Crack-Type Boundary Value Problems of Electro-Elasticity

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Dedicated to the memory of Erhard Meister

Abstract. Dirichlet, Neumann and mixed crack-type boundary value problems of statics are considered in three-dimensional bounded domains filled with a homogeneous anisotropic electro-elastic medium. Applying the method of the potential theory and the theory of pseudodifferential equations, we prove the existence and uniqueness theorems in Besov and Bessel potential spaces, and derive full asymptotic expansion of solutions near the crack edge.

Introduction

The recent years have shown an ever-growing interest in the investigation of models of an anisotropic elastic medium which take into account the influence of various physical fields such as thermal, electric, magnetic etc. A rather strong motivation for such studies is the creation of new artificial materials which possess non-standard properties. Among them are piezoelectric materials that form the core of modern structures and instruments.

Mathematical models of piezoelectric (electro-elastic) bodies and relevant boundary value problems have been studied with sufficient completeness (see, e.g., [BG2, No1, Pa1, To1] and the references therein). Of special interest is the case where the considered body contains cracks or cuts with an edge having a bihedral angle 2π (cuspidal edge)(see [Ch2, CD3]). In that case the presence of an electric field essentially changes the pattern of stress distribution near the cut or crack edge (see [DNS1, Pa1]).

In this work, Dirichlet, Neumann and mixed boundary value problems of statics are considered for a homogeneous anisotropic piezoelectric body with a crack. The existence and uniqueness of solutions of the considered problems are proved in Bessel potential $\mathbb{H}^s_p$ (and Besov $\mathbb{B}^s_{p,q}$) spaces. Complete asymptotic expansion of a solution near the crack edge is obtained. These results are important in the analysis of a stress field in electro-elastic bodies with cracks.

1. Formulation of Boundary Value Problems

Let $\Omega = \Omega_0$ and Ω_1 ($\overline{\Omega}_1 \subset \Omega$) be bounded domains in the three-dimensional Euclidean space $\mathbb{R}^3$ with infinitely smooth boundaries $\partial\Omega = \partial\Omega_0$ and $\partial\Omega_1$ respectively (we use the alternative notation $\Omega_0 = \Omega$ for conciseness of forthcoming formulae). The boundary $\partial\Omega_1$ of the domain Ω_1 (called interface) is divided in two parts: $\partial\Omega_1 = \overline{S}_0 \cup S_1$ with a smooth common boundary $\mathscr{E} := \partial S_0 = \partial S_1$. Let $\Omega_2 = \Omega\backslash\overline{\Omega}_1$.

We assume, that the domain $\Omega\backslash S_1$ is filled with homogeneous anisotropic electro-elastic material, having a crack at S_1 .

We use the following notation for function spaces: $\mathbb{W}^s_p(\Omega)$, $\mathbb{W}^s_p(\partial\Omega_i)$ for the Sobolev-Slobodetskij spaces; $\mathbb{H}^s_p(\Omega)$, $\mathbb{H}^s_p(\partial\Omega_i)$ for the Bessel potential spaces; $\mathbb{B}^s_{p,q}(\Omega)$, $\mathbb{B}^s_{p,q}(\partial\Omega_i)$ for the Besov spaces ($i = 1, 2$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$), (see [Tr1, Tr2] for the definitions and properties of these spaces). We use the common simplified notation $\mathbb{H}^s(\partial\Omega_i) = \mathbb{H}^s_2(\partial\Omega_i)$. Let further,

$$\mathbb{H}^s_p(S_j) = \{r_{S_j} u : \quad u \in \mathbb{H}^s_p(\partial\Omega_1)\} ,$$
$$\widetilde{\mathbb{H}}^s_p(S_j) = \{u \in \mathbb{H}^s_p(\partial\Omega_1) : \quad \mathrm{supp} u \in \overline{S_j}\} , \qquad j = 0, 1 ,$$

where $r_{S_j}\varphi := \varphi\Big|_{S_j}$ denotes the restriction operator onto the subset S_j . Similarly the Sobolev-Slobodetskii spaces $\mathbb{W}^s_p(S_j)$, $\widetilde{\mathbb{W}}^s_p(S_0)$, $\mathbb{B}^s_{p,q}(S_j)$ and $\widetilde{\mathbb{B}}^s_{p,q}(S_0)$ are defined.

We consider the system of static equations of electro-elasticity for a homogeneous anisotropic medium [No1]

$$\mathbf{A}(D)u(x) + F(x) = 0 , \qquad x \in \Omega\backslash S_1 , \tag{1.1}$$

where $u = (u_1, u_2, u_3, u_4)$; u_1, u_2, u_3 are displacement vector components, u_4 is an electric potential, F is a mass force. $\mathbf{A}(D)$ is a differential operator of the form

$$\begin{gathered} \mathbf{A}(D) = \|\mathbf{A}_{jk}(D)\|_{4\times 4} , \\ \mathbf{A}_{jk}(D) = c_{ijlk}\partial_i\partial_l , \quad j, k = 1, 2, 3, \\ \mathbf{A}_{j4}(D) = e_{kjl}\partial_k\partial_l , \quad j = 1, 2, 3, \\ \mathbf{A}_{4k}(D) = -e_{ikl}\partial_i\partial_l, \quad k = 1, 2, 3, \\ \mathbf{A}_{44}(D) = \varepsilon_{il}\partial_i\partial_l , \end{gathered} \tag{1.2}$$

where c_{ijlk} , e_{ikl} , ε_{ik} are the elastic, piezoelectric and dielectric constants, respectively.

Here and in what follows we use the standard convention: the summation is carried out over the repeated indices.

The constants in (1.2) satisfy the symmetry conditions

$$\begin{aligned} c_{ijlk} = c_{jilk} = c_{lkij}, \quad e_{kjl} = e_{klj}, \quad \varepsilon_{ik} = \varepsilon_{ki}, \\ i, j, k, l = 1, 2, 3 \end{aligned} \tag{1.3}$$

and the condition which provides positiveness of the internal energy:

$$\forall \xi_{ij}, \quad \eta_i, \quad \xi_{ij} = \xi_{ji}, \quad \exists c_0 > 0 \quad c_{ijkl}\xi_{ij}\xi_{kl} \geq c_0\xi_{ij}\xi_{ij}, \quad \varepsilon_{ij}\eta_i\eta_j \geq c_0\eta_i\eta_i. \tag{1.4}$$

The operator $\mathbf{A}(D)$ has a strongly elliptic symbol:

$$\mathbf{A}(\xi) := \|\mathbf{A}_{jk}(\xi)\|_{4\times 4},$$

$$\begin{aligned}
&\mathbf{A}_{jk}(\xi) = -c_{ijkl}\xi_i\xi_l, \quad j,k = 1,2,3,\\
&\mathbf{A}_{j4}(\xi) = -e_{kjl}\xi_k\xi_l, \quad j = 1,2,3,\\
&\mathbf{A}_{4k}(\xi) = e_{ikl}\xi_i\xi_l, \quad j = 1,2,3,\\
&\mathbf{A}_{44}(\xi) = -\varepsilon_{il}\xi_i\xi_l,
\end{aligned} \tag{1.5}$$

i.e. it satisfies the inequality

$$-\operatorname{Re}\sum_{i,k=1}^{4}\mathbf{A}_{ik}(\xi)\eta_k\overline{\eta}_i \geq c_0|\xi|^2|\eta|^2 \tag{1.6}$$

for some constant $c_0 > 0$ and arbitrary $\xi \in \mathbb{R}^3$ and $\eta \in \mathbb{C}^4$. Here the overbar $\overline{\eta}_i$ denotes the complex conjugate to η_i.

Inequality (1.6) follows from (1.3) and (1.4). Note that, in contrast to an analogous operator of classical elasticity [Fi1], the operator $\mathbf{A}(D)$ is neither a formally self adjoint nor a positive definite operator.

Let Ω be a bounded domain with a piecewise-smooth boundary, $u, v \in C^2(\overline{\Omega})$. Then the following Green formulae are valid:

$$\int\limits_{\Omega}\big[\overline{v}(x)\mathbf{A}(D)u(x) + E(u,v)\big]dx = \int\limits_{\partial\Omega}\overline{v}(y)T(\partial_y, n(y))u(y)dS, \tag{1.7}$$

$$\int\limits_{\Omega}\big[\overline{v}(x)\mathbf{A}(D)u(x) - u(x)\mathbf{A}^{\top}(D)\overline{v}(x)\big]dx$$
$$= \int\limits_{\partial\Omega}\big[\overline{v}(y)T(\partial_y, n(y))u(y) - u(y)\widetilde{T}(\partial_y, n(y))\overline{v}(y)\big]dS, \tag{1.8}$$

where $\mathbf{A}^{\top}$ is the transposed matrix to $\mathbf{A}$, $n(y) = (n_1(y), n_2(y), n_3(y))$ is the outward unit normal vector to Ω_2 at the point $y \in \partial\Omega \cup \partial\Omega_1$,

$$\begin{aligned}
T(\partial_y, n) &= \big\|T_{jk}(\partial_y, n)\big\|_{4\times 4},\\
T_{jk}(\partial_y, n) &= c_{ijlk}n_l\partial_i, \quad j,k = 1,2,3,\\
T_{j4}(\partial_y, n) &= e_{kjl}n_l\partial_k, \quad j = 1,2,3,\\
T_{4k}(\partial_y, n) &= -e_{ikl}n_i\partial_l, \quad k = 1,2,3,\\
T_{44}(\partial_y, n) &= \varepsilon_{ij}n_j\partial_i\,,
\end{aligned}$$

the operator $\widetilde{T}(\partial_y, n(y))$ is obtained from $T(\partial_y, n(y))$ by substituting e_{ijk} with $-e_{ijk}$, and $E(u,v)$ denotes the sesquilinear form

$$E(u,v) \quad = c_{ijkl}\partial_i\overline{v}_j\partial_l u_k + e_{kjl}\partial_l\overline{v}_j\partial_k u_4 - e_{ikl}\partial_i\overline{v}_4\partial_l u_k \quad +\varepsilon_{il}\partial_i\overline{v}_4\partial_l u_4\,. \tag{1.9}$$

Note, that the Green formulae (1.7) and (1.8) are also valid for unbounded domains with a compact boundary provided u and v meet the following constraints at infinity:

$$v_i(y)\partial_k u_j(y) = o(|y|^{-2}) \text{ and } u_i(y)\partial_k v_j(y) = o(|y|^{-2}), \qquad (1.10)$$
$$i,j = 1,2,3,4,5, \quad k = 1,2,3.$$

Let $r^k u := r_{\Omega_k} u$ denote the restriction operator to the domain Ω_k, $k = 1,2$, and γ^i -the trace operator from the domain Ω_i to the boundary $\partial\Omega_i$, $i = 0,1,2$. We remind that if $u \in \mathbb{W}^1_p(\Omega_i)$ then the trace $\gamma^i u \in \mathbb{W}^{1/p'}_p(\partial\Omega_1) = \mathbb{B}^{1/p'}_{p,p}(\partial\Omega_1)$ ($\gamma^i u \in \mathbb{H}^{1/2}(\partial\Omega_1)$ when $p = 2$), $i = 1,2$, $p' = p/(p-1)$ (see [Tr1]). Moreover, if $\mathbf{A}(D)u \in \mathbb{L}_p(\Omega)$ the trace $\gamma^i T(\partial_y, n)(r^i u)$ exists and is defined through the Green formula (1.7) (see [DNS1, NCS1]). In particular, if $u \in \mathbb{W}^1_p(\Omega)$ is a solution of equation (1.1) with $F \in \mathbb{L}_q(\Omega)$, $q \geq np/(n+p)$, then $\mathbf{A}(D)u \in \mathbb{L}_q(\Omega)$ and the traces

$$\gamma^i u = r_{S_1}\{\gamma^i(r^i u)\},$$
$$\gamma^i T(\partial_y, n)u = r_{S_1}\{\gamma^i T(\partial_y, n)(r^i u)\}, \quad i = 1,2,$$

are defined correctly. The trace $\gamma_{\partial\Omega} T(\partial_y, n)u$ on $\partial\Omega$ is defined similarly.

