xi\to x}\left\{a(\xi)\mathfrak{M}_{y\to\xi}[\varphi(y)]\right\}(x)\,, \quad x\in\mathbb{R}^+\,,\ \xi\in\mathbb{R}\,, \tag{3.13}$$

and $\mathfrak{M}^{\pm1} := \mathfrak{M}^{\pm1}_{\xi\to x}$ are the Mellin transforms

$$\begin{aligned} \mathfrak{M}\varphi(\xi) &:= \int_0^\infty y^{\frac{1}{p}-i\xi}\varphi(y)\frac{dy}{y}\,, \quad \xi\in\mathbb{R}, \\ \mathfrak{M}^{-1}\psi(x) &= (2\pi)^{-1}\int_{-\infty}^\infty x^{i\xi-\frac{1}{p}}\psi(\xi)\,d\xi, \qquad x\in\mathbb{R}^+ \end{aligned} \tag{3.14}$$

(see [Du6, Du9, Du5, DLS1]). Therefore the operator $A=\mathfrak{M}^0_{\mathcal{A}_p}$ in (3.11) is invertible if and only if

$$\mathcal{A}_p(\xi)\neq 0 \quad \Longleftrightarrow \quad -\frac{1}{p}<\alpha<1-\frac{1}{p} \tag{3.15}$$

(note that $\mathcal{A}_p(\xi)=0$ only for $\xi=0$).

Let us prove that the Mellin convolution operator

$$B = S^{(\alpha)}_{\mathbb{R}^+} - i\cot\pi\alpha I = \mathfrak{M}^0_{\mathcal{B}_p}\,, \tag{3.16}$$

$$\mathcal{B}_p(\xi) := \coth\pi\left(\frac{i}{p}+i\alpha+\xi\right) - i\cot\pi\alpha$$

is the inverse of A. Indeed,

$$\mathcal{B}_p(\xi) := \coth\pi\left(\frac{i}{p} + i\alpha + \xi\right) - i\cot\pi\alpha = -i\left[\cot\pi\left(\frac{1}{p} + \alpha - i\xi\right) - \cot\pi\alpha\right]$$

$$= i\frac{1 + \cot^2\pi\alpha}{\cot\pi(1/p - i\xi) + \cot\pi\alpha} = i\frac{\sin^{-2}\pi\alpha}{\cot\pi(1/p - i\xi) + \cot\pi\alpha} = \mathcal{A}_p^{-1}(\xi)$$

and hence we get $AB = BA = \mathfrak{M}^0_{\mathcal{A}_p\mathcal{A}_p^{-1}} = \mathfrak{M}^0_1 = I$.

The operator A in (3.11) can also be regarded as a Fourier convolution operator:

$$A := -i\sin\pi\alpha\cos\pi\alpha + \sin^2\pi\alpha\, S_{\mathbb{R}^+} = W_{a_p} \ : \ \widetilde{\mathbb{H}}^\nu_p(\Gamma) = \mathbb{H}^\nu_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^\nu_p(\mathbb{R}^+)\,,$$

$$a_p(\xi) = -i\sin\pi\alpha\cos\pi\alpha - \sin^2\pi\alpha \ \mathrm{sign}\ \xi \tag{3.17}$$

(see [Du6, Lemma 1.35]). Here

$$W^0_a\varphi(x) := \mathcal{F}^{-1}_{\xi\to x}\left\{a(\xi)\mathcal{F}_{y\to\xi}[\varphi(y)]\right\}(x)\,, \quad x \in \mathbb{R} \tag{3.18}$$

is a convolution on $\mathbb{R}$ ($\mathcal{F}^{\pm 1} = \mathcal{F}^{\pm 1}_{\xi\to x}$ are the Fourier transformations; see (2.4)).

$$W_a := r_+ W^0_a \ : \ \widetilde{\mathbb{X}}(\mathbb{R}^+) \longrightarrow \mathbb{X}(\mathbb{R}^+)$$

is the restriction of W^0_a to the positive half-line (r_+ denotes the restriction of a function from $\mathbb{R}$ to $\mathbb{R}^+$).

It is well known that the operators W^0_a and $\mathfrak{M}^0_a$ are isomorphic:

$$\mathfrak{M}^0_a = \mathbf{Z}^{-1} W^0_a \mathbf{Z}\,,$$

$$\mathbf{Z} \ : \ \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R})\,, \qquad \mathbf{Z}\varphi(x) = e^{-\frac{1}{p}x}\varphi(e^{-x})\,,$$

$$\mathbf{Z}^{-1} \ : \ \mathbb{L}_p(\mathbb{R}) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)\,, \quad \mathbf{Z}^{-1}\psi(t) = t^{-\frac{1}{p}}\psi(-\log t)$$

(see [Du6, § 8]).

Next we apply a lifting procedure to the operator A in (3.17). For this we recall that the Bessel potential operators

$$\Lambda^r_+ := W^0_{\lambda^r_+} \ : \ \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \widetilde{\mathbb{H}}^{s-r}_p(\mathbb{R}^+)\,, \quad \lambda^r_+(\xi) = (\xi + i)^r\,,$$

$$r_+\Lambda^r_-\ell := W_{\lambda^r_-}\ell \ : \ \mathbb{H}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{s-r}_p(\mathbb{R}^+)\,, \quad \lambda^r_-(\xi) = (\xi - i)^r\,, \tag{3.19}$$

where ℓ denotes an arbitrary extension from $\mathbb{R}^+$ to $\mathbb{R}$, arrange isomorphisms of the indicated spaces for arbitrary $s, r \in \mathbb{R}$ (see, e.g., [Du5, DS1, Es1, St1]) and that

$$r_+\Lambda^r_-\ell W_a\varphi = W_{a\lambda^r_-}\varphi\,, \qquad W_a r_+\Lambda^r_+\varphi = W_{a\lambda^r_+}\varphi\,, \qquad \chi_+\Lambda^r_+\varphi = \Lambda^r_+\varphi \tag{3.20}$$

for $\varphi \in \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$. In particular, we have the isomorphisms

$$\Lambda^{-\nu}_+ \ : \ \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \widetilde{\mathbb{H}}^\nu_p(\mathbb{R}^+)\,,$$

$$r_+\Lambda^\nu_-\ell \ : \ \mathbb{H}^\nu_p(\mathbb{R}^+) \longrightarrow r_+\mathbb{L}_p(\mathbb{R})\,, \tag{3.21}$$

and since the spaces $r_+\mathbb{L}_p(\mathbb{R})$ and $\mathbb{L}_p(\mathbb{R}^+)$ can be identified (extending functions $\varphi \in \mathbb{L}_p(\mathbb{R}^+)$ by 0), we can lift the operator $A = W_{a_p}$ in (3.17). As a result we get the equivalent operator

$$r_+\Lambda_-^{\nu}\ell W_{a_p}\Lambda_+^{-\nu} = W_{\lambda_-^{\nu}a_p\lambda_+^{-\nu}} = W_{a_{p,\nu}}\,, \qquad a_{p,\nu}(\xi) = a_p(\xi)\left(\frac{\xi - i}{\xi + i}\right)^{\nu}. \tag{3.22}$$

Since

$$\begin{aligned} a_{p,\nu}(+\infty) &= \frac{2e^{-i\pi\alpha}\sin^2\pi\alpha}{e^{i\pi\alpha} + e^{-i\pi\alpha}}\,, \\ a_{p,\nu}(-\infty) &= \frac{2e^{i\pi(\alpha-2\nu)}\sin^2\pi\alpha}{e^{i\pi\alpha} + e^{-i\pi\alpha}}\,, \\ a_{p,\nu}(0-0) &= \frac{2e^{i\pi\alpha}\sin^2\pi\alpha}{e^{i\pi\alpha} + e^{-i\pi\alpha}}\,, \\ a_{p,\nu}(0+0) &= \frac{2e^{-i\pi\alpha}\sin^2\pi\alpha}{e^{i\pi\alpha} + e^{-i\pi\alpha}}\,, \end{aligned} \tag{3.23}$$

we get

$$\begin{aligned} \frac{1}{2\pi}\arg\frac{a_{p,\nu}(+\infty)}{a_{p,\nu}(-\infty)} &= \frac{1}{2\pi}\arg e^{2\pi(\nu-\alpha)i} = \nu - \alpha\,, \\ \frac{1}{2\pi}\arg\frac{a_{p,\nu}(0+0)}{a_{p,\nu}(0-0)} &= \frac{1}{2\pi}\arg e^{-2\pi\alpha i} = -\alpha\,, \end{aligned}$$

and the conditions in the third line of (3.10) ensure the invertibility of the lifted convolution operator (3.22) in $\mathbb{L}_p(\mathbb{R}^+)$ (see [Du6, Lemma 4.1, Theorem 4.2]). Therefore, conditions (3.15) guarantee the invertibility of the operator $A = W_{a_p}$ in (3.17), and since B in (3.16) is its inverse, B is a bounded operator in $\mathbb{H}_p^{\nu}(\mathbb{R}^+)$. Thus, $S_{\mathbb{R}^+}^{(-\gamma)} = B - i\cot\pi\gamma I$ is a bounded operator in $\mathbb{H}_p^{\nu}(\mathbb{R}^+)$.

To prove the boundedness of the operator $N_{\mathbb{R}^+,\gamma}^{(\alpha)}$ (see (3.10)) it is equivalent to consider the operator $N_{\mathbb{R}^+,\gamma} := N_{\mathbb{R}^+,\gamma}^{(0)}$ in $\mathbb{H}_p^{\nu}(\mathbb{R}^+, x^{\alpha})$, due to the definition of the weighted spaces.

Let us apply the trick described in [Sc1]: there exists a sufficiently small and negative $\beta < 0$ such that

$$-\frac{1}{p} < \alpha + \beta - \nu,\ \alpha + \beta < 1 - \frac{1}{p}\,. \tag{3.24}$$

As already proved, these conditions ensure the boundedness of the operator $S_{\mathbb{R}^+}^{(\beta+i\vartheta)}$ in $\mathbb{H}_p^{\nu}(\mathbb{R}^+, x^{\alpha})$, because $\|S_{\mathbb{R}^+}^{(\beta+i\vartheta)}\| = \|S_{\mathbb{R}^+}^{(\beta)}\|$ for all $\vartheta \in \mathbb{R}$. So the operator

$$R_{\mathbb{R}^+,\gamma} := \frac{e^{-i\gamma} - 1}{2}\int_{-\infty}^{\infty}\frac{e^{i(\pi-\gamma)(\beta+i\vartheta)}}{\sin\pi(\beta + i\vartheta)}S_{\mathbb{R}^+}^{(\beta+i\vartheta)}d\vartheta$$

is also bounded in $\mathbb{H}_p^{\nu}(\mathbb{R}^+, x^{\alpha})$ because the integral is absolutely convergent (cf. the last inequality in (3.10)).

