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INTERFACE CRACKS PROBLEMS IN COMPOSITES WITH PIEZOELECTRIC AND THERMAL EFFECTS

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ABSTRACT

We investigate three–dimensional interface crack problems (ICP) for metallic-piezoelectric composite bodies with regard to thermal effects. We give a mathematical formulation of the physical problem when the metallic and piezoelectric bodies are bonded along some proper parts of their boundaries where interface cracks occur. By potential methods the ICP is reduced to an equivalent strongly elliptic system of pseudodifferential equations (ΨDEs) on overlapping manifolds with boundary, which have no analogues in mathematical literature. We study the solvability of obtained ΨDEs on overlapping manifolds with boundary by reduction to ΨDEs on non-overlapping manifolds with boundary in different function spaces. These general results are applied to prove the uniqueness and the existence theorems for the original ICP-Problem.

1 FORMULATION OF THE PROBLEM

1.1 Geometrical description of the composite configuration

Let us start by explaining some notation used in the paper: The bar $\overline{\Omega}$ denotes the closure of the set Ω, the equality $:=$ reads as "by the definition", the square "$\Box$" denotes the end of the proof, while "$A \setminus B$" denotes the "setminus", i.e., all elements of the set A which does not appear in the set B.

Let $\Omega^{(m)}$ and Ω be bounded disjoint domains of the three-dimensional Euclidean space $\mathbb{R}^3$ with boundaries $\partial\Omega^{(m)}$ and $\partial\Omega$, respectively (cf. Fig. 1). Moreover, let $\partial\Omega$ and $\partial\Omega^{(m)}$ have a nonempty, simply connected intersection $\overline{\Gamma^{(m)}}$ with a positive surface measure, i.e., $\partial\Omega \cap \partial\Omega^{(m)} = \overline{\Gamma^{(m)}}$, $\text{meas}\Gamma^{(m)} > 0$. From now on $\Gamma^{(m)}$ will be referred to as an interface surface. Throughout the paper n and $\nu = n^{(m)}$ stand for the outward unit normal vectors to $\partial\Omega$ and to $\partial\Omega^{(m)}$, respectively. Evidently, $n(x) = -\nu(x)$ for $x \in \Gamma^{(m)}$.

Further, let $\overline{\Gamma^{(m)}} = \overline{\Gamma^{(m)}_T} \cup \overline{\Gamma^{(m)}_C}$, where $\Gamma^{(m)}_C$ is an open, simply connected proper part of $\Gamma^{(m)}$. Moreover, $\Gamma^{(m)}_T \cap \Gamma^{(m)}_C = \varnothing$ and $\partial\Gamma^{(m)} \cap \overline{\Gamma^{(m)}_C} = \varnothing$.

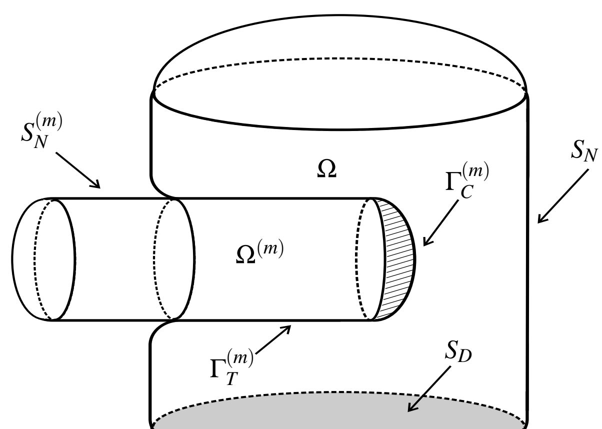

Fig. 1: Metallic-piezoelectric composite

We set $S_N^{(m)} := \partial\Omega^{(m)} \setminus \overline{\Gamma^{(m)}}$ and $S^* := \partial\Omega \setminus \overline{\Gamma^{(m)}}$. Further, we denote by S_D some open, nonempty, proper sub-manifold of S^* and let $S_N := S^* \setminus \overline{S_D}$. Thus, we have the following dissec-

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tions of the boundary surfaces (see Figure 1) $\partial\Omega = \overline{\Gamma_T^{(m)}} \cup \overline{\Gamma_C^{(m)}} \cup \overline{S_N} \cup \overline{S_D}$, $\partial\Omega^{(m)} = \overline{\Gamma_T^{(m)}} \cup \overline{\Gamma_C^{(m)}} \cup \overline{S_N^{(m)}}$. Throughout the paper, for simplicity, we assume that $\partial\Omega^{(m)}$, $\partial\Omega$, $\partial S_N^{(m)}$, $\partial\Gamma_T^{(m)}$, $\partial\Gamma_C^{(m)}$, ∂S_D, ∂S_N are C^∞-smooth and $\partial\Omega^{(m)} \cap \overline{S_D} = \varnothing$, if not otherwise stated. Some results, obtained in the paper, hold also true when these manifolds and their boundaries are Lipschitz and we formulate them separately.

Let Ω be filled by an anisotropic homogeneous piezoelectric medium (ceramic matrix) and $\Omega^{(m)}$ be occupied by an isotropic or anisotropic homogeneous elastic medium (metallic inclusion). These two bodies interact to each other along the interface $\Gamma^{(m)}$, where the interface crack $\Gamma_C^{(m)}$ occurs. Moreover, it is assumed that the composed body is fixed along the sub-surface S_D (the Dirichlet part of the boundary), while the sub-manifolds $S_N^{(m)}$ and S_N are the Neumann parts of the boundary. In the metallic domain $\Omega^{(m)}$ we have a four-dimensional thermoelastic field described by the displacement vector $u^{(m)} = (u_1^{(m)}, u_2^{(m)}, u_3^{(m)})^\top$ and temperature distribution function $u_4^{(m)} = \vartheta^{(m)}$, while in the piezoelectric domain Ω we have a five-dimensional physical field described by the displacement vector $u = (u_1, u_2, u_3)^\top$, temperature distribution function $u_4 = \vartheta$ and the electric potential $u_5 = \varphi$.

1.2 Thermoelastic and thermopiezoelastic field equations

In the present subsection we expose partial differential equations of the linear theory of thermoelasticity and thermopiezoelasticity for a general anisotropic case (see [14, 17, 18] for details).

Throughout the paper the Einstein convention about the summation over the repeated indices is meant from 1 to 3, unless stated otherwise.

From the constitutive relations, Fourier law, equations of motion and the equation of the entropy balance we derive the linear system of pseudo-oscillation equations of thermoelasticity in a matrix form

$$A^{(m)}(\partial,\tau)\,U^{(m)} + \tilde{X}^{(m)} = 0 \quad \text{in} \quad \Omega^{(m)}, \tag{1}$$

where $U^{(m)} := (u^{(m)}, \vartheta^{(m)})^\top$ is the unknown vector, $\tilde{X}^{(m)} = (X_1^{(m)}, X_2^{(m)}, X_3^{(m)}, X_4^{(m)})^\top$ is a given vector ($X_j^{(m)}$, $j = 1,2,3$, are the mass force densities and $X_4^{(m)}$ is the heat source density) and the nonselfadjoint matrix differential operator $A^{(m)}(\partial,\tau)$ reads as

$$A^{(m)}(\partial,\tau) = \Big[A_{jk}^{(m)}(\partial,\tau)\Big]_{4\times4}, \tag{2}$$

$$A_{jk}^{(m)}(\partial,\tau) = c_{ijlk}^{(m)}\partial_i\partial_l - \rho^{(m)}\tau^2\delta_{jk},$$
$$A_{4k}^{(m)}(\partial,\tau) = -\tau T_0^{(m)}\gamma_{kl}^{(m)}\partial_l, \quad A_{j4}^{(m)}(\partial,\tau) = -\gamma_{ij}^{(m)}\partial_i,$$
$$A_{44}^{(m)}(\partial,\tau) = \varkappa_{il}^{(m)}\partial_i\partial_l - \alpha^{(m)}\tau, \quad j,k = 1,2,3$$

where δ_{jk} stands for Kronecker's delta. $\tau = \sigma + i\omega$ is a complex oscillation parameter, $T_0^{(m)}$ is the temperature in the natural state; $\vartheta^{(m)}$ is the temperature increment; parameters $c_{ijlk}^{(m)}$, $\rho^{(m)}$, $\gamma_{kl}^{(m)}$, $\varkappa_{il}^{(m)}$, $\alpha^{(m)}$ are material constants of thermoelastic medium. Constants involved in equations (1)-(2) have the following properties (see, e.g., [14, 17]) (c_0 and c_1 are positive constants):

$$c_{ijkl}^{(m)} = c_{jikl}^{(m)} = c_{klij}^{(m)}, \quad \gamma_{ij}^{(m)} = \gamma_{ji}^{(m)}, \quad \varkappa_{ij}^{(m)} = \varkappa_{ji}^{(m)}, \quad i,j,k,l = 1,2,3, \tag{3}$$

$$c_{ijkl}^{(m)}\xi_{ij}\xi_{kl} \geq c_0\xi_{ij}\xi_{ij} \quad \text{for all} \quad \xi_{ij} = \xi_{ji} \in \mathbb{R}, \tag{4}$$

$$\varkappa_{ij}^{(m)}\eta_i\eta_j \geq c_1\eta_i\eta_i, \quad \text{for all} \quad \eta \in \mathbb{R}^3. \tag{5}$$

Note that for an isotropic medium the thermomechanical coefficients are

$$c_{ijlk}^{(m)} = \lambda^{(m)}\delta_{ij}\delta_{lk} + \mu^{(m)}(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}),$$
$$\gamma_{ij}^{(m)} := \gamma^{(m)}\delta_{ij}, \quad \varkappa_{ij}^{(m)} = \varkappa^{(m)}\delta_{ij},$$

where $\lambda^{(m)}$ and $\mu^{(m)}$ are the Lamé constants.

With the help of the symmetry conditions and inequalities (3)-(5) it can easily be shown that the principal part $A^{(m,0)}(\partial)$ of the operator $A^{(m)}(\partial,\tau)$ is a selfadjoint elliptic operator with positive definite symbol matrix

$$A^{(m,0)}(\xi)\,\eta\cdot\eta \geq c^{(m)}|\xi|^2|\eta|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^3 \text{ and for all } \eta \in \mathbb{C}^4$$

with some positive constant $c^{(m)} > 0$ depending on the material parameters.

