EXTENDED NORMAL VECTOR FIELD AND THE WEINGARTEN MAP ON HYPERSURFACES*

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Dedicated to the memory of Prof. J. L. Lions

Abstract

In this article we present a different method for studying the Weingarten map for a hypersurface in Euclidean space $\mathbb{R}^n$. Applying the cartesian coordinates of the ambient space and tangential Günter's derivatives we obtain a simple matrix representation formula for the Weingarten map for the implicit hypersurfaces, which can be applied, for example, to calculate the mean and Gauß's curvatures without passing to intrinsic coordinates.

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Introduction

Analysis on Riemannian manifolds usually applies intrinsic coordinates, involving metric tensor and Christoffel symbols. But if we deal with a hypersurface, the cartesian coordinates of the ambient space can be applied. This seemingly trivial idea often simplifies notions of the differential geometry. An advantage of this approach can be clearly observed, for example, by studying partial differential equations on hypersurfaces. It turns out that in this case the form of many classical differential equations on the surface are much simpler, cf. [Du1, DMM1]. Analysis of this type is connected tightly with the Weingarten map, the mean and Gauß's principal curvatures. The main purpose of the paper is to express the Weingarten matrix through the cartesian coordinates of the ambient space and obtain formulas for the mean and Gauß's curvatures. Such results are important in applications when differential equations (of elastic hypersurfaces, of shells etc.; cf. [Du1, DMM1]) are studied in the same setting, involving the cartesian coordinates of the ambient space.

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Let $\mathscr{S}$ be a C^2-smooth hypersurface in $\mathbb{R}^n$ and ν be the unit normal vector to $\mathscr{S}$. It is well known that the matrix representation of the Weingarten map $\mathscr{W}_{\mathscr{S}}$ depends on the choice of the frame in the tangential space $\mathbb{T}\mathscr{S}$. For example, written in local contravariant frame the Weingarten map is a matrix-function of order $n-1$ and coincides with the mixed curvature tensor $\mathscr{B}_{\mathscr{S}}(x) = \left\|b_j^k\right\|_{(n-1)\times(n-1)}$. Written in local covariant frame, it coincides with the curvature tensor $B_{\mathscr{S}}(x) = \left\|b_{jk}\right\|_{(n-1)\times(n-1)}$ (cf. [Ca1, Ci1, Ci2] and (1.15), (1.16) below).

Our approach applies Günter's derivatives $\left\{\mathscr{D}_j\right\}_{j=1}^n$, which are derivatives along the vector fields $\left\{d_j\right\}_{j=1}^n$, representing projections on the surface $\mathscr{S}$ of the cartesian unit vectors $\left\{e_j\right\}_{j=1}^n$. The system of vector fields $\left\{d_j\right\}_{j=1}^n$ is full, but linearly dependent in the space of tangential vector fields. In (2.1) we write the Weingarten map in the system $\left\{d_j\right\}_{j=1}^n$ as a $n \times n$ matrix function. Although the order of the matrix is increased, the obtained representation formula displays some simplicity and provides a possibility to calculate the mean and Gauß's curvatures without passing to intrinsic coordinates.

The paper is organized as follows. In Section 1 we recall some definitions and introduce basic notation from differential geometry for hypersurfaces. The Section 2 contains the necessary tools and material from linear algebra. Also there we express the Weingarten map through Günter's derivatives. In Section 3 we extend the unit vector field and using results of Section 2 obtain the representation formula of the Weingarten matrix. In the conclusion we illustrate the result by calculating Weingarten map and curvatures for the ellipsoid and the saddle surface.

1 Covariant derivative and the Weingarten map

Henceforth $\operatorname{matr}[u_1, \ldots, u_k]$ refers to the matrix with the listed vectors $u_1, \ldots, u_k$ as columns.

Definition 1.1 *A Subset $\mathscr{S} \subset \mathbb{R}^n$ of the Euclidean space is called hypersurface if it has a covering $\mathscr{S} = \bigcup_{j=1}^M \mathscr{S}_j$ and coordinate mappings*

$$\Theta_j \ : \ \omega_j \to \mathscr{S}_j = \Theta_j(\omega_j) \subset \mathbb{R}^n, \qquad \omega_j \subset \mathbb{R}^{n-1}, \quad j = 1, \ldots, M, \tag{1.1}$$

such that the corresponding differentials

$$D\Theta_j(x) := \operatorname{matr}\left[\partial_1\Theta_j(x), \ldots, \partial_{n-1}\Theta_j(x)\right], \tag{1.2}$$

have the full rank

$$\operatorname{rank} D\Theta_j(x) = n-1 \quad \text{for all} \quad x \in \omega_j\,, \quad k = 1, \ldots, n\,, \quad j = 1, \ldots, M.$$

Such mapping is called an immersion as well.

The columns of the Jacobi matrix $D\Theta_j(x)$ in (1.2) represent tangential vectors

$$g_1(x) := \partial_1\Theta_j(x), \ldots, g_{n-1}(x) := \partial_{n-1}\Theta_j(x) \in \mathbb{T}_x\mathscr{S}\,, \qquad \text{for all } x \in \omega_j\,. \tag{1.3}$$

Since $D\Theta_j(x)$ has the full rank, the above collection is a natural covariant frame (a basis) in the tangential space $\mathbb{T}_x\mathscr{S}$.

The hypersurface is called smooth if the corresponding coordinate diffeomorphisms Θ_j in (1.1) are all smooth (C^∞-smooth). Similarly is defined Lipschitz and k-smooth hypersurfaces, provided Θ_j in (1.1) are all Lipschitz continuous or k-smooth, respectively.

Next we expose yet another definition of a hypersurface which provides a powerful source of hypersurfaces.

Definition 1.2 *Let $k \geq 1$ an $U \subset \mathbb{R}^n$ be a compact domain. An implicit C^k-smooth (an implicit Lipschitz) hypersurface in $\mathbb{R}^n$ is defined as the set*

$$\mathscr{S} = \Big\{p \in U \ : \ \Psi_{\mathscr{S}}(p) = 0\Big\}, \tag{1.4}$$

where $\Psi_{\mathscr{S}} \ : \ U \to \mathbb{R}$ is a C^k-mapping (is a C^1-mapping, respectively) which has the non-vanishing gradient $\nabla\Psi_{\mathscr{S}}(p) \neq 0$, for all $p \in U$.

Note that by taking a single function $\Psi_{\mathscr{S}}$ for the implicit definition of a hypersurface $\mathscr{S}$ we does not restrict the generality and definitions of smooth hypersurface $\mathscr{S}$, since Definition 1.1 and Definition 1.2 are equivalent (a direct consequence of the implicit function theorem).

Example 1.3 *Let $R > 0$, $a \in \mathbb{R}^n$ be fixed. The sphere of radius R centered at a*

$$\mathbb{S}^{n-1}_R(a) := \Big\{x = (x_1, \dots, x_n)^\top \in \mathbb{R}^n \ : \ \Psi_{R,a}(x) = |x-a|^2 - R^2 = 0\Big\} \tag{1.5}$$

defines a hypersurface.

Similarly, for a fixed point $a = (a_1, \dots, a_n)^\top \in \mathbb{R}^n$, and a vector $r = (r_1, \dots, r_n)^\top$ with positive components $r_j > 0, \dots, r_n > 0$ the ellipsoid

$$\mathscr{E}^{n-1}_{r,a} := \left\{x = (x_1, \dots, x_n)^\top \in \mathbb{R}^n \ : \ \Psi_{r,a}(x) = \sum_{j=1}^n \left(\frac{x_j - a_j}{r_j}\right)^2 - 1 = 0\right\} \tag{1.6}$$

is a hypersurface in $\mathbb{R}^n$.

For vector fields $U, V \in \mathbb{T}\mathbb{R}^n = \mathscr{V}(\mathbb{R}^n)$ define the *first order differential operator*

$$\partial_U V(p) := \lim_{h\to 0} \frac{V\left(\mathscr{F}^h_U(p)\right) - V(p)}{h} = \frac{d}{dt}\, V\left(\mathscr{F}^t_U(p)\right)\Big|_{t=0}, \quad -\varepsilon < t < \varepsilon\,, \tag{1.7}$$

where $p \in \mathbb{R}^n$ and

$$y = y(t,p) = \mathscr{F}^t_U(p) \ : \ \mathbb{R}^n \to \mathbb{R}^n$$

is the solution of the following initial value problem

$$y' = U(y)\,, \qquad y(0) = p\,, \qquad -\varepsilon < t < \varepsilon$$

is called the *flow* (or the *orbit*) generated by the vector field U.