Let $0 < t < 1$. By changing the order of integration and applying the residue theorem in the right complex half-plane Re $\pi(\beta + i\vartheta) > 0$, where the integrand has simple poles at $\beta = 0, 1, \ldots$, one finds that

$$R_{\mathbb{R}^+,\gamma}\varphi(x) = \frac{1}{\pi i}\int_0^\infty K\left(\frac{x}{y}\right)\frac{\varphi(y)dy}{y-x}\,,$$

$$\begin{aligned}K(t) &= \frac{e^{-i\gamma}-1}{2}\int_{-\infty}^{\infty}\frac{t^{\beta+i\vartheta}e^{i(\pi-\gamma)(\beta+i\vartheta)}}{\sin\,\pi(\beta+i\vartheta)}d\vartheta\\ &= i(e^{-i\gamma}-1)\int_{-\infty}^{\infty}\frac{t^{\beta+i\vartheta}e^{i(\pi-\gamma)(\beta+i\vartheta)}}{e^{i\pi(\beta+i\vartheta)}-e^{-i\pi(\beta+i\vartheta)}}d\vartheta = \pi i\frac{e^{-i\gamma}-1}{2}\sum_{k=0}^{\infty}\frac{2it^k e^{i(\pi-\gamma)k}}{(-1)^{k+1}\pi}\\ &= (1-e^{-i\gamma})\sum_{k=0}^{\infty}t^k e^{-i\gamma k} = \frac{1-e^{-i\gamma}}{1-te^{-i\gamma}}\,.\end{aligned}$$

If $1 < t < \infty$, one can apply the residue theorem in the left complex half-plane Re $\pi(\beta + i\vartheta) < 0$, where the integrand has simple poles at $\beta = -1, -2, \ldots$. Similarly to the foregoing case we get

$$\begin{aligned}K(t) &= \frac{e^{-i\gamma}-1}{2}\int_{-\infty}^{\infty}\frac{t^{\beta+i\vartheta}e^{i(\pi-\gamma)(\beta+i\vartheta)}}{\sin\,\pi(\beta+i\vartheta)}d\vartheta = (e^{-i\gamma}-1)\sum_{k=1}^{\infty}t^{-k}e^{i\gamma k}\\ &= -\frac{t^{-1}e^{i\gamma}(1-e^{-i\gamma})}{1-t^{-1}e^{i\gamma}} = \frac{1-e^{-i\gamma}}{1-te^{-i\gamma}}\,.\end{aligned}$$

Therefore,

$$R_{\mathbb{R}^+,\gamma}\varphi(x) = \frac{1}{\pi i}\int_0^\infty \frac{1-e^{-i\gamma}}{1-\frac{x}{y}e^{-i\gamma}}\frac{\varphi(y)dy}{y-x} \tag{3.25}$$

and hence, by virtue of the equality $R_{\mathbb{R}^+,\gamma} = N_{\mathbb{R}^+,\gamma} - e^{-i\gamma}S_{\mathbb{R}^+}$, the boundedness of $N_{\mathbb{R}^+,\gamma}$ in the space $\mathbb{H}^\nu_p(\mathbb{R}^+, x^\alpha)$ follows.

To prove that the operator $S_{\Gamma,w}$ is bounded in the modified weighted Besov space $\widetilde{\mathbb{B}}^\nu_{p,q}(\Gamma,\rho) = \mathbb{B}^\nu_{p,q}(\Gamma,\rho)$ we can employ the interpolation method: if conditions (3.7) hold, the operator $S_{\Gamma,w\rho}$ is bounded not only in $\mathbb{H}^\nu_p(\Gamma)$, but also in $\mathbb{H}^{\nu\pm\varepsilon}_p(\Gamma,\rho)$ for a small $\varepsilon > 0$. Due to the Interpolation Theorem 2.2.iii, $S_{\Gamma,w\rho}$ is bounded in $\mathbb{B}^\nu_{p,q}(\Gamma)$ and, therefore, $S_{\Gamma,w}$ is bounded in $\mathbb{B}^\nu_{p,q}(\Gamma,\rho)$.

Now we will prove the boundedness result for a positive integer $m \in \mathbb{N}$. The assertions can be reformulated as follows: the operator

$$S_\Gamma \;:\; \mathbb{X}^m(\Gamma, w\rho) \longrightarrow \mathbb{X}^m(\Gamma, w\rho) \tag{3.26}$$

is bounded provided

$$\begin{aligned} \mathbb{X}^m(\Gamma, w\rho) &= \mathbb{K}\mathbb{W}_p^m(\Gamma, w\rho) && \text{and (3.5) holds}\,, \\ \mathbb{X}^m(\Gamma, w\rho) &= \mathbb{K}\mathbb{Z}^0_{m+\nu}(\Gamma, w\rho) && \text{and (3.6) holds}\,, \\ \mathbb{X}^m(\Gamma, w\rho) &= \mathbb{K}\mathbb{H}_p^{m+\nu}(\Gamma, w\rho) && \text{and (3.7) holds}\,, \\ \mathbb{X}^m(\Gamma, w\rho) &= \mathbb{K}\mathbb{B}_{p,q}^{m+\nu}(\Gamma, w\rho) && \text{and (3.7) holds}\,. \end{aligned} \tag{3.27}$$

The following is easy to verify (cf. (3.3)):

$$(\vartheta^k \partial_t^k)(S_\Gamma \varphi)(t) = \vartheta^k (S_\Gamma \partial_\tau^k \varphi)(t) = S_\Gamma(\vartheta^k \partial_\tau^k \varphi)(t) + \sum_{j,r=0}^{k} (B_{j,r}\varphi)(t)\,, \tag{3.28}$$

where the functionals

$$\begin{aligned} &B_{j,k} \;:\; \mathbb{X}^m(\Gamma, w\rho) \longrightarrow \mathbb{C}\,, \qquad j = 1, \dots, k\,, \\ &B_{j,k}\varphi := (-1)^j \frac{t^{k-j}}{\pi i} \int_\Gamma \tau^{j-1} \partial_\tau^k \varphi(\tau) d\tau\,, \end{aligned}$$

are bounded (note that even if Γ contains open arcs, partial integration in (3.28) does not generate any summands at the boundary points, because these summands are eliminated by the factor $\vartheta^k(t)$).

In fact, from the corresponding conditions (3.5)-(3.7) we conclude that

$$\partial_\tau^k \varphi \in \mathbb{X}^{m-k}(\Gamma, w\rho) \subset \mathbb{L}_1(\Gamma)$$

and that the embedding is continuous, i.e.,

$$\|\partial_\tau^k \varphi \big| \mathbb{L}_1(\Gamma)\| \le C'_k \|\partial_\tau^k \varphi \big| \mathbb{X}^{m-k}(\Gamma, w\rho)\| \le C_k \|\varphi \big| \mathbb{X}^m(\Gamma, w\rho)\|\,.$$

Thus,

$$|B_{j,k}\varphi| \le C_j \|\varphi \big| \mathbb{X}^m(\Gamma, \rho)\| \qquad \forall\, j = 1, \dots, m\,. \tag{3.29}$$

Since the singular integral operator

$$S_\Gamma \;:\; \mathbb{X}(\Gamma, w\rho) := \mathbb{X}^0(\Gamma, w\rho) \longrightarrow \mathbb{X}(\Gamma, w\rho) \tag{3.30}$$

is bounded (see the first part of the proof) and, by definition, $\vartheta^k \partial_t^k \varphi \in \mathbb{X}^{m-k}(\Gamma, w\rho)$, from (3.28)-(3.30) we get

$$\begin{aligned} \|S_\Gamma \varphi \mid \mathbb{X}^m(\Gamma, w\rho)\| &= \sum_{k=0}^m \|\vartheta^k \partial_t^k S_\Gamma \varphi \mid \mathbb{X}(\Gamma, w\rho)\| \\ &\le \sum_{k=0}^m \|S_\Gamma \vartheta^k \partial_t^k \varphi \mid \mathbb{X}(\Gamma, w\rho)\| + \sum_{k=0}^m \sum_{j=0}^k |B_{j,k}\varphi| \\ &\le C \sum_{k=0}^m \|\vartheta^k \partial_t^k \varphi \mid \mathbb{X}(\Gamma, w\rho)\| = C\|\varphi \mid \mathbb{X}^m(\Gamma, w\rho)\|\,. \end{aligned} \tag{3.31}$$

The boundedness of $S_{\Gamma,w}$ in the modified spaces $\mathbb{K}\mathbb{X}^m(\Gamma, \rho)$ for negative $m = -1, -2, \dots$ (excluding the Hölder-Zygmund $\mathbb{K}\mathbb{Z}^0_\mu(\Gamma, \rho)$ spaces, which are defined only for positive $\mu > 0$) follows by duality.

In fact, let the operator $S_{\Gamma,w}$ be bounded in the modified weighted spaces $\mathbb{KW}_p^m(\Gamma,\rho)$, in $\mathbb{KH}_p^m(\Gamma,\rho)$, or in $\mathbb{KB}_{p,q}^m(\Gamma,\rho)$ and suppose that the conditions (3.5)-(3.7) hold. The dual operator, $S_{\Gamma,w^{-1}}$, to $S_{\Gamma,w}$ is defined by the bilinear form

$$(\varphi, S_{\Gamma,w^{-1}}\psi) := (S_{\Gamma,w}\varphi,\psi)\,, \qquad (\varphi,\psi) := \int_\Gamma \varphi(\tau)\psi(\tau)d\tau\,, \tag{3.32}$$

and is thus bounded in the dual spaces $\mathbb{KW}_{p'}^{-m}(\Gamma,\rho^{-1})$, $\mathbb{KH}_{p'}^{-m}(\Gamma,\rho^{-1})$, and $\mathbb{KB}_{p',q'}^{-m}(\Gamma,\rho^{-1})$, respectively, because $-m\in\mathbb{N}$ is already positive and the parameters p', q', $-\alpha_j, -\beta_j$ satisfy the corresponding conditions (3.5)-(3.7). □

Theorem 3.1 enables us to establish a Fredholm criterion and an index formula for a singular integral operator with complex conjugation:

$$\begin{aligned} &A\varphi := a\varphi + bS_\Gamma\varphi + cVS_\Gamma V\varphi = f\,, \quad V\varphi(t) := \overline{\varphi(t)}\,, \\ &a,b,c\in\mathbb{KPC}^{m+1}(\Gamma,\mathcal{T}_\Gamma) \\ &\qquad\text{for}\quad \varphi,\ f\in\mathbb{X}^m(\Gamma,\rho)=\mathbb{KW}_p^m(\Gamma,\rho),\ \mathbb{KH}_p^{m+\nu}(\Gamma,\rho)\,,\ \mathbb{KB}_{p,q}^{m+\nu}(\Gamma,\rho)\,, \\ &a,b,c\in\mathbb{KPZ}_{m+\nu}(\Gamma,\mathcal{T}_\Gamma)\quad\text{for}\quad \varphi,\ f\in\mathbb{X}^m(\Gamma,\rho)=\mathbb{KZ}_{m+\nu}^0(\Gamma,\rho)\,, \end{aligned} \tag{3.33}$$

where $\mathbb{X}^m(\Gamma,\rho)$ is defined by (3.27).

Although the coefficients of the operator A are $N\times N$ matrix functions and equation (3.33) is considered in weighted N-vector spaces, we use the same notation for spaces and classes of functions as in the scalar case $N=1$ for the sake of simplicity.