Components of the mechanical thermostress vector acting on a surface element with a normal $\nu = (\nu_1,\nu_2,\nu_3)$ read as follows

$$\sigma_{ij}^{(m)}\nu_i = c_{ijlk}^{(m)}\nu_i\partial_l u_k^{(m)} - \gamma_{ij}^{(m)}\nu_i\vartheta^{(m)}, \quad j = 1,2,3,$$

while the normal component of the heat flux vector (with opposite sign) has the form $-q_i^{(m)}\nu_i = \varkappa_{il}^{(m)}\nu_i\partial_l\vartheta^{(m)}$. We introduce the following generalized thermostress operator

$$\mathcal{T}^{(m)}(\partial,\nu) = [\,\mathcal{T}_{jk}^{(m)}(\partial,\nu)\,]_{4\times4}, \tag{6}$$

$$\mathcal{T}_{jk}^{(m)}(\partial,\nu) = c_{ijlk}^{(m)}\nu_i\partial_l, \quad \mathcal{T}_{j4}^{(m)}(\partial,\nu) = -\gamma_{ij}^{(m)}\nu_i,$$
$$\mathcal{T}_{4k}^{(m)}(\partial,\nu) = 0, \quad \mathcal{T}_{44}^{(m)}(\partial,\nu) = \varkappa_{il}^{(m)}\nu_i\partial_l, \quad j,k = 1,2,3. \tag{7}$$

For a four–vector $U^{(m)} = (u^{(m)}, \vartheta^{(m)})^\top$ we have

$$\mathcal{T}^{(m)}U^{(m)} = (\sigma_{i1}^{(m)}\nu_i,\ \sigma_{i2}^{(m)}\nu_i,\ \sigma_{i3}^{(m)}\nu_i,\ -q_i^{(m)}\nu_i)^\top. \tag{8}$$

Clearly, the components of the vector $\mathcal{T}^{(m)}U^{(m)}$ given by (8) have the physical sense: the first three components correspond to the mechanical stress vector in the theory of thermoelasticity, while the forth one is the normal component of the heat flux vector (with opposite sign).

From the constitutive relations, Fourier law, equations of motion, equation of the entropy balance and the equation of static electric field we derive the linear system of pseudo-oscillation equations of thermopiezoelasticity in a matrix form

$$A(\partial,\tau)\,U(x) + \widetilde{X}(x) = 0 \ \text{ in } \ \Omega, \tag{9}$$

where $U := (u,\vartheta,\varphi)^\top$, $\widetilde{X} = (X_1,X_2,X_3,X_4,X_5)^\top$ and the matrix differential operator $A(\partial,\tau)$ is

$$A(\partial,\tau) = \big[A_{jk}(\partial,\tau)\big]_{5\times 5}, \tag{10}$$

$$\begin{aligned}
&A_{jk}(\partial,\tau) = c_{ijlk}\,\partial_i\partial_l - \rho\tau^2\delta_{jk},\\
&A_{j4}(\partial,\tau) = -\gamma_{ij}\,\partial_i, \quad A_{j5}(\partial,\tau) = e_{lij}\partial_l\partial_i,\\
&A_{4k}(\partial,\tau) = -\tau T_0\,\gamma_{kl}\,\partial_l, \quad A_{44}(\partial,\tau) = \varkappa_{il}\,\partial_i\partial_l - \alpha\tau,\\
&A_{45}(\partial,\tau) = \tau T_0\,g_i\,\partial_i, \quad A_{5k}(\partial,\tau) = -e_{ikl}\,\partial_i\partial_l,\\
&A_{54}(\partial,\tau) = -g_i\partial_i, \quad A_{55}(\partial,\tau) = \varepsilon_{il}\,\partial_i\partial_l, \quad j,k = 1,2,3.
\end{aligned}$$

Here $X = (X_1,X_2,X_3)^\top$ is a mass force density, X_4 is a heat source density, X_5 is a charge density and g_i are pyroelectric constants characterizing the relation between thermodynamic processes and piezoelectric effects. Parameters c_{ijlk}, ρ, γ_{kl}, $e_{(lij)}$, $\varkappa_{il}$, α, g_i involved in (9)-(10) are material constants of thermopiezoelastic medium. They have the following properties (see, e.g., [17])

$$\begin{aligned}
&c_{ijkl} = c_{jikl} = c_{klij}, \;\; e_{ijk} = e_{ikj}, \;\; \varepsilon_{ij} = \varepsilon_{ji},\\
&\gamma_{ij} = \gamma_{ji}, \;\; \varkappa_{ij} = \varkappa_{ji}, \;\; i,j,k,l = 1,2,3,
\end{aligned}$$

$$c_{ijkl}\,\xi_{ij}\,\xi_{kl} \ge c_0\,\xi_{ij}\,\xi_{ij} \;\text{ for all }\; \xi_{ij} = \xi_{ji} \in \mathbb{R}, \tag{11}$$

$$\varepsilon_{ij}\,\eta_i\,\eta_j \ge c_1\,\eta_i\,\eta_i, \;\; \varkappa_{ij}\,\eta_i\,\eta_j \ge c_2\,\eta_i\,\eta_i \;\; \forall \eta \in \mathbb{R}^3. \tag{12}$$

and c_0, c_1, and c_2 are positive constants. With the help of the symmetry conditions and inequalities (11)–(12) it can easily be shown that the principal part $A^{(0)}(\partial)$ of the operator $A(\partial,\tau)$ is nonselfadjoint, although is strongly elliptic, that is,

$$\operatorname{Re} A^{(0)}(\xi)\,\eta\cdot\eta \ge c\,|\xi|^2\,|\eta|^2 \text{ for all } \xi \in \mathbb{R}^3 \text{ and for all } \eta \in \mathbb{C}^4$$

with some positive constant $c > 0$ depending on the material parameters.

In the theory of thermopiezoelasticity the components of the three-dimensional mechanical stress vector acting on a surface element with a normal $n = (n_1,n_2,n_3)$ have the form

$$\sigma_{ij}\,n_i = c_{ijlk}\,n_i\,\partial_l u_k + e_{lij}\,n_i\,\partial_l\varphi - \gamma_{ij}\,n_i\,\vartheta \;\text{ for }\; j = 1,2,3,$$

while the normal components of the electric displacement vector and the heat flux vector (with opposite sign) read as

$$-D_i\,n_i = -e_{ikl}\,n_i\,\partial_l u_k + \varepsilon_{il}\,n_i\,\partial_l\varphi - g_i\,n_i\,\vartheta, \;\; -q_i\,n_i = \varkappa_{il}\,n_i\,\partial_l\vartheta.$$

Let us introduce the following matrix differential operator

$$\mathcal{T}(\partial,n) = \big[\,\mathcal{T}_{jk}(\partial,n)\,\big]_{5\times 5}, \tag{13}$$

$$\begin{aligned}
&\mathcal{T}_{jk}(\partial,n) = c_{ijlk}\,n_i\,\partial_l, \quad \mathcal{T}_{j4}(\partial,n) = -\gamma_{ij}\,n_i,\\
&\mathcal{T}_{j5}(\partial,n) = e_{lij}\,n_i\,\partial_l, \quad \mathcal{T}_{4k}(\partial,n) = 0, \qquad\qquad (14)\\
&\mathcal{T}_{44}(\partial,n) = \varkappa_{il}\,n_i\,\partial_l, \quad \mathcal{T}_{45}(\partial,n) = 0, \quad \mathcal{T}_{54}(\partial,n) = -g_i\,n_i,\\
&\mathcal{T}_{5k}(\partial,n) = -e_{ikl}\,n_i\,\partial_l, \quad \mathcal{T}_{55}(\partial,n) = \varepsilon_{il}\,n_i\,\partial_l, \quad j,k = 1,2,3.
\end{aligned}$$

For a vector $U = (u,\varphi,\vartheta)^\top$ we have

$$\mathcal{T}(\partial,n)\,U = (\sigma_{i1}\,n_i,\ \sigma_{i2}\,n_i,\ \sigma_{i3}\,n_i,\ -q_i\,n_i,\ -D_i\,n_i)^\top. \tag{15}$$

Clearly, the components of the vector $\mathcal{T}U$ given by (15) have the physical sense: the first three components correspond to the mechanical stress vector in the theory of thermoelectroelasticity, the forth and fifth ones are the normal components of the heat flux vector and the electric displacement vector (with opposite sign), respectively.

Let us introduce some further notation. Throughout the paper the symbol $\{\,\cdot\,\}^+$ denotes the interior one-sided trace operator on $\partial\Omega$ (respectively $\partial\Omega^{(m)}$) from Ω (respectively $\Omega^{(m)}$). Similarly, $\{\,\cdot\,\}^-$ denotes the exterior one-sided trace operator on $\partial\Omega$ (respectively $\partial\Omega^{(m)}$) from the exterior of Ω (respectively $\Omega^{(m)}$).

By L_p, W_p^r, H_p^s, and $B_{p,q}^s$ (with $r \ge 0$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$) we denote the well-known Lebesgue, Sobolev-Slobodetski, Bessel potential, and Besov function spaces, respectively (see, e.g., [19]). Recall that $H_2^r = W_2^r = B_{2,2}^r$ for any $r \ge 0$, $H_2^s = B_{2,2}^s$ for any $s \in \mathbb{R}$, $W_p^t = B_{p,p}^t$ for any positive and non-integer t, and $H_p^k = W_p^k$ for any non-negative integer k.

Let $\mathcal{M}_0$ be a smooth surface without boundary. For a smooth sub-manifold $\mathcal{M} \subset \mathcal{M}_0$ by $\widetilde{H}_p^s(\mathcal{M})$ and $\widetilde{B}_{p,q}^s(\mathcal{M})$ we denote the subspaces of $H_p^s(\mathcal{M}_0)$ and $B_{p,q}^s(\mathcal{M}_0)$, respectively,

$$\begin{aligned}
&\widetilde{H}_p^s(\mathcal{M}) = \{g : g \in H_p^s(\mathcal{M}_0),\ \operatorname{supp} g \subset \overline{\mathcal{M}}\},\\
&\widetilde{B}_{p,q}^s(\mathcal{M}) = \{g : g \in B_{p,q}^s(\mathcal{M}_0),\ \operatorname{supp} g \subset \overline{\mathcal{M}}\}.
\end{aligned}$$

By $r_{\mathcal{M}} f$ denote the restriction of f onto a submanifold $\mathcal{M}$ and introduce the spaces: $H_p^s(\mathcal{M}) = \{r_{\mathcal{M}} H_p^s(\mathcal{M}_0)\}$ and $B_{p,q}^s(\mathcal{M}) = \{r_{\mathcal{M}} B_{p,q}^s(\mathcal{M}_0)\}$.

1.3 Formulation of the ICP-Problem and the uniqueness of a solution

In what follows, without loss of generality, we assume that the mass force density, heat source density and charge density vanish in the corresponding regions. Otherwise, we can write particular solutions to the nonhomogeneous differential equations explicitly, in the form of volume Newtonian potentials. Therefore, we will consider the homogeneous versions of the above differential equations. However, we have to take into consideration that the original mechanical and thermo-electrical homogeneous boundary and transmission conditions become then nonhomogeneous, in general.