Based on the above arguments, we will consider the following boundary value problems (BVPs) in the domain $\Omega\backslash S_1$: we look for a function $u \in \mathbb{W}^1(\Omega)$ [1] which solves the following BVPs: :

the Dirichlet BVP:

$$\begin{cases} \mathbf{A}(D)u = 0 & \text{in } \Omega\backslash S_1, \\ \gamma^0 T(\partial_y, n)u = \psi & \text{on } \partial\Omega, \\ \gamma^i u = \varphi_i, \quad i = 1,2, & \text{on } S_1, \end{cases} \qquad (1.11)$$

where $\varphi_i \in \mathbb{H}^{1/2}(S_1)$, $i = 1,2$, $\psi \in \mathbb{H}^{-1/2}(\partial\Omega)$;

the Neumann BVP:

$$\begin{cases} \mathbf{A}(D)u = 0 & \text{in } \Omega\backslash S_1, \\ \gamma^0 T(\partial_y, n)u = \psi & \text{on } \partial\Omega, \\ \gamma^i T(\partial_y, n)u = \psi_i, \quad i = 1,2, & \text{on } S_1, \end{cases} \qquad (1.12)$$

where $\psi_i \in \mathbb{H}^{-1/2}(S_1)$, $i = 1,2$, $\psi \in \mathbb{H}^{-1/2}(\partial\Omega)$.

the mixed BVP:

$$\begin{cases} \mathbf{A}(D)u = 0 & \text{in } \Omega\backslash S_1, \\ \gamma^0 T(\partial_y, n)u = \psi & \text{on } \partial\Omega, \\ \gamma^1 u = \varphi_1, & \text{on } S_1, \\ \gamma^2 T(\partial_y, n)u = \psi_2, & \text{on } S_1, \end{cases} \qquad (1.13)$$

where $\varphi_1 \in \mathbb{H}^{1/2}(S_1)$, $\psi_2 \in \mathbb{H}^{-1/2}(S_1)$, $\psi \in \mathbb{H}^{-1/2}(\partial\Omega)$.

Note, that in the above BVPs (1.11) - (1.13) boundary conditions on the surface S_1 differ, while they are the same on the boundary $\partial\Omega$.

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[1] For simplicity we drop the case $p \neq 2$, i.e. when $u \in \mathbb{W}^1_p(\Omega)$); these cases can be considered as in [DNS1, NCS1, Ch2].

Theorem 1.1. *The Dirichlet and the mixed BVPs have a unique solution* $u \in \mathbb{W}^1(\Omega \backslash S_1)$ *, and the solution of the Neumann boundary value problem is defined up to a summand* $u = (u_1, u_2, u_3, u_4)$ *:*

$$u_i = \varepsilon_{ijk} a_j x_k + b_i, \quad i = 1, 2, 3, \quad v_4 = b_4 \,. \tag{1.14}$$

Here a_j *,* $j = 1, 2, 3$ *,* b_i *,* $i = 1, 2, 3, 4$ *, are arbitrary constants,* ε_{ijk} *are the* ε *-tensor components (the Levi-Civita symbol) .*

Proof. The proof is based on the Green formula (1.6) and is standard (see similar proofs in [BG1, DNS1, NCS1, Fi1] etc.). ■

2. Properties of Potentials

Denote by H a fundamental solution of the operator $\mathbf{A}(D)$ and consider the simple layer potentials

$$V^{(k)}(g)(x) = \int\limits_{\partial\Omega_k} H(x-y) g(y) d_y S, \quad x \in \Omega_1 \cup \Omega_2, \quad k = 0, 1 \,.$$

Theorem 2.1. *Let* $\partial\Omega, \partial\Omega_1$ *be* C^β *-smooth,* $\beta \geq |s| + 1 + 1/p$ *,* $1 < p < \infty$ *,* $1 \leq q < \infty$ *. Then the operators* $V^{(1)}$ *,* $V^{(0)}$ *extend to the continuous operators*

$$V^{(1)} \;\; : \;\; \mathbb{B}^s_{p,q}(\partial\Omega_1) \to \mathbb{B}^{s+1+1/p}_{p,q}(\Omega_i) \bigcap \mathbb{H}^{s+1+1/p}_p(\Omega_i), \quad i = 1, 2,$$

$$V^{(0)} \;\; : \;\; \mathbb{B}^s_{p,q}(\partial\Omega) \to \mathbb{B}^{s+1+1/p}_{p,q}(\Omega) \bigcap \mathbb{H}^{s+1+1/p}_p(\Omega).$$

We consider the following integral operators on the surfaces

$$V^{(1)}_{-1}(g)(z) = \int\limits_{\partial\Omega_1} H(z-y) g(y) d_y S,$$

$$V^{(1)}_0(g)(z) = \int\limits_{\partial\Omega_1} T(\partial_z, n(z)) H(z-y) g(y) d_y S, \;\; z \in \partial\Omega_1 = \partial\Omega_2,$$

$$V^{(0)}_{-1}(h)(z) = \int\limits_{\partial\Omega} H(z-y) h(y) d_y S,$$

$$V^{(0)}_0(h)(z) = \int\limits_{\partial\Omega} T(\partial_z, n(z)) H(z-y) h(y) d_y S, \;\; z \in \partial\Omega,$$

which are the direct values of the corresponding layer potentials. Then for the traces of the layer potentials we have the following Plemelj formulae.

Theorem 2.2. *Let* $\partial\Omega, \partial\Omega_1$ *be* C^β *-smooth,* $\beta \geq 3$ *,* $g \in C^3(\partial\Omega_1)$ *,* $h \in C^3(\partial\Omega_1)$ *. Then*

$$\gamma^i V^{(1)}(g)(z) = V_{-1}^{(1)}(g)(z), \quad z \in \partial\Omega_1;$$

$$\gamma^0 V^{(0)}(h)(z) = V_{-1}^{(0)}(h)(z), \quad z \in \partial\Omega\,.$$

$$\gamma^i\big(T(\partial_z, n(z))V^{(1)}(g)\big)(z) = \frac{(-1)^{i+1}}{2}g(z) + V_0^{(1)}(g)(z), \qquad z \in \partial\Omega_1,$$

$$\gamma^0\big(T(\partial_z, n(z))V^{(0)}(h)\big)(z) = -\frac{1}{2}h(z) + V_0^{(0)}(h)(z), \qquad z \in \partial\Omega,$$

Theorem 2.3. *Let* $\partial\Omega, \partial\Omega_1$ *be* C^β *-smooth,* $\beta \geq |s| + 3$ *,* $1 < p < \infty$ *,* $1 \leq q \leq \infty$ *. Let* $\mathbb{X}^s = \mathbb{H}_p^s$ *or* $\mathbb{X}^s = \mathbb{B}_{p,q}^s$ *. Then the operators* $V_{-1}^{(i)}$ *,* $V_0^{(i)}$ *extend to the following continuous operators:*

$$\begin{aligned} &V_k^{(1)} : \mathbb{X}^s(\partial\Omega_1) \to \mathbb{X}^{s-k}(\partial\Omega_1), \\ &V_k^{(0)} : \mathbb{X}^s(\partial\Omega) \to \mathbb{X}^{s-k}(\partial\Omega) \qquad \text{for} \quad k = 0, -1\,. \end{aligned} \tag{2.1}$$

Theorem 2.4. *The operators in* (2.1) *are strongly elliptic*

$$\begin{aligned} &\mathrm{Re}\langle V_{-1}^{(1)}g, g\rangle \leq 0 \qquad \forall g \in \mathbb{H}_2^{-1/2}(\partial\Omega_1)\,, \\ &\mathrm{Re}\langle V_{-1}^{(0)}h, h\rangle \leq 0 \qquad \forall h \in \mathbb{H}_2^{-1/2}(\partial\Omega) \end{aligned} \tag{2.2}$$

and the equalities in (2.3) *are achieved only for* $g = 0, h = 0$ *.*

Moreover, the operators $V_{-1}^{(1)}$ *and* $V_{-1}^{(0)}$ *in* (2.1) *are invertible.*

Although the operators $V_{-1}^{(i)}$, $i = 0, 1$ are not self-adjoint as in the theory of elasticity, the proofs of Theorems 2.1 - 2.4 does not differ from those proved in [DNS1, Sh2] for the classical elasticity case.

The operator $r_{S_0}\mathbf{A} = r_{S_0}\big(V_{-1}^{(1)}\big)^{-1}$ is strongly elliptic and the following is true (cf. [DNS1, CD1]).

Theorem 2.5. *Let* $1 < p < \infty$ *,* $1 \leq q \leq \infty$ *. Then the operator*

$$\begin{aligned} r_{S_0}\mathbf{A} \quad &: \quad \widetilde{\mathbb{H}}_p^s(S_0) \to \mathbb{H}_p^{s-1}(S_0) \\ &: \quad \widetilde{\mathbb{B}}_{p,q}^s(S_0) \to \mathbb{B}_{p,q}^{s-1}(S_0) \end{aligned} \tag{2.3}$$

has the Fredholm property if and only if the conditions

$$\frac{1}{p} - \frac{1}{2} < s < \frac{1}{p} + \frac{1}{2} \tag{2.4}$$

hold. Moreover, if (2.4) holds, the operator (2.3) is invertible.

Theorem 2.6. (see [BG2, NCS1]). *Let* $1 < p < \infty$, $1 \le q \le \infty$, $s \in \mathbb{R}$. *The singular integral operator*

$$\begin{aligned} -\frac{1}{2}I + V_0^{(0)} \quad &: \quad \widetilde{\mathbb{H}}^s_p(\partial\Omega) \to \mathbb{H}^s_p(\partial\Omega) \\ &: \quad \widetilde{\mathbb{B}}^s_{p,q}(\partial\Omega) \to \mathbb{B}^s_{p,q}(\partial\Omega) \end{aligned} \tag{2.5}$$

is Fredholm with index 0 .