Also for conciseness, we assume that Γ is a closed piecewise smooth curve with smooth arcs Γ_{j-1}, Γ_j, having in common the knot t_j where they meet under the interior angle $\pi\gamma_j$ (measured from the bounded domain Ω^+ enclosed by Γ). Therefore, $0\le\gamma_j\le 2$, $j=1,\dots,n$, while the values $\gamma_j=0,2$ correspond to a cusp at t_j. This assumption simplifies the symbol of operator (3.33). In the general case the symbol can be written down in a similar but more complicated form (see [Du3, Du4, Du5, DLS1, RS1]).

When Γ has no cusps, $0<\gamma_j<2$, the symbol of the operator A in the space $\mathbb{X}^m(\Gamma,\rho)$ is defined as follows:

$$\mathcal{A}_{\mathbb{X}^m(\Gamma,\rho)}(t,\xi) := \widetilde{a}(t) + \widetilde{b}(t)S_{\mathbb{X}^m(\Gamma,\rho)}(t,\xi) + \widetilde{c}(t)\overline{S_{\mathbb{X}^m(\Gamma,\rho)}(t,-\xi)}\,, \tag{3.34}$$

where

$$S_{\mathbb{X}^m(\Gamma,\rho)}(t,\xi) := \begin{bmatrix} \coth\pi(i\beta_t+\xi) & -\dfrac{e^{\pi(\gamma_t-1)(i\beta_t+\xi)}}{\sinh\pi(i\beta_t+\xi)} \\ \dfrac{e^{\pi(1-\gamma_t)(i\beta_t+\xi)}}{\sinh\pi(i\beta_t+\xi)} & -\coth\pi(i\beta_t+\xi) \end{bmatrix}, \quad \xi\in\mathbb{R}\,, \tag{3.35}$$

$$\beta_t := \begin{cases} 1/p + \alpha_j - \nu & \text{if } t \in \Gamma\,, & \mathbb{X}^m(\Gamma,\rho) = \mathbb{KH}_p^{m+\nu}(\Gamma,\rho)\,, \\ & & \qquad\qquad \mathbb{KB}_{p,q}^{m+\nu}(\Gamma,\rho)\,, \\ 1/p & \text{if } t \neq t_1,\dots,t_n\,, & \mathbb{X}^m(\Gamma,\rho) = \mathbb{KW}_p^m(\Gamma,\rho)\,, \\ 1/2 & \text{if } t \neq t_1,\dots,t_n\,, & \mathbb{X}^m(\Gamma,\rho) = \mathbb{KZ}_{m+\nu}^0(\Gamma,\rho)\,, \\ 1/p + \alpha_j & \text{if } t = t_j\,, & \mathbb{X}^m(\Gamma,\rho) = \mathbb{KW}_p^m(\Gamma,\rho)\,, \\ \alpha_j - \nu & \text{if } t = t_j\,, & \mathbb{X}^m(\Gamma,\rho) = \mathbb{KZ}_{m+\nu}^0(\Gamma,\rho)\,, \end{cases} \tag{3.36}$$

$$\widetilde{d}(t) := \begin{bmatrix} d(t+0) & 0 \\ 0 & d(t-0) \end{bmatrix}, \qquad d \in \mathbb{PC}(\Gamma, \mathcal{T}_\Gamma)\,, \quad t \in \Gamma\,,$$

$$\gamma_t := \begin{cases} 1 & \text{if} \quad t \neq t_1,\dots,t_n\,, \\ \gamma_j & \text{if} \quad t = t_j\,. \end{cases} \tag{3.37}$$

Let us note that the symbol would be a full matrix function if the corresponding operator contains the terms VS_Γ, VaI, aV, or $S_\Gamma V$ (see Remark 3.5).

Due to assumptions (3.5)-(3.7) and (3.27) we have $0 < \beta_t < 1$ for all $t \in \Gamma$ and the symbol $\mathcal{A}_{\mathbb{X}^m(\Gamma,\rho)}(t,\xi)$ represents a piecewise continuous uniformly bounded function of the variables $(t,\xi) \in \Gamma \times \mathbb{R}$.

Theorem 3.2. *Let Γ have no cusps, i.e., $0 < \gamma_j < 2$, $j = 1,\dots,n$ and let $\mathbb{X}^m(\Gamma,\rho)$ be defined by* (3.27). *Then equation* (3.33) *is Fredholm in the space $\mathbb{X}^m(\Gamma,\rho)$ if and only if*

$$\inf_{t\in\Gamma,\ \xi\in\mathbb{R}} \left|\det \mathcal{A}_{\mathbb{X}^m(\Gamma,\rho)}(t,\xi)\right| > 0\,. \tag{3.38}$$

If condition (3.38) *holds, then*

$$\operatorname{Ind} A = -\frac{1}{2\pi}\left\{ \left[\arg\det \mathcal{A}_{\mathbb{X}^m(\Gamma,\rho)}(t,+\infty)\right]_\Gamma + \sum_{j=1}^n \left[\arg\det \mathcal{A}_{\mathbb{X}^m(\Gamma,\rho)}(t_j,\xi)\right]_\mathbb{R} \right\}.$$

If, in particular, $c = 0$ and the operator $A = aI + bS_\Gamma$ has scalar coefficients $(N = 1)$, then A is invertible in $\mathbb{X}^m(\Gamma,\rho)$ from the left or the right in dependence on whether $\operatorname{Ind} A \leq 0$ *or* $\operatorname{Ind} A \geq 0$, *respectively.*

Proof. If a singular integral operator is bounded in the space $\mathbb{X}^0(\Gamma,\rho)$, it is bounded in $\mathbb{X}^m(\Gamma,\rho)$ (see Theorem 3.1). This is also valid for any inverse operator and any regularizer to the canonical operator $A = aI + bS_\Gamma$. The same is true if $\Gamma = \mathbb{R}$ and $\rho(x) \equiv 1$, or $\Gamma = \mathbb{R}^+$ and $\rho(x) = x^\alpha$.

A similar simultaneous boundedness property, for all values of the parameter $m \in \mathbb{N}_0$, holds also for Mellin convolution operators $\mathfrak{M}_g^0$ in the spaces $\mathbb{X}^m(\mathbb{R}^+, x^\alpha)$ (for general boundedness of Mellin convolution operators we refer to J. Elschner's results in [El1] and in [PS1, Ch. 5]).

Thus, it suffices to prove the theorem for $m = 0$. For this case we apply quasi-localization (see [Du5, DLS1, Si1, Sp1, Ra1]). Note that localization in the weighted Hölder space is a special case (see [Po1, Sc1]). Let us expose here a short description of the approach. If $\mathcal{L}(\mathbb{X})$ denotes the algebra of all bounded operators in a Banach space $\mathbb{X}$ and $\mathfrak{S}(\mathbb{X}) \subset \mathcal{L}(\mathbb{X})$ is the ideal of all compact operators, then in the quotient algebra $\mathcal{L}(\mathbb{X})/\mathfrak{S}(\mathbb{X})$ (the Calkin algebra) the essential norm of an operator,

$$|||B\big|\mathcal{L}(\mathbb{X})||| := \inf_{T\in\mathfrak{S}(\mathbb{X})} \|B + T\big|\mathcal{L}(\mathbb{X})\|\,, \tag{3.39}$$

defines a norm of the coset which contains this operator.

K. Kuratowski introduced the measure of non-compactness $||\mathbb{Y}||_d$ (the Kuratowski measure) of a bounded set $\mathbb{Y} \subset \mathbb{X}$ as the minimal value of all numbers ε for which $\mathbb{Y}$ can be covered by an ε-net of a finite number of elements. The Kuratowski measure of the image of the unit sphere under an operator B,

$$||B\big|\mathcal{L}(\mathbb{X})||_d := ||B\mathcal{B}_{\mathbb{X}}(0,1)||\,, \quad \mathcal{B}_{\mathbb{X}}(0,1) := \{x \in \mathbb{X}\ :\ \|x\| = 1\}, \tag{3.40}$$

is called the measure of non-compactness of the operator B (see [AKPRS1]). Obviously, $\|B\big|\mathcal{L}(\mathbb{X})\|_d \le |||B\big|\mathcal{L}(\mathbb{X})|||$, while the equality $||B\big|\mathcal{L}(\mathbb{X})||_d = |||B\big|\mathcal{L}(\mathbb{X})|||$ holds for $\mathbb{X} = \mathbb{L}_p$ and does not hold for the Hölder-Zygmund spaces, where we have the inequality $|||B\big|\mathcal{L}(\mathbb{X})||| \le C\|B\big|\mathcal{L}(\mathbb{X})\|_d$ with some constant C independent of the operator B (see [AKPRS1, Po1]).

In [Po1], R. Pöltz proved that the Kuratowski measure of a multiplication operator gI, $g \in H_\nu(\Gamma)$, in the weighted Hölder space $H^0_\nu(\Gamma,\rho)$ coincides with the supremum-norm,

$$\begin{aligned} |||gI\big|\mathcal{L}(H^0_\nu(\Gamma,\rho))||| &\le C\|gI\big|\mathcal{L}(H^0_\nu(\Gamma,\rho))\|_d\,, \\ \|gI\big|\mathcal{L}(H^0_\nu(\Gamma,\rho))\|_d &= \|gI\big|C(\Gamma)\| := \sup_{t\in\Gamma}|g(t)|\,, \end{aligned} \tag{3.41}$$

although for the usual norm we have the inequality

$$\|gI\big|\mathcal{L}(H^0_\nu(\Gamma,\rho))\| \le \|g\big|\mathbb{P}H_\nu(\Gamma,\mathcal{T}_\Gamma)\| := \max_{j=1,\dots,n}\|g\big|H_\nu(\Gamma_j)\|\,.$$

Property (3.21) enables a localization of coefficients similar to the case of the space $\mathbb{L}_p(\Gamma,\rho)$, exposed in [DLS1] (see also [Du7, RS1]).

A local quasi-equivalent representative of A at a point $t_0 \in \Gamma$ has the form

$$\begin{aligned} A_{t_0} &:= \widetilde{a}(t_0)I + \widetilde{b}(t_0)\begin{bmatrix} S_{\mathbb{R}^+} & -N_{\mathbb{R}^+,-\gamma_{t_0}} \\ N_{\mathbb{R}^+,\gamma_{t_0}} & -S_{\mathbb{R}^+}\end{bmatrix} \\ &\quad + \widetilde{c}(t_0)V\begin{bmatrix} S_{\mathbb{R}^+} & -N_{\mathbb{R}^+,-\gamma_{t_0}} \\ N_{\mathbb{R}^+,\gamma_{t_0}} & -S_{\mathbb{R}^+}\end{bmatrix}V \\ &= \widetilde{a}(t_0)I + \widetilde{b}(t_0)\begin{bmatrix} S_{\mathbb{R}^+} & -N_{\mathbb{R}^+,-\gamma_{t_0}} \\ N_{\mathbb{R}^+,\gamma_{t_0}} & -S_{\mathbb{R}^+}\end{bmatrix} - \widetilde{c}(t_0)\begin{bmatrix} S_{\mathbb{R}^+} & -N_{\mathbb{R}^+,\gamma_{t_0}} \\ N_{\mathbb{R}^+,-\gamma_{t_0}} & -S_{\mathbb{R}^+}\end{bmatrix}, \end{aligned}$$

$$S_{\mathbb{R}^+}\varphi(x) := \frac{1}{\pi i}\int\limits_0^\infty \frac{\varphi(y)dy}{y-x}\,, \quad N_{\mathbb{R}^+,\gamma}\varphi(x) := \frac{1}{\pi i}\int\limits_0^\infty \frac{\varphi(y)dy}{y-e^{i\gamma}x}\,, \quad x\in\mathbb{R}^+ \tag{3.42}$$

(see, e.g., [DLS1]). We prove that the local quasi-equivalent representative A_{t_0} is locally invertible in the space $\mathbb{L}_p(\mathbb{R}^+, x^{\alpha_{t_0}})$ for the Sobolev space, in $H_\nu(\mathbb{R}^+, x^{\alpha_{t_0}})$ for the weighted Hölder space, and in $\mathbb{H}_p^\nu(\mathbb{R}^+, x^{\alpha_{t_0}})$ for the weighted Bessel potential space. Let us consider these cases separately and, afterwards, the case of Besov spaces.