ICP-Problem: We consider the problem when *the crack gap is thermally and electrically conductive*. Find vector-functions

$$\begin{aligned}
&U^{(m)} = (u_1^{(m)}, u_2^{(m)}, u_3^{(m)}, u_4^{(m)})^\top \;:\; \Omega^{(m)} \to \mathbb{C}^4\\
&\text{and }\; U = (u_1,u_2,u_3,u_4,u_5)^\top \;:\; \Omega \to \mathbb{C}^5
\end{aligned}$$

belonging respectively to the spaces $[W_p^1(\Omega^{(m)})]^4$ and $[W_p^1(\Omega)]^5$ with $1 < p < \infty$ and satisfying:

(i) *the systems of partial differential equations*:

$$\begin{aligned}
&\big[A^{(m)}(\partial,\tau)\,U^{(m)}\big]_j = 0 \quad \text{in} \quad \Omega^{(m)}, \;\; j = 1,2,3,4,\\
&[A(\partial,\tau)\,U]_k = 0 \quad \text{in} \quad \Omega, \;\; k = 1,2,3,4,5,
\end{aligned} \tag{16}$$

(ii) *the boundary conditions*:

$$r_{S_N^{(m)}}\big\{[\mathcal{T}^{(m)}(\partial,\nu)U^{(m)}]_j\big\}^+ = Q_j^{(m)} \;\text{ on }\; S_N^{(m)},\;\; j=1,2,3,4,$$
$$r_{S_N}\{[\mathcal{T}(\partial,n)U]_k\}^+ = Q_k \;\text{ on }\; S_N,\;\; k=1,2,3,4,5,$$
$$r_{S_D}\{u_k\}^+ = f_k \;\text{ on }\; S_D,\;\; k=1,2,3,4,5, \tag{17}$$
$$r_{\Gamma^{(m)}}\{u_5\}^+ = f_5^{(m)} \;\text{ on }\; \Gamma^{(m)},$$

(iii) *the transmission conditions*:

$$r_{\Gamma_T^{(m)}}\{u_l\}^+ - r_{\Gamma_T^{(m)}}\{u_l^{(m)}\}^+ = f_l^{(m)} \;\text{ on }\; \Gamma_T^{(m)},\; l=\overline{1,3},$$
$$r_{\Gamma_T^{(m)}}\big\{[\mathcal{T}(\partial,n)U]_l\big\}^+ + r_{\Gamma_T^{(m)}}\big\{[\mathcal{T}^{(m)}(\partial,\nu)U^{(m)}]_l\big\}^+ = F_l^{(m)} \quad\text{on }\; \Gamma_T^{(m)},\; l=\overline{1,3}, \tag{18}$$
$$r_{\Gamma^{(m)}}\{u_4\}^+ - r_{\Gamma^{(m)}}\{u_4^{(m)}\}^+ = f_4^{(m)} \;\text{ on }\; \Gamma^{(m)},$$
$$r_{\Gamma^{(m)}}\big\{[\mathcal{T}(\partial,n)U]_4\big\}^+ + r_{\Gamma^{(m)}}\big\{[\mathcal{T}^{(m)}(\partial,\nu)U^{(m)}]_4\big\}^+ = F_4^{(m)} \quad\text{on }\; \Gamma^{(m)},$$

(iv) *the interface crack conditions*:

$$r_{\Gamma_C^{(m)}}\big\{[\mathcal{T}^{(m)}(\partial,\nu)U^{(m)}]_l\big\}^+ = \widetilde{Q}_l^{(m)} \;\text{ on }\; \Gamma_C^{(m)},\;\; l=1,2,3,$$
$$r_{\Gamma_C^{(m)}}\{[\mathcal{T}(\partial,n)U]_l\}^+ = \widetilde{Q}_l \;\text{ on }\; \Gamma_C^{(m)},\;\; l=1,2,3. \tag{19}$$

Here $Q_j^{(m)}$, Q_k, $\widetilde{Q}_l^{(m)}$, $\widetilde{Q}_l$, $f_k^{(m)}$, f_k, $F_j^{(m)}$, $j=1,\dots,4$, $k=1,\dots,5$, are given data.

Next we formulate the uniqueness result. The proof is standard, based on the Green formula and we drop the details.

Theorem 1.1. *Let $\Omega^{(m)}$ and Ω be Lipschitz and either $\tau=\sigma+i\omega$ with $\sigma>0$ or $\tau=0$. The above formulated interface crack ICP-problem has at most one solution in the space $[W_2^1(\Omega^{(m)})]^4\times[W_2^1(\Omega)]^5$, provided* $\operatorname{meas} S_D>0$.

2 REPRESENTATION OF SOLUTIONS

Here we derive integral representation formulas of solutions to the homogeneous equations (1) by means of the layer potentials and certain boundary integral (pseudodifferential) operators generated by them.

2.1 Layer potentials

Let $\Psi^{(m)}(\cdot,\tau) = \big[\Psi_{kj}^{(m)}(\cdot,\tau)\big]_{4\times4}$ and $\Psi(\cdot,\tau) = \big[\Psi_{kj}(\cdot,\tau)\big]_{5\times5}$ be the fundamental matrix-functions of the differential operators $A^{(m)}(\partial,\tau)$ and $A(\partial,\tau)$ and introduce the single and the double layer potentials:

$$V_\tau(h)(x) = \int_{\partial\Omega}\Psi(x-y,\tau)\,h(y)\,d_yS, \tag{20}$$
$$W_\tau(h)(x) = \int_{\partial\Omega}\big[\widetilde{\mathcal{T}}(\partial,n(y),\overline{\tau})\,[\Psi(x-y,\tau)]^\top\big]^\top h(y)\,d_yS,$$

where $\widetilde{\mathcal{T}}(\partial,n(y)=\big[\widetilde{\mathcal{T}}_{jk}(\partial,n,\tau)\big]_{5\times5}$ is the boundary operator, associated with the formally adjoint differential operator $A^*(\partial,\tau)$:

$$\widetilde{\mathcal{T}}_{jk}(\partial,n,\tau) = c_{ijlk}\,n_i\,\partial_l,\quad \widetilde{\mathcal{T}}_{j4}(\partial,n,\tau) = \overline{\tau}\,T_0\,\gamma_{ij}\,n_i,$$
$$\widetilde{\mathcal{T}}_{j5}(\partial,n,\tau) = -e_{lij}\,n_i\,\partial_l,\quad \widetilde{\mathcal{T}}_{4k}(\partial,n,\tau) = 0,$$
$$\widetilde{\mathcal{T}}_{44}(\partial,n,\tau) = \varkappa_{il}\,n_i\,\partial_l,\quad \widetilde{\mathcal{T}}_{45}(\partial,n,\tau) = 0,$$
$$\widetilde{\mathcal{T}}_{54}(\partial,n,\tau) = -\tau\,T_0\,g_i\,n_i,\quad \widetilde{\mathcal{T}}_{5k}(\partial,n,\tau) = e_{ikl}\,n_i\,\partial_l,$$
$$\widetilde{\mathcal{T}}_{55}(\partial,n,\tau) = \varepsilon_{il}\,n_i\,\partial_l,\quad j,k=1,2,3.$$

Similarly, using the boundary operator $\widetilde{\mathcal{T}}^{(m)}(\partial,n(y) = \big[\widetilde{\mathcal{T}}_{jk}^{(m)}(\partial,n,\tau)\big]_{5\times5}$, associated with the formally adjoint differential operator $A^{(m)*}(\partial,\tau)$

$$\widetilde{\mathcal{T}}_{jk}^{(m)}(\partial,\nu,\tau) = c_{ijlk}^{(m)}\nu_i\,\partial_l,\quad \widetilde{\mathcal{T}}_{j4}^{(m)}(\partial,\nu,\tau) = \overline{\tau}\,T_0^{(m)}\gamma_{ij}^{(m)}\nu_i,$$
$$\widetilde{\mathcal{T}}_{4k}^{(m)}(\partial,\nu,\tau) = 0,\quad \widetilde{\mathcal{T}}_{44}^{(m)}(\partial,\nu,\tau) = \varkappa_{il}^{(m)}\nu_i\,\partial_l,\quad j,k=1,2,3,$$

one defines the layer potentials $V_\tau^{(m)}(h^{(m)})$ and $W_\tau^{(m)}(h^{(m)})$; here $h^{(m)} = (h_1^{(m)},h_2^{(m)},h_3^{(m)},h_4^{(m)})^\top$ and $h=(h_1,h_2,h_3,h_4,h_5)^\top$ are densities of the potentials.

For the boundary integral (pseudodifferential) operators generated by the layer potentials we will employ the following notation:

$$\mathcal{H}_\tau(h)(x) := \int_{\partial\Omega}\Psi(x-y,\tau)\,h(y)\,d_yS,$$
$$\mathcal{K}_\tau(h)(x) := \int_{\partial\Omega}\big[\mathcal{T}(\partial,n(x))\Psi(x-y,\tau)\big]\,h(y)\,d_yS,$$
$$\widetilde{\mathcal{K}}_\tau^*(h)(x) := \int_{\partial\Omega}\big[\widetilde{\mathcal{T}}(\partial,n(y),\overline{\tau})[\Psi(x-y,\tau)]^\top\big]^\top h(y)\,d_yS,$$
$$\mathcal{L}_\tau(h)(x) := \big\{\mathcal{T}(\partial,n(x))W_\tau(h)(x)\big\}^\pm,\quad x\in\partial\Omega.$$

Similarly, with the help of the fundamental solution $\Psi^{(m)}(x-y,\tau)$ are defined the boundary integral (pseudodifferential) operators $\mathcal{H}_\tau^{(m)}$, $\mathcal{K}_\tau^{(m)}$, $\widetilde{\mathcal{K}}_\tau^{(m)*}$ and $\mathcal{L}_\tau^{(m)}$.

The layer boundary operators $\mathcal{H}_\tau^{(m)}$, $\mathcal{H}_\tau$ and $\mathcal{L}_\tau^{(m)}$, $\mathcal{L}_\tau$ are pseudodifferential operators of order -1 and 1, respectively, while the operators $\mathcal{K}_\tau^{(m)}$, $\widetilde{\mathcal{K}}_\tau^{(m)*}$, $\mathcal{K}_\tau$ and $\widetilde{\mathcal{K}}_\tau^*$ are singular integral operators (pseudodifferential operators of order 0) (for details see [2–6, 12, 16]).

2.2 Auxiliary problems and representation of solutions

Here we assume that $\operatorname{Re}\tau=\sigma>0$ and consider two auxiliary boundary value problems needed for our further purposes.

Auxiliary problem I: Find a vector function $U^{(m)} = (u_1^{(m)},\, u_2^{(m)},\, u_3^{(m)},\, u_4^{(m)})^\top : \Omega^{(m)}\to\mathbb{C}^4$ which belongs to the space $[W_2^1(\Omega^{(m)})]^4$ and satisfies the following conditions:

$$A^{(m)}(\partial,\tau)U^{(m)} = 0 \quad \text{in} \quad \Omega^{(m)}, \qquad (21)$$
$$\{\mathcal{T}^{(m)}U^{(m)}\}^+ = \chi^{(m)} \quad \text{on} \quad \partial\Omega^{(m)},$$

where $\chi^{(m)} = (\chi_1^{(m)},\chi_2^{(m)},\chi_3^{(m)},\chi_4^{(m)})^\top \in \big[H_2^{-\frac{1}{2}}(\partial\Omega^{(m)})\big]^4$. With the help of Green's formula it can easily be shown that the homogeneous version of this auxiliary BVP possesses only the trivial solution. Moreover, we have the following existence result.