∂_U is a *derivation*, i.e., a linear mapping

$$\partial_U \ : \ C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n) \tag{1.8}$$

with the property

$$\partial_{fU} V = f\partial_U V \,, \qquad \partial_U(fg) = g\partial_U f + f\partial_U g \,, \tag{1.9}$$
$$\forall\, V \in \mathscr{V}(\mathbb{R}^n) \,, \quad \forall\, f, g \in C_0^\infty(\mathbb{R}^n) \,.$$

Now if U and V are tangent vector fields on on a hypersurface $\mathscr{S}$ in $\mathbb{R}^n$, i.e., $\mathbb{T}\mathscr{C}$, and U, V are both defined only at points of $\mathscr{S}$, the derivative $\partial_U V$ still makes sense as long as U is tangent to $\mathscr{S}$. As usual, $\partial_U V$ measures the rate of change of V in U direction, for more details cf. [Ta1, v. I, § 1.7], [Ne1, Ch. 5], etc. Note that even if U is a tangent vector field, the derivative $\partial_U V$ need not be tangent to $\mathscr{S}$.

Let ν denote the unit normal vector filed on a hypersurface $\mathscr{S}$, orthogonal to the frame of the tangent space

$$\langle \nu(p), g_k^*(p)\rangle = 0 \,, \qquad g^*(p) := g(\Theta^{-1}(p)), \qquad k = 1, \dots, n-1. \tag{1.10}$$

If the hypersurface is defined by immersions $\Theta_j \in C^1(\omega_j)$, $j = 1, \dots, M$, as in Definition 1.1, the normal vector field is then defined as the normed vector product of all tangent vectors (cf. (1.3))

$$\nu(p) := \pm \frac{g_1^*(p) \wedge \cdots \wedge g_{n-1}^*(p)}{|g_1^*(p) \wedge \cdots \wedge g_{n-1}^*(p)|} \tag{1.11}$$

at $p \in \mathscr{S}$, known in differential geometry as the *Gauß mapping*.

If the hypersurface is defined by an implicit function $\Psi_{\mathscr{S}}$, as in Definition 1.2, the normed gradient

$$\nu(p) := \frac{\nabla\Psi_{\mathscr{S}}(p)}{|\nabla\Psi_{\mathscr{S}}(p)|} \,, \qquad p \in \mathscr{S} \tag{1.12}$$

is then the normal vector field.

It is easy to show that the derivative $\partial_U \nu$ of the normal vector field on a hypersurface with respect to a tangent vector U at p is a tangent to the hypersurface at p. Indeed, differentiating the relation $1 \equiv \langle \nu, \nu\rangle$, we get

$$0 = \langle \partial_U \nu, \nu\rangle + \langle \nu, \partial_U \nu\rangle = \langle 2\, \partial_U \nu, \nu\rangle.$$

Thus $\partial_U \nu$ is orthogonal to $\nu(p)$.

Definition 1.4 *The linear map*

$$\mathscr{W}_{\mathscr{S}}(p) : \mathbb{T}_p\mathscr{S} \to \mathbb{T}_p\mathscr{S}$$

defined for a fixed $p \in \mathscr{S}$ *by*

$$\mathscr{W}_{\mathscr{S}}(p)V(p) = -\partial_{V(p)}\nu(p),$$

is called the Weingarten map or the shape operator of $\mathscr{S}$ *at* $p \in \mathscr{S}$.

The Weingarten map, applied to a tangent vector, can be interpreted as the rate of change of the unit normal vector field ν along the direction of a given tangent vector. Note that if ν is replaced by $-\nu$, then $\mathscr{W}_{\mathscr{S}}$ changes to $-\mathscr{W}_{\mathscr{S}}$.

The fundamental forms on $\mathscr{S}$ can be now defined in terms of $\mathscr{W}_{\mathscr{S}}$ and the inner product:

$$\mathrm{I}_p(V,U) = \langle V(p), U(p)\rangle \quad \text{and} \quad \mathrm{II}_p(V,U) = \langle \mathscr{W}_{\mathscr{S}}(p)V(p), U(p)\rangle$$

for $V, U \in \mathbb{T}_p(\mathscr{S})$. The first fundamental form is obviously a positive definite symmetric bilinear function on the tangent space. An important property of the Weingarten map and the second fundamental form is stated in the following theorem.

Proposition 1.5 (see [Ca1, Ta1] and cf. Corollary 2.4). *The second fundamental form is symmetric, i.e.,*

$$\langle \mathscr{W}_{\mathscr{S}}(p)V, U\rangle = \langle V, \mathscr{W}_{\mathscr{S}}(p)U\rangle \quad \text{for } \; V, U \in \mathbb{T}_p\mathscr{S}, \quad p \in \mathscr{S}.$$

In other words, the Weingarten map is self-adjoint with respect to the natural scalar product inherited from the ambient Euclidean space.

Moreover, using the pointwise principle we can take the derivative of a vector field V, i.e., define $\partial_V Z(p) = \partial_{V(p)}Z$. This leads to the following extension of the Weingarten map

$$\mathscr{W}_{\mathscr{S}} : \mathbb{T}\mathscr{S} \longrightarrow \mathbb{T}\mathscr{S} := \bigcup_{p\in\mathscr{S}} \mathbb{T}_p\mathscr{S}, \tag{1.13}$$

where

$$\langle \mathscr{W}_{\mathscr{S}}V, U\rangle = \mathrm{II}(V,U) = -\langle \partial_V\nu, U\rangle = -\langle \partial_U\nu, V\rangle \quad \text{for all} \quad V, U \in \mathbb{T}\mathscr{S}\,. \tag{1.14}$$

The matrix representation of the Weingarten map $\mathscr{W}_{\mathscr{S}}$ depends on the choice of the frame in $\mathbb{T}\mathscr{S}$. For example, written in a local contravariant frame $\left\{g^j\right\}_{j=1}^{n-1}$ it is a matrix-function of order $n-1$ coincides with the *mixed curvature tensor*:

$$\mathscr{B}_{\mathscr{S}}(x) = \left\|b_j^k(x)\right\|_{(n-1)\times(n-1)}, \quad \text{for all } \; x \in \Omega := \bigcup_{j=1}^{M}\omega_j\,, \tag{1.15}$$

$$b_j^k(x) := \langle \partial_j g^k(x), \nu\left(\Theta_k(x)\right)\rangle = -\langle g^k(x), \partial_j\nu\left(\Theta_k(x)\right)\rangle, \quad x \in \omega_k\,.$$

If the Weingarten map is written in the covariant frame $\left\{g_j\right\}_{j=1}^{n-1}$, coincides with the *curvature tensor*:

$$B_{\mathscr{S}}(x) = \left\|b_{jk}(x)\right\|_{(n-1)\times(n-1)}, \quad \text{for all } \; x \in \Omega \tag{1.16}$$

$$b_{jk}(x) := \langle \partial_k g_j(x), \nu\left(\Theta_k(x)\right)\rangle = -\langle g_k(x), \partial_j\nu\left(\Theta_k(x)\right)\rangle, \quad x \in \omega_k\,.$$

Note that b_{jk} are the coefficients of the *second fundamental form* and we have the both claimed representations:

$$\mathrm{II}(V,U)=\langle\mathscr{B}_{\mathscr{S}}V,U\rangle=\langle B_{\mathscr{S}}V,U\rangle\qquad\text{for all}\quad V,U\in\mathbb{T}\mathscr{S}\,. \tag{1.17}$$

Let us remind some very well known properties of the covariant and the contravariant frames. First due to the orthogonality, the covariant $\{g_k\}_{k=1}^{n-1}$ and the contravariant $\{g^k\}_{k=1}^{n-1}$ frames are related by the equalities

$$g_j=\sum_{k=1}^{n-1}g_{jk}g^k\,,\qquad g^j=\sum_{k=1}^{n-1}g^{jk}g_k\,,\qquad j,k=1,\ldots,n-1\,, \tag{1.18}$$

where

$$G_{\mathscr{S}}(p)=\big[g_{jk}(p)\big]_{(n-1)\times(n-1)}\,,\qquad g_{jk}=\langle g_j,g_k\rangle\,,\quad p\in\mathscr{S} \tag{1.19}$$

is the *covariant metric tensor* and

$$G^{-1}_{\mathscr{S}}(p)=\big[g^{jk}(p)\big]_{(n-1)\times(n-1)}\,,\qquad g^{jk}=\langle g^j,g^k\rangle\,,\quad p\in\mathscr{S} \tag{1.20}$$

is the *contravariant metric tensor* (the inverse to the covariant metric tensor):

$$G_{\mathscr{S}}G^{-1}_{\mathscr{S}}=I\quad\text{or}\quad g_{jm}(p)g^{mk}(p)=g^{jm}(p)g_{mk}(p)=\delta_{jk}\,. \tag{1.21}$$