Theorem 2.7. *Let* $1 < p < \infty$, $1 \le q \le \infty$, $s \in \mathbb{R}$. *Then the pseudodifferential operator*

$$\begin{aligned} \mathbf{B}^{(i)}_N &:= (-V^{(1)}_{-1})^N + \Big(-\frac{1}{2}\,I + (-1)^i V^{(1)}_0\Big)(V^{(1)}_{-1})^{-1} \\ \mathbf{B}^{(i)}_N \quad &: \quad \widetilde{\mathbb{H}}^s_p(\partial\Omega_1) \to \mathbb{H}^{s-1}_p(\partial\Omega_1) \\ &: \quad \widetilde{\mathbb{B}}^s_{p,q}(\partial\Omega_1) \to \mathbb{B}^{s-1}_{p,q}(\partial\Omega_1) \end{aligned} \tag{2.6}$$

is invertible for $N = 0, 1, \dots$ *and* $i = 1, 2$. *Moreover under the conditions*

$$\frac{1}{p} - \frac{1}{2} < s < \frac{1}{p} + \frac{1}{2} \tag{2.7}$$

the following operator is invertible:

$$\begin{aligned} r_{S_0}\Big[\big(\mathbf{B}^{(1)}_N\big)^{-1} + \big(\mathbf{B}^{(2)}_N\big)^{-1}\Big] \quad &: \quad \widetilde{\mathbb{H}}^{s-1}_p(S_0) \to \mathbb{H}^s_p(S_0) \\ &: \quad \widetilde{\mathbb{B}}^{s-1}_{p,q}(S_0) \to \mathbb{B}^s_{p,q}(S_0)\,. \end{aligned} \tag{2.8}$$

Proof. The proof of the first part is similar to Lemmata 5.2 and 6.2, in [Ch2].

The second part follows because the pseudodifferential operator $r_{S_0}\big(\mathbf{B}^{(i)}_N\big)^{-1}$, $i = 1, 2$, is strongly elliptic. ■

Let us consider a $N \times N$ system of pseudodifferential equations on $\mathbb{R}^n_+$

$$\mathbf{R}(x, D)\chi = \Psi \tag{2.9}$$

with a matrix symbol $\sigma_R(x, \xi)$, $x \in \overline{\mathbb{R}}^n_+$, $\xi \in \mathbb{R}^n$ from the Hörmander class $\mathbb{S}^r_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Let $\lambda_j(x')$, $j = 1, \dots, N$ be the eigenvalues of the matrix

$$\big(\sigma_R(x', 0, 0, +1)\big)^{-1}\sigma_R(x', 0, 0, -1), \quad x' \in \mathbb{R}^{n-1} = \partial\mathbb{R}^n_+\,.$$

Lemma 2.8. *Let the symbol* $\sigma_R(x, \xi)$ *be strongly elliptic and*

$$\delta(x') = \sup_{1 \le j \le N} \frac{1}{2\pi}\big|\arg \lambda_j(x')\big|$$

$$1 < p < \infty\,, \qquad 1 \le q \le \infty\,, \qquad \frac{1}{p} - 1 + \frac{r}{2} + \delta(x') < s < \frac{1}{p} + \frac{r}{2} - \delta(x')\,.$$

Then the operator

$$\begin{aligned} \mathbf{R}(t, D) \quad &: \quad \widetilde{\mathbb{H}}^s_p(\mathbb{R}^n_+) \oplus \widetilde{\mathbb{H}}^s_p(\mathbb{R}^n_+) \to \mathbb{H}^{s-r}_p(\mathbb{R}^n_+) \oplus \mathbb{H}^{s-r}_p(\mathbb{R}^n_+) \\ &: \quad \widetilde{\mathbb{B}}^s_{p,q}(\mathbb{R}^n_+) \oplus \widetilde{\mathbb{B}}^s_{p,q}(\mathbb{R}^n_+) \to \mathbb{B}^{s-r}_{p,q}(\mathbb{R}^n_+) \oplus \mathbb{B}^{s-r}_{p,q}(\mathbb{R}^n_+) \end{aligned}$$

is invertible.

Proof. The result follows from the general theory of pseudodifferential equations on manifolds with boundary in [CD1, Du1, Sh1, Sh2]. ∎

3. The Dirichlet Problem

Let $\Phi_0^{(i)} \in \mathbb{H}^{1/2}(\partial\Omega_1)$ be some extension of the function $\varphi_i \in \mathbb{H}^{1/2}(S_1)$ on $\partial\Omega_1$, $i = 1, 2$. Then any extension $\Phi^{(i)}$ of the function φ_i on $\partial\Omega_1$ has the form

$$\Phi^{(i)} = \Phi_0^{(i)} + \varphi_0^{(i)}, \quad \varphi_0^{(i)} \in \widetilde{\mathbb{H}}^{1/2}(S_0).$$

We will seek a solution of all considered problems in the following form:

$$\begin{aligned} r^1 u = V^{(1)} g_1 \quad \text{in} \quad \Omega_1\,, \qquad & g_1 \in \mathbb{H}^{1/2}(\partial\Omega_1) \\ r^2 u = V^{(1)} g_2 + V^{(0)} h \quad \text{in} \quad \Omega_2\,, \qquad & g_2 \in \mathbb{H}^{1/2}(\partial\Omega_1)\,, \quad h \in \mathbb{H}^{1/2}(\partial\Omega)\,. \end{aligned} \tag{3.1}$$

Since the operator $\mathbf{A}(D)$ is elliptic, any generalized solution of the homogeneous equation (1.1) is an analytic function in the domain $\Omega \backslash S_1$. Then

$$\begin{cases} \gamma^1\{r^1 u\} - \gamma^2\{r^2 u\} = 0 & \text{on } S_0, \\ \gamma^1\{T(r^1 u)\} - \gamma^2\{T(r^2 u)\} = 0 & \text{on } S_0,\ . \end{cases} \tag{3.2}$$

Due to the boundary conditions of the Dirichlet problem we have

$$\begin{cases} \Big(-\frac{1}{2}I + V_0^{(0)}\Big)h + \gamma_{\partial\Omega}\Big(TV^{(1)}\Big)g_2 = \psi & \text{on } \partial\Omega, \\ r_{S_1} V_{-1}^{(1)} g_1 = \varphi_1 & \text{on } S_1, \\ r_{S_1} V_{-1}^{(1)} g_2 + \gamma_{S_1}\Big(V^{(0)}\Big)h = \varphi_2 & \text{on } S_1, \end{cases}$$

Taking into account (3.2) we get the system of pseudodifferential equations with respect to $(h, g_1, g_2, \varphi_0^{(1)}, \varphi_0^{(2)})$:

$$\begin{cases} \Big(-\frac{1}{2}I + V_0^{(0)}\Big)h + \gamma_{\partial\Omega}\Big(TV^{(1)}\Big)g_2 = \psi & \text{on } \partial\Omega, \\ V_{-1}^{(1)} g_1 - \varphi_0^{(1)} = \Phi_0^{(1)} & \text{on } \partial\Omega_1, \\ V_{-1}^{(1)} g_2 - \varphi_0^{(2)} + \gamma_{\partial\Omega_1}\Big(V^{(0)}\Big)h = \Phi_0^{(2)} & \text{on } \partial\Omega_1, \\ \varphi_0^{(1)} - \varphi_0^{(2)} = -r_{S_0}\Phi_0^{(1)} + r_{S_0}\Phi_0^{(2)} & \text{on } S_0, \\ r_{S_0}\Big(\frac{1}{2}I + V_0^{(1)}\Big)g_1 - r_{S_0}\Big(-\frac{1}{2}I + V_0^{(1)}\Big)g_2 - \gamma_{S_0}\Big(TV^{(0)}\Big)h = 0 & \text{on } S_0, \end{cases} \tag{3.3}$$

where the integral operators $\gamma_{\partial\Omega}\Big(TV^{(1)}\Big)$, $\gamma_{\partial\Omega_1}\Big(V^{(0)}\Big)$ and $\gamma_{S_0}\Big(TV^{(0)}\Big)$ have infinitely smooth kernels (the integration and the outer variable vary on disjoint sets) and are therefore compact. Now we formulate the basic theorems about the existence and uniqueness of a solution to the Dirichlet problem.

Theorem 3.1. *The Dirichlet problem has a unique solution in the class* $\mathbb{W}^1(\Omega\backslash S_1)$ *, written as follows*

$$\begin{aligned} r^1 u &= V^{(1)}\big(\big(V_{-1}^{(1)}\big)^{-1}(\Phi_0^{(1)}+\varphi_0^{(1)})\big), \\ r^2 u &= V^{(1)}\big(\big(V_{-1}^{(1)}\big)^{-1}(\Phi_0^{(2)}+\varphi_0^{(2)}-r_{\partial\Omega_1}V^{(0)}h\big)+V^{(0)}h\,. \end{aligned} \tag{3.4}$$

Here $\Phi_0^{(i)}\in\mathbb{H}^{1/2}(\partial\Omega_1)$ *is a fixed extension of the function* φ_i *on* $\partial\Omega_1$ *,* $h\in\mathbb{H}^{-1/2}(\partial\Omega)$ *and* $\varphi_0^{(1)},\varphi_0^{(2)}\in\widetilde{\mathbb{H}}^{1/2}(S_0)$ *can be found from the system (3.3), which is uniquely solvable.*

Moreover, let $1<r<\infty$ *,* $1\le q\le\infty$ *,* $\frac{1}{r}-\frac{1}{2}<s<\frac{1}{r}+\frac{1}{2}$ *and* $u\in\mathbb{W}^1(\Omega\backslash S_1)$ *be the solution of the Dirichlet problem. Then*

$$\begin{aligned} &u\in\mathbb{H}_r^{s+1/r}(\Omega\backslash S_1)\ \text{ if }\ \psi\in\mathbb{B}_{r,r}^{s-1}(\partial\Omega),\ \ \varphi_i\in\mathbb{B}_{r,r}^{s}(S_1),\ \ i=1,2, \\ &u\in\mathbb{B}_{r,q}^{s+1/r}(\Omega\backslash S_1)\ \text{ if }\ \psi\in\mathbb{B}_{r,q}^{s-1}(\partial\Omega),\ \ \varphi_i\in\mathbb{B}_{r,q}^{s}(S_1),\ \ i=1,2\,. \end{aligned} \tag{3.5}$$

Proof. The system (3.3) can be rewritten in the matrix form

$$\mathbf{M}\begin{bmatrix} h \\ g_1 \\ g_2 \\ \varphi_0^{(1)} \\ \varphi_0^{(2)} \end{bmatrix} = \begin{bmatrix} \Psi \\ \Phi_0^{(1)} \\ \Phi_0^{(2)} \\ -r_{S_0}\Phi_0^{(1)}+r_{S_0}\Phi_0^{(2)} \\ 0 \end{bmatrix},$$

where

$$\mathbf{M}=\begin{bmatrix} -\frac{1}{2}I+V_0^{(0)} & 0 & \gamma_{\partial\Omega}\Big(TV^{(1)}\Big) & 0 & 0 \\ 0 & V_{-1}^{(1)} & 0 & -I & 0 \\ \gamma_{\partial\Omega_1}\Big(V^{(0)}\Big) & 0 & V_{-1}^{(1)} & 0 & -I \\ 0 & 0 & 0 & I & -I \\ -\gamma_{S_0}\Big(TV^{(0)}\Big) & r_{S_0}\Big(\frac{1}{2}I+V_0^{(1)}\Big) & -r_{S_0}\Big(-\frac{1}{2}I+V_0^{(1)}\Big) & 0 & 0 \end{bmatrix}.$$

Obviously,

$$\mathbf{M}=\mathbf{Q}+\mathbf{T}_{-\infty}\,,\qquad \mathbf{Q}=\begin{bmatrix} -\frac{1}{2}I+V_0^{(0)} & 0 \\ 0 & \mathbf{P} \end{bmatrix}, \tag{3.6}$$

where $\mathbf{T}_{-\infty}$, comprised by the operators $\gamma_{\partial\Omega}\Big(TV^{(1)}\Big)$, $\gamma_{\partial\Omega_1}\Big(V^{(0)}\Big)$ and $\gamma_{S_0}\Big(TV^{(0)}\Big)$, is an infinitely smoothing compact operator, while

$$\mathbf{P}=\begin{bmatrix} V_{-1}^{(1)} & 0 & -I & 0 \\ 0 & V_{-1}^{(1)} & 0 & -I \\ 0 & 0 & I & -I \\ r_{S_0}\Big(\frac{1}{2}I+V_0^{(1)}\Big) & -r_{S_0}\Big(-\frac{1}{2}I+V_0^{(1)}\Big) & 0 & 0 \end{bmatrix}.$$

The entry $-\frac{1}{2}I+V_0^{(0)}$ of the operator $\mathbf{Q}$ is a Fredholm singular operator with index 0 (see Theorem 2.6).