Lemma 2.1. *Let* $\mathrm{Re}\,\tau = \sigma > 0$ *and* $1 < p < \infty$. *An arbitrary solution vector* $U^{(m)} \in [W_p^1(\Omega^{(m)})]^4$ *to the homogeneous equation* (21) *is uniquely represented by the single layer potential*

$$U^{(m)}(x) = V_\tau^{(m)}\big(\big[\mathcal{P}_\tau^{(m)}\big]^{-1}\chi^{(m)}\big)(x), \quad x \in \Omega^{(m)}, \qquad (22)$$

where

$$\mathcal{P}_\tau^{(m)} := -2^{-1}I_4 + \mathcal{K}_\tau^{(m)}$$
$$\text{and} \quad \chi^{(m)} = \{\mathcal{T}^{(m)}U^{(m)}\}^+ \in \big[B_{p,p}^{-\frac{1}{p}}(\partial\Omega^{(m)})\big]^4. \qquad (23)$$

Proof. Evidently, if $\chi^{(m)} \in \big[B_{p,p}^{-\frac{1}{p}}(\partial\Omega^{(m)})\big]^4$ then the vector function (22) solves the auxiliary BVP and belongs to the space $[W_p^1(\Omega^{(m)})]^4$ due to standard theorems on the mapping properties of potentials and Sokhotsky-Plemelji formulae, describing their traces on the boundary (cf. [2,4–6] and [10] for a most general theorems). The uniqueness follows from the general integral representation formula

$$U^{(m)}(x) = W_\tau^{(m)}\big(\{U^{(m)}\}^+\big)(x) - V_\tau^{(m)}\big(\{\mathcal{T}^{(m)}U^{(m)}\}^+\big)(x)$$

for all $x \in \Omega^{(m)}$, and a standard application of the Green formulae (cf., e.g., [2,4–6] for similar proofs). □

Auxiliary problem II: Find a vector function $U = (u_1,u_2,u_3,u_4,u_5)^\top : \Omega \to \mathbb{C}^5$ which belongs to the space $[W_2^1(\Omega)]^5$ and satisfies the following conditions:

$$A(\partial,\tau)U = 0 \quad \text{in} \quad \Omega,$$
$$\{\mathcal{T}U\}^+ + \beta\{U\}^+ = \chi \quad \text{on} \quad \partial\Omega, \qquad (24)$$

where $\chi := (\chi_1,\chi_2,\chi_3,\chi_4,\chi_5)^\top \in \big[H_2^{-\frac{1}{2}}(\partial\Omega)\big]^5$, β is a smooth real valued scalar function which does not vanish identically and

$$\beta \geq 0, \quad \mathrm{supp}\,\beta \subset S_D. \qquad (25)$$

By standard arguments, involving the Green formulae, we can easily show that the homogeneous version of this boundary value problem possesses only the trivial solution in the space $[W_2^1(\Omega)]^5$.

We look for a solution to the auxiliary BVP (24) as a single layer potential, $U(x) = V_\tau(f)(x)$, where $f = (f_1,f_2,f_3,f_4,f_5)^\top \in \big[H_2^{-\frac{1}{2}}(\partial\Omega)\big]^5$ is an unknown density. The boundary condition in (24) leads then to the system of equations:

$$\big(-2^{-1}I_5 + \mathcal{K}_\tau\big)f + \beta\,\mathcal{H}_\tau f = \chi \ \text{ on } \partial\Omega.$$

Denote the matrix operator generated by the left hand side expression of this equation by $\mathcal{P}_\tau$ and rewrite the system as

$$\mathcal{P}_\tau f = \chi \ \text{ on } \partial\Omega,$$
$$\mathcal{P}_\tau := -2^{-1}I_5 + \mathcal{K}_\tau + \beta\,\mathcal{H}_\tau. \qquad (26)$$

Lemma 2.2. *Let* $\mathrm{Re}\,\tau = \sigma > 0$. *The operators*

$$\mathcal{P}_\tau : \big[H_p^s(\partial\Omega)\big]^5 \to \big[H_p^s(\partial\Omega)\big]^5,$$
$$: \big[B_{p,t}^s(\partial\Omega)\big]^5 \to \big[B_{p,t}^s(\partial\Omega)\big]^5, \qquad (27)$$

are invertible for all $1 < p < \infty$, $1 \leq t \leq \infty$, *and* $s \in \mathbb{R}$.

Proof. From the uniqueness result for the auxiliary BVP (24) it follows that the operator (27) is injective for $p = 2$ and $s = -1/2$. The operator $\mathcal{H}_\tau : \big[H_2^{-\frac{1}{2}}(\partial\Omega)\big]^5 \to \big[H_2^{-\frac{1}{2}}(\partial\Omega)\big]^5$ is compact. By a standard theorem on a perturbation of a Fredholm operator we then conclude that the index of the operator (27) equals to zero. Since $\mathcal{P}_\tau$ is an injective singular integral operator of normal type with zero index it follows that it is surjective. Thus the operator (27) is invertible for $p = 2$ and $s = -1/2$.

The invertibility of the operators (27) for all $1 < p < \infty$, $1 \leq t \leq \infty$, and $s \in \mathbb{R}$ then follows by standard duality and interpolation arguments for the $C^\infty-$regular surface $\partial\Omega$ from [1,13] (see [3,4,16] for similar arguments). □

Lemma 2.3. *Let* $\mathrm{Re}\,\tau = \sigma > 0$ *and* $1 < p < \infty$. *An arbitrary solution* $U \in [W_p^1(\Omega)]^5$ *to the homogeneous equation* (24) *can be uniquely represented by the single layer potential* $U(x) = V_\tau\big(\mathcal{P}_\tau^{-1}\chi\big)(x)$, *where* $\chi = \{\mathcal{T}U\}^+ + \beta\{U\}^+ \in \big[B_{p,p}^{-\frac{1}{p}}(\partial\Omega)\big]^5$.

Remark 2.4. *For* $p = 2$ *the above results remain true for Lipschitz domains* $\Omega^{(m)}$ *and* Ω (*cf.* [15]).

3 EXISTENCE AND REGULARITY OF A SOLUTION TO ICP-PROBLEM

The ICP-Problem from Subsection 1.2 is reduced to a complicated, nonclassical system of boundary pseudodifferential equations which needs a special analysis.

3.1 Reduction to boundary equations

For the data of the problem in (17), (18), (19) we assume that

$$\begin{aligned}
&Q_j^{(m)} \in B_{p,p}^{-1/p}(S_N^{(m)}), \quad Q_k \in B_{p,p}^{-1/p}(S_N),\\
&f_l^{(m)} \in B_{p,p}^{1/p'}(\Gamma_T^{(m)}), \quad f_t^{(m)} \in B_{p,p}^{1/p'}(\Gamma^{(m)}),\\
&F_l^{(m)} \in B_{p,p}^{-1/p}(\Gamma_T^{(m)}), \quad F_4^{(m)} \in B_{p,p}^{-1/p}(\Gamma^{(m)}),\\
&\widetilde{Q}_l \in B_{p,p}^{-1/p}(\Gamma_C^{(m)}), \quad \widetilde{Q}_l^{(m)} \in B_{p,p}^{-1/p}(\Gamma_C^{(m)}),\\
&f_k \in B_{p,p}^{1/p'}(S_D), \quad j = \overline{1,4}, \ k = \overline{1,5}, \ t = 4,5, \ l = 1,2,3.
\end{aligned} \qquad (28)$$

Further, let

$$G_l := \begin{cases} Q_l \text{ on } S_N, \\ \widetilde{Q}_l \text{ on } \Gamma_C^{(m)}, \end{cases} \quad G_l^{(m)} := \begin{cases} Q_l^{(m)} \text{ on } S_N^{(m)}, \\ \widetilde{Q}_l^{(m)} \text{ on } \Gamma_C^{(m)}, \end{cases} \qquad (29)$$
$$G_t := Q_t \text{ on } S_N, \qquad G_4^{(m)} := Q_4^{(m)} \text{ on } S_N^{(m)},$$
$$l = 1,2,3, \quad t = 4,5, \quad k = \overline{1,5}, \quad j = \overline{1,4},$$

and let $G_{0k}\in B^{-1/p}_{p,p}(\partial\Omega)$, $G^{(m)}_{0j}\in B^{-1/p}_{p,p}(\partial\Omega^{(m)})$ be some fixed extensions of the functions G_k and $G^{(m)}_j$ respectively onto $\partial\Omega$ and $\partial\Omega^{(m)}$ preserving the space. Denote

$$\begin{aligned} &G_0 := (G_{01},\cdots,G_{05})^\top\in[B^{-1/p}_{p,p}(\partial\Omega)]^5,\\ &G^{(m)}_0 := (G^{(m)}_{01},\cdots,G^{(m)}_{04})^\top\in[B^{-1/p}_{p,p}(\partial\Omega^{(m)})]^4. \end{aligned}\tag{30}$$

It is evident that arbitrary extensions G^*_j and $G^{(m)*}_j$ of the same functions can be represented then as $G^*_k=G_{0k}+\psi_k+h_k$ for $k=\overline{1,5}$ and $G^{(m)*}_j=G^{(m)}_{0j}+h^{(m)}_j$ for $j=\overline{1,4}$, where $\psi_k\in\widetilde{B}^{-1/p}_{p,p}(S_D)$, for $k=\overline{1,5}$, $h_l\in\widetilde{B}^{-1/p}_{p,p}(\Gamma^{(m)}_T)$ for $l=1,2,3$, $h_t\in\widetilde{B}^{-1/p}_{p,p}(\Gamma^{(m)})$ for $t=4,5$, $h^{(m)}_l\in\widetilde{B}^{-1/p}_{p,p}(\Gamma^{(m)}_T)$ for $l=1,2,3$, $h^{(m)}_4\in\widetilde{B}^{-1/p}_{p,p}(\Gamma^{(m)})$ are arbitrary functions. We set

$$\begin{aligned} &\psi:=(\psi_1,\cdots,\psi_5)^\top\in[\widetilde{B}^{-1/p}_{p,p}(S_D)]^5,\\ &h=(h_1,\cdots,h_5)^\top\in[\widetilde{B}^{-1/p}_{p,p}(\Gamma^{(m)}_T)]^3\times[\widetilde{B}^{-1/p}_{p,p}(\Gamma^{(m)})]^2,\\ &h^{(m)}=(h^{(m)}_1,\cdots,h^{(m)}_4)^\top\in[\widetilde{B}^{-1/p}_{p,p}(\Gamma^{(m)}_T)]^3\times\widetilde{B}^{-1/p}_{p,p}(\Gamma^{(m)}). \end{aligned}\tag{31}$$

We apply the indirect boundary integral equations method and, in accordance with Lemmas 3.1 and 3.3, look for a solution vectors $U^{(m)}=(u^{(m)},\cdots,u^{(m)}_4)^\top$ and $U=(u_1,\cdots,u_5)^\top$ of the interface crack problem (16)-(19) in the form of single layer potentials

$$\begin{aligned} &U^{(m)}=V^{(m)}_\tau\big(\big[\mathcal{P}^{(m)}_\tau\big]^{-1}\big[G^{(m)}_0+h^{(m)}\big]\big)\quad\text{in}\quad\Omega^{(m)},\\ &U=V_\tau\big(\mathcal{P}^{-1}_\tau\big[G_0+\psi+h\big]\big)\quad\text{in}\quad\Omega, \end{aligned}\tag{32}$$

where $\mathcal{P}^{(m)}_\tau$ and $\mathcal{P}_\tau$ are given by (23) and (26), G_0 and $G^{(m)}_0$ are the above introduced known vector-functions, and $h^{(m)}$, h and ψ are unknown vector-functions satisfying the inclusions (31).