I. *The weighted Lebesgue space* $\mathbb{L}_p(\mathbb{R}^+, x^{\alpha_{t_0}})$.

Since the operator A_{t_0} is dilation invariant, i.e.,

$$\mathcal{D}_\lambda A_{t_0} = A_{t_0}\mathcal{D}_\lambda\,, \qquad \mathcal{D}_\lambda\varphi(x) := \varphi(\lambda x)\,, \quad \forall\ \lambda,\ x\in\mathbb{R}^+\,, \tag{3.43}$$

A_{t_0} is locally invertible at $0\in\mathbb{R}^+$ if and only if it is invertible in $\mathbb{L}_p(\mathbb{R}^+, x^{\alpha_{t_0}})$ (see [Du8]).

Next we replace A_{t_0} by the equivalent operator

$$\begin{aligned} A^0_{t_0} &:= x^{\alpha_{t_0}+\frac{1}{p}} A_{t_0} x^{-\alpha_{t_0}-\frac{1}{p}} I = \widetilde{a}(t_0)I + \widetilde{b}(t_0)\mathfrak{M}^0_{S_{\mathbb{X}^m(\Gamma,\rho)}(t_0,\cdot)} + \widetilde{c}(t_0)\mathfrak{M}^0_{\overline{S_{\mathbb{X}^m(\Gamma,\rho)}(t_0,\cdot)}} \\ &= \mathfrak{M}^0_{\mathcal{A}_{\mathbb{X}^m(\Gamma,\rho)}(t_0,\cdot)} \end{aligned} \tag{3.44}$$

in the weighted space $\mathbb{L}_p(\mathbb{R}^+, x^{-1})$ (see [Du5, DLS1] and [RS1]). We recall that the Mellin convolution operator $\mathfrak{M}^0_g$ is invertible in $\mathbb{L}_p(\mathbb{R}^+, x^{-1})$ if and only if the symbol $g(\xi)$, which belongs to the L_p-multiplier class $\mathcal{M}_p(\mathbb{R})$, does not vanish, $\inf|g(\xi)|\neq 0$ for $\xi\in\mathbb{R}$, and that the inverse is $\mathfrak{M}^0_{g^{-1}}$ (see [Du6, DLS1, RS1]). The proof is completed in a standard way by application of the local principle: condition (3.38) is necessary and sufficient for A to have the Fredholm property in $\mathbb{L}_p(\Gamma,\rho)$. The index formula is proved by a standard homotopy argument (see [DLS1]).

II. *The weighted Bessel potential space* $\mathbb{H}^\nu_p(\mathbb{R}^+, x^{\alpha_{t_0}})$, $1/p - 1 < \nu - \alpha_{t_0} < 1/p$. We apply the lifting procedure (see (3.19)-(3.22)). As a result we get an equivalent operator,

$$\Lambda^\nu_- A_{t_0} \Lambda^{-\nu}_+ = \widetilde{a}(t_0) I + \widetilde{b}(t_0) \begin{bmatrix} S^\nu_{\mathbb{R}^+} & -N^\nu_{\mathbb{R}^+, -\gamma_{t_0}} \\ N^\nu_{\mathbb{R}^+, \gamma_{t_0}} & -S^\nu_{\mathbb{R}^+} \end{bmatrix}$$

$$- \widetilde{c}(t_0) \begin{bmatrix} S^\nu_{\mathbb{R}^+} & -N^\nu_{\mathbb{R}^+, \gamma_{t_0}} \\ N^\nu_{\mathbb{R}^+, -\gamma_{t_0}} & -S^\nu_{\mathbb{R}^+} \end{bmatrix}, \tag{3.45}$$

$$S^\nu_{\mathbb{R}^+} := \Lambda^\nu_- S_{\mathbb{R}^+} \Lambda^{-\nu}_+ , \qquad N^\nu_{\mathbb{R}^+, \gamma_{t_0}} := \Lambda^\nu_- N_{\mathbb{R}^+, \gamma_{t_0}} \Lambda^{-\nu}_+ .$$

It is known that all Mellin convolution operators (e.g., $N_{\mathbb{R}^+, \gamma_{t_0}}$) belong to the Banach algebra generated by the Fourier convolution operators W_a with symbols discontinuous at infinity [2], i.e., $a(-\infty) \neq a(+\infty)$. Moreover, the algebra $\mathfrak{M}(L_p(\mathbb{R}^+, x^{\alpha_{t_0}}))$ of all Mellin convolution operators in $L_p(\mathbb{R}^+, x^{\alpha_{t_0}})$ is generated by only two operators: by the identity I and by the Cauchy singular integral operator,[3] which is, at the same time, a convolution operator:

$$S_{\mathbb{R}^+} = \mathfrak{M}^0_{g_p} = W_{-\operatorname{sign} \xi} , \qquad g_p(\xi) := \coth \pi \left(\frac{i}{p} + \xi \right), \quad \xi \in \mathbb{R} \tag{3.46}$$

(see (3.22) and [Du9, Lemma 2.2]).

Since we need only the local invertibility of the lifted convolution operator

$$\Lambda^\nu_- W_a \Lambda^{-\nu}_+ = W_{a \varkappa_\nu} , \quad \varkappa_\nu(\xi) := \left(\frac{\xi - i}{\xi + i} \right)^\nu , \quad \xi \in \mathbb{R} \tag{3.47}$$

at $0 \in \mathbb{R}^+$, we can work with any local representative and its symbol. Since the operator $g W_h$ with $g(\pm\infty) = h(\pm\infty) = 0$ is compact in $L_p(\mathbb{R})$ (see, e.g., [Du9]), and $\varkappa_\nu(-\infty) = e^{-2\pi i \nu}$, $\varkappa_\nu(+\infty) = 1$, it is easy to ascertain the local equivalence

$$W_{a \varkappa_\nu} \overset{0}{\sim} W_{a_\nu} , \qquad a_\nu = e^{-2\pi i \nu} a_- \chi_-(\xi) + a_+ \chi_+(\xi) , \tag{3.48}$$

---

[2] We have already proved, by a different method, that all four entries of the lifted matrix operator $\Lambda^\nu_- A_{t_0} \Lambda^{-\nu}_+$ in (3.45) belong to the algebra generated by $S_{\mathbb{R}^+}$ and I (see (3.10)-(3.16) and (3.24)-(3.25) above).

[3] This result is proved in [Du9, Lemma 2.2] for $p = 2$, but extends easily to arbitrary $1 < p < \infty$ because the closed sub-algebra $\mathfrak{M}(L_p(\mathbb{R}^+, x^{\alpha_{t_0}})) \subset \mathfrak{M}(L_2(\mathbb{R}^+, x^{\alpha_{t_0}}))$ is generated by the same operators I and $S_{\mathbb{R}^+}$.

where $\chi_\pm(\xi)$ are the characteristic functions of the half-lines $\mathbb{R}^\pm$ and $a(\pm\infty) = a_\pm$. The symbol $\mathcal{W}_{a_\nu}(\xi)$ of W_{a_ν} is

$$\begin{aligned}\mathcal{W}_{a_\nu}(\xi) &= e^{-2\pi i\nu} a_- [1 + \coth \pi\Xi] + a_+ [1 - \coth \pi\Xi] \\ &= \mathcal{G}_p(\xi) \{a_- [1 + \coth \pi(\Xi - \nu i)] + a_+ [1 - \coth \pi(\Xi - \nu i)]\} \,,\end{aligned} \tag{3.49}$$

$$\mathcal{G}_p(\xi) := \frac{e^{-\pi\nu i} \sinh \pi(\Xi - \nu i)}{\sinh \pi\Xi} \neq 0 \,, \quad \xi \in \mathbb{R} \,, \qquad \mathcal{G}_p(\pm\infty) = e^{-\pi\nu i} \,, \tag{3.50}$$

$$\Xi := \frac{i}{p} + i\alpha_{t_0} + \xi \,,$$

and (3.50) holds since $0 < 1/p + \alpha_{t_0}$ and $1/p + \alpha_{t_0} - \nu < 1$.

As can be seen from (3.49), (3.50), to write down the symbol of the lifted operator $\Lambda^\nu_- B \Lambda^{-\nu}_+$, where B belongs to the Banach algebra generated by convolutions, we should detach the common non-vanishing factor $\mathcal{G}_p(\xi)$ and replace $1/p$ by $1/p - \nu$ in the definition of the symbol. This can also be interpreted as considering operators in the weighted space $\mathbb{L}_p(\mathbb{R}^+, x^{\alpha_{t_0} - \nu})$ instead of $\mathbb{L}_p(\mathbb{R}^+, x^{\alpha_{t_0}})$. The same holds for all four entries of the lifted operator $\Lambda^\nu_- A_{t_0} \Lambda^{-\nu}_+$ in (3.45), and the symbol of this operator acquires the form described in (3.34)-(3.37) as the symbol of A in the weighted Bessel potential space $\mathbb{K}\mathbb{H}^\nu_p(\Gamma, \rho)$. The index formula is proved by a standard homotopy argument (see [DLS1]).

III. *The weighted Hölder space* $H^0_\nu(\mathbb{R}^+, x^{\alpha_{t_0}})$, $0 < \nu < 1$, $\nu < \alpha_{t_0} < \nu + 1$. The weighted Hölder space on the half-line is defined as follows:

$$H^0_\nu(\mathbb{R}^+, x^{\alpha_{t_0}}) := \{\varphi_0 = x^{\alpha_{t_0}} \varphi \in H_\nu(\mathbb{R}^+) \ : \ \varphi_0(0) = 0\} \,,$$

$$\|\psi \mid H_\nu(\mathbb{R}^+)\| := \sup_{x \in \mathbb{R}^+} |\psi(x)| + \sup_{\substack{x_1, x_2 \in \mathbb{R}^+ \\ x_1 \neq x_2}} \frac{|\psi(x_2) - \psi(x_1)|}{\left| \dfrac{x_2}{x_2 + i} - \dfrac{x_1}{x_1 + i} \right|^\nu} \,. \tag{3.51}$$

Absolutely similar to the case of weighted L_p-spaces we prove that the operators $S^{(\alpha)}_{\mathbb{R}^+}$ and $N^{(\alpha)}_{\mathbb{R}^+,\gamma}$ (see (3.10)) belong to the Banach algebra of operators in the space $H^0_\nu(\mathbb{R}^+, x^{\alpha_{t_0}})$ generated by the two operators $S^+_\mathbb{R}$ and I provided the conditions

$$0 < \alpha + \alpha_{t_0} - \nu, \ \alpha_{t_0} < 1 \,, \quad 0 < |\gamma| < \pi \tag{3.52}$$

hold (see (3.10)-(3.16) and (3.24)-(3.25)).