Consider the system corresponding to the operator $\mathbf{P}$

$$\begin{cases} V_{-1}^{(1)}\widetilde{g}_i - \widetilde{\varphi}_0^{(i)} = F^{(i)}, \ i = 1, 2, & \text{on } \partial\Omega_1, \\ \widetilde{\varphi}_0^{(1)} - \widetilde{\varphi}_0^{(2)} = g & \text{on } S_0, \\ r_{S_0}\Big(\frac{1}{2}I + V_0^{(1)}\Big)\widetilde{g}_1 - r_{S_0}\Big(-\frac{1}{2}I + V_0^{(1)}\Big)\widetilde{g}_2 = f & \text{on } S_0 \end{cases}$$

with respect to the unknowns $\widetilde{g}_i, \ \widetilde{\varphi}_0^{(i)}, \ i = 1, 2$. Note that they differ from $g_i, \ \varphi_0^{(i)}$ by infinitely differentiable functions.

Since the operator $V_{-1}^{(1)}$ is invertible,

$$\widetilde{g}_i = \big(V_{-1}^{(1)}\big)^{-1}(F^{(i)} + \widetilde{\varphi}_0^{(i)}), \ \ i = 1, 2.$$

On substituting these equalities in the last two equations in the foregoing system, we obtain a system of pseudodifferential equations on S_0 with respect to $\widetilde{\varphi}_0^{(1)}$ and $\widetilde{\varphi}_0^{(2)}$

$$\begin{cases} \widetilde{\varphi}_0^{(1)} - \widetilde{\varphi}_0^{(2)} = g, \\ r_{S_0}\Big(\frac{1}{2}I + V_0^{(1)}\Big)\big(V_{-1}^{(1)}\big)^{-1}\widetilde{\varphi}_0^{(1)} - r_{S_0}\Big(-\frac{1}{2}I + V_0^{(1)}\Big)\big(V_{-1}^{(1)}\big)^{-1}\widetilde{\varphi}_0^{(2)} = G, \end{cases} \tag{3.7}$$

where

$$G = f + r_{S_0}\Big(-\frac{1}{2}I + V_0^{(1)}\Big)\big(V_{-1}^{(1)}\big)^{-1}F^{(2)} - r_{S_0}\Big(\frac{1}{2}I + V_0^{(1)}\Big)\big(V_{-1}^{(1)}\big)^{-1}F^{(1)}.$$

The system (3.7) is thus reduced to a pseudodifferential equation on the open manifold

$$r_{S_0}\mathbf{A}\widetilde{\varphi}_0^{(1)} = \Psi \quad \text{on } S_0,$$

where $\Psi \in \mathbb{H}_p^{s-1}(S_0)$, $\big(\Psi \in \mathbb{B}_{p,q}^{s-1}(S_0),\big)$ and

$$\mathbf{A} = \Big(\frac{1}{2}I + V_0^{(1)}\Big)\big(V_{-1}^{(1)}\big)^{-1} - \Big(-\frac{1}{2}I + V_0^{(1)}\Big)\big(V_{-1}^{(1)}\big)^{-1} = \big(V_{-1}^{(1)}\big)^{-1}.$$

The operator $r_{S_0}\mathbf{A} = r_{S_0}\big(V_{-1}^{(1)}\big)^{-1}$ is strongly elliptic and, due to Theorem 2.5, the first part of the theorem is proved.

The second part, the solvability properties (3.5), is a consequence of the first part and mapping properties of the potential operators (see [DNS1, NCS1, Ch2] for similar considerations in elasticity). ■

Let us look at the asymptotics of the solution to the Dirichlet problem near the boundary $\mathscr{E} = \partial S_1$. We assume that the boundary conditions of the Dirichlet problem are sufficiently smooth, namely: $\varphi_i \in \mathbb{H}_r^{(\infty,s+2N+1),\infty}(S_1)$, $i = 1, 2$ (see [CD1] for the definition and details).

The principal symbol $\sigma_{\mathbf{A}}(x', \xi')$ of the pseudodifferential operator $\mathbf{A}$ in (2.3) is written as (see [CD1])

$$\sigma_{\mathbf{A}}(x', \xi') = \sigma^{-1}_{-V_{-1}^{(1)}}(x', \xi'), \ \ x' \in S_0.$$

Moreover, the principal symbol $\sigma_{-V_{-1}^{(1)}}(x', \xi')$ of the operator $-V_{-1}^{(1)}$ is even with respect to the variable ξ' and, therefore, all eigenvalues of the matrix

$$(\sigma_{\mathbf{A}}(x', 0, 0, +1))^{-1}\sigma_{\mathbf{A}}(x', 0, 0, -1) = I$$

are trivial:

$$\lambda_j(x') = 1, \quad j = 1,2,3,4, \quad x' \in \mathscr{E}.$$

In the normal plane $\Pi_{x'}$ to $\mathscr{E}$, containing $x' \in \mathscr{E}$, we consider the polar coordinates (r,θ), where $r \geq$ denotes the distance from $x = (x', r, \theta)$ to the boundary $\mathscr{E}$ and $-\pi \leq \theta \leq \pi)$ is the angular parameter. Then the points $(x', r, \pm\pi) \in S_1^{\pm}$ belong to the different faces of the surface S_1 and $x' = (x', 0, \theta)$ belongs to the boundary $x' \in \mathscr{E}$ for all $\theta \in [-\pi, \pi]$.

Applying Theorem 2.1 from [CD1] and taking into account the first equality in (3.7), we obtain the asymptotic expansion of the function $\varphi_0^{(i)}$, $i = 1, 2$,

$$\varphi_0^{(i)}(x', r) = c_0(x') r^{\frac{1}{2}} + \sum_{k=1}^{N} c_k(x') r^{\frac{1}{2}+k} + \varphi_{N+1}^{(i)}(x', r), \tag{3.8}$$

where $c_k \in C^\infty(\mathscr{E}),\ k = 0, 1, ..N,$ and the remainder term $\varphi_{N+1}^{(i)}$ belongs to the space $\mathbb{H}_r^{(\infty, s+N+1), \infty}(\mathscr{E}_\varepsilon^+) \subset C^{s+N}([0,\varepsilon], C^\infty(\mathscr{E}))$, $\mathscr{E}_\varepsilon^+ = \mathscr{E} \times [0, \varepsilon]$.

As we can see from (3.8) logarithms are absent in the entire asymptotic due to the properties of the symbol $\sigma_{\mathbf{A}}(x', \xi')$ (see [CDD1]).

From Theorem 3.1 it follows that the solution of a Dirichlet problem can be written as a simple-layer potential.

For any $x' \in \mathscr{E}$, let $\tau_1(x'), \ldots, \tau_\ell(x')$ be all different roots of the polynomial equation

$$\det \mathbf{A}\big((\mathscr{J}_\varkappa^\top(x'))^{-1}(0,1,\tau)^\top\big) = 0, \qquad x' \in \mathscr{E}, \quad \operatorname{Im} \tau < 0. \tag{3.9}$$

We recall that $(0, 1, \tau)$ represents the value of the dual variable ξ and that $\mathscr{I}_\varkappa(x')$ is the Jacobian of the local coordinate diffeomorphisms $\varkappa$ (see [CD1]).

We assume that it is possible to enumerate $\tau_1(x'), \ldots, \tau_\ell(x')$ so that the multiplicities $n_1, \ldots, n_\ell$ of $\tau_1(x'), \ldots, \tau_\ell(x')$ are constant on $\mathscr{E}$. Therefore the functions $\tau_1, \ldots \tau_\ell$ can be chosen smoothly $\tau_m \in C^\infty(\mathscr{E})$.

Since $\mathbf{A}$ is a 4×4 elliptic system of order two, $n_1 + \ldots + n_\ell = 4$ and $\overline{\tau_m(x')}$ are the roots of (3.9) with $\operatorname{Im} \tau > 0$. Let us define the following functions

$$\begin{aligned} &\psi_{m,-1}(x', \theta) := \cos\theta + \tau_m(x') \sin\theta, \\ &\psi_{m,+1}(x', \theta) := \cos\theta + \overline{\tau}_m(x') \sin\theta, \qquad x' \in \mathscr{E}, \ \ m = 1, \ldots, \ell. \end{aligned} \tag{3.10}$$

Under the assumption $\Phi_0^{(i)} \in \mathbb{H}_r^{(\infty, s+2N+1), \infty}(\partial\Omega_1)$ the solutions of the Dirichlet problem has the following asymptotic form in the vicinity of the crack front $\mathscr{E}$ (see [CD2, Theorems 2.2 and 2.3] and [CDD1, Theorem B.8.1]):

$$\begin{aligned} \boldsymbol{u} = r^{\frac{1}{2}} \sum_{m=1}^{\ell} \sum_{\omega=\pm 1} \chi(r) \Bigg[&\sum_{j=0}^{n_m-1} \sin^j\theta\, \psi_{m,\omega}^{\frac{1}{2}-j}(x', \theta)\, \boldsymbol{d}_{m,\omega}^j(x') \\ &+ \sum_{k=1}^{N-1} \sum_{j=0}^{p(m,k)} \sum_{|\alpha| \leq N(m,k)} r^k\, \psi_{m,\omega}^{\frac{1}{2}-j+k}(x', \theta) \sin^{\alpha_1}\theta \cos^{\alpha_2}\theta\, \boldsymbol{d}_{m,\omega}^{k,j,\alpha}(x') \Bigg] + \boldsymbol{u}_{\mathrm{rem},N} \end{aligned} \tag{3.11}$$

where $\boldsymbol{u}_{\mathrm{rem},N} \in \mathbb{H}^{\frac{1}{2}+N}_{\mathrm{loc}}(\mathbb{R}^3)$ and the coefficients $\boldsymbol{d}^{j}_{m,\omega}$ and $\boldsymbol{d}^{j,k,\alpha}_{m,\omega}$ are $C^\infty(\mathscr{E})$. $p(m,k)$ and $N(m,k)$ are some positive integers.