By Lemmas 2.1, 2.3 and the property (25) we see that the homogeneous differential equations (16), the first two boundary conditions in (17) and the crack conditions (19) are satisfied automatically.

The remaining boundary and transmission conditions in (17), (18) lead to the following system of pseudodifferential equations with respect to the unknown vector-functions ψ, h and $h^{(m)}$:

$$r_{S_D}[\mathcal{A}_\tau\psi]_k+r_{S_D}[\mathcal{A}_\tau h]_k=\widetilde{f}_k\quad\text{on } S_D,\quad k=\overline{1,5},\tag{33}$$

$$\begin{cases} r_{\Gamma^{(m)}_T}[\mathcal{A}_\tau\psi]_l+r_{\Gamma^{(m)}_T}[(\mathcal{A}_\tau+\mathcal{B}^{(m)}_\tau)h]_l=\widetilde{g}^{(m)}_l\\ r_{\Gamma^{(m)}_T}h^{(m)}_l+r_{\Gamma^{(m)}_T}h_l=\widetilde{F}^{(m)}_l,\quad\text{on }\Gamma^{(m)}_T\quad l=\overline{1,3}, \end{cases}\tag{34}$$

$$\begin{cases} r_{\Gamma^{(m)}}[\mathcal{A}_\tau\psi]_4+r_{\Gamma^{(m)}}[(\mathcal{A}_\tau+\mathcal{B}^{(m)}_\tau)h]_4=\widetilde{g}^{(m)}_4,\\ r_{\Gamma^{(m)}}[\mathcal{A}_\tau\psi]_5+r_{\Gamma^{(m)}}[\mathcal{A}_\tau h]_5=\widetilde{g}^{(m)}_5\\ r_{\Gamma^{(m)}}h^{(m)}_4+r_{\Gamma^{(m)}}h_4=\widetilde{F}^{(m)}_4\qquad\text{on }\Gamma^{(m)}, \end{cases}\tag{35}$$

where

$$\begin{aligned} &\widetilde{g}^{(m)}_l:=\widetilde{f}^{(m)}_l+r_{\Gamma^{(m)}_T}\big[\mathcal{H}^{(m)}_\tau\big[\mathcal{P}^{(m)}_\tau\big]^{-1}\widetilde{F}^{(m)}\big]_l\in B^{1-1/p}_{p,p}(\Gamma^{(m)}_T),\\ &\widetilde{f}^{(m)}_l:=f^{(m)}_l+r_{\Gamma^{(m)}_T}\big[\mathcal{H}^{(m)}_\tau\big[\mathcal{P}^{(m)}_\tau\big]^{-1}G^{(m)}_0\big]_l,\\ &\qquad -r_{\Gamma^{(m)}_T}\big[\mathcal{H}_\tau\mathcal{P}^{-1}_\tau G_0\big]_l\in B^{1-1/p}_{p,p}(\Gamma^{(m)}_T),\quad l=\overline{1,3},\\ &\widetilde{g}^{(m)}_4:=\widetilde{f}^{(m)}_4+r_{\Gamma^{(m)}_T}\big[\mathcal{H}^{(m)}_\tau\big[\mathcal{P}^{(m)}_\tau\big]^{-1}\widetilde{F}^{(m)}\big]_4\in B^{1-1/p}_{p,p}(\Gamma^{(m)}),\\ &\widetilde{f}^{(m)}_4:=f^{(m)}_4+r_{\Gamma^{(m)}}\big[\mathcal{H}^{(m)}_\tau\big[\mathcal{P}^{(m)}_\tau\big]^{-1}G^{(m)}_0\big]_4\\ &\qquad -r_{\Gamma^{(m)}}\big[\mathcal{H}_\tau\mathcal{P}^{-1}_\tau G_0\big]_4\in B^{1-1/p}_{p,p}(\Gamma^{(m)}),\\ &\widetilde{g}^{(m)}_5=\widetilde{f}^{(m)}_5\in B^{1-1/p}_{p,p}(\Gamma^{(m)}),\\ &\widetilde{f}_k:=f_k-r_{S_D}[\mathcal{H}_\tau\mathcal{P}^{-1}_\tau G_0]_k\in B^{1-1/p}_{p,p}(S_D),\quad k=\overline{1,5},\\ &\widetilde{f}^{(m)}_5:=f^{(m)}_5-r_{\Gamma^{(m)}}\big[\mathcal{H}_\tau\mathcal{P}^{-1}_\tau G_0\big]_5\in B^{1-1/p}_{p,p}(\Gamma^{(m)}), \end{aligned}\tag{36}$$

$$\begin{aligned} &\widetilde{F}^{(m)}_l:=F^{(m)}_l-r_{\Gamma^{(m)}_T}G_{0l}-r_{\Gamma^{(m)}_T}G^{(m)}_{0l}\in r_{\Gamma^{(m)}_T}\widetilde{B}^{-1/p}_{p,p}(\Gamma^{(m)}_T),\\ &\qquad\qquad l=\overline{1,3},\\ &\widetilde{F}^{(m)}_4:=F^{(m)}_4-r_{\Gamma^{(m)}}G_{04}-r_{\Gamma^{(m)}}G^{(m)}_{04}\in r_{\Gamma^{(m)}}\widetilde{B}^{-1/p}_{p,p}(\Gamma^{(m)}). \end{aligned}\tag{37}$$

and

$$\begin{aligned} &\mathcal{A}_\tau:=\mathcal{H}_\tau\mathcal{P}^{-1}_\tau,\\ &\mathcal{B}^{(m)}_\tau:=\begin{bmatrix}[\mathcal{H}^{(m)}_\tau\big[\mathcal{P}^{(m)}_\tau\big]^{-1}]_{4\times4} & [0]_{4\times1}\\ [0]_{1\times4} & [0]_{1\times1}\end{bmatrix}_{5\times5}. \end{aligned}\tag{38}$$

The inclusions in (37) are the *compatibility conditions* for ICP-Problem due to the relations (31). Therefore, in what follows we assume that $\widetilde{F}^{(m)}_l$ and $\widetilde{F}^{(m)}_4$ are extended from $\Gamma^{(m)}_T$ and $\Gamma^{(m)}$, respectively, onto $\partial\Omega^{(m)}$ by zero, i.e., $\widetilde{F}^{(m)}_l\in\widetilde{B}^{-1/p}_{p,p}(\Gamma^{(m)}_T)$, $l=\overline{1,3}$, and $\widetilde{F}^{(m)}_4\in\widetilde{B}^{-1/p}_{p,p}(\Gamma^{(m)})$.

Note that the above systems are non-classical systems of pseudodifferential equations since the sub-manifolds $\Gamma^{(m)}_T$ and $\Gamma^{(m)}_C$ are proper parts of $\Gamma^{(m)}$. We will discuss this problem in detail in the next subsection.

3.2 Existence theorems

Here we show that the system of pseudodifferential equations (33)-(35) is uniquely solvable. To this end we introduce the suitable spaces:

$$\begin{aligned} \mathbb{X}^s_p:=&[\widetilde{H}^s_p(S_D)]^5\times[\widetilde{H}^s_p(\Gamma^{(m)}_T)]^3\times[\widetilde{H}^s_p(\Gamma^{(m)})]^2\times\\ &\times[\widetilde{H}^s_p(\Gamma^{(m)}_T)]^3\times\widetilde{H}^s_p(\Gamma^{(m)}),\\ \mathbb{Y}^s_p:=&[H^{s+1}_p(S_D)]^5\times[H^{s+1}_p(\Gamma^{(m)}_T)]^3\times[H^{s+1}_p(\Gamma^{(m)})]^2\times\\ &\times[\widetilde{H}^s_p(\Gamma^{(m)}_T)]^3\times\widetilde{H}^s_p(\Gamma^{(m)}) \end{aligned}$$

and similarly the spaces $\mathbb{X}^s_{p,q}$ and $\mathbb{Y}^s_{p,q}$, where H^s_p spaces are replaced by $B^s_{p,q}$-spaces. Notice that $\mathbb{X}^s_{2,2} = \mathbb{X}^s_2$ and $\mathbb{Y}^s_{2,2} = \mathbb{Y}^s_2$.

Let us rewrite the system (33)-(35) as the operator equation

$$\mathcal{N}_\tau^{(B)} \Phi = Y, \tag{39}$$

where the vector $\Phi := (\psi,\ h,\ h^{(m)})^\top \in \mathbb{X}^{-1/p}_{p,p}$ is unknown, while $Y := (\widetilde{f},\ \widetilde{g}^{(m)},\ \widetilde{F}^{(m)})^\top \in \mathbb{Y}^{-1/p}_{p,p}$ is a given vector with $\widetilde{f} = (\widetilde{f}_1, \cdots, \widetilde{f}_5)^\top$, $\widetilde{g}^{(m)} := (\widetilde{g}^{(m)}_1, \cdots, \widetilde{g}^{(m)}_5)^\top$, and $\widetilde{F}^{(m)} := (\widetilde{F}^{(m)}_1, \cdots, \widetilde{F}^{(m)}_4)^\top$.