From (1.15), (1.16) and (1.18) follows that:

$$b^k_j=\sum_{m=1}^{n-1}g^{km}b_{mj}\,,\qquad b_{jk}=\sum_{m=1}^{n-1}g_{km}b^m_j\,,\qquad j,k=1,\ldots,n-1\,. \tag{1.22}$$

The covariant curvature tensor is symmetric $b_{jk}=b_{kj}$, while the contravariant curvature tensor b^k_j is not in general. Thus,

$$\begin{aligned}\mathscr{B}_{\mathscr{S}}(p)&:=\big[b^k_j(p)\big]_{(n-1)\times(n-1)}=G^{-1}_{\mathscr{S}}(p)B_{\mathscr{S}}(p)\,,\\ B_{\mathscr{S}}(p)&:=\big[b_{jk}(p)\big]_{(n-1)\times(n-1)}=B^{\top}_{\mathscr{S}}(p)\,,\quad p\in\mathscr{S}.\end{aligned} \tag{1.23}$$

Our approach applies Günter's derivatives (cf. Section 2 for details), which are derivatives with respect to the full but linearly dependent system $\{d_j\}_{j=1}^{n}$ of tangential vector fields in (2.1). Before we go into details let us introduce further notation.

Denote by $\kappa_1(p),\ldots,\kappa_{n-1}(p)$ the eigenvalues of the mixed curvature tensor $\mathscr{B}_{\mathscr{S}}(p)$. They are called the principal curvatures, while

$$\begin{gathered}\mathscr{H}_{\mathscr{S}}(p):=\frac{\operatorname{Tr}\mathscr{B}_{\mathscr{S}}(p)}{n-1}\,,\qquad \operatorname{Tr}\mathscr{B}_{\mathscr{S}}(p):=\sum_{j=1}^{n-1}b^j_j(p)=\sum_{j=1}^{n-1}\frac{\kappa_j(p)}{n-1}\,,\\ \mathscr{K}_{\mathscr{S}}(p):=\det\,\mathscr{B}_{\mathscr{S}}(p)=\prod_{j=1}^{n-1}\kappa_j(p)\,,\qquad p\in\mathscr{S}\end{gathered} \tag{1.24}$$

are the *mean curvature* and the *Gaußian curvature* of $\mathscr{S}$, respectively.

2 Günter's derivatives

Günter's derivatives are defined as follows

$$\mathscr{D}_{\mathscr{S}} := (\mathscr{D}_1, \dots, \mathscr{D}_n)^\top, \quad \mathscr{D}_j := \partial_j - \nu_j(p)\partial_\nu = \partial_{d_j}, \qquad d_j := \pi_{\mathscr{S}}\, e_j = e_j - \nu_j\nu, \tag{2.1}$$

where $\partial_\nu := \sum_{j=1}^n \nu_j\partial_j$ denotes the normal derivative. They represent tangent derivatives along the tangent vector fields. Here, for each $1 \le j \le n$, the first-order differential operator $\mathscr{D}_j = \partial_{d_j}$ is the directional derivative along the tangential vector d_j. The operator (the matrix)

$$\pi_{\mathscr{S}} : \mathbb{R}^n \to \mathbb{T}\mathscr{S}, \quad \pi_{\mathscr{S}}(p) = I - \nu(p)\nu^\top(p) = \left[\delta_{jk} - \nu_j(p)\nu_k(p)\right]_{n\times n}, \quad p \in \mathscr{S} \tag{2.2}$$

defines the canonical orthogonal projection $\pi^2_{\mathscr{S}} = \pi_{\mathscr{S}}$ onto the tangential space $\mathbb{T}_p\mathscr{S}$ to $\mathscr{S}$ at the point $p \in \mathscr{S}$:

$$(\nu, \pi_{\mathscr{S}} v) = \sum_j \nu_j v_j - \sum_{j,k} \nu_j^2 \nu_k v_k = 0 \qquad \text{for all} \quad v = (v_1, \dots, v_n)^\top \in \mathbb{R}^n$$

and, as usual, $e_j = (\delta_{jk})_{1\le k\le n} \in \mathbb{R}^n$, with the Kronecker's symbol δ_{jk}.

A derivative ∂_U is called *covariant* if it maps tangential vector fields to tangential fields $\partial_U : \mathbb{T}\mathscr{S} \to \mathbb{T}\mathscr{S}$. Note that Günter's derivatives in (2.1) are not covariant $\mathscr{D}_j : \mathbb{T}\mathscr{S} \not\to \mathbb{T}\mathscr{S}$, i.e., not always $V \in \mathbb{T}\mathscr{S}$ implies $\mathscr{D}_j V \in \mathbb{T}\mathscr{S}$. To make Günter's derivatives covariant we apply the projection:

$$\mathscr{D}^{\mathscr{S}}_j V := \pi_{\mathscr{S}}\mathscr{D}_j V = \mathscr{D}_j V - \langle \nu, \mathscr{D}_j V\rangle \nu \tag{2.3}$$

or, in general,

$$\partial^{\mathscr{S}}_U V := \pi_{\mathscr{S}}\partial_U V \qquad \text{for} \quad U, V \in \mathbb{T}\mathscr{S}.$$

Then,

$$\mathscr{D}^{\mathscr{S}}_j V = \mathscr{D}_j V + (\mathscr{W}_{\mathscr{S}} V)_j \nu = \mathscr{D}_j V + (\mathscr{D}_V \nu_j)\nu, \tag{2.4}$$

where $\mathscr{W}_{\mathscr{S}} := \left[\mathscr{D}_j \nu_k\right]_{n\times n}$, $\mathscr{D}_V\varphi := \sum_{k=1}^n V_k\mathscr{D}_k\varphi$, because

$$\begin{aligned}\langle \nu, \mathscr{D}_j V\rangle &= \sum_{m=1}^n \nu_m \mathscr{D}_j V_m = \sum_{m=1}^n \left[\mathscr{D}_j(\nu_m V_m) - V_m \mathscr{D}_j \nu_m\right] \\ &= -\sum_{m=1}^n V_m \mathscr{D}_j \nu_m = -(\mathscr{W}_{\mathscr{S}} V)_j = -\sum_{m=1}^n V_m \mathscr{D}_m \nu_j = -\mathscr{D}_V \nu_j .\end{aligned}$$

It is easy to check that the operators $\mathscr{D}^{\mathscr{S}}_j$ are automorphisms of the tangential vector space

$$\mathscr{D}^{\mathscr{S}}_j : \mathbb{T}\mathscr{S} \to \mathbb{T}\mathscr{S}. \tag{2.5}$$

Any tangential vector field $U \in \mathbb{T}\mathscr{S}$ can be represented by the full system $\{d_j\}_{j=1}^n$ of tangential fields

$$U = \sum_{j=1}^n U_j e_j = \sum_{j=1}^n U_j d_j \quad \Longleftrightarrow \quad \partial_U V = \sum_{j=1}^n U_j \partial_j V = \sum_{j=1}^n U_j \mathscr{D}_j V \tag{2.6}$$

since $\sum_{j=1}^n U_j \nu_j = 0$. Then for the corresponding covariant derivative (for the Levi-Civita connection) $\partial_U^{\mathscr{S}}$ we get the following representation by the full system $\{\mathscr{D}_j^{\mathscr{S}}\}_{j=1}^n$ of covariant derivatives:

$$\partial_U^{\mathscr{S}} V := \partial_U V - \langle \nu, \partial_U V\rangle \nu = \sum_{j=1}^n U_j \mathscr{D}_j^{\mathscr{S}} V \qquad \text{for all} \quad V \in \mathbb{T}\mathscr{S}\,. \tag{2.7}$$

Then for the second fundamental form of $\mathscr{S}$ we have (cf. [Ta1]):

$$\begin{aligned} \mathrm{II}(U(p), V(p))\nu(p) &= \partial_U V(p) - \partial_U^{\mathscr{S}} V(p) = (I - \pi_{\mathscr{S}})\partial_U V(p) \\ &= \langle \nu(p), \partial_U V(p)\rangle \nu(p), \quad \text{for all } p \in \mathscr{S}\,, \quad U, V \in \mathbb{T}\mathscr{S}. \end{aligned}$$

