Continuing functions from a hypersurface with the boundary

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We extend m-tuples of functions from the Besov spaces on both face of a smooth hypersurface $\mathscr{S}$ with the smooth boundary $\Gamma = \partial\mathscr{S} \neq \emptyset$ in $\mathbb{R}^n$ into the ambient domain slit by the hypersurface $\mathbb{R}^n_{\mathscr{S}} := \mathbb{R}^n \setminus \overline{\mathscr{S}}$. These tuples satisfy a compatibility conditions on the boundary Γ. The traces are defined by arbitrary Dirichlet system of boundary operators and extension is performed by two different methods, one implicit and one explicit. Explicit extension is based on the solution to the Dirichlet BVP for the poly harmonic equation and permits the extension of distributions from the Besov space $\mathbb{B}^s_{p,p}(\mathscr{S})$ with a negative $s < 0$.

Coretractions have essential applications in boundary value problems for partial differential equations when, for example, it is necessary to reduce a BVP with non-homogeneous boundary conditions to a BVP with the homogeneous boundary conditions.

The extended version of the present paper will appear in [Du2].

Preliminaries

We distinguish two faces $\mathscr{S}^-$ and $\mathscr{S}^+$ of the hypersurface and the normal vector field $\boldsymbol{\nu}$ is directed, as usual, from the face $\mathscr{S}^+$ to $\mathscr{S}^-$. Let $\mathbb{R}^n_{\mathscr{C}} := \mathbb{R}^n \setminus \overline{\mathscr{C}}$ be the "cutted" space by a hypersurface $\mathscr{S}$. Let $\mathscr{S}$ be a subsurface of an equally smooth surface $\mathscr{C}$ without boundary $\partial\mathscr{C} = \emptyset$, which borders a compact inner domain Ω^+. Let $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$ be the complemented domain and $\boldsymbol{\nu}$-the outer unit normal vector field, which extends the existed field from the subsurface $\mathscr{S}$.

For the definition of the Bessel potential $\mathbb{H}^\theta_p(\mathscr{C})$, $\widetilde{\mathbb{H}}^\theta_p(\mathscr{S})$ and the Besov $\mathbb{B}^\theta_{p,p}(\mathscr{C})$, $\widetilde{\mathbb{B}}^\theta_{p,p}(\mathscr{S})$ spaces used below, we refer, e.g., to [Tr1].

For a pair of Besov spaces we introduce the following shortcut $\overset{2}{\mathbb{B}}{}^s_{p,p}(\mathscr{S}) := \mathbb{B}^s_{p,p}\mathscr{S}) \otimes \widetilde{\mathbb{B}}^s_{p,p}(\mathscr{S})$. The notation $[s]^- \in \mathbb{Z}$ is used for the largest positive or negative integer less than s, i.e., $s - 1 \leqslant [s]^- < s$.

A partial differential operator with matrix coefficients $\boldsymbol{A}(x, D)$ is called **normal** on $\mathscr{S}$ if it's homogeneous principal symbol is non-degenerated on the normal vector field $\inf |\det\mathscr{A}_{\rm pr}(\mathscr{x}, \boldsymbol{\nu}(\mathscr{x}))| \neq 0$ for all $\mathscr{x} \in \mathscr{S}$. Obviously, the class of elliptic operators $\inf |\det\mathscr{A}_{\rm pr}(\mathscr{x}, \xi))| \neq 0$ for all $\mathscr{x} \in \mathscr{S}$, and all $|\xi| = 1$, is a subclass of the class of the normal operators. For operators with constant symbols these classes coincide.

It is well known (see [Tr1]) that the trace operators $\gamma^\pm_{\mathscr{S}}\boldsymbol{B}(x, D)$ map the following spaces continuously

$$\gamma^\pm_{\mathscr{S}}\boldsymbol{B}(x, D)\ :\ \mathbb{H}^s_{p,loc}(\overline{\mathbb{R}^n_{\mathscr{S}}}) \longrightarrow \mathbb{B}^{s-\frac{1}{p}-r}_{p,p}(\mathscr{S}), \quad 1 < p < \infty\,,$$