Moreover, the explicit formulae expressing the coefficients $\boldsymbol{d}^{j}_{m,\omega}(x')$ by the first coefficients $c_0(x')$, $m = 1, \dots l$, $j = 0, \dots, n_m - 1$, of the surface expansion (3.8) are available as well (see [CD2]).

4. The Neumann Problem

Let $\Psi^{(i)}_0 \in \mathbb{H}^{-1/2}(\partial\Omega_1)$ be some fixed extension of the function $\psi_i \in \mathbb{H}^{-1/2}(S_1)$ on $\partial\Omega_1 = S_1 \cup \overline{S}_0$. Then any extension $\psi_i \in \mathbb{H}^{-1/2}(S_1)$ of the function ψ_i on $\partial\Omega_1$ has the form

$$\Psi^{(i)} = \Psi^{(i)}_0 + \psi^{(i)}_0,$$

where $\psi^{(i)}_0 \in \widetilde{\mathbb{H}}^{-1/2}(S_0)$, $i = 1, 2$. Solutions of the Neumann boundary value problem will be sought in the form (3.1). Due to the boundary conditions (1.12) we have

$$\begin{cases} \Big(-\dfrac{1}{2}I + V^{(0)}_0\Big)h + r_{\partial\Omega}\Big(TV^{(1)}\Big)g_2 = \psi & \text{on } \partial\Omega, \\ r_{S_1}\Big(\frac{1}{2}I + V^{(1)}_0\Big)g_1 = \varphi_1 & \text{on } S_1, \\ r_{S_1}\Big(-\dfrac{1}{2}I + V^{(1)}_0\Big)g_2 + r_{S_1}\Big(TV^{(1)}\Big)h = \varphi_2 & \text{on } S_1, \end{cases} \tag{4.1}$$

Taking into account (3.2) we obtain the following system of equations

$$\begin{cases} \Big(-\dfrac{1}{2}\,I + V^{(0)}_0\Big)h + r_{\partial\Omega}(TV^{(1)})g_2 = \psi & \text{on } \partial\Omega, \\ \Big(\dfrac{1}{2}\,I + V^{(1)}_0\Big)g_1 - \psi^{(1)}_0 = \Psi^{(1)}_0 & \text{on } \partial\Omega_1, \\ \Big(-\dfrac{1}{2}\,I + V^{(1)}_0\Big)g_2 - \psi^{(2)}_0 + r_{\partial\Omega_1}(TV^{(0)})h = \Psi^{(2)}_0 & \text{on } \partial\Omega_1, \\ r_{S_0}V^{(1)}_{-1}g_1 - r_{S_0}V^{(1)}_{-1}g_2 - r_{S_0}V^{(0)}h = 0 & \text{on } S_0, \\ \psi^{(1)}_0 - \psi^{(2)}_0 = -r_{S_0}\Psi^{(1)}_0 + r_{S_0}\Psi^{(2)}_0 & \text{on } S_0 \end{cases} \tag{4.2}$$

with respect to the known and unknown vector-function

$$\begin{aligned} (\psi, \Psi^{(1)}_0, \Psi^{(2)}_0, \Phi^{(1)}, \Phi^{(2)}) &\in \mathbb{H}^{1/2}(\partial\Omega) \times \mathbb{H}^{1/2}(\partial\Omega_1) \times \mathbb{H}^{1/2}(\partial\Omega_1) \\ &\quad \times \mathbb{H}^{1/2}(S_0) \times \mathbb{H}^{-1/2}(S_0), \\ (h, g_1, g_2, \psi^{(1)}_0, \psi^{(2)}_0) &\in \mathbb{H}^{1/2}(\partial\Omega) \times \mathbb{H}^{1/2}(\partial\Omega_1) \times \mathbb{H}^{1/2}(\partial\Omega_1) \\ &\quad \times \widetilde{\mathbb{H}}^{-1/2}(S_0) \times \widetilde{\mathbb{H}}^{-1/2}(S_0). \end{aligned} \tag{4.3}$$

Now we can formulate the basic theorem about the existence and uniqueness of a solution of the Neumann problem.

Theorem 4.1. *The Neumann problem has a solution of the class* $\mathbb{W}^1(\Omega\backslash S_1)$ *in the bounded domain* Ω *if and only if the equality*

$$\int\limits_{\partial\Omega}\Big[\varepsilon_{ijk}\psi_i a_j z_k + \sum_{k=1}^{4} b_k\psi_k\Big] d_z S = 0 \tag{4.4}$$

is fulfilled for any constant vectors $a = (a_1, a_2, a_3)$ *and* $b = (b_1, b_2, b_3, b_4)$.

Solutions of the Neumann problem are represented as

$$\begin{cases} r^1 u = V^{(1)}\big(V_{-1}^{(1)}\big)^{-1} g_1 \quad \text{in} \quad \Omega_1, \\ r^2 u = V^{(1)}\big(V_{-1}^{(1)}\big)^{-1} g_1 + V^{(0)} h \quad \text{in} \quad \Omega_2, \end{cases} \tag{4.5}$$

where h, g_1, g_2 *are solutions of system* (4.1) .

Moreover, let

$$1 < r < \infty\,, \qquad 1 \le q \le \infty\,, \qquad \frac{1}{r} - \frac{1}{2} < s < \frac{1}{r} + \frac{1}{2}\,. \tag{4.6}$$

If $u \in \mathbb{W}^1(\Omega\backslash S_1)$ *is a solution of the Neumann problem then*

$$\begin{aligned} &u \in \mathbb{H}_r^{s+\frac{1}{r}}(\Omega\backslash S_1) \quad \text{if} \quad \psi \in \mathbb{W}_r^{s-1}(\partial\Omega), \quad \psi_i \in \mathbb{W}_r^{s-1}(S_1), \quad i = 1, 2, \\ &u \in \mathbb{B}_{r,q}^{s+\frac{1}{r}}(\Omega\backslash S_1) \quad \text{if} \quad \psi \in \mathbb{B}_{r,q}^{s-1}(\partial\Omega), \quad \psi_i \in \mathbb{B}_{r,q}^{s-1}(S_1), \quad i = 1, 2\,. \end{aligned} \tag{4.7}$$

Proof. The system (4.2) can be rewritten in the matrix form

$$\mathbf{M}\begin{bmatrix} h \\ g_1 \\ g_2 \\ \psi_0^{(1)} \\ \psi_0^{(2)} \end{bmatrix} = \begin{bmatrix} \Psi \\ \Psi_0^{(1)} \\ \Psi_0^{(2)} \\ 0 \\ r_{S_0}\Psi_0^{(2)} - r_{S_0}\Psi_0^{(1)} \end{bmatrix}, \tag{4.8}$$

where

$$\mathbf{M} = \begin{bmatrix} -\frac{1}{2}\,I + V_0^{(0)} & 0 & r_{\partial\Omega}(TV^{(1)}) & 0 & 0 \\ 0 & \frac{1}{2}\,I + V_0^{(1)} & 0 & -I & 0 \\ r_{\partial\Omega_1}(TV^{(0)}) & 0 & -\frac{1}{2}\,I + V_0^{(1)} & 0 & -I \\ -r_{S_0}V^{(0)} & r_{S_0}V_{-1}^{(1)} & -r_{S_0}V_{-1}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & I & -I \end{bmatrix}.$$

The operator **M** has the same representation (3.6) with a compact and smoothing $T_{-\infty}$, but yet different $\mathbf{P}$:

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2}\,I + V_0^{(1)} & 0 & -I & 0 \\ 0 & -\frac{1}{2}\,I + V_0^{(1)} & 0 & -I \\ r_{S_0}V_{-1}^{(1)} & -r_{S_0}V_{-1}^{(1)} & 0 & 0 \\ 0 & 0 & I & -I \end{bmatrix}.$$

Let us introduce the operators

$$\mathbf{P}_N = \begin{bmatrix} (-V_{-1}^{(1)})^N + \mathscr{M}_+ (V_{-1}^{(1)})^{-1} & 0 & -I & 0 \\ 0 & (-V_{-1}^{(1)})^N + \mathscr{M}_- (V_{-1}^{(1)})^{-1} & 0 & -I \\ -r_{S_0} I & -r_{S_0} I & 0 & 0 \\ 0 & 0 & I & -I \end{bmatrix},$$

$$\mathscr{M}_+ := \Big(\frac{1}{2}\, I + V_0^{(1)}\Big), \qquad \mathscr{M}_- := \Big(-\frac{1}{2}\, I + V_0^{(1)}\Big), \qquad N = 1, 2, \dots ,$$

and

$$\mathbf{D} = \begin{bmatrix} -(V_{-1}^{(1)})^{-1} & 0 & 0 & 0 \\ 0 & (V_{-1}^{(1)})^{-1} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

Then $\mathbf{P} \circ D$ differs from $\mathbf{P}_N$ only modulo a compact operator $\mathbf{T}_{-N}$:

$$\mathbf{P} \circ \mathbf{D} - \mathbf{P}_N = \mathbf{T}_{-N}.$$

Consider the following system

$$\begin{cases} \mathbf{B}_N^{(i)} \widetilde{g}_i - \widetilde{\psi}_0^{(i)} = \widetilde{F}^{(i)}, \ \ i = 1, 2, & \text{on } \ \partial\Omega_1, \\ -r_{S_0} \widetilde{g}_1 - r_{S_0} \widetilde{g}_2 = \widetilde{G}_1 & \text{on } \ S_0, \\ \widetilde{\psi}_0^{(1)} - \widetilde{\psi}_0^{(2)} = \widetilde{G}_2 & \text{on } \ S_0, \end{cases} \tag{4.9}$$

where

$$\mathbf{B}_N^{(i)} = (-V_{-1}^{(1)})^N + \Big(-\frac{1}{2}\, I + (-1)^i V_0^{(1)}\Big)(V_{-1}^{(1)})^{-1}, \ \ i = 1, 2.$$

$$\widetilde{g}_i = (\mathbf{B}_N^{(i)})^{-1} \widetilde{\psi}_0^{(i)} + (\mathbf{B}_N^{(i)})^{-1} F^{(i)}, \ \ i = 1, 2, \qquad N = 1, 2, \dots .$$

The systems (4.1) (i.e., the system (4.2)) and (4.9) are Fredholm-equivalent: are Fredholm or are not Fredholm only simultaneously and have equal indices. Due to Theorem 2.7 the pseudodifferential operator

$$\begin{aligned} \mathbf{B}_N^{(i)} \quad &: \quad \widetilde{\mathbb{H}}_p^s(S_0) \to \mathbb{H}_p^{s-1}(S_0) \\ &: \quad \widetilde{\mathbb{B}}_{p,q}^s(S_0) \to \mathbb{B}_{p,q}^{s-1}(S_0) \end{aligned}$$

is invertible.