For the Weingarten map the obtained equality gives

$$\langle \mathscr{W}_{\mathscr{S}} U, V\rangle = \mathrm{II}(U, V) = \langle \nu, \partial_U V\rangle = -\langle \partial_U \nu, V\rangle = -\langle \partial_U^{\mathscr{S}} \nu, V\rangle \tag{2.8}$$
$$\text{for all } U, V \in \mathbb{T}\mathscr{S}\,.$$

We have applied that $V \in \mathbb{T}\mathscr{S}$ yields $\langle \nu, V\rangle \equiv 0$ and, by differentiating,

$$\langle \partial_U \nu, V\rangle + \langle \nu, \partial_U V\rangle \equiv 0 \qquad \text{for all} \quad U \in \mathbb{T}\mathscr{S}\,.$$

For each pair of frames $H := \{h_j\}_{j=1}^n$ and $D := \{d_j\}_{j=1}^n$ in a finite dimensional Banach space $\mathfrak{B}$ there exists the *matrix of frame transformation* $D = \mathscr{A}_{H\to D} H$, $\mathscr{A}_{H\to D} = \big[h_{jk}\big]_{n\times n}$. Due to the linear independence of frames the matrix of frame transformation is invertible $\mathscr{A}_{D\to H} = \mathscr{A}_{H\to D}^{-1} = \big[h^{jk}\big]_{n\times n}$ and the inverse is responsible for the inverse frame transformation $H = \mathscr{A}_{D\to H} D$.

By fixing a frame $H := \{h_j\}_{j=1}^n$ in a finite dimensional Banach space $\mathfrak{B}$, a linear operator $A : \mathfrak{B} \to \mathfrak{B}$ can be represented in the matrix form $A = \big[a_{jk}\big]_{n\times n}$, where the entries a_{jk} are the coefficients of the representations

$$A h_j = \sum_{k=1}^n a_{kj} h_k\,, \qquad a_{kj} = \langle h^k, A h_j\rangle\,, \qquad j = 1, \dots, n\,, \tag{2.9}$$

and $H^\perp := \big\{h^k\big\}_{k=1}^n$ is the biorthogonal frame to H: $\langle h_j, h^k\rangle = \delta_{jk}$, $j, k = 1, \dots, n$.

The matrix representation of an operator A depends on the choice of a frame:

$$\widehat{A}_D = \mathscr{A}_{H\to D} \widehat{A}_H \mathscr{A}_{D\to H} \quad \text{or} \quad \big[b_{jk}\big]_{n\times n} = \big[h_{jk}\big]_{n\times n} \big[a_{jk}\big]_{n\times n} \big[h^{jk}\big]_{n\times n}\,. \tag{2.10}$$

Since $\mathscr{A}_{D\to H} = \mathscr{A}_{H\to D}^{-1}$, an immediate consequence of (2.10) is that the determinants of the representation matrices are independent of the choice of frames D and H in $\mathfrak{B}$, i.e., $\det \widehat{A}_D = \det \widehat{A}_H$. Moreover, also the trace is invariant and the following assertion is well-known in the operator theory.

Lemma 2.1 *Let $A : \mathfrak{B} \to \mathfrak{B}$ be a linear operator in a finite dimensional Banach space $\mathfrak{B}$ with a frame $H := \{h_j\}_{j=1}^n$ and $\widehat{A}_H = [a_{jk}]_{n\times n}$ be the corresponding matrix representation of A. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A counted with their multiplicities.*

The trace of A is independent of a choice of frame H in $\mathfrak{B}$:

$$\mathrm{Tr}A = \sum_{j=1}^n a_{jj} = \sum_{j=1}^n \langle h^j, Ah_j\rangle = \sum_{j=1}^n \lambda_j. \tag{2.11}$$

In the next Lemma 2.2 we consider a special case, important for the present consideration. Let $H := \{h_j\}_{j=1}^n$, $|h_j| = 1$, be a frame in n-dimensional Banach space $\mathfrak{B}$. Consider the hyperspace $\mathfrak{B}_\nu := \{u \in \mathfrak{B} : \langle u, \nu\rangle = 0\}$ orthogonal to a vector $\nu \in \mathfrak{B}$, $|\nu| \neq 0$. The system

$$d_j := h_j - \nu_j \nu\,, \quad \nu_j := \langle \nu, h_j\rangle \qquad j = 1, \dots, n\,, \tag{2.12}$$

is full in $\mathfrak{B}_\nu$ but linearly dependent and thus can not be a frame.

Lemma 2.2 *If a linear operator $A : \mathfrak{B} \to \mathfrak{B}_\nu$ with $A\nu = 0$ has the representation $\widehat{A}_H := [a_{jk}]_{n\times n}$ in the frame $H := \{h_j\}_{j=1}^n \subset \mathfrak{B}$, the restricted operator to the subspace $A^\nu := A\big|_{\mathfrak{B}_\nu}$ has the same representation $\widehat{A}^\nu_D = [a_{jk}]_{n\times n}$ in the systems $D := \{d_j\}_{j=1}^n \subset \mathfrak{B}_\nu$.*

If $\widehat{A}^\nu_B = [b_{jk}]_{(n-1)\times(n-1)}$ is the representation of the restricted operator in some frame $B := \{b_j\}_{j=1}^{n-1} \subset \mathfrak{B}_\nu$, all eigenvalues $\lambda_1, \dots \lambda_{n-1}$ of the matrix $\widehat{A}^\nu_B = [b_{jk}]_{(n-1)\times(n-1)}$ are the eigenvalues of $\widehat{A}_H = \widehat{A}^\nu_D = [a_{jk}]_{n\times n}$, which has the extra eigenvalue $\lambda_n = 0$ as well. Consequently these matrices have the same traces and

$$\det\, \widehat{A}^\nu_B = \lambda_1 \cdots \lambda_{n-1} = -\frac{d}{d\lambda}\det(\widehat{A}_H - \lambda\, I)\bigg|_{\lambda=0}. \tag{2.13}$$

Proof: Let us notice that

$$\sum_{k=1}^n a_{jk}\nu_k = \sum_{k=1}^n a_{kj}\nu_k = 0 \qquad \text{for all} \quad j = 1, \dots, n\,,$$

where the first equality is equivalent to $A\nu = 0$ and the second one-to $\langle \nu, A\xi\rangle = 0$ for all $\xi \in \mathfrak{B}$. Applying the obtained equalities we find that

$$A\hat{h}_j = Ah_j - \nu_j A\nu = \sum_{k=1}^n a_{kj}h_k = \sum_{k=1}^n a_{kj}\hat{h}_k + \sum_{k=1}^n a_{kj}\nu_k\nu = \sum_{k=1}^n a_{kj}\hat{h}_k$$

which entails $\widetilde{a}_{kj} = a_{kj}$ (cf. (2.9)).

If $B := \{b_j\}_{j=1}^{n-1}$ is a frame in $\mathfrak{B}_\nu$, the extended system $\widetilde{B} := \{b_j\}_{j=1}^n$, $b_n := \nu$ is a frame in the entire space $\mathfrak{B}$ and

$$\widehat{A}_{\widetilde{B}} := [b_{jk}]_{n\times n}, \qquad b_{nj} = b_{jn} = 0\,, \quad j = 1, \dots, n, \tag{2.14}$$

is the representation of A in this frame. The matrix $\widehat{A}_{\widetilde{B}} = \big[b_{jk}\big]_{n\times n}$ has the common eigenvalues $\lambda_1, \dots, \lambda_{n-1}$ with the matrix $\widehat{A}^{\nu}_{B} = \big[b_{jk}]_{(n-1)\times(n-1)}$ and one extra eigenvalue $\lambda_n = 0$. On the other hand, due to the foregoing Lemma 2.1 the matrices $\widehat{A}_{\widetilde{B}} = \big[b_{jk}]_{n\times n}$ and $\widehat{A}_H = \big[a_{jk}]_{n\times n}$ are equivalent and thus their all eigenvalues coincide. $\square$