The symbol of the singular integral operator $S_{\mathbb{R}_+}$ (see (3.42)) in the space $H^0_\nu(\mathbb{R}^+, x^{\alpha_{t_0}})$, given in [Du3, Du4], can be rewritten in the equivalent form

$$S_{H^0_\nu(\mathbb{R}^+, x^{\alpha_{t_0}})} := \coth \pi(i(\alpha_{t_0} - \nu + i\xi)) \,, \qquad \xi \in \mathbb{R} \,, \tag{3.53}$$

where $t = t_0$, $\beta_{t_0} = \alpha_{t_0} - \nu$ (cf. the diagonal terms in (3.35); this corresponds to the symbol of $S_{\mathbb{R}^+}$ in $L_p(\mathbb{R}^+)$ with $p = (\alpha_{t_0} - \nu)^{-1}$). It is easy to ascertain that the symbols of the entries $N_{\mathbb{R}^+, \mp\gamma_{t_0}}$ in (3.45) are exactly those which are inserted as the off-diagonal terms at $t = t_0$ in the symbol of $S_{\mathbb{Z}^0_\mu(\Gamma,\rho)}$ in (3.35). Obviously, $VS_{\mathbb{R}^+}V = -S_{\mathbb{R}^+}$ and $VN_{\mathbb{R}^+,\gamma}V = -N_{\mathbb{R}^+,-\gamma}$. Therefore we can easily write the symbol $\mathcal{A}_{t_0}(\xi)$ of the operator A_{t_0} in $H^0_\nu(\mathbb{R}^+, x^{\alpha_{t_0}})$, which coincides with the symbol $\mathcal{A}_{\mathbb{Z}^0_\mu(\Gamma,\rho)}(t_0,\xi)$ in (3.34). This accomplishes the proof for the Hölder-Zygmund space $\mathbb{Z}^0_\mu(\Gamma,\rho)$ because the condition $\inf|\det\mathcal{A}_{\mathbb{Z}^0_\mu(\Gamma,\rho)}(t_0,\xi)|\neq 0$ provides the criterion of the invertibility of the local operator A_{t_0} in the local space $H^0_\nu(\mathbb{R}^+, x^{\alpha_{t_0}})$ for all $t_0\in\Gamma$. By the local principle, this coincides with the Fredholm criterion of A in $\mathbb{H}^0_\nu(\Gamma,\rho)$ and, by the above considerations, in $\mathbb{Z}^0_\mu(\Gamma,\rho)$. The index formula is proved by a standard homotopy argument (see [DLS1]).

IV. *The weighted Besov space* $\mathbb{B}^\nu_{p,q}(\Gamma,\rho)$.
Let condition (3.38) for the space $\mathbb{X}^m(\Gamma,\rho)=\mathbb{B}^\mu_{p,q}(\Gamma,\rho)$, $\mu=m+\nu$, hold. Then, as already proved, A is Fredholm in $\mathbb{H}^\nu_p(\Gamma,\rho)$. Hence A has a regularizer R: $AR = I+T_r$, $RA=I+T_\ell$ in $\mathbb{H}^\nu_p(\Gamma,\rho)$, where T_r, T_ℓ are compact operators. There exists a sufficiently small $\varepsilon>0$ such that A has a regularizer in $\mathbb{H}^{\nu\pm\varepsilon}_p(\Gamma,\rho)$. This implies that R and the operators T_r, T_ℓ are all bounded in the spaces $\mathbb{H}^{\nu\pm\varepsilon}_p(\Gamma,\rho)$ and, due to the Interpolation Theorem 2.2.iii, R, T_r and T_ℓ are all bounded in the space $\mathbb{B}^\nu_{p,q}(\Gamma,\rho)$. Moreover, due to the interpolation property of compact operators (see Lemma 2.1), T_r, T_ℓ are both compact in $\mathbb{B}^\nu_{p,q}(\Gamma,\rho)$. Therefore, A is Fredholm in the space $\mathbb{B}^\nu_{p,q}(\Gamma,\rho)$.

If appropriate conditions hold and A is invertible in $\mathbb{H}^\nu_p(\Gamma,\rho)$, then A is invertible in $\mathbb{H}^{\nu\pm\varepsilon}_p(\Gamma,\rho)$ for a sufficiently small ε and the inverse A^{-1} is bounded in $\mathbb{B}^\nu_{p,q}(\Gamma,\rho)$ (see the foregoing case). Therefore the operator A is invertible in $\mathbb{B}^\nu_{p,q}(\Gamma,\rho)$. The index formula is proved, again, by a standard homotopy argument (see [DLS1]). □

Corollary 3.3. *Let*

$$A_0=a_0I+a_1S_\Gamma=(a_0+a_1)(P_++GP_-),\quad P_\pm:=\frac{1}{2}(I\pm S_\Gamma),\quad G:=\frac{a_0-a_1}{a_0+a_1}\,.$$

Then condition (3.38) *holds if and only if the following two conditions are satisfied:*

i. $\inf_{t\in\Gamma}|a_0(t)\pm a_1(t)|>0$*;*

ii′. $-2\pi\beta_{t_j}<\arg\dfrac{G(t_j-0)}{G(t_j+0)}<2\pi(1-\beta_{t_j})$, $j=1,\dots,n$, *where* β_{t_j} *is defined by* (3.36).

Furthermore, condition ii' *is equivalent to the following:*

ii″. $G(t)$ *has the representation*

$$G(t)=G_0(t)\prod_{j=1}^n(t-z_0)^{\nu_j}_{t_j}\,,\quad G_0\in C(\Gamma)\,,$$

$$z_0\in\Omega^+\,,\quad -\beta_{t_j}<\nu_j<1-\beta_{t_j}\,,\quad j=1,\dots,n$$

and $(t-z_0)^{\nu_j}_{t_j}$ *is taken as a branch of* $(t-z_0)^{\nu_j}$ *which has a jump only at the point* $t_j\in\Gamma$.

If conditions i *and* ii' (*or* i *and* ii") *hold, then* $\operatorname{Ind}A_0=\operatorname{ind}G$.

Remark 3.4. ¿From Theorem 3.2 we find that the Fredholm properties and the index of the operator A (see (3.33)) in the space $\mathbb{X}^m(\Gamma,\rho)$ are independent of the smoothness parameter $m \in \mathbb{N}_0$. This means that if equation (3.33) has a solution $\varphi \in \mathbb{X}^0(\Gamma,\rho)$ for a given $f \in \mathbb{X}^m(\Gamma,\rho)$, then automatically $\varphi \in \mathbb{X}^m(\Gamma,\rho)$.

Remark 3.5. Equations more general than (3.33), such as

$$\widetilde{A}\varphi := a\varphi + bV\varphi + cS_\Gamma\varphi + dVS_\Gamma\varphi + eS_\Gamma V\varphi + gVS_\Gamma V\varphi = f\,, \tag{3.54}$$

are linear in the space $\mathbb{X}^m(\Gamma,\rho)$ over the field of the real numbers $\mathbb{R}$. After "doubling" the equation by adding the composition $V\widetilde{A}\varphi = Vf$ and introducing new vector-functions $\Phi := (\varphi, V\varphi)$, $F := (f, Vf)$, we get the equivalent equation

$$\begin{bmatrix} a & b \\ \overline{b} & \overline{a} \end{bmatrix}\Phi + \begin{bmatrix} c & e \\ \overline{d} & \overline{g} \end{bmatrix} S_\Gamma\Phi + \begin{bmatrix} g & d \\ \overline{e} & \overline{c} \end{bmatrix} VS_\Gamma V\Phi = F\,, \tag{3.55}$$

which is linear (the same as in (3.33)) and can be treated in the space $\mathbb{X}^m(\Gamma,\rho)$ over the field of complex numbers $\mathbb{C}$ (see [DL1, Li1]). We will only indicate the symbol of the operator $\widetilde{A}$ because the Fredholm properties and the index are defined by the symbol as in Theorem 3.2 (note that we do not need to double the size of the symbol of the operator $\widetilde{A}$ as this was done for the operator A in order to characterize the Fredholm property and the index of A). Namely,

$$\begin{aligned}\mathcal{A}_{\mathbb{X}^m(\Gamma,\rho)}(t,\xi) &:= \widetilde{a}(t) + \widetilde{b}(t)\mathcal{V} + \widetilde{c}(t)S_{\mathbb{X}^m(\Gamma,\rho)}(t,\xi) \\ &+\widetilde{d}(t)\mathcal{V}S_{\mathbb{X}^m(\Gamma,\rho)}(t,\xi) + \widetilde{e}(t)S_{\mathbb{X}^m(\Gamma,\rho)}(t,\xi)\mathcal{V} + \widetilde{g}\mathcal{V}S_{\mathbb{X}^m(\Gamma,\rho)}(t,\xi)\mathcal{V}\,,\end{aligned} \tag{3.56}$$

where, in addition to (3.34)-(3.37), we have to indicate the symbol $\mathcal{V} = \mathcal{V}_{\mathbb{X}^m(\Gamma,\rho)}$ of the complex conjugate operator:

$$\mathcal{V} := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

which is independent of the point $t \in \Gamma$ and the space $\mathbb{X}^m(\Gamma,\rho)$.

Note that if $\mathcal{B}_{\mathbb{X}^m(\Gamma,\rho)}(t,\xi)$ is the symbol of B, the symbol for the operator VBV is

$$(\mathcal{VBV})_{\mathbb{X}^m(\Gamma,\rho)}(t,\xi) = \overline{\mathcal{B}(t,-\xi)}$$

(see [DLS1, § 1]). Therefore, $\mathcal{V}S_{\mathbb{X}^m(\Gamma,\rho)}(t,\xi)\mathcal{V} = \overline{S_{\mathbb{X}^m(\Gamma,\rho)}(t,-\xi)}$ (cf. (3.35)).

Remark 3.6. The readers familiar with [Du3, Du5, DLS1] will find differences in writing the symbol of the operators A (cf. (3.34)-(3.37)): the symbol of the operator A_0 defined in [Du3, Du5, DLS1] has a block-diagonal form

$$\mathcal{A}_{\mathbb{X}^m(\Gamma,\rho)}(t,\xi) = \begin{bmatrix} (\mathcal{A}_0)_{\mathbb{X}^m(\Gamma,\rho)}(t,\xi) & 0 \\ 0 & \overline{(\mathcal{A}_0)_{\mathbb{X}^m(\Gamma,\rho)}(t,-\xi)} \end{bmatrix}.$$

It turns out that it is sufficient to consider only the first block as a symbol of A_0. It is obvious that this does not influence the Fredholm criterion

$$\inf\det(\mathcal{A}_0)_{\mathbb{X}^m(\Gamma,\rho)}(t,\xi)\neq 0,$$

while for the index formula with $\mathcal{A}_{\mathbb{X}^m(\Gamma,\rho)}(t,\xi)$ we need to add the factor 1/2.