Let us define $\widetilde{g}_1$ and $\widetilde{g}_2$ from the first two equations of system (4.9) and insert them into the third equation. We obtain the system of differential equations on the open manifold S_0

$$\begin{cases} r_{S_0} (\mathbf{B}_N^{(1)})^{-1} \widetilde{\psi}_0^{(1)} + r_{S_0} (\mathbf{B}_N^{(2)})^{-1} \widetilde{\psi}_0^{(2)} = F, \\ \widetilde{\psi}_0^{(1)} - \widetilde{\psi}_0^{(2)} = G_2, \end{cases}$$

or

$$\begin{cases} r_{S_0} \mathbf{B} \widetilde{\psi}_0^{(1)} = \widetilde{\Psi}, \\ \widetilde{\psi}_0^{(2)} = \widetilde{\psi}_0^{(1)} - G_2, \end{cases} \tag{4.10}$$

where

$$\mathbf{B} = (\mathbf{B}_N^{(1)})^{-1} + (\mathbf{B}_N^{(2)})^{-1}.$$

The pseudodifferential operator $r_{S_0}\mathbf{B}$ is strongly elliptic and, due to Theorem 2.7 the operator

$$r_{S_0}\mathbf{B} : \widetilde{\mathbb{H}}_p^{s-1}(S_0) \to \mathbb{H}_p^s(S_0) \quad \big(\widetilde{\mathbb{B}}_{p,q}^{s-1}(S_0) \to \mathbb{B}_{p,q}^s(S_0)\big)$$

is invertible provided the conditions (4.6) hold.

Summarizing we find that if the conditions (4.6) hold, the operator

$$\mathbf{N} \;:\; \begin{array}{c} \mathbb{H}_p^{s-1}(\partial\Omega) \\ \oplus \\ \mathbb{H}_p^{s-1}(\partial\Omega_1) \\ \oplus \\ \mathbb{H}_p^{s-1}(\partial\Omega_1) \end{array} \;\to\; \begin{array}{c} \mathbb{H}_p^{s-1}(\partial\Omega) \\ \oplus \\ \mathbb{H}_p^{s-1}(S_1) \\ \oplus \\ \mathbb{H}_p^{s-1}(S_1) \end{array} \quad \left(\begin{array}{c} \mathbb{B}_{p,q}^{s-1}(\partial\Omega) \\ \oplus \\ \mathbb{B}_{p,q}^{s-1}(\partial\Omega_1) \\ \oplus \\ \mathbb{B}_{p,q}^{s-1}(\partial\Omega_1) \end{array} \;\to\; \begin{array}{c} \mathbb{B}_{p,q}^{s-1}(\partial\Omega) \\ \oplus \\ \mathbb{B}_{p,q}^{s-1}(S_1) \\ \oplus \\ \mathbb{B}_{p,q}^{s-1}(S_1) \end{array} \right)$$

defined by the left-hand side of (4.1) is Fredholm and Ind $\mathbf{N} = 0$. It is well known that the kernel, the cokernel and, therefore, the index of the operator **N** are independent of the parameters of the spaces where it is Fredholm (see [Ag1, DNS1, Ka1]). Therefore it suffices to find Coker $\mathbf{N}$ for $s = 1/2$ and $p = 2$. Thus, we consider the dual (adjoint) operator in the spaces:

$$\mathbf{N}^* \;:\; \begin{array}{c} \mathbb{H}_2^{1/2}(\partial\Omega) \\ \oplus \\ \widetilde{\mathbb{H}}_2^{1/2}(S_1) \\ \oplus \\ \widetilde{\mathbb{H}}_2^{1/2}(S_1) \end{array} \;\to\; \begin{array}{c} \mathbb{H}_2^{1/2}(\partial\Omega) \\ \oplus \\ \mathbb{H}_2^{1/2}(\partial\Omega_1) \\ \oplus \\ \mathbb{H}_2^{1/2}(\partial\Omega_1) \end{array} \;.$$

If $(\widetilde{h}, \chi_0^{(1)}, \chi_0^{(2)})$ is the solution of homogeneous equation

$$\mathbf{N}^*(\widetilde{h}, \chi_0^{(1)}, \chi_0^{(2)}) = 0\,, \qquad \mathrm{supp}\chi_0^{(1)} \subset \overline{S}_1, \quad \mathrm{supp}\chi_0^{(2)} \subset \overline{S}_1\,.$$

Then we have

$$\begin{cases} \Big(-\dfrac{1}{2}I + \widetilde{V}_0^{(0)}\Big)\widetilde{h} + r_{\partial\Omega}(\widetilde{U}^{(1)})\chi_0^{(2)} = 0 & \text{on } \partial\Omega, \\ \Big(\dfrac{1}{2}I + \widetilde{V}_0^{(1)}\Big)\chi_0^{(1)} = 0 & \text{on } \partial\Omega_1, \\ r_{\partial\Omega_1}(\widetilde{U}^{(0)})\widetilde{h} + \Big(-\dfrac{1}{2}I + \widetilde{V}_0^{(1)}\Big)\chi_0^{(2)} = 0 & \text{on } \partial\Omega_1. \end{cases} \tag{4.11}$$

Here $\widetilde{U}^{(i)}$ are the double-layer potentials:

$$\widetilde{U}^{(i)}(g)(x) = \int\limits_{\partial\Omega_i} \Big[\widetilde{T}(\partial_y, n(y))H^T(x-y)\Big]^T g(y)d_yS\,, \qquad i = 0, 1\,.$$

Let

$$v^{(2)} = \widetilde{U}^{(0)}\widetilde{h} + \widetilde{U}^{(1)}\chi_0^{(2)},$$
$$v^{(1)} = \widetilde{U}^{(1)}\chi_0^{(1)}.$$

Due to (4.11) $v^{(i)}, i=1,2,$ satisfy the boundary conditions:

$$\begin{cases} \gamma^- v^{(2)} = 0 \text{ on } \partial\Omega, \\ \gamma^1 v^{(1)} = 0 \text{ on } \partial\Omega_1, \\ \gamma^2 v^{(2)} = 0 \text{ on } \partial\Omega_1\,. \end{cases}$$

Here γ^- denotes the trace on $\partial\Omega$ from the outer domain $\mathbb{R}^3 \setminus \overline{\Omega}$. Therefore $v^{(2)}$ is the solution of the following BVP:

$$\begin{cases} \mathbf{A}^T(D) v^{(2)} = 0 \text{ in } \mathbb{R}^3 \backslash \overline{\Omega}_2, \\ \gamma^- v^{(2)} = 0 \text{ on } \partial\Omega, \\ \gamma^1 v^{(2)} = 0 \text{ on } \partial\Omega_1, \end{cases}$$

which have only the trivial solution in $\mathbb{R}^3 \backslash \overline{\Omega}_2$. Consequently, $v^{(2)} \equiv 0$ in $\mathbb{R}^3 \backslash \overline{\Omega_2}$ and from boundary condition

$$\gamma^0 \left(\widetilde{T} v^{(2)}\right) - \gamma^- \widetilde{T} v^{(2)} = 0 \text{ on } \partial\Omega;$$

$$\gamma^2 \left(\widetilde{T} v^{(2)}\right) - \gamma^1 \left(\widetilde{T} v^{(2)}\right) = 0 \text{ on } \partial\Omega_1;$$

it follows that

$$\gamma^0 \left(\widetilde{T} v^{(2)}\right) = 0 \text{ on } \partial\Omega, \;\; \gamma^2 \left(\widetilde{T} v^{(2)}\right) = 0 \text{ on } \partial\Omega_1.$$

Thus, $v^{(2)}$ is the solution of the next BVP in Ω_2 :

$$\begin{cases} \mathbf{A}^T(D) v^{(2)} = 0 \text{ in } \Omega_2, \\ \gamma^0 \left(\widetilde{T} v^{(2)}\right) = 0 \text{ on } \partial\Omega, \\ \gamma^2 \left(\widetilde{T} v^{(2)}\right) = 0 \text{ on } \partial\Omega_1, \end{cases}$$

which have only solution of the form $v^{(2)} = a \cdot x + b,$ where a is an antisymmetric matrix and b is an arbitrary constant vector. Due to the boundary conditions:

$$\gamma^0 v^{(2)} - \gamma^- v^{(2)} = \widetilde{h} \text{ on } \partial\Omega,$$

$$\gamma^2 v^{(2)} - \gamma^1 v^{(2)} = \chi_0^{(2)} \text{ on } \partial\Omega_1$$

it follows that

$$\widetilde{h} = a \cdot x + b, \;\; \chi_0^{(2)} = a \cdot x + b,$$

Since $\operatorname{supp} \chi_0^{(2)} \subset \overline{S}_1$, we get $\chi_0^{(2)} = 0.$ Analogously we obtain $\chi_0^{(1)} = 0$.

The second part, the solvability properties (4.7), are consequences of the first part and the mapping properties of the potential operators (see [DNS1, NCS1, Ch2] for similar considerations in elasticity). ∎

Let us investigate the asymptotic behavior of the solution to the Neumann problem in the vicinity of the edge $\mathscr{E}$. Assume that the corresponding boundary data are sufficiently smooth. Namely, $\psi_i \in \mathbb{H}_r^{(\infty,s+2N),\infty}(S_1)$.

We can rewrite the symbol $\sigma_B(x',\xi')$ as

$$\begin{aligned}\sigma_B(x',\xi') &= \Big[\Big(-\frac{1}{2}I+\sigma_{V_0^{(1)}}(x',\xi')\Big)(\sigma_{V_{-1}^{(1)}}(x',\xi'))^{-1}\Big]^{-1}\\ &\quad -\Big[\Big(\frac{1}{2}I+\sigma_{V_0^{(1)}}(x',\xi')\Big)(\sigma_{V_{-1}^{(1)}}(x',\xi'))^{-1}\Big]^{-1}.\end{aligned}$$

The principal symbol $\sigma_{V_0^{(1)}}(x',\xi')$ is odd with respect to the variable ξ', while the symbol $\sigma_{V_{-1}^{(1)}}(x',\xi')$ is even; therefore, $\sigma_B(x',\xi')$ is even with respect to ξ':

$$\sigma_B(x',-\xi')=\sigma_B(x',\xi'),\;\; x'\in S_0$$

and all eigenvalues of the matrix $(\sigma_B(x',0,0,+1))^{-1}\sigma_B(x',0,0,-1)=I$ are trivial: $\lambda_j=1$, $j=1,2,3,4$, $x'\in C^\infty(\mathscr{E})$.

Consider the above-described local coordinate system $x=(x',r)\in S_0$. Using the theory of strongly elliptic pseudodifferential equations (see [CD1, Theorem 2.1]) we get the following asymptotic expansion of solutions of equation (4.8):

$$\psi_0^{(i)}(x',r)=c_0(x')r^{-\frac{1}{2}}+\sum_{k=1}^{N}c_k(x')r^{-\frac{1}{2}+k}+\psi_{N+1}^{(i)}(x',r)\,,\qquad i=1,2\,,\tag{4.12}$$

$$x'\in\mathscr{E}\,,\quad 0<r<\varepsilon\,,\quad \psi_{N+1}^{(i)}\in\mathbb{H}_r^{(\infty,s+N),\infty}(\mathscr{E}_\varepsilon^+)\subset C^{s+N}([0,\varepsilon],C^\infty(\mathscr{E})\,,$$

where $c_k\in C^\infty(\mathscr{E}),\ k=0,1,..N$.

And again, logarithms are absent in the entire asymptotic representation (4.12).