Theorem 2.3 *Let $\mathscr{S}$ be a C^1-smooth hypersurface in $\mathbb{R}^n$ and ν be the outer unit normal vector to $\mathscr{S}$. In the full but linearly dependent system $\big\{d_j\big\}_{j=1}^n$ in* (2.1) *we have the following representation of the Weingarten map:*

$$\mathscr{W}_{\mathscr{S}} V = -\partial^{\mathscr{S}}_V \nu = -(\mathscr{D}\nu)V\,, \quad \mathscr{D}\nu := \big[\mathscr{D}_j\nu_k\big]_{n\times n} \quad \text{on } \mathscr{S} \qquad (2.15)$$

$$\text{for all } V \in \mathbb{T}\mathscr{S}\,.$$

The eigenvalues $\{\kappa_j(p)\}_{1\leq j\leq n-1}$ of $\mathscr{W}_{\mathscr{S}}(p)$ at $p \in \mathscr{S}$, except the last one which is trivial $\kappa_n(p) = 0$, are the principal curvatures at $p \in \mathscr{S}$; their arithmetical mean is the mean curvature and the product $\kappa_1(p)\cdots\kappa_{n-1}(p)$, except the last trivial one $\kappa_n(p) = 0$, is Gauß's principal curvature at $p \in \mathscr{S}$ (cf. (1.24)*).*

Proof: We are under the scope of the foregoing Lemma 2.2. In fact, let $\mathfrak{B} = \mathbb{R}^n$ and $H := \big\{e^j = e_j\big\}_{j=1}^n$ be the canonical frame in the Euclidean space $\mathbb{R}^n$. $\nu(p)$ be the normal vector to the surface at $p \in \mathscr{S}$ and $D_p := \big\{d_j(p)\big\}_{j=1}^n$ be the projection of the frame H to the tangential space $\mathbb{T}_p\mathscr{S}$. The system D_p is orthogonal to the normal vector field ν, is full in the tangential space $\mathbb{T}\mathscr{S} \subset \mathbb{R}^n$ and generates Günter's derivatives (cf. (2.1)).

Since the Weingarten operator $\mathscr{W}_{\mathscr{S}}$ in (2.8) annihilates the normal vector (cf. (2.17)), due to the foregoing Lemma 2.2 it has the same representation in the systems H and D_p:

$$\langle \mathscr{W}_{\mathscr{S}} U, V\rangle = \mathrm{II}(U,V) = \langle -\partial^{\mathscr{S}}_U \nu, V\rangle = -\sum_{j,k=1}^n \big(U_k \mathscr{D}_k \nu_j\big)V_j = \langle -(\mathscr{D}\nu)U, V\rangle\,.$$

Since a vector field $V \in \mathbb{T}\mathscr{S}$ is arbitrary, the latter equality implies (2.15).

Let us consider the contravariant frame $\big\{g^j\big\}_{j=1}^{n-1}$ in the tangential space $\mathbb{T}\mathscr{S}$ and its extension $\big\{g^j\big\}_{j=1}^{n}$ in $\mathbb{R}^n$ by the normal unit vector field $g^n := \nu$ on $\mathscr{S}$. The Weingarten operator is represented in the frame $\big\{g^j\big\}_{j=1}^{n-1}$ by the matrix $\mathscr{B}_{\mathscr{S}}(x) = \big\|b^k_j(x)\big\|_{(n-1)\times(n-1)}$ (cf. (1.15)). Thus, we are indeed under the scope of the foregoing Lemma 2.2 and the matrices $\mathscr{B}_{\mathscr{S}}$ of order $n-1$ and $\mathscr{W}_{\mathscr{S}}$ of order n, have $n-1$ eigenvalues $\{\kappa_j(p)\}_{1\leq j\leq n-1}$ in common. The last extra eigenvalue of $\mathscr{W}_{\mathscr{S}}$ is trivial $\kappa_n = 0$. $\square$

A similar assertion for the case $n = 3$ is proved in [Gu1, Ch. 1, § 3] by a different approach.

Corollary 2.4 *The Weingarten matrix is symmetric*

$$\mathscr{W}_{\mathscr{S}} = -\big[\mathscr{D}_j\nu_k\big]_{n\times n} = \mathscr{W}_{\mathscr{S}}^{\top} \tag{2.16}$$

and has linearly dependent columns

$$\mathscr{W}_{\mathscr{S}}(p)\nu(p) = 0. \tag{2.17}$$

The latter equality implies:

$$\nu(p) \in \mathbf{Ker}\,\mathscr{W}_{\mathscr{S}}(p) \quad \text{and} \quad \det\,\mathscr{W}_{\mathscr{S}}(p) \equiv 0 \quad \text{for all} \quad p \in \mathscr{S}. \tag{2.18}$$

Proof: : (2.16) is a consequence of equality

$$\mathscr{D}_j\nu_k = \partial_j\nu_k = \partial_k\nu_j = \mathscr{D}_k\nu_j \qquad \text{for all} \quad j,k = 1,\dots,n, \tag{2.19}$$

proved in [DMM1, Proposition 3.1.ii] (can also be derived from Lemma 3.3 below), while (2.17) is proved as follows:

$$(\mathscr{W}_{\mathscr{S}}\nu)_k = -2\sum_j \nu_j\mathscr{D}_k\nu_j = -\mathscr{D}_k|\nu|^2 = -\partial_k 1 = 0. \qquad \square$$

3 Extension of the unit normal vector field to a hypersurface

Lemma 3.1 *Let $\mathscr{S} \subset \mathbb{R}^n$ be a k-smooth hypersurface, $k = 1, 2, \dots$, given implicitly $\Psi_{\mathscr{S}}(p) = 0$ by the function $\Psi_{\mathscr{S}} \in C^k(U_{\mathscr{S}})$ defined in some neighborhood of the surface $\mathscr{S} \subset U_{\mathscr{S}} \subset \mathbb{R}^n$. The unit vector field*

$$\mathscr{N} := \frac{\nabla\Psi_{\mathscr{S}}}{|\nabla\Psi_{\mathscr{S}}|} = \{\mathscr{N}_1,\dots,\mathscr{N}_n\}^{\top}, \qquad \mathscr{N}_j = \frac{\partial_j\Psi_{\mathscr{S}}}{|\nabla\Psi_{\mathscr{S}}|}, \quad j = 1,\dots,n \tag{3.1}$$

is C^{k-1}-smooth and restricted to the surface coincides with the normal vector field on $\mathscr{S}$

$$|\mathscr{N}| = 1 \;\; \textit{near} \;\; \mathscr{S} \quad \textit{and} \;\; \mathscr{N}\Big|_{\mathscr{S}} = \nu\,. \tag{3.2}$$

Moreover, if $k \geq 2$ the following equality holds:

$$\nu = \nabla\frac{\Psi_{\mathscr{S}}}{|\nabla\Psi_{\mathscr{S}}|}\Bigg|_{\mathscr{S}} \quad \text{or, componentwise,} \quad \nu_j = \partial_j\frac{\Psi_{\mathscr{S}}}{|\nabla\Psi_{\mathscr{S}}|}\Bigg|_{\mathscr{S}}, \; j = 1,\dots,n. \tag{3.3}$$

Proof: Let $\{\mathscr{S}_j, \Theta_j\}_{j=1}^M$ be an atlas which defines $\mathscr{S}$. The pull-back functions $\Psi_j^*(x) = (\Theta_{j,*}\Psi_{\mathscr{S}})(x) = \Psi_j(\Theta_j(x))$, $x \in \Omega_j \subset \mathbb{R}^{n-1}$, are immersions: the corresponding gradient has maximal rank

$$\begin{gathered}\nabla\Psi_j^*(x) := \text{matr}\,[\partial_1\Psi_j^*(x),\dots,\partial_{n-1}\Psi_j^*(x)]\,,\\ \text{rank}\,\nabla\Psi_j^*(x) = n-1 \quad \text{for all } x \in \Omega_j\,, \quad j = 1,\dots,M\,.\end{gathered} \tag{3.4}$$

Since $\Psi_j^*(x) \equiv 0$ for $x \in \Omega_j$, the chain rule provides

$$\partial_k \Psi_j^*(x) = \sum_{m=1}^{n-1} (\partial_m \Psi_{\mathscr{S}})(\Theta_j(x))(\partial_k \Theta_j)_m(x) = 0\,, \quad k = 1, \dots, n-1$$

and justifies that the gradient of the hypograph function is orthogonal to all tangential vectors

$$\big\langle \partial_k \Theta_j(x), (\nabla \Psi_{\mathscr{S}})(\Theta_j(x)) \big\rangle \equiv 0 \quad \forall x \in \Omega_j\,, \quad k = 1, \dots, n,\ j = 1, \dots, M. \tag{3.5}$$

Therefore, the normed gradient

$$\mathscr{N}(x) = \frac{(\nabla \Psi_{\mathscr{S}})(x)}{|(\nabla \Psi_{\mathscr{S}})(x)|}\,, \qquad x \in \mathscr{S} \tag{3.6}$$

coincides with the unit outer normal vector on the surface.