4. Application: an oblique derivative problem for the Laplacian in a domain with a piecewise smooth boundary

Throughout this section Γ is a closed, oriented, simple (i.e., without self-intersections), piecewise Ljapunov curve in the complex plane $\mathbb{C}$, which borders a bounded domain Ω^+ as well as an unbounded domain Ω^- and has knots at $\mathcal{T}_\Gamma := \underbrace{\{t_1,\dots,t_n\}}\subset\Gamma$. The boundary $\Gamma=\partial\Omega^+=\partial\Omega^-$ consists of n arcs $\Gamma_j := [t_j,t_{j+1}] = t_j t_{j+1}$, $j=1,\dots,n$, which are μ-smooth (see § 1) and oriented, with μ as in (1.4). Let $\pi\gamma_j$ be the interior angle with respect to Γ between Γ_{j-1} and Γ_j at the knot $t_j\in\mathcal{T}_\Gamma$ ($0\le\gamma_j\le 2$, $j=1,...,n$). When $\gamma_j=0$ or $\gamma_j=2$, the domain Ω^+ has an outward or an inward peak, respectively, or, what is the same, the boundary curve Γ has a cusp. As usual, $\vec{\nu}(t):=(\nu_1(t),\nu_2(t))$ denotes the outer unit normal vector to Ω^+ at the point $t\in\Gamma\setminus\mathcal{T}_\Gamma$.

The main objective of the present section is to study the following boundary value problem (BVP): find a harmonic function

$$\Delta u(x)=0\,,\quad x\in\Omega^\pm \tag{4.1}$$

with given oblique derivative (also known as the Poincaré problem)

$$(\partial_{\vec{\ell}(t)}u)^\pm(t)+c(t)u^\pm(t)=f(t)\,,\qquad t\in\Gamma, \tag{4.2}$$

$$\partial_{\vec{\ell}(t)} := \ell_1(t)\partial_{t_1}+\ell_2(t)\partial_{t_2},$$

where the coefficients are piecewise smooth such that

$$\mathrm{Im}\ \ell_1(t)\equiv\mathrm{Im}\ \ell_2(t)\equiv\mathrm{Im}\ c(t)\equiv 0\,,\qquad \ell_1,\ell_2,c\in\mathbb{KPC}^m(\Gamma,\mathcal{T}_\Gamma)\,, \tag{4.3}$$

and the space $\mathbb{KPC}^m(\Gamma,\mathcal{T}_{\Gamma_1})$ of piecewise m-smooth functions is defined similarly to (1.16).

It is common to write the oblique derivative boundary condition (4.2) in the form

$$a(t)(\partial_{\vec{\nu}(t)}u)^\pm(t)+b(t)(\partial_{\vec{s}(t)}u)^\pm(t)+c(t)u^\pm(t)=f(t)\,,\qquad t\in\Gamma\,, \tag{4.4}$$

where

$$a(t)=\ell_1(t)\cos\vartheta_t+\ell_2(t)\sin\vartheta_t\,,\qquad b(t)=-\ell_1(t)\sin\vartheta_t+\ell_2(t)\cos\vartheta_t\,, \tag{4.5}$$

and

$$\partial_{\vec{\nu}(t)} := \cos\vartheta_t\partial_{t_1}+\sin\vartheta_t\partial_{t_2}\,,\qquad \partial_{\vec{s}(t)} := -\sin\vartheta_t\partial_{t_1}+\cos\vartheta_t\partial_{t_2} \tag{4.6}$$

are the normal and the tangential derivatives, respectively, i.e., the derivatives with respect to the outer unit normal vector and the positively directed tangent vector at $t \in \Gamma$,

$$\vec{\nu}(t) := (\cos\vartheta_t, \sin\vartheta_t)\,, \qquad \vec{s}(t) := (-\sin\vartheta_t, \cos\vartheta_t)\,. \tag{4.7}$$

As usual, ϑ_t denotes the inclination of the outer unit normal vector with respect to the abscissa axis (see, e.g., [Mu1, § 74] and the recent book [Pa1]).

In the particular case where the oblique derivative vector $\vec{\ell}(t) = (\ell_1(t), \ell_2(t))$ coincides with the outer unit normal vector $\vec{\ell}(t) = \vec{\nu}(t)$ and $c(t) \equiv 0$, we get the Neumann BVP. If $\vec{\ell}(t) \equiv 0$ and $c(t) \equiv 1$ we have the Dirichlet BVP.

It is known that the usual function spaces, $\mathbb{W}^1_2(\Omega^\pm)$ for the solutions and $\mathbb{W}_2^{-\frac{1}{2}}(\Gamma)$ for the right-hand sides, cannot ensure solvability and uniqueness of solutions of BVPs in domains with outward peaks (see [DS1]). To describe suitable function spaces for the solutions and boundary data we recall the modified Smirnov-Sobolev space $\mathcal{KW}^m_p(\overline{\Omega^\pm}, \rho)$. This space consists of all functions in $\Omega^\pm$ which have finite norm

$$\|\psi\big|\mathcal{KW}^m_p(\overline{\Omega^\pm}, \rho)\| := \sup_{0<r<1} \|\psi\big|\mathbb{KW}^m_p(\Gamma^{(r)}, \rho)\|\,,$$

where $\Gamma^{(r)} := \{z = \omega(r\zeta) \;:\; |\zeta| = 1\}$ are the images of the concentric circles of radius r under the conformal mapping of the unit disk $\mathcal{D}_1$ onto the domain $\Omega^\pm$,

$$\omega \;:\; \mathcal{D}_1 \longrightarrow \Omega^\pm\,. \tag{4.8}$$

The weight function $\rho(t)$ is defined by (1.12) and we assume that the following conditions hold:

$$1 < p < \infty\,, \quad m = 0, \pm 1, \ldots, \,, \quad -\frac{1}{p} < \alpha_j < 1 - \frac{1}{p}\,, \quad j = 1, \ldots, n\,. \tag{4.9}$$

An equivalent definition of the modified Smirnov-Sobolev spaces $\mathcal{KW}^m_p(\overline{\Omega^\pm}, \rho)$ is the following: $\Phi \in \mathcal{KW}^m_p(\overline{\Omega^\pm}, \rho)$ if and only if $\Phi(z)$ is represented by the Cauchy integral in the form

$$\begin{aligned} &\Phi(z) = c_0 + C_\Gamma\varphi(z)\,, \quad c_0 = \text{const}\,, \quad \varphi \in \mathbb{KW}^m_p(\Gamma, \rho)\,, \\ &C_\Gamma\varphi(z) := \frac{1}{2\pi i}\int_\Gamma \frac{\varphi(\tau)d\tau}{\tau - z}\,, \quad z \in \Omega^\pm\,, \end{aligned} \tag{4.10}$$

and for a compact Ω^+ one can take $c_0 = 0$ (cf. [Pv1]).

We know that a function $\Phi \in \mathbb{W}^m_p(\overline{\Omega^\pm}, \rho)$ in general has traces $\Phi^\pm$ on the boundary Γ only for $m \geq 1$, see [Tr1]; the same is true for the modified spaces $\Phi \in \mathbb{KW}^m_p(\overline{\Omega^\pm}, \rho)$. In contrast to this fact, a function from the modified Smirnov-Sobolev space $\Phi \in \mathcal{KW}^m_p(\Gamma, \rho)$, represented by the Cauchy integral in (4.10), has the traces

$$\Phi^\pm(t) = c_0 \pm \frac{1}{2}\varphi(t) + \frac{1}{2}S_\Gamma\varphi(t)\,, \qquad \Phi^\pm \in \mathbb{KW}^m_p(\Gamma, \rho) \tag{4.11}$$

for arbitrary $m = 0, \pm1, \ldots$, $1 < p < \infty$ provided $\rho(t)$ is defined in (1.12) and conditions (4.9) hold. For a negative $m = -1, -2, \ldots$, the space $\mathbb{KW}^m_p(\Omega^\pm, \rho)$ is defined as the dual space to $\mathbb{KW}^{-m}_{p'}(\Omega^\pm, \rho^{-1})$, where $p' := p/(p-1)$.

The Sokhotski-Plemelj [Mu1] formulae (4.11) are well known for Hölder continuous (see [Mu1]) and Lebesgue integrable functions (see [GK1]) and for functions $\varphi \in \mathbb{KW}^m_p(\Gamma, \rho)$, which follows from Theorem 3.1.i since the mentioned spaces are dense in $\mathbb{KW}^m_p(\Gamma, \rho)$ under the asserted conditions.

If $\vec{\ell}(t) \not\equiv 0$ we take the right-hand side of (4.2) in the modified Sobolev space $\mathbb{KW}^{m-1}_p(\Gamma, \rho)$ and look for the solutions in the corresponding Smirnov-Sobolev space $\mathcal{KW}^m_{p,loc}(\overline{\Omega^\pm}, \rho)$,

$$\begin{aligned} &f \in \mathbb{KW}^{m-1}_p(\Gamma, \rho)\,, \qquad u \in \mathcal{KW}^m_p(\overline{\Omega^+}, \rho) \quad \text{for} \quad \Omega^+\,, \\ &u \in \mathcal{KW}^m_{p,loc}(\overline{\Omega^-}, \rho)\,, \quad u(x) = \mathcal{O}(1) \quad \text{as} \quad |x| \to \infty \quad \text{for} \quad \Omega^-\,. \end{aligned} \tag{4.12}$$

We suppose that $\rho(t)$ is defined by (1.12) and that conditions (4.9) hold.

If $\vec{\ell}(t) \equiv 0$ we get the Dirichlet problem and replace (4.12) by

$$\begin{aligned} &f \in \mathbb{KW}^{m}_p(\Gamma, \rho)\,, \qquad u \in \mathcal{KW}^m_p(\overline{\Omega^+}, \rho) \quad \text{for} \quad \Omega^+\,, \\ &u \in \mathcal{KW}^m_{p,loc}(\overline{\Omega^-}, \rho)\,, \quad u(x) = \mathcal{O}(1) \quad \text{as} \quad |x| \to \infty \quad \text{for} \quad \Omega^-\,. \end{aligned} \tag{4.13}$$

If the domain $\Omega^\pm$ has no outward peak, conditions (4.12) can be replaced by the following equivalent conditions, which are simpler:

$$\begin{aligned} &f \in \mathbb{KW}^{m-1}_p(\Gamma, \rho)\,, \qquad u \in \mathbb{KW}^m_p(\overline{\Omega^+}, \rho) \quad \text{for} \quad \Omega^+\,, \\ &u \in \mathbb{KW}^m_{p,loc}(\overline{\Omega^-}, \rho)\,, \quad u(x) = \mathcal{O}(1) \quad \text{as} \quad |x| \to \infty \quad \text{for} \quad \Omega^-\,, \end{aligned}$$

and similarly for (4.13) (see [DSi1]).

Theorem 4.1. *Let Γ be piecewise $\mathbb{C}^m$-smooth, let the weight function $\rho(t)$ be defined by* (1.12) *and let conditions* (4.9) *be satisfied. Let $\ell_1, \ell_2, c \in \mathbb{KPC}^m(\Gamma)$* (*see* (4.3)) *and introduce $G(t) := \ell_1(t) + i\ell_2(t)$.*

Further, let $\mathbb{T} := \{\zeta \in \mathbb{C} \;:\; |\zeta| = 1\}$ be the unit circle and $\mathcal{T}_{\mathbb{T}} := \{\zeta_j \;:\; \omega(\zeta_j) = t_j\,,\ j = 1, \ldots, n\}$ (*see* (4.3)) *be the pre-image of all knots* (*the angular points and peaks*) *of Γ under the conformal mapping $\omega(z)$ of* (4.8).