Let $(h,g_1,g_2,\psi_0^{(1)},\psi_0^{(2)})$ be a solution of the system (4.2):

$$\begin{gathered}\mathbf{M}(h,g_1,g_2,\psi_0^{(1)},\psi_0^{(2)})=\Psi,\\ \Psi=(\psi,\Psi_0^{(1)},\Psi_0^{(2)},0,-(r_{S_0}\Phi_0^{(1)}-r_{S_0}\Phi_0^{(2)})),\end{gathered}$$

then $(g_1,g_2,\psi_0^{(1)},\psi_0^{(2)})$ satisfies the equation

$$\mathbf{P_N}\big(-V_{-1}^{(1)}g_1,V_{-1}^{(1)}g_2,\psi_0^{(1)},\psi_0^{(2)}\big)=\overset{*}{\Psi},\tag{4.13}$$

where

$$\begin{aligned}\overset{*}{\Psi}=\big(\Psi_0^{(1)}+(V_{-1}^{(1)})^{2N}g_1,\Psi_0^{(2)}+(V_{-1}^{(1)})^{2N}g_2-r_{\partial\Omega_1}(TV^{(0)})h,\\ r_{S_0}Vh,-r_{S_0}\Phi_0^{(1)}+r_{S_0}\Phi_0^{(2)}\big).\end{aligned}$$

Equation (4.8) (i.e., equation (4.2)) can be written in the form

$$\begin{cases}\mathbf{B_{2N}^{(1)}}(-V_{-1}^{(1)})g_1-\psi_0^{(1)}=\Psi_0^{(1)}+(V_{-1}^{(1)})^{2N}g_1,\\ \mathbf{B_{2N}^{(1)}}(V_{-1}^{(1)})g_2-\psi_0^{(2)}=\Psi_0^{(2)}+(V_{-1}^{(1)})^{2N}g_1-r_{\partial\Omega_1}(TV^{(0)})h,\\ -r_{S_0}g_1-r_{S_0}g_2=r_{S_0}V^{(0)}h,\\ \psi_0^{(1)}-\psi_0^{(2)}=-r_{S_0}\Phi_0^{(1)}+r_{S_0}\Phi_0^{(2)}.\end{cases}\tag{4.14}$$

where

$$\mathbf{B}^{(\mathbf{1})}_{\mathbf{2N}} = \big(V^{(1)}_{-1}\big)^{2N} - \Big(\tfrac{1}{2}I + V^{(1)}_0\Big)\big(V^{(1)}_{-1}\big)^{-1},$$
$$\mathbf{B}^{(\mathbf{2})}_{\mathbf{2N}} = \big(V^{(1)}_{-1}\big)^{2N} + \Big(-\tfrac{1}{2}I + V^{(1)}_0\Big)\big(V^{(1)}_{-1}\big)^{-1}.$$

As we have seen before, equation (4.14) can be reduced to a pseudodifferential equation with a strongly elliptic operator.

Using the first two equations of system (4.14), we find

$$g_1 = \big(-V^{(1)}_{-1}\big)^{-1}\big(\mathbf{B}^{(\mathbf{1})}_{\mathbf{2N}}\big)^{-\mathbf{1}}\psi^{(\mathbf{1})}_{\mathbf{0}} + \mathbf{F_1},$$
$$g_2 = \big(V^{(1)}_{-1}\big)^{-1}\big(\mathbf{B}^{(\mathbf{2})}_{\mathbf{2N}}\big)^{-\mathbf{1}}\psi^{(\mathbf{2})}_{\mathbf{0}} + \mathbf{F_2}.$$

Note that

$$F_i \in \mathbb{H}^{(\infty,s+2N+1),\infty}_r(\partial\Omega_i), \quad i = 1,2.$$

Hence we obtain a representation of the solutions of the Neumann BVP by potential-type functions:

$$r^1 u = V^{(1)}\big(-V^{(1)}_{-1}\big)^{-1}\big(\mathbb{B}^{(1)}_{2N}\big)^{-1}\psi^{(1)}_0 + G_1,$$
$$r^2 u = V^{(1)}\big(V^{(1)}_{-1}\big)^{-1}\big(\mathbb{B}^{(2)}_{2N}\big)^{-1}\psi^{(2)}_0 + G_2,$$

where $G_i \in C^{N+1}(\overline{\Omega}_i)$ $i = 1,2$.

Thus, using the asymptotic expansion (4.12) of the function $\psi^{(i)}_0$, $i = 1,2$ and the asymptotic expansion of functions that can be represented by potential operators (see [CD2, Theorems 2.2 and 2.3]) we obtain the asymptotics of solutions to the Neumann problem in terms of the local coordinates (see § 3) in a neighborhood of the crack boundary:

$$\boldsymbol{u} = \sum_{m=1}^{\ell}\sum_{\omega=\pm1}\chi(r)\Bigg[\sum_{j=0}^{n_m-1} r^{\frac{1}{2}}\sin^j\theta\,\psi^{\frac{1}{2}-j}_{m,\omega}(x',\theta)\,\boldsymbol{d}^j_{m,\omega}(x') \tag{4.15}$$
$$+\sum_{k=1}^{N-1}\sum_{j=0}^{p(m,k)}\sum_{|\alpha|\le N(m,k)} r^{\frac{1}{2}+k}\,\psi^{\frac{1}{2}-j+k}_{m,\omega}(x',\theta)\sin^{\alpha_1}\theta\cos^{\alpha_2}\theta\,\boldsymbol{d}^{k,j,\alpha}_{m,\omega}(x')\Bigg] + \boldsymbol{u}_{\mathrm{rem},N}\,,$$

where $\boldsymbol{u}_{\mathrm{rem},N} \in \mathbb{H}^{\frac{1}{2}+N}_{\mathrm{loc}}(\mathbb{R}^3)$ and the coefficients $\boldsymbol{d}^j_{m,\omega}$ and $\boldsymbol{d}^{j,k,\alpha}_{m,\omega}$ are $C^\infty(\mathscr{E})$ (see (3.11) for a similar notation).

As in § 3, the coefficients $\boldsymbol{d}^j_{m,\omega} \in C^\infty(\mathscr{E})$ of the asymptotic formula (4.15) can be expressed by the first coefficients c_0 of the surface expansion (4.12) (see [CD2]).

5. The Mixed Problem

Any extension $\Phi^{(1)} \in \mathbb{H}^{1/2}(\partial\Omega_1)$ of the function φ_1 to the whole boundary $\partial\Omega_1 = S_1 \cup \overline{S}_0$ has the form

$$\Phi^{(1)} = \Phi^{(1)}_0 + \varphi^{(1)}_0,$$

where $\Phi^{(1)}_0$ is a fixed extension of the function φ_1, and $\varphi^{(1)}_0 \in \widetilde{\mathbb{H}}^{1/2}(S_0)$.

Any extension $\Psi^{(2)} \in \mathbb{H}^{-1/2}(\partial\Omega_1)$ of the function ψ_2 to the whole boundary $\partial\Omega_1 = S_1 \cup \overline{S}_0$ has the form

$$\Psi^{(2)} = \Psi_0^{(2)} + \psi_0^{(2)},$$

where $\Psi_0^{(2)}$ is a fixed extension of the function ψ_2, and $\psi_0^{(2)} \in \widetilde{\mathbb{H}}^{-1/2}(S_0)$.

Solutions of the mixed boundary value problem will be sought in the form (3.1). Bearing in mind boundary conditions of the mixed problem and equalities (3.2) for the function u, we get a system of equations with respect to h, g_1, g_2, $\varphi_0^{(1)}$, $\varphi_0^{(2)}$:

$$\begin{cases} \Big(-\frac{1}{2}I + V_0^{(0)}\Big)h + r_{\partial\Omega}\Big(TV^{(1)}\Big)g_2 = \psi & \text{on } \partial\Omega, \\ V_{-1}^{(1)}g_1 - \varphi_0^{(1)} = \Phi_0^{(1)} & \text{on } \partial\Omega_1, \\ \Big(-\frac{1}{2}\,I + V_0^{(1)}\Big)g_2 - \psi_0^{(2)} + r_{\partial\Omega_1}(TV^{(0)})h = \Psi_0^{(2)} & \text{on } \partial\Omega_1, \\ -r_{S_0}V^{(0)}h - r_{S_0}V_{-1}^{(1)}g_2 + \varphi_0^{(1)} = -r_{S_0}\Phi_0^{(1)} & \text{on } S_0, \\ r_{S_0}\Big(\frac{1}{2}I + V_0^{(1)}\Big)g_1 - \psi_0^{(2)} = r_{S_0}\Psi_0^{(2)} & \text{on } S_0, \end{cases} \tag{5.1}$$

Now we can formulate the basic theorem of the existence and uniqueness of a solution of the mixed problem.

Theorem 5.1. *The mixed boundary value problem has a unique solution in the space* $\mathbb{W}^1(\Omega\backslash S_1)$ *, which is given by the potential-type functions*

$$u^{(1)} = V^{(1)}(V_{-1}^{(1)})^{-1}\varphi_0^{(1)} + R_1,$$
$$u^{(2)} = V^{(1)}(\mathbf{B}_{2N+1}^{(1)})^{-1}V_{-1}^{(1)}\psi_0^{(2)} + R_2,$$

where

$$R_i \in C^{N+1}(\overline{\Omega}_i), \quad i = 1, 2,$$
$$\mathbf{B}_{2N+1}^{(1)} = -(V_{-1}^{(1)})^{2N+1} + V_{-1}^{(1)}\Big(-\frac{1}{2}\,I + V_0^{(1)}\Big)$$

and $(-L_+^{-1}\psi_0^{(2)}, \varphi_0^{(1)})$ *is the solution of the strongly elliptic pseudodifferential equation (5.3).*

Moreover, let

$$\delta := \sup_{\substack{1\le j\le 8\\ x'\in\mathscr{E}}} \frac{1}{2\pi}\left|\arg \lambda_j(x')\right| ,$$
$$1 < t < \infty\,, \qquad 1 \le r \le \infty\,, \qquad \frac{1}{t} - \frac{1}{2} + \delta < s < \frac{1}{t} + \frac{1}{2} - \delta\,. \tag{5.2}$$

and $u \in \mathbb{W}^1(\Omega\backslash S_1)$ *be the solution of the mixed BVP. Then:*

If $\varphi_1 \in \mathbb{B}_{t,t}^s(S_1)$ *,* $\psi_2 \in \mathbb{W}_t^{s-1}(S_1)$ *,* $\psi \in \mathbb{W}_t^{s-1}(\partial\Omega)$ *, we have* $u \in \mathbb{H}_t^{s+1/t}(\Omega\backslash S_1)$ *.*

If $\varphi_1 \in \mathbb{B}_{t,r}^s(S_1)$ *,* $\psi_2 \in \mathbb{B}_{t,r}^{s-1}(S_1)$ *,* $\psi \in \mathbb{B}_{t,r}^{s-1}(\partial\Omega)$ *, we have* $u \in \mathbb{B}_{t,r}^{s+1/t}(\Omega\backslash S_1)$ *.*

Proof. The operator corresponding to the system (5.1) will be denoted by $\mathbf{M}$. It has the form

$$\mathbf{M} = \begin{bmatrix} \left(-\frac{1}{2}I + V_0^{(0)}\right) & 0 & r_{\partial\Omega}(TV^{(1)}) & 0 & 0 \\ 0 & V_{-1}^{(1)} & 0 & -I & 0 \\ r_{\partial\Omega_1}(TV^{(0)}) & 0 & -\frac{1}{2}I + V_0^{(1)} & 0 & -I \\ -r_{S_0}V^{(0)} & 0 & -r_{S_0}V_{-1}^{(1)} & I & 0 \\ 0 & r_{S_0}\left(\frac{1}{2}I + V_0^{(1)}\right) & 0 & 0 & -I \end{bmatrix}.$$

And again, operator **M** is decomposed in the form (3.6) with a compact and smoothing $T_{-\infty}$, but yet different P:

$$\mathbf{P} = \begin{bmatrix} V_{-1}^{(1)} & 0 & -I & 0 \\ 0 & -\frac{1}{2}I + V_0^{(1)} & 0 & -I \\ 0 & -r_{S_0}V_{-1}^{(1)} & I & 0 \\ r_{S_0}\left(\frac{1}{2}I + V_0^{(1)}\right) & 0 & 0 & -I \end{bmatrix}.$$

We consider the composition $\mathbf{D}\circ\mathbf{P}$ of $\mathbf{P}$ with the invertible operator

$$\mathbf{D} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & V_{-1}^{(1)} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}.$$

For the invertibility of the pseudodifferential operator $V_{-1}^{(1)}$ see Theorem 2.4.