The equality (3.3) follows taking into account that $\Psi_{\mathscr{S}}\big|_{\mathscr{S}} \equiv 0$:

$$\partial_j \frac{\Psi_{\mathscr{S}}}{|\nabla \Psi_{\mathscr{S}}|}\bigg|_{\mathscr{S}} = \frac{\partial_j \Psi_{\mathscr{S}}}{|\nabla \Psi_{\mathscr{S}}|}\bigg|_{\mathscr{S}} - \Psi_{\mathscr{S}}\big|_{\mathscr{S}} \frac{\partial_j |\nabla \Psi_{\mathscr{S}}|}{|\nabla \Psi_{\mathscr{S}}|^2}\bigg|_{\mathscr{S}} = \frac{\partial_j \Psi_{\mathscr{S}}}{|\nabla \Psi_{\mathscr{S}}|}\bigg|_{\mathscr{S}} = \nu_j\,. \qquad \square$$

Remark 3.2 *In [DMM1, § 3] the foregoing Lemma 3.1 was proved for a particular implicit function-for the signed distance*

$$\Psi_{\mathscr{S}}(x) := \pm \mathrm{dist}(x, \mathscr{S}) \qquad x \in U_{\mathscr{S}}\,, \tag{3.7}$$

where the signs "+" and "−" are chosen when x is "above" $\mathscr{S}$ (in the direction of the unit normal vector) and "below" $\mathscr{S}$, respectively.

The next Lemma 3.3 is identical to [DMM1, Proposition 3.1]. Here we present a slightly simplified proof.

Lemma 3.3 *For any unitary extension $\mathscr{N} \in C^1(U_{\mathscr{S}})$ of the unitary outer normal vector field ν in a neighborhood $U_{\mathscr{S}}$ of $\mathscr{S}$ the following conditions are equivalent:*

i. $\partial_{\mathscr{N}} \mathscr{N}\big|_{\mathscr{S}} = 0$, *i.e.,* $\partial_{\mathscr{N}} \mathscr{N}_j(x) \to 0$ *for* $x \to p \in \mathscr{S}$ *and* $j = 1, 2, ..., n$;

ii. $[\partial_k \mathscr{N}_j - \partial_j \mathscr{N}_k]\big|_{\mathscr{S}} = 0$ *for* $k, j = 1, 2, \dots, n$.

Proof: The implication $(ii) \Rightarrow (i)$ follows readily by writing

$$\begin{aligned}\partial_{\mathscr{N}} \mathscr{N}\big|_{\mathscr{S}} &= \left\{ \sum_{j=1}^n \mathscr{N}_j \partial_j \mathscr{N}_k \right\}_{k=1}^n \Bigg|_{\mathscr{S}} = \left\{ \sum_{j=1}^n \mathscr{N}_j \partial_k \mathscr{N}_j \right\}_{k=1}^n \Bigg|_{\mathscr{S}} \\ &= \frac{1}{2} \nabla_x |\mathscr{N}|^2 \Big|_{\mathscr{S}} = \frac{1}{2} \nabla_x 1 = 0\,.\end{aligned} \tag{3.8}$$

As for the inverse implication, we first observe that, in general,

$$\partial_V \mathscr{N}\Big|_{\mathscr{S}} = 0 \quad \& \quad \mathscr{N}\Big|_{\mathscr{S}} = \nu \text{ imply } \partial_V \mathscr{N}\Big|_{\mathscr{S}} \quad \text{depends only on} \quad \nu \tag{3.9}$$

and does not depend on a particular extension $\mathscr{N}$ for arbitrary vector field V. In the sequel we shall tacitly assume that the projection $\pi_{\mathscr{S}}$ in (2.2) has been extended to the neighborhood $U_{\mathscr{S}}$

$$\widetilde{\pi}_{\mathscr{S}}(x) = \left[\delta_{jk} - \mathscr{N}_j(x)\mathscr{N}_k(x)\right]_{n\times n}, \quad \widetilde{\pi}^2_{\mathscr{S}} = \widetilde{\pi}_{\mathscr{S}}, \quad x \in U_{\mathscr{S}}. \tag{3.10}$$

Note that $U = \widetilde{\pi}_{\mathscr{S}}U + \langle U, \mathscr{N}\rangle\mathscr{N}$ for arbitrary field U in the neighborhood $U_{\mathscr{S}}$. Then

$$\partial_U\mathscr{N}\Big|_{\mathscr{S}} = \partial_{\widetilde{\pi}_{\mathscr{S}}U}\mathscr{N}\Big|_{\mathscr{S}} + (U, \mathscr{N})\partial_{\mathscr{N}}\mathscr{N}\Big|_{\mathscr{S}} = \partial_{\widetilde{\pi}_{\mathscr{S}}U}\mathscr{N}\Big|_{\mathscr{S}} = \partial_{\pi_{\mathscr{S}}U}\nu\,,$$

because $\partial_{\mathscr{N}}\mathscr{N}\Big|_{\mathscr{S}} = 0$ and $\pi_{\mathscr{S}}U$ is a tangential field to $\mathscr{S}$. Thus, we can dwell on the particular extension (3.3) and observe

$$\partial_k\mathscr{N}_j|_{\mathscr{S}} = \partial_k\partial_j\frac{\Psi_{\mathscr{S}}}{|\nabla\Psi_{\mathscr{S}}|}\Bigg|_{\mathscr{S}} = \partial_j\partial_k\frac{\Psi_{\mathscr{S}}}{|\nabla\Psi_{\mathscr{S}}|}\Bigg|_{\mathscr{S}} = \partial_j\mathscr{N}_k|_{\mathscr{S}}\ ,$$

which proves the implication $(i) \Rightarrow (ii)$. □

Remark 3.4 *It is clear that a normal vector field and it's (non-unique) extension exists for arbitrary Lipschitz surface, but almost everywhere on $\mathscr{S}$.*

Moreover to enjoy the properties listed in Lemma 3.3, we have to consider smoother than Lipschitz surfaces and assume C^2-smoothness of $\mathscr{S}$. □

Definition 3.5 *Let $\mathscr{S}$ be a surface in $\mathbb{R}^n$ with the unit normal vector field ν. A vector filed $\mathscr{N} \in C^1(U_{\mathscr{S}})$ in a neighborhood $U_{\mathscr{S}}$ of $\mathscr{S}$ will be referred to as a* **proper extension** *if $\mathscr{N}\Big|_{\mathscr{S}} = \nu$, $|\mathscr{N}| = 1$ in $U_{\mathscr{S}}$, and if $\mathscr{N}$ satisfies one of the (equivalent) conditions listed in Lemma 3.3.*

For our purposes below the conditions of Lemma 3.3 play a crucial role. Now we are going to construct the extension of the unit normal field, which is not proper extension, but satisfies the both conditions of Lemma 3.3.

Lemma 3.6 *Let $\Psi_{\mathscr{S}}(x) \in C^2(U_{\mathscr{S}})$ and $\mathscr{N}$ be unit vector field be as in Lemma 3.1. Then the vector field*

$$\widehat{\mathscr{N}} = \{\widehat{\mathscr{N}_1}, \ldots, \widehat{\mathscr{N}_n}\}^\top, \qquad \widehat{\mathscr{N}_j} := \mathscr{N}_j - \frac{\Psi_{\mathscr{S}}}{|\nabla\Psi_{\mathscr{S}}|}\,\partial_{\mathscr{N}}\mathscr{N}_j \tag{3.11}$$

is C^{k-1}-smooth and restricted to the surface coincides with the normal vector field on $\mathscr{S}$, i.e., $\widehat{\mathscr{N}}\big|_{\mathscr{S}} = \mathscr{N}\big|_{\mathscr{S}} = \nu$ and we have $\partial_{\widehat{\mathscr{N}}}\widehat{\mathscr{N}}\big|_{\mathscr{S}} = 0$.