The oblique derivative problem (4.1), (4.4) (*or* (4.1), (4.2)) *is Fredholm if and only if one of the following conditions* **A** *or* **B** *is satisfied.*

A. $\inf_{t\in\Gamma} |G(t)| \neq 0$, *conditions* (4.12) *hold, and the following singular integral equation on the unit circle is Fredholm:*

$$P^+_{\mathbb{T}}\varphi(\zeta) + F(\zeta)P^-_{\mathbb{T}}\varphi(\zeta) = f_0(\zeta)\,, \qquad \zeta \in \mathbb{T}\,, \tag{4.14}$$

where

$$P^{\pm}_{\mathbb{T}} := \frac{1}{2}(I \pm S_{\mathbb{T}}) , \qquad f_0, \varphi \in \mathbb{KW}^{m-1}(\mathbb{T}, \mathcal{T}_{\mathbb{T}}) ,$$

$$F(\zeta) := \rho(\omega(\zeta))\overline{G(\omega(\zeta))}[G(\omega(\zeta))\overline{\rho(\omega(\zeta))}]^{-1}[\omega'(\zeta)]^{\frac{1}{p}}[\overline{\omega'(\zeta)}]^{-\frac{1}{p}} , \tag{4.15}$$

$$f_0(\zeta) := 2[G(\omega(\zeta))]^{-1}\rho(\omega(\zeta))[\omega'(\zeta)]^{\frac{1}{p}} f(\omega(\zeta)) , \quad \zeta \in \mathbb{T} . \tag{4.16}$$

B. $G(t) \equiv 0$, $\inf_{t\in\Gamma} |c(t)| \neq 0$, *conditions* (4.13) *hold and the following singular integral equation on the unit circle is Fredholm:*

$$P^{+}_{\mathbb{T}}\psi(\zeta) + F(\zeta)P^{-}_{\mathbb{T}}\psi(\zeta) + \frac{F(\zeta)-1}{2}K_0\psi(\zeta) = f_0(\zeta) , \tag{4.17}$$

$$F(\zeta) := \rho(\omega(\zeta))[\overline{\rho(\omega(\zeta))}]^{-1}[\omega'(\zeta)]^{\frac{1}{p}}[\overline{\omega'(\zeta)}]^{-\frac{1}{p}} , \qquad f_0, \psi \in \mathbb{KW}^{m}_{p}(\mathbb{T}, \mathcal{T}_{\mathbb{T}}) ,$$

$$f_0(\zeta) := 2[c(\omega(\zeta))]^{-1}\rho(\omega(\zeta))[\omega'(\zeta)]^{\frac{1}{p}} f(\omega(\zeta)) , \quad \zeta \in \mathbb{T} .$$

If one of these two conditions is satisfied we have furthermore in the corresponding case:

A. *The indices of the BVP* (4.1), (4.4) *and of the integral equation* (4.14) *are equal.*

The coefficient $F(\zeta)$ *in* (4.14) *is piecewise smooth, i.e.,* F *is in* $\mathbb{KPC}^m(\mathbb{T}, \mathcal{T}_{\mathbb{T}})$ (*see Remark* 4.2).

If $c(t) \equiv 0$, *the BVP* (4.1), (4.4) *and the modified* (*with the help of the one-dimensional operator* K_0) *integral equations*

$$\begin{cases} P^{+}_{\mathbb{T}}\varphi(\zeta) + F(\zeta)P^{-}_{\mathbb{T}}\varphi(\zeta) = f_0(\zeta) , \\ K_0\varphi(\zeta) := \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \varphi^{-}(e^{i\vartheta})d\vartheta = 0 , \end{cases} \qquad for \quad \Omega^{-} , \tag{4.18}$$

$$P^{+}_{\mathbb{T}}\varphi(\zeta) + F(\zeta)P^{-}_{\mathbb{T}}\varphi(\zeta) + \frac{F(\zeta)-1}{2}K_0\varphi(\zeta) = f_0(\zeta) , \quad for \quad \Omega^{+}$$

are equivalent in the sense that there is a one-to-one correspondence between their solutions.

B. *The indices of the BVP* (4.1), (4.4) *and of the integral equation* (4.17) *are equal and, moreover, they are equivalent in the sense that there is a one-to-one correspondence between their solutions.*

The coefficient $F(\zeta)$ *in* (4.17) *is piecewise smooth, that is,* F *is in* $\mathbb{KPC}^m(\mathbb{T}, \mathcal{T}_{\mathbb{T}})$ (*see Remark* 4.2).

Proof. The oblique derivative problem (4.1), (4.4) (or (4.1), (4.2)) can also be written as follows (see [Mu1, § § 74,75]):

$$\mathrm{Re}\,[G(t)(\Psi')^{\pm}(t) + c(t)\Psi^{\pm}(t)] = f(t) , \quad t \in \Gamma ,$$

$$u(x) = \mathrm{Re}\,\Psi(x) , \quad \Psi \in \mathcal{KW}^{m}_{p}(\overline{\Omega^{\pm}}, \rho) , \quad x \in \Omega^{\pm} , \tag{4.19}$$

$$G(t) = \ell_1(t) + i\ell_2(t) = e^{i\vartheta_t}a(t) + ie^{i\vartheta_t}b(t) = e^{i\vartheta_t}a(t) + e^{i\frac{\pi}{2}+i\vartheta_t}b(t)$$

(see (4.5)). Indeed, since

$$\Psi = u + iv \in \mathcal{KW}^m_p(\overline{\Omega^\pm}, \rho)\,, \quad \Psi' := \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \in \mathcal{KW}^{m-1}_p(\overline{\Omega^\pm}, \rho)\,,$$

with the help of (4.5) and (4.6) we get

$$\begin{aligned} \mathrm{Re}\left[G(t)(\Psi')^\pm(t) + c(t)\Psi^\pm(t)\right] &= \ell_1(t)(\partial_{t_1} u)^\pm(t) + \ell_2(t)(\partial_{t_2} u)^\pm(t) + c(t)u^\pm(t) \\ &= a(t)(\partial_{\vec{\nu}(t)} u)^\pm(t) + b(t)(\partial_{\vec{s}(t)} u)^\pm(t) + c(t)u^\pm(t) \end{aligned}$$

and (4.19) follows.

The case **B.** Thus, we suppose $G(t) \equiv 0$ and follow the scheme of [DSi1, Theorem 1.16]. The analytic function defined by

$$\Phi(z) := \begin{cases} \rho(\omega(z))[\omega'(z)]^{\frac{1}{p}}\Psi(\omega(z)) & \text{for} \quad |z| < 1\,, \\ \overline{\rho\left(\omega\left(\frac{1}{\overline{z}}\right)\right)}\,\left[\overline{\omega'\left(\frac{1}{\overline{z}}\right)}\right]^{\frac{1}{p}}\,\overline{\Psi\left(\omega\left(\frac{1}{\overline{z}}\right)\right)} & \text{for} \quad |z| > 1 \end{cases} \tag{4.20}$$

belongs to the space $\mathcal{KW}^m_p(\overline{\mathcal{D}_1}, \mathcal{T}_\mathbb{T})$. This can be verified straightforwardly with the help of the following property of the conformal mapping ω:

$$\prod_{\zeta_j \in \Theta} (z - \zeta_j)^k \partial^k_z \omega \in C(\overline{\mathcal{D}_1}) \tag{4.21}$$

for all $k = 1, \ldots, m$, where $m \in \mathbb{N}$. Notice that property (4.21) was already proved in [DSi2, Theorem 5.1].

For the analytic function $\Phi(z)$ in (4.20) the boundary condition (4.19) acquires the form

$$\mathrm{Re}\left[c(\omega(\zeta))\Psi^\pm(\omega(\zeta))\right] = \frac{c(\omega(\zeta))}{2}\left[\frac{\Phi^+(\zeta)}{\rho(\omega(\zeta))[\omega'(\zeta)]^{\frac{1}{p}}} - \frac{\Phi^-(\zeta)}{\overline{\rho(\omega(\zeta))}\,\overline{[\omega'(\zeta)]}^{\frac{1}{p}}}\right] = f(\omega(\zeta))\,,$$

which can also be written as follows:

$$\Phi^+(\zeta) - F(\zeta)\Phi^-(\zeta) = f_0(\zeta)\,, \quad \zeta \in \mathbb{T}\,, \tag{4.22}$$

with $F(\zeta)$ and $f_0(\zeta)$ defined by (4.17). It is easy to verify by having recourse to (4.21) that $f_0 \in \mathbb{KW}^m_p(\mathbb{T}, \mathcal{T}_\mathbb{T})$.

Since $\Phi \in \mathcal{KW}^m_p(\overline{\mathcal{D}_1}, \mathcal{T}_\mathbb{T})$, it has a representation of the form

$$\Phi(z) = -\frac{i}{2}K_0\psi + C_\mathbb{T} i\psi(z) = -\frac{i}{4\pi}\int_{-\pi}^{\pi}\psi(e^{i\vartheta})d\vartheta + \frac{1}{2\pi}\int_{|\tau|=1}\frac{\psi(\tau)d\tau}{\tau - z} \tag{4.23}$$

for all $|z| \neq 1$ with a density $i\psi$, $\psi \in \mathbb{KW}^m_p(\mathbb{T}, \mathcal{T}_\mathbb{T})$. If we apply the Sokhotski-Plemelj formulae for the boundary values of Φ (see (4.11)) we obtain (for a density ψ)

$$\Phi^\pm(\zeta) = -\frac{1}{2}K_0\psi \pm \frac{1}{2}\left[\psi(\zeta) \pm S_\mathbb{T}\psi(\zeta)\right] = -\frac{1}{2}K_0\psi \pm P^\pm_\mathbb{T}\psi(\zeta)\,, \quad \zeta \in \mathbb{T}\,,$$

and inserting this into (4.22) we get (4.17) for the density $\psi \in \mathbb{KW}_p^m(\mathbb{T}, \mathcal{T}_\mathbb{T})$.