Note that, as in previous cases, the difference

$$\mathbf{T}_N = \mathbf{D}\circ\mathbf{P} - \mathbf{P}_N, \quad N = 2, 3, \ldots, ,$$

where

$$\mathbf{P}_N = \begin{pmatrix} V_{-1}^{(1)} & 0 & -I & 0 \\ 0 & (-V_{-1}^{(1)})^N + V_{-1}^{(1)}(-\frac{1}{2}I + V_0^{(1)}) & 0 & -V_{-1}^{(1)} \\ 0 & -r_{S_0}V_{-1}^{(1)} & I & 0 \\ r_{S_0}(-\frac{1}{2}I + V_0^{(1)}) & 0 & 0 & I \end{pmatrix}$$

is a compact operator. So it is sufficient for us to investigate the operator $\mathbf{P}_N$ which acts in the following spaces:

$$\mathbf{P}_N : \begin{matrix} \mathbb{H}_p^{s-1}(\partial\Omega_1) \\ \oplus \\ \mathbb{H}_p^{s-1}(\partial\Omega_1) \\ \oplus \\ \widetilde{\mathbb{H}}_p^{s}(S_0) \\ \oplus \\ \widetilde{\mathbb{H}}_p^{s-1}(S_0) \end{matrix} \longrightarrow \begin{matrix} \mathbb{H}_p^{s}(\partial\Omega_1) \\ \oplus \\ \mathbb{H}_p^{s}(\partial\Omega_1) \\ \oplus \\ \mathbb{H}_p^{s}(S_0) \\ \oplus \\ \mathbb{H}_p^{s-1}(S_0) \end{matrix} \left(\begin{matrix} \mathbb{B}_{p,r}^{s-1}(\partial\Omega_1) \\ \oplus \\ \mathbb{B}_{p,r}^{s-1}(\partial\Omega_1) \\ \oplus \\ \widetilde{\mathbb{B}}_{p,r}^{s}(S_0) \\ \oplus \\ \widetilde{\mathbb{B}}_{p,r}^{s-1}(S_0) \end{matrix} \longrightarrow \begin{matrix} \mathbb{B}_{p,r}^{s}(\partial\Omega_1) \\ \oplus \\ \mathbb{B}_{p,r}^{s}(\partial\Omega_1) \\ \oplus \\ \mathbb{B}_{p,r}^{s}(S_0) \\ \oplus \\ \mathbb{B}_{p,r}^{s-1}(S_0) \end{matrix} \right)$$

$$(\beta \geq [|s|] + 3, \quad 1 < p < \infty, \quad 1 \leq t \leq \infty).$$

Now let us consider a system of equations that corresponds to the operator $\mathbf{P}_N$ with respect to h_1, h_2, $\psi_0^{(1)}$, $\psi_0^{(2)}$

$$\begin{cases} V_{-1}^{(1)}\widetilde{g}_1 - \widetilde{\varphi}_0^{(1)} = \widetilde{\Phi}_0^{(1)}, \\ \big[(-V_{-1}^{(1)})^N + V_{-1}^{(1)}\big(-\frac{1}{2}I + V_0^{(1)}\big)\big]\widetilde{g}_2 - V_{-1}^{(1)}\widetilde{\psi}_0^{(2)} = \widetilde{\Psi}_0^{(2)}, \\ -r_{S_0}V_{-1}^{(2)}\widetilde{g}_2 + \widetilde{\varphi}_0^{(1)} = F_1, \\ r_{S_0}\big(-\frac{1}{2}I + V_0^{(1)}\big)\widetilde{g}_1 + \widetilde{\psi}_0^{(2)} = F_2\,. \end{cases} \tag{5.3}$$

Similarly as it has been done in [Ch2] system (5.3) is reduced equivalently to a strongly elliptic pseudodifferential equation on S_0

$$\mathbf{R}(x', D')\chi = \Psi\,, \tag{5.4}$$

where

$$\mathbf{R}(x', D') = \begin{bmatrix} L_- \circ r_{S_0}\mathbf{A}_2(x', D') \circ L_+ & L_- \\ -L_+ & r_{S_0}\mathbf{A}_1(x', D') \end{bmatrix},$$

$$\mathbf{A}_1 = \Big(-\frac{1}{2}\,I + V_0^{(1)}\Big)(V_{-1}^{(1)})^{-1},$$

$$\mathbf{A}_2 = \Big[-(V_{-1}^{(1)})^{2N+1} - \Big(\frac{1}{2}\,I + V_0^{(1)}\Big)(V_{-1}^{(1)})^{-1}\Big]^{-1},$$

$$L_- = r^+\mathrm{diag}\Lambda_-\ell, \quad L_+ = \mathrm{diag}\Lambda_+.$$

Here $\Lambda_\pm$ are the pseudodifferential operators with symbol $\Lambda_\pm(\xi') = \xi_2 \pm i \pm i|\xi_1|$, r^+ denotes the operator of restriction on $\mathbb{R}^2_+$ and ℓ is an extension operator.

Since the operator has the same kernel and cokernel in all spaces where it is Fredholm and thus it has the same index (see [Ag1, Ka1, DNS2]), from Lemma 2.8 and Theorem 1.1 it follows that the operator $\mathbf{N}$ is invertible in the corresponding Besov and Bessel potential spaces (cf. similar proofs in [DNS1, DNS2]).

The second part, the solvability properties, are consequences of the first part and the mapping properties of the potential operators (see [DNS1, NCS1, Ch2] for similar considerations in elasticity). ■

We drop the detailed asymptotic expansion of a solution to mixed BVP, because we can not suggest any simplification in contrast to the Dirichlet and Neumann BVPs considered above. Asymptotic formulae for general PsDOs on a smooth surface with a smooth boundary are exposed in detail in [CD1, CDD1]. Spatial asymptotic representations of the solution to a BVP, based on surface asymptotics of a solution to the corresponding boundary pseudodifferential equation is exposed in [CD2, CDD1].

6. An example: media with cubic symmetry

Consider an electro-elastic medium with cubic symmetry [No1, To1]. In this case the entries of the corresponding operator (1.2) are of the form:

$$
\begin{aligned}
\mathbf{A}_{11}(D) &= c_{11}\partial_1^2 + c_{44}\partial_2^2 + c_{44}\partial_3^2,\\
\mathbf{A}_{22}(D) &= c_{44}\partial_1^2 + c_{11}\partial_2^2 + c_{44}\partial_3^2,\\
\mathbf{A}_{33}(D) &= c_{44}\partial_1^2 + c_{44}\partial_2^2 + c_{11}\partial_3^2,\\
\mathbf{A}_{ik}(D) &= (c_{12}+c_{44})\partial_i\partial_k, \quad i,k=1,2,3, \quad i\neq k,\\
\mathbf{A}_{14}(D) &= -\mathbf{A}_{41}(D) = 2e_{14}\partial_2\partial_3,\\
\mathbf{A}_{24}(D) &= -\mathbf{A}_{42}(D) = 2e_{14}\partial_1\partial_3,\\
\mathbf{A}_{34}(D) &= -\mathbf{A}_{43}(D) = 2e_{14}\partial_1\partial_2,\\
\mathbf{A}_{44}(D) &= \varepsilon_{11}\Delta.
\end{aligned}
$$

Condition (1.4) of internal energy positiveness imposes the following restrictions on the coefficients c_{ij} :

$$c_{11} > 0, \quad c_{44} > 0, \quad \varepsilon_{11} > 0, \quad -\frac{1}{2} < \frac{c_{12}}{c_{11}} < 1.$$

We can calculate the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_8$ with the help of the following theorem (see [Ch2, Theorem 6.6]):

Theorem 6.1. *Let* $\lambda_k(x')$, $k=1,\ldots,8,$ *be the eigenvalues of the matrix* $b_{\mathscr{R}}$ *. Then*

$$\lambda_k(x') = \begin{cases} i\sqrt{\dfrac{1-2\beta_k(x')}{1+2\beta_k(x')}}, & \text{if } \ k=1,\ldots,4,\\[2ex] -i\sqrt{\dfrac{1-2\beta_{k-4}(x')}{1+2\beta_{k-4}(x')}}, & \text{if } \ k=5,\ldots,8, \quad x'\in\mathscr{E}, \end{cases}$$

where $\beta_k \in \left]-\frac{1}{2};\frac{1}{2}\right[$ *are the eigenvalues of the matrix* $\sigma_{\overset{*}{\mathbf{V}}_0}$ *.*

After calculating the matrix $\sigma_{V_0^{(1)}}$ we obtain

$$\sigma_{V_0^{(1)}} = \begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & i\,\dfrac{(c_{11}-c_{12})}{2c_{11}\sqrt{2(\widetilde{c}+1)}} & 0\\ 0 & -i\,\dfrac{(c_{11}-c_{12})}{2c_{11}\sqrt{2(\widetilde{c}+1)}} & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$\widetilde{c} = \frac{c_{11}^2 - c_{12}^2}{2c_{11}c_{44}} - \frac{c_{12}}{c_{44}} > -1.$$

So

$$\beta_1 = \beta_2 = 0, \quad \beta_{3,4} = \pm\frac{c_{11}-c_{12}}{2c_{11}\sqrt{2(\widetilde{c}+1)}}$$

and the eigenvalues λ_j are purely imaginary.

Moreover, in this special case exponents of the leading term of asymptotic are $\frac{1}{4}$ and $\frac{1}{4} + i\delta$; exponents of further terms increase by order $\frac{1}{2}$ and not by 1 as it was in (3.8), 3.11), (4.12) and (4.15). A similar asymptotic encounters in mixed problems (see [CD3] and also [Ch2, DN1]). In case of general anisotropy however, these exponents depend on the elastic constants as well as on the geometry of the crack edge.

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