The operator $\mathcal{N}_\tau^{(B)}$ in the left hand side of the system (33)-(35) (rearranged properly) is a 14×14 matrix operator

$$\mathcal{N}_\tau^{(B)} := \tag{40}$$

$$= \begin{bmatrix} r_{S_D}\, \mathcal{A}_\tau & r_{S_D}\, \mathcal{A}_\tau & [0]_{5\times 4} \\ r_{\Gamma_T^{(m)}} [(\mathcal{A}_\tau)_{l,k}]_{3\times 5} & r_{\Gamma_T^{(m)}} [(\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)})_{l,k}]_{3\times 5} & [0]_{3\times 4} \\ r_{\Gamma^{(m)}} [(\mathcal{A}_\tau)_{t,k}]_{2\times 5} & r_{\Gamma^{(m)}} [(\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)})_{t,k}]_{2\times 5} & [0]_{2\times 4} \\ [0]_{3\times 5} & r_{\Gamma_T^{(m)}} I_{3\times 5} & r_{\Gamma_T^{(m)}} I_{3\times 4} \\ [0]_{1\times 5} & r_{\Gamma^{(m)}} I_{1\times 5} & r_{\Gamma^{(m)}} I_{1\times 4} \end{bmatrix},$$

where $\mathcal{A}_\tau$ and $\mathcal{B}_\tau^{(m)}$) are defined in (38); the indices appearing in the block matrices take the following values $k = \overline{1,5}$, $l = 1,2,3$, and $t = 4,5$; $\mathcal{A}_\tau$ and $\mathcal{B}_\tau^{(m)}$ are the Steklov-Poincaré type 5×5 matrix pseudodifferential operators; the symbol $[0]_{N\times M}$ stands for the zero matrix of dimension $N \times M$, while

$$I_{3\times 5} := \begin{bmatrix} 1\,0\,0\,0\,0 \\ 0\,1\,0\,0\,0 \\ 0\,0\,1\,0\,0 \end{bmatrix},\ I_{3\times 4} := \begin{bmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,1\,0 \end{bmatrix},\ \begin{matrix} I_{1\times 5} = (0,0,0,1,0), \\ I_{1\times 4} = (0,0,0,1). \end{matrix}$$

Applying the results on boundedness of potential and the related boundary pseudodifferential operators, invoking Lemma 2.2 we easily establish the following mapping properties

$$\begin{aligned} \mathcal{N}_\tau^{(B)} &: \mathbb{X}^s_p \to \mathbb{Y}^s_p, \quad s \in \mathbb{R}, \\ &: \mathbb{X}^s_{p,q} \to \mathbb{Y}^s_{p,q}, \quad 1 < p < \infty, \quad 1 \le q \le \infty \end{aligned} \tag{41}$$

(see, e.g., [3–5, 12, 16] etc. and also [10] for the boundedness results of general layer potentials).

Our goal is to establish Fredholm properties and invertibility of the operator (41). For this we have to define some constants. Let $\lambda^{(1)}_j(x)$, $j = \overline{1,5}$, be the eigenvalues of the matrix $[\mathfrak{S}(\mathcal{A}_\tau)(x,0,+1)]^{-1}\ \mathfrak{S}(\mathcal{A}_\tau)(x,0,-1)$ for $x \in \partial S_D$, where $\mathfrak{S}(\mathcal{A}_\tau)(x,\xi_1,\xi_2)$ is the homogeneous principal symbol matrix of the pseudodifferential operator $\mathcal{A}_\tau$ in (38). Let

$$\gamma_1{}' := \frac{1}{2\pi} \inf \arg \lambda^{(1)}_j(x), \quad \gamma_1{}'' := \frac{1}{2\pi} \sup \arg \lambda^{(1)}_j(x), \tag{42}$$

where "infimum" and "supremum" are taken over the sets $x \in \partial S_D$, $1 \le j \le 5$.

Similarly, $\lambda^{(2)}_j(x)$ and $\lambda^{(3)}_j(z)$, $j = \overline{1,5}$, be the eigenvalues of the matrices $[\mathfrak{S}(\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)})(x,0,+1)]^{-1}\ \mathfrak{S}(\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)})(x,0,-1)$ and $[\mathfrak{S}(D_{\tau,y})(z,0,+1)]^{-1}\ \mathfrak{S}(D_{\tau,y})(z,0,-1)$ for $x \in \partial\Gamma^{(m)}$, $z \in \partial\Gamma_C^{(m)}$ where $\mathfrak{S}(\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)})(x,\xi_1,\xi_2)$ and $\mathfrak{S}(D_{\tau,y})(x,\xi_1,\xi_2)$ are the homogeneous principal symbol matrices of the pseudodifferential operators $\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)}$ in (38) and $D_{\tau,y}$ in (55). Let

$$\gamma_2{}' := \frac{1}{2\pi} \inf \arg \lambda^{(2)}_j(x), \quad \gamma_2{}'' := \frac{1}{2\pi} \sup \arg \lambda^{(2)}_j(x), \tag{43}$$

$$\gamma_3{}' := \frac{1}{2\pi} \inf \arg \lambda^{(3)}_j(z), \quad \gamma_3{}'' := \frac{1}{2\pi} \sup \arg \lambda^{(3)}_j(z), \tag{44}$$

where "infimum" and "supremum" is taken over the sets $x \in \partial\Gamma^{(m)}$ and $z \in \partial\Gamma_C^{(m)}$ respectively, and $1 \le j \le 5$.

Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be Banach spaces and $\mathfrak{B} := \mathfrak{B}_1 \times \mathfrak{B}_2$ be their direct product, consisting of pairs $U = (u', u'')^\top \in \mathfrak{B}$, where $u' \in \mathfrak{B}_1$ and $u'' \in \mathfrak{B}_2$. Further, let $\mathfrak{B}^*_j$ be the adjoint spaces to $\mathfrak{B}_j$, $j = 1,2$, and $\mathfrak{B}^* := \mathfrak{B}^*_1 \times \mathfrak{B}^*_2$. The notation $\langle F, u\rangle$ with $F \in \mathfrak{B}^*_j$ and $u \in \mathfrak{B}_j$ (or $F \in \mathfrak{B}^*$ and $u \in \mathfrak{B}$) is used for the duality pairing between the adjoint spaces. It is obvious that the bounded operator $\mathbf{A}\ :\ \mathfrak{B} \to \mathfrak{B}^*$ has the matrix form $\mathbf{A} = \left[\mathbf{A}_{jk}\right]$, where the operators $\mathbf{A}_{jk}\ :\ \mathfrak{B}_k \to \mathfrak{B}^*_j$ are all bounded.

Lemma 3.1. *Let the operator* $\mathbf{A} = \left[\mathbf{A}_{jk}\right]$ *be strongly coercive* $\operatorname{Re}\langle \mathbf{A}U, U\rangle \ge C\|U\|^2_{\mathfrak{B}}\ \ \forall U \in \mathfrak{B}$ *(or be positive definite* $\langle \mathbf{A}U, U\rangle \ge C\|U\|^2_{\mathfrak{B}}\ \forall U \in \mathfrak{B}$*). Then the operators* $\mathbf{A}_{11}$ *and* $\mathbf{A}_{22}$ *are both strongly coercive (are positive definite, respectively) and thus invertible. Moreover, the operators*

$$\mathbf{B} := \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\ :\ \mathfrak{B}_1 \to \mathfrak{B}^*_1, \tag{45}$$

$$\mathbf{D} := \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\ :\ \mathfrak{B}_2 \to \mathfrak{B}^*_2 \tag{46}$$

are strongly coercive (are positive definite) and, thus, invertible.

Proof: The strong coercivity (the positive definiteness) of $\mathbf{A}_{11}$ and of $\mathbf{A}_{22}$ follows by taking consecutively $U = (u,0)^\top \in \mathfrak{B}$ and $U = (0,v)^\top \in \mathfrak{B}$, $u \in \mathfrak{B}_1$, $v \in \mathfrak{B}_2$. The strong coercivity implies the invertibility.

Recall that $\|u\|^2_{\mathfrak{B}_1} \le \|u\|^2_{\mathfrak{B}_1} + \|v\|^2_{\mathfrak{B}_2} = \|U\|^2_{\mathfrak{B}}$ for $U = (u,v)^\top \in \mathfrak{B}$, $u \in \mathfrak{B}_1$, $v \in \mathfrak{B}_2$. The strong coercivity (the positive definiteness) of $\mathbf{B}$ and of $\mathbf{D}$ in (45), (46) follows if we introduce consecutively $v = -\mathbf{A}_{22}^{-1}\mathbf{A}_{21}u$ and $u = -\mathbf{A}_{11}^{-1}\mathbf{A}_{12}v$ into the equality $\langle \mathbf{A}U, U\rangle = \langle \mathbf{A}_{11}u, u\rangle + \langle \mathbf{A}_{12}v, u\rangle + \langle \mathbf{A}_{21}u, v\rangle + \langle \mathbf{A}_{22}v, v\rangle$ and apply the strong coercivity (the positive definiteness) of $\mathbf{A}$. □

Theorem 3.2. *The operator* $\mathcal{N}_\tau^{(B)}$ *in* (40) *and* (41) *is invertible provided the following constraints hold*

$$\frac{1}{p} - \frac{3}{2} + \max\{\gamma_1{}'', \gamma_2{}'', \gamma_3{}''\} < r < \frac{1}{p} - \frac{1}{2} + \min\{\gamma_1{}', \gamma_2{}', \gamma_3{}'\}. \tag{47}$$

Proof. Note, that the operators $r_{S_D}\mathcal{A}_\tau : [\widetilde{B}^s_{p,q}(\Gamma^{(m)})]^5 \to [B^{s+1}_{p,q}(S_D)]^5$ and $r_{\Gamma^{(m)}}\mathcal{A}_\tau : [\widetilde{B}^s_{p,q}(S_D)]^5 \to [B^{s+1}_{p,q}(\Gamma^{(m)})]^5$ are compact for $1 < p < +\infty$, $s \in \mathbb{R}$ and $1 \le q \le +\infty$ since the domains are disjoint: $\overline{S_D} \cap \overline{\Gamma^{(m)}} = \varnothing$. Then the pseudodifferential operator

$$\mathcal{N}^{(B,0)}_\tau := \begin{bmatrix} r_{S_D}\mathcal{A}_\tau & [0]_{5\times5} & [0]_{5\times4} \\ [0]_{3\times5} & r_{\Gamma^{(m)}_T}[(\mathcal{A}_\tau + \mathcal{B}^{(m)}_\tau)_{l,k}]_{3\times5} & [0]_{3\times4} \\ [0]_{2\times5} & r_{\Gamma^{(m)}}[(\mathcal{A}_\tau + \mathcal{B}^{(m)}_\tau)_{t,k}]_{2\times5} & [0]_{2\times4} \\ [0]_{3\times5} & r_{\Gamma^{(m)}_T} I_{3\times5} & r_{\Gamma^{(m)}_T} I_{3\times4} \\ [0]_{1\times5} & r_{\Gamma^{(m)}} I_{1\times5} & r_{\Gamma^{(m)}} I_{1\times4} \end{bmatrix}_{14\times14}$$

is a compact perturbation of the operator $\mathcal{N}^{(B)}_\tau$ and has the same mapping property as $\mathcal{N}^{(B)}_\tau$ in (41). $\mathcal{N}^{(B,0)}_\tau$ is of block-lower triangular form

$$\mathcal{N}^{(B,0)}_\tau := \begin{bmatrix} r_{S_D}\mathcal{A}_\tau & [0]_{5\times5} & [0]_{5\times4} \\ [0]_{5\times5} & \mathcal{N}^{(2)}_\tau & [0]_{5\times4} \\ [0]_{4\times5} & I_{4\times5} & I_{4\times4} \end{bmatrix}_{14\times14}, \tag{48}$$

$$\mathcal{N}^{(2)}_\tau := \begin{bmatrix} r_{\Gamma^{(m)}_T}[(\mathcal{A}_\tau + \mathcal{B}^{(m)}_\tau)_{l,k}]_{3\times5} \\ r_{\Gamma^{(m)}}[(\mathcal{A}_\tau + \mathcal{B}^{(m)}_\tau)_{t,k}]_{2\times5} \end{bmatrix}_{5\times5}, \tag{49}$$

where $k = \overline{1,5}$, $l = 1,2,3$, and $t = 4,5$. Further, the operators

$$\begin{aligned} r_{S_D}\mathcal{A}_\tau &: [\widetilde{H}^r_p(S_D)]^5 \to [H^{r+1}_p(S_D)]^5, \\ &: [\widetilde{B}^r_{p,q}(S_D)]^5 \to [B^{r+1}_{p,q}(S_D)]^5 \end{aligned} \tag{50}$$

are invertible if

$$\frac{1}{p} - 1 + \gamma_1{}'' < r + \frac{1}{2} < \frac{1}{p} + \gamma_1{}', \tag{51}$$

where $\gamma_1{}'$ and $\gamma_1{}''$ are defined in (42). The proof is based on the results of [7] and similar proofs for ΨDOs with positive definite symbols are available e.g. in [3–5, 7, 11, 12, 16] and many other papers.