Note that the field $\widehat{\mathscr{N}}$ is not, in contrast to (3.1) the unit normal vector field, since $|\widehat{\mathscr{N}}(x)| \neq 1$ for $x \notin \mathscr{S}$.
Proof of Lemma 3.6: First note that

$$\partial_k\widehat{\mathscr{N}_j}\big|_{\mathscr{S}} = (\partial_k\mathscr{N}_j - \mathscr{N}_k\partial_{\mathscr{N}}\mathscr{N}_j)\big|_{\mathscr{S}}. \tag{3.12}$$

Then

$$\partial_{\widehat{\mathscr N}}\widehat{\mathscr N}\Big|_{\mathscr S}=\sum_{k=1}^{n}\widehat{\mathscr N_k}\partial_k\widehat{\mathscr N_j}\Big|_{\mathscr S}=\sum_{k=1}^{n}\mathscr N_k\big(\partial_k\mathscr N_j-\mathscr N_k\partial_{\mathscr N}\mathscr N_j\big)\Big|_{\mathscr S}$$
$$=\sum_{k=1}^{n}\mathscr N_k\partial_k\mathscr N_j\Big|_{\mathscr S}-\partial_{\mathscr N}\mathscr N_j\sum_{k=1}^{n}\mathscr N_k^2\Big|_{\mathscr S}=\partial_{\mathscr N}\mathscr N_j\big|_{\mathscr S}-\partial_{\mathscr N}\mathscr N_j\big|_{\mathscr S}=0.$$

□

Let us consider the matrix-function

$$\widetilde{\mathscr W_{\mathscr S}}(x):=-\nabla\mathscr N(x)=-\big[\partial_j\mathscr N_k(x)\big]_{n\times n},\qquad x\in U_{\mathscr S}. \tag{3.13}$$

Lemma 3.7 *Let $\mathscr S$ be a hypersurface in $\mathbb R^n$ and fix a properly extended unit field $\mathscr N$ in a neighborhood $U_{\mathscr S}$ of $\mathscr S$. Then for the matrix function $\widetilde{\mathscr W_{\mathscr S}}(x)$ in (3.13) the following are true:*

i. $\widetilde{\mathscr W_{\mathscr S}}\mathscr N=0$ *in* $U_{\mathscr S}$;

ii. when restricted to the hypersurface $\widetilde{\mathscr W_{\mathscr S}}$ coincides with the Weingarten mapping of $\mathscr S$:
$$\widetilde{\mathscr W_{\mathscr S}}\Big|_{\mathscr S}=\mathscr W_{\mathscr S}\,; \tag{3.14}$$

iii. the trace on the surface $\mathscr W_{\mathscr S}=\widetilde{\mathscr W_{\mathscr S}}\big|_{\mathscr S}$ only depends on $\mathscr S$ and not on the choice of the extension $\mathscr N$;

iv. $\mathrm{Tr}\,(\widetilde{\mathscr W_{\mathscr S}})\big|_{\mathscr S}=\mathrm{Tr}\,\mathscr W_{\mathscr S}=\mathscr H^0_{\mathscr S}$;

v. $\mathscr W_{\mathscr S}V$ is tangential to $\mathscr S$ for any vector field $V:\mathscr S\to\mathbb R^n$.

Proof: First, $\widetilde{\mathscr W_{\mathscr S}}\mathscr N=\nabla\|\mathscr N\|^2=0$ in $U_{\mathscr S}$, justifying (i). Next, (ii) and (iv) follow from (3.13) and Lemma 2.3, whereas (iii) is a direct consequences of (3.9).

Next, (v) is a consequence of (3.14), for each $V\in\mathbb T\mathscr S$ we write

$$\mathscr W_{\mathscr S}V=-\partial^{\mathscr S}_V\mathscr N\Big|_{\mathscr S}=-\pi_{\mathscr S}(\partial_V\mathscr N)=-\partial_V\mathscr N=-\widetilde{\mathscr W_{\mathscr S}}V$$

since, as we have just seen, $\partial_V\mathscr N=\widetilde{\mathscr W_{\mathscr S}}V$ is tangential to $\mathscr S$. □

Let us consider a derivative

$$\widehat{\mathscr D}_k=\partial_k-\widehat{\mathscr N_k}\partial_{\widehat{\mathscr N}},\qquad 1\le k\le n.$$

which is an extended Günter's derivative: while restricted to the surface it coincides with $\mathscr D_k=\partial_k-\nu_k\partial_\nu$ on $\mathscr S$. By virtue of Lemma 3.6, Lemma 3.3 and formula (3.12) we conclude that:

$$\widehat{\mathscr D}_k\widehat{\mathscr N_j}\Big|_{\mathscr S}=\partial_k\widehat{\mathscr N_j}\Big|_{\mathscr S}=\mathscr D_k\mathscr N_j\big|_{\mathscr S}=\mathscr D_k\nu_j=\mathscr D_j\nu_k=\mathscr D_j\mathscr N_k\big|_{\mathscr S}$$
$$=\partial_j\widehat{\mathscr N_k}\Big|_{\mathscr S}=\widehat{\mathscr D}_j\widehat{\mathscr N_k}\Big|_{\mathscr S}. \tag{3.15}$$

Theorem 3.8 *Let $\mathscr{S} \subset \mathbb{R}^n$ be a hypersurface given implicitly $\Psi_{\mathscr{S}}(p) = 0$ by the function $\Psi_{\mathscr{S}}(p) \in C^2(U_{\mathscr{S}})$. Then at $p \in \mathscr{S}$ the Weingarten matrix is*

$$\mathscr{W}_{\mathscr{S}}(p) = (\mathscr{N}(p)\mathscr{N}^\top(p) - I_n)W_{\mathscr{S}}(p)(I_n - \mathscr{N}(p)\mathscr{N}(p)^\top), \tag{3.16}$$

where $\mathscr{N}$ is given by the formula (3.1) *and $W_{\mathscr{S}}(p)$ is, up to the multiplier, the Hessian of $\Psi_{\mathscr{S}}$:*

$$W_{\mathscr{S}}(p) = \frac{1}{|\nabla\Psi_{\mathscr{S}}(p)|}\begin{pmatrix} \partial_1^2\Psi_{\mathscr{S}}(p) & \cdots & \partial_1\partial_n\Psi_{\mathscr{S}}(p) \\ \vdots & \ddots & \vdots \\ \partial_n\partial_1\Psi_{\mathscr{S}}(p) & \cdots & \partial_n^2\Psi_{\mathscr{S}}(p) \end{pmatrix}.$$

Proof: Due to (3.14) for the Weingarten matrix at $p \in \mathscr{S}$ we have

$$\mathscr{W}_{\mathscr{S}}(p) = -\big[\partial_k\widehat{\mathscr{N}_j}\big|_{p\in\mathscr{S}}\big]_{n\times n}.$$

For conciseness, we will drop the sign of restriction $|_{\mathscr{S}}$ when it does not leads to a confusion. Applying (3.12) we then obtain

$$\mathscr{W}_{\mathscr{S}} = (\mathscr{N}\mathscr{N}^\top - I_n)\begin{pmatrix} \partial_1\mathscr{N}_1 & \cdots & \partial_1\mathscr{N}_n \\ \vdots & \ddots & \vdots \\ \partial_n\mathscr{N}_1 & \cdots & \partial_n\mathscr{N}_n \end{pmatrix}. \tag{3.17}$$

Further, we have

$$\begin{aligned} \partial_k\mathscr{N}_j\big|_{\mathscr{S}} = \partial_k\frac{\partial_j\Psi_{\mathscr{S}}}{|\nabla\Psi_{\mathscr{S}}|}\big|_{\mathscr{S}} &= \frac{\partial_k\partial_j\Psi_{\mathscr{S}}}{|\nabla\Psi_{\mathscr{S}}|} + \partial_k\Big(\frac{1}{|\nabla\Psi_{\mathscr{S}}|}\Big)\partial_j\Psi_{\mathscr{S}} \\ &= \frac{1}{|\nabla\Psi_{\mathscr{S}}|}\big(\partial_j\partial_k\Psi_{\mathscr{S}} - \mathscr{N}_j\,\partial_{\mathscr{N}}\partial_k\Psi_{\mathscr{S}}\big) \end{aligned}$$

and therefore we get

$$\begin{pmatrix} \partial_1\mathscr{N}_1 & \cdots & \partial_1\mathscr{N}_n \\ \vdots & \ddots & \vdots \\ \partial_n\mathscr{N}_1 & \cdots & \partial_n\mathscr{N}_n \end{pmatrix} = \frac{1}{|\nabla\Psi_{\mathscr{S}}|}\begin{pmatrix} \partial_1^2\Psi_{\mathscr{S}} & \cdots & \partial_1\partial_n\Psi_{\mathscr{S}} \\ \vdots & \ddots & \vdots \\ \partial_n\partial_1\Psi_{\mathscr{S}} & \cdots & \partial_n^2\Psi_{\mathscr{S}} \end{pmatrix}(I_n - \mathscr{N}\mathscr{N}^\top).$$

From the obtained equality and from (3.17) follows (3.16). $\square$

If $\theta(p) \in \mathbb{T}_p\mathscr{S}$, then

$$\mathscr{W}_{\mathscr{S}}(p)\theta(p) = \big\{\mathscr{N}(p)\mathscr{N}^\top(p) - I_n\big\}\,W_{\mathscr{S}}(p)\theta(p).$$

Thus the eigenvalues of the matrices $\mathscr{W}_{\mathscr{S}}(p)$ and of

$$\mathscr{A}_{\mathscr{S}}(p) := \big\{\mathscr{N}(p)\mathscr{N}^\top(p) - I_n\big\}\,W_{\mathscr{S}}(p) \tag{3.18}$$

coincide. Let denote them by $\{\kappa_j(p)\}_{1\le j\le n}$, setting the last one zero $\kappa_n(p) \equiv 0$.