Let us remind that we need only the real-valued solution $\psi = \operatorname{Re}\psi$ of (4.17). To this end let us verify that if $\psi \in \mathbb{KW}_p^m(\mathbb{T}, \mathcal{T}_\mathbb{T})$ is a solution, then $\overline{\psi}$ is a solution as well. In fact, applying the relations

$$\overline{\zeta} = \frac{1}{\zeta}\,, \quad |\zeta| = 1\,, \quad \overline{\tau} = \frac{1}{\tau}\,, \quad d\overline{\tau} = -\frac{d\tau}{\tau^2}\,, \quad \frac{d\tau}{\tau} = id\vartheta \ \text{ for } \ \tau = e^{i\vartheta}\,, \quad -\pi < \vartheta < \pi$$

we find that

$$\begin{aligned}
&\overline{F(\zeta)} = F^{-1}(\zeta)\,, \quad \overline{f_0(\zeta)} = F^{-1}(\zeta) f_0(\zeta) \quad \text{since} \quad \overline{f} = f\,,\\
&\overline{P_\mathbb{T}^\pm \psi(\zeta)} = \frac{1}{2}\overline{\psi(\zeta)} \mp \frac{1}{2\pi i} \int\limits_{|\tau|=1} \frac{\overline{\psi(\tau)}\overline{d\tau}}{\overline{\tau} - \overline{\zeta}} = \frac{1}{2}\overline{\psi(\zeta)} \mp \frac{1}{2\pi i} \int\limits_{|\tau|=1} \frac{\zeta}{\tau} \frac{\overline{\psi(\tau)} d\tau}{\tau - \zeta}\\
&\qquad = P_\mathbb{T}^\mp \overline{\psi}(\zeta) \pm \frac{1}{2\pi i} \int\limits_{|\tau|=1} \overline{\psi(\tau)} \frac{d\tau}{\tau} = P_\mathbb{T}^\mp \overline{\psi}(\zeta) \pm K_0 \overline{\psi}\,.
\end{aligned} \tag{4.24}$$

Now, if $\psi_0 \in \mathbb{KW}_p^m(\mathbb{T}, \mathcal{T}_\mathbb{T})$ is a solution of equation (4.17), taking the complex conjugate and invoking (4.24) we get the same equality for $\overline{\psi}_0$:

$$P_\mathbb{T}^+ \overline{\psi}_0(\zeta) + F(\zeta) P_\mathbb{T}^- \overline{\psi}_0(\zeta) + \frac{F(\zeta) - 1}{2} K_0 \overline{\psi}_0 = f_0(\zeta)\,, \quad \zeta \in \mathbb{T}\,.$$

Therefore, the real-valued function $\psi := \operatorname{Re}\psi = \frac{1}{2}(\psi_0 + \overline{\psi}_0)$ is a solution that we look for.

With a solution $\psi = \operatorname{Re}\psi$ of (4.17) at hand we find $\Phi(z)$ from (4.22), but the latter has the following symmetry property:

$$\Phi_*(z) := \overline{\Phi\left(\frac{1}{\overline{z}}\right)} = \Phi(z)\,, \quad z \in \Omega^+ \cup \Omega^-\,,$$

as it follows from the definition (4.20). This property can be verified similarly to (4.24):

$$\begin{aligned}
\Phi_*(z) &= \overline{\Phi\left(\frac{1}{\overline{z}}\right)} = \frac{i}{2} K_0 \psi + \frac{1}{2\pi} \int\limits_{|\tau|=1} \frac{\psi(\tau)\overline{d\tau}}{\overline{\tau} - \frac{1}{z}} = \frac{i}{2} K_0 \psi + \frac{1}{2\pi} \int\limits_{|\tau|=1} \frac{z}{\tau} \frac{\psi(\tau) d\tau}{\tau - z}\\
&= -\frac{i}{2} K_0 \psi + \frac{1}{2\pi} \int\limits_{|\tau|=1} \frac{\psi(\tau) d\tau}{\tau - z} = -\frac{i}{2} K_0 \psi + i C_\mathbb{T} \psi(z) = \Phi(z)\,.
\end{aligned} \tag{4.25}$$

Inserting $\Phi(z)$ in (4.20) we find first $\Psi(z)$ and afterwards $u = \operatorname{Re}\Psi$.

Conversely, if $\psi(\zeta)$ is a solution of (4.17) we easily ascertain that $\Psi(z)$ defined by (4.23) and (4.20) solves the BVP (4.19) and $u(z) = \operatorname{Re}\,\Psi(z)$ solves the Dirichlet BVP (4.1), (4.2), (4.13) with $\vec{\ell} \equiv 0$.

The case **A.** In this case we can ignore $c(t)$ (take $c(t) \equiv 0$) because, after equivalent reduction, the corresponding summand in the integral equation has a weakly singular kernel (the corresponding operator is compact) and has no influence on the Fredholm property and the index of the equation. In the rest of the proof we follow the scheme of [DSi1, Theorem 1.17].

The analytic function

$$\Phi(z) := \begin{cases} \rho(\omega(z))[\omega'(z)]^{\frac{1}{p}}\,\Psi'(\omega(z)) & \text{for} \quad |z| < 1\,, \\ \overline{\rho\left(\omega\left(\frac{1}{\overline{z}}\right)\right)}\,\overline{\left[\omega'\left(\frac{1}{\overline{z}}\right)\right]}^{\frac{1}{p}}\,\overline{\Psi'\left(\omega\left(\frac{1}{\overline{z}}\right)\right)} & \text{for} \quad |z| > 1\,, \end{cases} \tag{4.26}$$

belongs to the space $\mathcal{KW}_p^{m-1}(\overline{\mathcal{D}_1}, \mathcal{T}_{\mathbb{T}})$. This can be verified straightforwardly with the help of (4.21).

For the analytic function $\Phi(z)$ in (4.26) we get the following BVP:

$$\Phi^+(\zeta) - F(\zeta)\Phi^-(\zeta) = f_0(\zeta)\,, \quad \zeta \in \mathbb{T}\,, \tag{4.27}$$

where $f_0(\zeta)$ and $F(\zeta)$ are defined in (4.15)-(4.16). It is easy to see, by applying (4.21), that $f_0 \in \mathbb{KW}_p^{m-1}(\mathbb{T}, \mathcal{T}_{\mathbb{T}})$.

Since $\Phi \in \mathcal{KW}_p^{m-1}(\overline{\mathcal{D}_1}, \mathcal{T}_{\mathbb{T}})$, it has a representation by the Cauchy integral

$$\Phi(z) = -\frac{i}{2}K_0\varphi + C_{\mathbb{T}}i\varphi(z) = -\frac{i}{4\pi}\int_{-\pi}^{\pi}\varphi(e^{i\vartheta})d\vartheta + \frac{1}{2\pi}\int_{|\tau|=1}\frac{\varphi(\tau)d\tau}{\tau - z} \tag{4.28}$$

for all $|z| \neq 1$ with the density $i\varphi$, $\varphi \in \mathbb{KW}_p^{m-1}(\mathbb{T}, \mathcal{T}_{\mathbb{T}})$. If we apply the Sokhotski-Plemelj formulae for the boundary values (see (4.11)) we get equation (4.18).

Note that for the domain Ω^- we have to require in addition (see the condition in (4.18)) that

$$K_0\varphi = \frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(e^{i\vartheta})d\vartheta = 0\,.$$

To justify this we remind that $\Psi \in \mathcal{KW}_p^m(\overline{\Omega^\pm}, \rho)$ and that the derivative must vanish at infinity, i.e., $\Psi'(\infty) = 0$ (see (4.12)); therefore (see (4.26), (4.28))

$$\int_{-\pi}^{\pi}\varphi(e^{i\vartheta})d\vartheta = 2\pi\Phi(0) = 2\pi\rho(\omega(0))[\omega'(0)]^{\frac{1}{p}}\,\Psi'(\omega(0)) = 0$$

because $\omega(0) = \infty$.

Since we need only real-valued solutions $\varphi = \mathrm{Re}\,\varphi$ of (4.18), we verify by analogy to (4.24) that, together with φ_0, equations (4.18) have the solution $\overline{\varphi_0}$. Therefore the real-valued solution $\varphi := \mathrm{Re}\,\varphi_0 = \frac{1}{2}(\varphi_0 + \overline{\varphi_0})$ is the one we look for.

The function $\Phi(z)$ in (4.28) must have the symmetry property $\Phi_*(z) = \Phi(z)$ (cf. (4.25) and (4.26)). This can also be verified with the help of properties similar to (4.24) (see (4.25)).

Conversely, if $\varphi = \mathrm{Re}\,\varphi$ is a real-valued solution of (4.18), then the function $\Phi(z)$ defined by (4.26) solves the BVP (4.27), which implies that $u(x) := \mathrm{Re}\ \Phi(z)$ solves the BVP (4.1), (4.2) and (4.12) with $c(t) \equiv 0$. □

Remark 4.2. The coefficient $F(\zeta)$ in (4.14) and in (4.17) is piecewise smooth, namely $F \in \mathbb{KPC}^m(\mathbb{T}, \mathcal{T}_\mathbb{T})$. Moreover, we can indicate the jumps at the knots:

$$\begin{aligned}
\frac{F(\zeta_j - 0)}{F(\zeta_j + 0)} &= \frac{G(t_j + 0)}{G(t_j - 0)} \overline{\left[\frac{G(t_j - 0)}{G(t_j + 0)}\right]} \frac{\rho(\omega(\zeta_j - 0))}{\rho(\omega(\zeta_j + 0))} \overline{\left[\frac{\rho(\omega(\zeta_j + 0))}{\rho(\omega(\zeta_j - 0))}\right]} \\
&\quad \times \left[\frac{\omega'(\zeta_j - 0)}{\omega'(\zeta_j + 0)}\right]^{\frac{1}{p}} \overline{\left[\frac{\omega'(\zeta_j + 0)}{\omega'(\zeta_j - 0)}\right]^{\frac{1}{p}}} \\
&= \exp\left\{2i[\arg G(t_j + 0) - \arg G(t_j - 0)] - 2\pi i \left(\frac{1}{p} + \alpha_j\right)(1 - \gamma_j)\right\} \qquad (4.29) \\
&= \exp\left\{2i[\arg G(t_j + 0) - \arg G(t_j - 0)] - \frac{2\pi i}{p} - 2\pi i \alpha_j + 2\pi i \left(\frac{1}{p} + \alpha_j\right)\gamma_j\right\},
\end{aligned}$$

where $\pi\gamma_j$ is the interior angle at the knot $t_j \in \mathcal{T}_\Gamma$ and α_j is the exponent of the weight at the same t_j. In fact,

$$\begin{aligned}
\frac{\rho(\omega(\zeta_j - 0))}{\rho(\omega(\zeta_j + 0))} &= \lim_{\varepsilon \to 0} \left[\frac{\omega(e^{-i\varepsilon}\zeta_j) - t_j}{\omega(e^{i\varepsilon}\zeta_j) - t_j}\right]^{\alpha_j} = \lim_{\varepsilon \to 0} \left[\frac{\dfrac{\omega(e^{-i\varepsilon}\zeta_j) - \omega(\zeta_j)}{e^{-i\varepsilon}\zeta_j - \zeta_j}}{\dfrac{\omega(e^{i\varepsilon}\zeta_j) - \omega(\zeta_j)}{e^{i\varepsilon}\zeta_j - \zeta_j}}\right]^{\alpha_j} \\
&= \left[\frac{\omega'(\zeta_j - 0)}{\omega'(\zeta_j + 0)}\right]^{\alpha_j} = \exp[-2\pi i \alpha_j (1 - \gamma_j)].
\end{aligned}$$

¿From (4.29) and Corollary 3.3 it is clear that even if $G(t)$ is continuous at one of the outward peaks,

$$G(t_j - 0) = G(t_j + 0) \quad \text{when} \quad \gamma_j = 0,$$

then the corresponding singular integral operator in (4.14) and (4.17) is not Fredholm (moreover, is not normally solvable, i.e., has non-closed image).

Due to Theorem 4.1 we are able to apply Theorem 3.2 to the oblique derivative problem (4.1), (4.4) (or to (4.1), (4.2); cf. [DSi1]).

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