To prove the invertibility of $\mathcal{N}^{(B,0)}_\tau$ it remains to investigate the operators

$$\begin{aligned} \mathcal{N}^{(2)}_\tau &: \widetilde{\mathbb{H}}^r_p \to \mathbb{H}^{r+1}_p, \\ &: \widetilde{\mathbb{B}}^r_{p,q} \to \mathbb{B}^{r+1}_{p,q} \end{aligned} \tag{52}$$

where the spaces are $\widetilde{\mathbb{H}}^r_p := [\widetilde{H}^r_p(\Gamma^{(m)}_T)]^3 \times [\widetilde{H}^r_p(\Gamma^{(m)})]^2$, $\mathbb{H}^{r+1}_p := [H^{r+1}_p(\Gamma^{(m)}_T)]^3 \times [H^{r+1}_p(\Gamma^{(m)})]^2$, $\widetilde{\mathbb{B}}^r_{p,q} := [\widetilde{B}^r_{p,q}(\Gamma^{(m)}_T)]^3 \times [\widetilde{B}^r_{p,q}(\Gamma^{(m)})]^2$, $\mathbb{B}^{r+1}_{p,q} := [B^{r+1}_{p,q}(\Gamma^{(m)}_T)]^3 \times B^{r+1}_{p,q}(\Gamma^{(m)})]^2$.

Since $\Gamma^{(m)}_T$ is a proper part of $\Gamma^{(m)}$ we can not apply standard theorems on Fredholm properties of the operators (52). Instead we will apply the local principle for para-algebras, exposed in the book [9]. To this end, let either $\mathbb{Z}^r_p := \mathbb{H}^r_p$ ($\widetilde{\mathbb{Z}}^r_p := \widetilde{\mathbb{H}}^r_p$) or $\mathbb{Z}^r_p := \mathbb{B}^r_{p,q}$ ($\widetilde{\mathbb{Z}}^r_p := \widetilde{\mathbb{B}}^r_{p,q}$). Consider the quotient para-algebra $\Psi'(\widetilde{\mathbb{Z}}^r_p, \mathbb{Z}^{r+1}_p) = [\Psi(\widetilde{\mathbb{Z}}^r_p, \mathbb{Z}^{r+1}_p)/\mathfrak{C}(\widetilde{\mathbb{Z}}^r_p, \mathbb{Z}^{r+1}_p)]_{2\times2}$ of all ΨDOs $\Psi(\widetilde{\mathbb{Z}}^r_p, \mathbb{Z}^{r+1}_p)$ acting between the indicated spaces factored by the space of all compact operators $\mathfrak{C}(\widetilde{\mathbb{Z}}^r_p, \mathbb{Z}^{r+1}_p)$. Further, for arbitrary point $y \in \overline{\Gamma^{(m)}}$ we define the following localizing class $\Delta_y := \{[g_y I_5], \quad g_y \in \mathbb{C}^\infty(\Gamma^{(m)}), \quad \operatorname{supp} g_y \subset W_y, \quad g_y(x) = 1 \quad \forall x \in \widetilde{W}_y\}$, where $\widetilde{W}_y \subset W_y \subset \overline{\Gamma^{(m)}}$ are arbitrarily small embedded neighborhoods of y. The symbol $[A]$ stands for the quotient class containing the operator A. It is obvious that the system $\{\Delta_y\}_{y\in\overline{\Gamma^{(m)}}}$ is covering and all its elements $[g_y I_5]$ commute with the class $[A]$ for arbitrary ΨDO $A \in \Psi(\widetilde{\mathbb{Z}}^r_p, \mathbb{Z}^{r+1}_p)$ (to justify the commutativity recall that a commutant $AgI - gA$, with the identity operator I, is compact for an arbitrary smooth function g).

The ΨDO $\mathcal{A}_\tau = \mathcal{H}_\tau \mathcal{P}^{-1}_\tau$ "lives" on the surface $\partial\Omega$ (see (38) and Section 2). Let us consider a similar operator $\mathcal{A}^{(m)}_\tau := \mathcal{H}_{(m),\tau}\mathcal{P}^{-1}_{(m),\tau}$ which "lives" on the surface $\partial\Omega^{(m)}$, composed of ΨDOs $\mathcal{H}_{(m),\tau}$ and $\mathcal{P}_{(m),\tau}$ representing the direct values of the potential operators, defined in Section 2 in the domain $\Omega^{(m)}$. The closed surfaces $\partial\Omega$ and $\partial\Omega^{(m)}$, where the operators $\mathcal{A}_\tau$ and $\mathcal{A}^{(m)}_\tau$ are defined, have in common the open surface $\Gamma^{(m)} = \partial\Omega \cap \partial\Omega^{(m)}$. On the other hand, an arbitrary ΨDO $A(x,D)$ and, in particular the operators $\mathcal{A}_\tau$ and $\mathcal{A}^{(m)}_\tau$, are of local type: if g_1 and g_2 are functions with disjoint supports $\operatorname{supp} g_1 \cap \operatorname{supp} g_2 = \varnothing$, then the operator $g_1 A(x,D) g_2 I$ is compact in the spaces where $A(x,D)$ is bounded. Applying the mentioned property, it is easy to check that the operators $\mathcal{A}_\tau$ and $\mathcal{A}^{(m)}_\tau$ are locally equivalent $[\mathcal{A}_\tau] \overset{\Delta_y}{\sim} [\mathcal{A}^{(m)}_\tau]$ for all $y \in \Gamma^{(m)}$. Applying the above local equivalence we can check the following local equivalences $[\mathcal{N}^{(2)}_\tau] \overset{\Delta_y}{\sim} [\mathcal{N}^{(2)}_{\tau,y}]$, where

$$\begin{aligned} \mathcal{N}^{(2)}_{\tau,y} &:= \mathcal{A}^{(m)}_\tau + \mathcal{B}^{(m)}_\tau \;:\; [H^r_p(\partial\Omega^{(m)})]^5 \\ &\qquad \to [H^{r+1}_p(\partial\Omega^{(m)})]^5 \quad \text{for} \quad y \in \Gamma^{(m)}_T, \end{aligned} \tag{53a}$$

$$\begin{aligned} \mathcal{N}^{(2)}_{\tau,y} &:= \Big[(\mathcal{A}^{(m)}_\tau + \mathcal{B}^{(m)}_\tau)_{t,q}\Big]_{2\times2} \;:\; [H^r_p(\partial\Omega^{(m)})]^2 \\ &\qquad \to [H^{r+1}_p(\partial\Omega^{(m)})]^2 \quad \text{for} \quad y \in \Gamma^{(m)}_C, \end{aligned} \tag{53b}$$

$$\begin{aligned} \mathcal{N}^{(2)}_{\tau,y} &:= \Big[r_{\Gamma^{(m)}}(\mathcal{A}^{(m)}_\tau + \mathcal{B}^{(m)}_\tau)\Big]_{5\times5} \;:\; [\widetilde{H}^r_p(\Gamma^{(m)})]^5 \\ &\qquad \to [H^{r+1}_p(\Gamma^{(m)})]^5 \quad \text{for} \quad y \in \partial\Gamma^{(m)}, \end{aligned} \tag{53c}$$

$$\mathcal{N}^{(2)}_{\tau,y} := \begin{bmatrix} r_{\Gamma^c_{(m)}}[(\mathcal{A}^{(m)}_\tau + \mathcal{B}^{(m)}_\tau)_{l,k}]_{3\times3} & r_{\Gamma^c_{(m)}}[(\mathcal{A}^{(m)}_\tau + \mathcal{B}^{(m)}_\tau)_{l,q}]_{3\times2} \\ [(\mathcal{A}^{(m)}_\tau + \mathcal{B}^{(m)}_\tau)_{t,k}]_{2\times3} & [(\mathcal{A}^{(m)}_\tau + \mathcal{B}^{(m)}_\tau)_{t,q}]_{2\times2} \end{bmatrix}$$
$$: \widetilde{\mathbb{V}}^r_p \to \mathbb{V}^{r+1}_p \quad \text{for} \quad y \in \partial\Gamma^{(m)}_C, \quad l,k = 1,2,3, \quad t,q = 4,5. \tag{53d}$$

Here $\partial\Omega^{(m)}$ is a closed surface, $\Gamma^c_{(m)} := \partial\Omega^{(m)} \setminus \overline{\Gamma^{(m)}_C} = \Gamma^{(m)}_T \cup \overline{S^{(m)}_N}$ and $\widetilde{\mathbb{V}}^r_p := [\widetilde{X}^r_p(\Gamma^c_{(m)})]^3 \times [X^r_p(\partial\Omega^{(m)})]^2$, $\mathbb{V}^{r+1}_p := [X^{r+1}_p(\Gamma^c_{(m)})]^3 \times$

$[X_p^{r+1}(\partial\Omega^{(m)})]^2$ with either $X_p^r = H_p^r$ or $X_p^r = B_{p,q}^r$.

Due to the local principle the operator $\mathcal{N}_\tau^{(2)}$ in (52) is Fredholm if and only if the operators $\mathcal{N}_{\tau,y}^{(2)}$ in (53a)-(53d) are Fredholm for all $y \in \overline{\Gamma^{(m)}}$.

The positive definite ΨDOs $\mathcal{N}_{\tau,y}^{(2)}$ in (53a) and in (53b) on the closed surface $\partial\Omega^{(m)}$ are Fredholm with index 0 for all $y \in \Gamma_C^{(m)} \cup \Gamma_T^{(m)}$.

The same positive definite ΨDOs $\mathcal{N}_{\tau,y}^{(2)}$ in (53c) but on the surface $\Gamma^{(m)}$ with the smooth boundary $\partial\Gamma^{(m)} \neq \varnothing$ is Fredholm if the following constraints hold

$$\frac{1}{p} - \frac{3}{2} + \gamma_2{}'' < r < \frac{1}{p} - \frac{1}{2} + \gamma_2{}' \qquad (54)$$

with $\gamma_2{}'$ and $\gamma_2{}''$ defined in (43) (see [7, 11]).