From this representation we immediately have that the eigenvalues $\{\kappa_j(p)\}_{1\leq j\leq n}$ are the solutions of the following equation

$$\det(\mathscr{A}_{\mathscr{S}}(p)-\kappa\, I)=0$$

and the mean curvature is

$$\mathscr{H}_{\mathscr{S}}(p)=\sum_{j=1}^{n-1}\frac{\kappa_j(p)}{n-1}=\frac{\mathrm{Tr}\mathscr{A}_{\mathscr{S}}(p)}{n-1}\quad\text{for}\quad p\in\mathscr{S}.$$

Gauß's principal curvature $\mathscr{K}_{\mathscr{S}}(p)$ at $p\in\mathscr{S}$ (cf. (1.24)) equals to the coefficient at λ of the polynomial $-\det(\mathscr{A}_{\mathscr{S}}(p)-\lambda\, I)$ (cf. (2.13)), i.e.:

$$\mathscr{K}_{\mathscr{S}}(p):=-\frac{d}{d\lambda}\det(\mathscr{A}_{\mathscr{S}}(p)-\lambda\, I)\Big|_{\lambda=0}. \tag{3.19}$$

Example 3.9 *Let us demonstrate some obtained results for the ellipsoid $\mathscr{E}^{n-1}_{r,0}$ defined in* (1.6)*. We have*

$$W_{\mathscr{E}^{n-1}_{r,0}}(p)=\left[\frac{\delta_{jk}}{R_{r,0}(p)r_k^2}\right]_{n\times n}=\mathrm{diag}(\alpha_1(p),\dots,\alpha_n(p))\,,$$
$$\alpha_j(p)=\frac{1}{r_j^2R_{r,0}(p)}\,,\qquad \nu_j(p)=p_j\alpha_j(p)=\frac{p_j}{r_j^2R_{r,0}(p)}\,,$$

where

$$R_{r,0}(p):=\left[\sum_{j=1}^{n}\left(\frac{p_j}{r_j^2}\right)^2\right]^{1/2}.$$

Then from (3.18) *and* (3.2) *we get*

$$\mathscr{A}_{\mathscr{E}^{n-1}_{r,0}}(p)=\left[\begin{array}{cccc}\alpha_1(\nu_1^2-1) & \alpha_2\nu_2\nu_1 & \dots & \alpha_n\nu_n\nu_1\\ \alpha_1\nu_1\nu_2 & \alpha_2(\nu_2^2-1) & \dots & \alpha_n\nu_n\nu_2\\ \vdots & \vdots & \dots & \vdots\\ \alpha_1\nu_1\nu_n & \alpha_2\nu_2\nu_n & \cdots & \alpha_n(\nu_n^2-1)\end{array}\right] \tag{3.20}$$

and the mean curvature is

$$\mathscr{H}_{\mathscr{E}^{n-1}_{r,0}}(p)=\sum_{j=1}^{n-1}\frac{\kappa_j(p)}{n-1}=\frac{\mathrm{Tr}\mathscr{A}_{\mathscr{E}^{n-1}_{r,0}}(p)}{n-1}=\sum_{j=1}^{n}\frac{\alpha_j(p)(\nu_j^2(p)-1)}{n-1}$$
$$\textit{for}\quad p\in\mathscr{E}_{r,0}.$$

Further, to avoid routine and voluminous calculation, let $n=3$. Then we obtain

$$-\det(\mathscr{A}_{\mathscr{E}^{n-1}_{r,0}}-\kappa\, I)=\kappa^3-[\alpha_1(\nu_1^2-1)+\alpha_2(\nu_2^2-1)+\alpha_3(\nu_3^2-1)]\kappa^2$$
$$+[\alpha_1\alpha_3\nu_2^2+\alpha_2\alpha_3\nu_1^2+\alpha_1\alpha_2\nu_3^2]\kappa$$

and, due to (3.19)*, Gauß's principal curvature at $p\in\mathscr{E}_{r,0}$ equals:*

$$\mathscr{K}_{\mathscr{E}^{n-1}_{r,0}}(p)=\alpha_1(p)\alpha_3(p)\nu_2^2(p)+\alpha_2(p)\alpha_3(p)\nu_1^2(p)+\alpha_1(p)\alpha_2(p)\nu_3^2(p)$$
$$=\frac{1}{r_1^2r_2^2r_3^2R^4_{r,0}(p)}\quad\textit{for}\quad p\in\mathscr{E}^{n-1}_{r,0}\,.$$

Example 3.10 *Let us calculate the mean and Gauß's principal curvatures of the saddle surface in $\mathbb{R}^3$, given implicitly by the function*

$$\Psi_{\mathscr{S}}(x_1, x_2, x_3) = x_3 - x_1 x_2 = 0.$$

Then

$$\mathscr{N}_1(x) = \frac{-x_2}{(1 + x_1^2 + x_2^2)^{\frac{1}{2}}}, \quad \mathscr{N}_2(x) = \frac{-x_1}{(1 + x_1^2 + x_2^2)^{\frac{1}{2}}}, \quad \mathscr{N}_3(x) = \frac{1}{(1 + x_1^2 + x_2^2)^{\frac{1}{2}}}$$

and

$$W_{\mathscr{S}}(p) = \frac{1}{(1 + p_1^2 + p_2^2)^{\frac{1}{2}}} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad p \in \mathscr{S}.$$

From (3.18) *and* (3.2) *we then obtain*

$$\mathscr{A}_{\mathscr{S}}(p) = \frac{-1}{(1 + p_1^2 + p_2^2)^{\frac{1}{2}}} \begin{pmatrix} \nu_1(p)\nu_2(p) & \nu_1^2(p) - 1 & 0 \\ \nu_2^2(p) - 1 & \nu_1(p)\nu_2(p) & 0 \\ \nu_3(p)\nu_2(p) & \nu_3(p)\nu_1(p) & 0 \end{pmatrix},$$

which immediately gives

$$\mathscr{K}_{\mathscr{S}}(p) = -\frac{1}{1 + p_1^2 + p_2^2}(1 - \nu_1^2(p) - \nu_2^2(p)) = \frac{-1}{(1 + p_1^2 + p_2^2)^2}$$

and

$$\mathscr{H}_{\mathscr{S}}(p) = \frac{\mathrm{Tr}\mathscr{A}_{\mathscr{S}}(p)}{n-1} = \frac{-1}{(1 + p_1^2 + p_2^2)^{\frac{1}{2}}}\nu_1\nu_2 = \frac{-p_1 p_2}{(1 + p_1^2 + p_2^2)^{\frac{3}{2}}} \quad \textit{for} \quad p \in \mathscr{S}.$$

Two principal curvatures of the surface coincide with the eigenvalues of the matrix $\mathscr{A}_{\mathscr{S}}$

$$\kappa_1(p) = -\frac{p_1 p_2 - \sqrt{(1 + p_1^2)(1 + p_2^2)}}{(1 + p_1^2 + p_2^2)^{\frac{3}{2}}},$$
$$\kappa_2(p) = -\frac{p_1 p_2 + \sqrt{(1 + p_1^2)(1 + p_2^2)}}{(1 + p_1^2 + p_2^2)^{\frac{3}{2}}},$$

for $p \in \mathscr{S}$, while the third eigenvalue is indeed 0.

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