To investigate the strongly elliptic ΨDO $\mathcal{N}_{\tau,y}^{(2)}$ in (53d) for $y \in \partial\Gamma_C^{(m)}$, first note that $\mathcal{G}_y := [(\mathcal{A}_\tau^{(m)} + \mathcal{B}_\tau^{(m)})_{t,q}]_{2\times 2}$ is defined on the closed surface $\partial\Omega^{(m)}$, is strongly elliptic due to Lemma 3.1 and, therefore, is Fredholm. Then the quotient class $[\mathcal{G}_y]$ is invertible and since $\operatorname{Ind}\mathcal{G}_y = 0$, there exists a compact operator T_y such that $\mathcal{G}_y + T_y$ is invertible for all $y \in \partial\Gamma_C^{(m)}$. For the quotient classes the equalities $[\mathcal{G}_y + T_y] = [\mathcal{G}_y]$ and $[\mathcal{G}_y + T_y]^{-1} = [\mathcal{G}_y]^{-1}$ hold.

Note that the quotient classes

$$[\mathcal{F}_\pm] := \begin{bmatrix} [I_{3\times 3}] & [[0]_{3\times 2}] \\ \pm[\mathcal{G}_y]^{-1}\left[[(\mathcal{N}_{\tau,y}^{(2)})_{t,k}]_{2\times 3}\right] & [I_{2\times 2}] \end{bmatrix}_{5\times 5}$$

are invertible $[\mathcal{F}_-][\mathcal{F}_+] = [\mathcal{F}_+][\mathcal{F}_-] = [I_{5\times 5}]$ and the composing with the quotient class $\left[\mathcal{N}_{\tau,y}^{(2)}\right]$ gives

$$\left[\widetilde{\mathcal{N}}_{\tau,y}^{(2)}\right] := \left[\mathcal{N}_{\tau,y}^{(2)}\right][\mathcal{F}_-] = \begin{bmatrix} [D_{\tau,y}] & \left[r_{\Gamma^c_{(m)}}[(\mathcal{N}_{\tau,y}^{(2)})_{l,q}]_{3\times 2}\right] \\ [[0]_{2\times 3}] & [\mathcal{G}_y] \end{bmatrix},$$

$$D_{\tau,y} := r_{\Gamma^c_{(m)}}\Big([(\mathcal{N}_{\tau,y}^{(2)})_{l,k}]_{3\times 3} - [(\mathcal{N}_{\tau,y}^{(2)})_{l,k}]_{3\times 2}[\mathcal{G}_y + T_y]^{-1}[(\mathcal{N}_{\tau,y}^{(2)})_{t,k}]_{2\times 3}\Big). \qquad (55)$$

$D_{\tau,y}$ is the positive definite ΨDO of order -1 due to Lemma 3.1. It is sufficient to prove that the composition $\left[\widetilde{\mathcal{N}}_{\tau,y}^{(2)}\right]$ is an invertible class. $\left[\widetilde{\mathcal{N}}_{\tau,y}^{(2)}\right]$ in upper block-triangular and the entry $[\mathcal{G}_y]$ on the diagonal is an invertible class. Moreover, the entries on the diagonal $D_{\tau,y}$ and $\mathcal{G}_y$ are ΨDOs and the corresponding quotient classes commute (actually, these entries are matrices of different dimension 3×3 and 2×2, but we can extend the entire matrix $\left[\widetilde{\mathcal{N}}_{\tau,y}^{(2)}\right]$ by identity on the diagonal and by zeros on the off-diagonal entries in the last row and the last column, without affecting the invertibility properties of the entire matrix and the diagonal entries. Then $[\mathcal{G}_y]$ extends to the matrix of the same dimension 3×3 as $[D_{\tau,y}]$). Therefore $\left[\widetilde{\mathcal{N}}_{\tau,y}^{(2)}\right]$ is invertible if and only if the quotient class $[D_{\tau,y}]$ is invertible. This is interpreted as follows: the operator $\widetilde{\mathcal{N}}_{\tau,y}^{(2)} : \widetilde{\mathbb{Z}}_p^r \to \mathbb{Z}_p^{r+1}$ is Fredholm if and only if the operator

$$D_{\tau,y} : [\widetilde{X}_p^r(\Gamma^c_{(m)})]^3 \to [X_p^{r+1}(\Gamma^c_{(m)})]^3 \qquad (56)$$

is Fredholm.

Let $\mathfrak{S}(D_{\tau,y})(x,\xi_1,\xi_2)$ be the principal homogeneous symbol matrix of the operator $D_{\tau,y}$ and $\lambda_j^{(3)}(x)$, $j = 1,2,3$, be the eigenvalues of the matrix $[\mathfrak{S}(D_{\tau,y})(x,0,+1)]^{-1}\,\mathfrak{S}(D_{\tau,y})(x,0,-1)$ for $x \in \partial\Gamma_C^{(m)}$. The operators $D_{\tau,y}$ in (56) and, therefore, the operator $\mathcal{N}_\tau^{(2)}$ in (52) are Fredholm if the following constraints are fulfilled

$$\frac{1}{p} - \frac{3}{2} + \gamma_3{}'' < r < \frac{1}{p} - \frac{1}{2} + \gamma_3{}', \qquad (57)$$

where $\gamma_3{}'$ and $\gamma_3{}''$ are defined in (44).

The conditions (51), (54) and (57) are equivalent to (47).

Next we have to prove that the operator $\mathcal{N}_\tau^{(2)}$ in (52) has zero index: $\operatorname{Ind}\mathcal{N}_\tau^{(2)} = 0$. To this end we consider the homotopy

$$B_\lambda := \lambda\mathcal{R} + (1-\lambda)\mathcal{N}_\tau^{(2)} : \widetilde{\mathbb{H}}_p^r \to \mathbb{H}_p^{r+1} \qquad 0 \le \lambda \le 1,$$

$$\mathcal{R} := \begin{bmatrix} \Lambda^{r,-1}_{\Gamma_T^{(m)}} I_3 & [0]_{3\times 2} \\ [0]_{2\times 3} & \Lambda^{r,-1}_{\Gamma^{(m)}} I_2 \end{bmatrix}_{5\times 5},$$

where $\Lambda^{(r,-1)}_{\Gamma_T^{(m)}}(x,D) := \Lambda^{-1-r}_{\Gamma_T^{(m)}}(x,D)\widetilde{\Lambda}^{-r}_{\Gamma_T^{(m)}}(x,D)$ and

$$\widetilde{\Lambda}^r_{\Gamma_T^{(m)}}(x,D) : \widetilde{H}_p^r(\Gamma_T^{(m)}) \to \widetilde{H}_p^0(\Gamma_T^{(m)}) = H_p^0(\Gamma_T^{(m)}),$$

$$\Lambda^{-1-r}_{\Gamma_T^{(m)}}(x,D) : H_p^0(\Gamma_T^{(m)}) \to H_p^{r+1}(\Gamma_T^{(m)})$$

are the Bessel potential operators, arranging isomorphism of the spaces. Therefore $\Lambda^{(r,-1)}_{\Gamma_T^{(m)}}(x,D)$ is an invertible ΨDO. Moreover, the potential operators $\Lambda^{-1-r}_{\Gamma_T^{(m)}}(x,D)$, $\widetilde{\Lambda}^{-r}_{\Gamma_T^{(m)}}(x,D)$ and therefore $\Lambda^{(r,-1)}_{\Gamma_T^{(m)}}(x,D)$, are positive definite (have the positive definite symbols; cf., e.g., [11]). The definition and the properties of the operator $\Lambda^{(r,-1)}_{\Gamma^{(m)}}(x,D)$ are verbatim.

The homotopy B_λ connects continuously the operator $B_0 = \mathcal{N}_\tau^{(2)}$ with the invertible operator $B_1 = \mathcal{R} : \widetilde{\mathbb{H}}_p^r \to \mathbb{H}_p^{r+1}$. The operator B_λ is positive definite for all $0 \le \lambda \le 1$ since represents the sum of the operators with positive definite and positive definite symbols (see Lemma 3.1). The operator B_λ is then Fredholm for all $0 \le \lambda \le 1$ and $p = 2$ and, therefore, $\operatorname{Ind}\mathcal{N}_\tau^{(2)} = \operatorname{Ind}B_0 = \operatorname{Ind}B_1 = \operatorname{Ind}\mathcal{R} = 0$.

From the results obtained above it follows that the ΨDO $\mathcal{N}_\tau^{(B,0)}$ is Fredholm with zero index in the space setting (41).

Therefore its compact perturbation, the operator $\mathcal{N}_\tau^{(B)}$ in (41) is Fredholm with zero index as well in the same space setting (41). Due to Theorem 1.1 the operator $\mathcal{N}_\tau^{(B)} : \mathbb{X}_2^{-1/2} \to \mathbb{Y}_2^{-1/2}$ has the trivial kernel and is invertible.

The invertibility of the ΨDO $\mathcal{N}_\tau^{(B,0)}$ in the space setting (41) and if the conditions (47) are fulfilled follows then by standard duality and interpolation arguments from [1] (see [3, 4, 16] for similar arguments). □

Corollary 3.3. *Let the compatibility conditions hold (see the inclusions in* (37)*). Let the inclusions* (28) *hold and* $\frac{4}{3-2\gamma''} < p < \frac{4}{1-2\gamma'}$ *with* $\gamma' := \min\{\gamma_1{}', \gamma_2{}', \gamma_3{}'\}$, $\gamma'' := \max\{\gamma_1{}'', \gamma_2{}'', \gamma_3{}''\}$.

Then the interface crack problem (16)–(19) *has a unique solution* $(U^{(m)}, U) \in [W_p^1(\Omega^{(m)})]^4 \times [W_p^1(\Omega)]^5$, *which can be represented by formulae* $U^{(m)} = V_\tau^{(m)}\big(\big[\mathcal{P}_\tau^{(m)}\big]^{-1}\big[G_0^{(m)} + h^{(m)}\big]\big)$ *in* $\Omega^{(m)}$ *and* $U = V_\tau\big(\mathcal{P}_\tau^{-1}\big[G_0 + \psi + h\big]\big)$ *in* Ω *where the densities* ψ, *h, and* $h^{(m)}$ *are to be determined from the system* (33)-(35)*.*

Moreover, the vector functions $G_0 + \psi + h$ *and* $G_0^{(m)} + h^{(m)}$ *are defined uniquely by the above systems.*

Remark 3.4. *Based on results from [7, 8] one can formulate and prove regularity results for solutions of the interface crack problem ICP and write their detailed asymptotic expansion near the crack edge (see [3, 4] etc. for similar formulations).*

Remark 3.5. *Theorem* 3.3 *with* $p = 2$ *remains valid for Lipschitz domains* $\Omega^{(m)}$ *and* Ω. *This follows with the help of lax-Milgram Lemma and Theorem* 1.1.

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