*The investigation was supported by the grant of the Georgian National Science Foundation

provided $s > r + 1/p$ and the "jump" function meets the compatibility condition:

$$[\gamma_{\mathscr{S}}\boldsymbol{B}(x,D)\Phi] := \gamma^+_{\mathscr{S}}\boldsymbol{B}(x,D)\Phi - \gamma^-_{\mathscr{S}}\boldsymbol{B}(x,D)\Phi \in \widetilde{\mathbb{B}}^{s-\frac{1}{p}-r}_{p,p}(\mathscr{S}), \quad \forall\,\Phi \in \mathbb{H}^s_{p,loc}(\overline{\mathbb{R}^n_{\mathscr{S}}}).$$

The trace operator can be viewed as a continuous mapping by the operator pairs

$$\mathscr{R}_{\mathscr{S}}(\boldsymbol{B})\Phi := \left\{\gamma^+_{\mathscr{S}}\boldsymbol{B}(x,D)\Phi + \gamma^-_{\mathscr{S}}\boldsymbol{B}(x,D)\Phi, \gamma^+_{\mathscr{S}}\boldsymbol{B}(x,D)\Phi - \gamma^-_{\mathscr{S}}\boldsymbol{B}(x,D)\Phi\right\} \tag{1}$$

$$\mathscr{R}_{\mathscr{S}}(\boldsymbol{B}) \;:\; \mathbb{H}^s_{p,loc}(\overline{\mathbb{R}^n_{\mathscr{S}}}) \longrightarrow \mathbb{B}^{s-\frac{1}{p}-r}_{p,p}(\mathscr{S}) \otimes \widetilde{\mathbb{B}}^{s-\frac{1}{p}-r}_{p,p}(\mathscr{S}). \tag{2}$$

1 An implicit extension from a hypersurface

For a pair of spaces we introduce the following shortcut $\overset{2}{\mathbb{B}}{}^s_{p,p}(\mathscr{S}) := \mathbb{B}^s_{p,p}\mathscr{S}) \otimes \widetilde{\mathbb{B}}^s_{p,p}(\mathscr{S})$. The notation $[s]^- \in \mathbb{Z}$ is used for the largest positive or negative integer less than s, i.e., $s - 1 \leqslant [s]^- < s$.

Theorem 1.1 *Let $\boldsymbol{A}(x,D)$ be a PDO of order $k \in \mathbb{N}_0$ and of normal type, $1 < p < \infty$, $k \in \mathbb{N}_0$, $s > 0$, $k \leqslant s + 1$ and denote $p_s := p$ if $s \neq 0, \pm 1, \ldots$, $p_s < p$ if $s = 0, \pm 1, \ldots$.*

Further let $\vec{\boldsymbol{B}}^{(k)}(x,D) := \{\mathbf{B}_0(x,D), \ldots, \mathbf{B}_{k-1}(x,D)\}^\top$ be a Dirichlet system of boundary operators and $\left\{\varphi^\pm_j\right\}_{j=0}^{k-1}$ be vector functions such that

$$\Phi_j := (\varphi^+_j + \varphi^-_j, \varphi^+_j - \varphi^-_j) \in \overset{2}{\mathbb{B}}{}^{s-j}_{p,p}(\mathscr{S}), \qquad \text{for all} \quad j = 0, 1, \ldots, k-1. \tag{3}$$

Then there exists a continuous linear operator

$$\mathscr{P}_{\boldsymbol{A}} \;:\; \bigotimes_{j=0}^{k-1} \overset{2}{\mathbb{B}}{}^{s-j}_{p,p}(\mathscr{S}) \to \mathbb{H}^{s+1/p_s}_{p_s,loc}(\mathbb{R}^n_{\mathscr{S}}) \tag{4}$$

such that $\gamma_{\mathscr{S}^\pm}\mathbf{B}_j\mathscr{P}_{\boldsymbol{A}}\Phi = \varphi^\pm_j$ for $j = 0, 1, \ldots, k-1$ and $\boldsymbol{A}\mathscr{P}_{\boldsymbol{A}}\Phi \in \widetilde{\mathbb{H}}^{s-k+1/p_s}_{p_s,loc}(\mathbb{R}^n_{\mathscr{S}})$, where $\Phi := \left\{\Phi_j\right\}_{j=0}^{k-1}$.

Proof: First we extend the pairs $(\varphi^+_j + \varphi^-_j, \varphi^+_j - \varphi^-_j)$ to the closed hypersurface $\mathscr{C}$, which contains $\mathscr{S}$. The second components are extended by 0. The obtained system is then extended into Ω^+ and Ω^- by coretraction for closed surfaces (see [Tr1]) and, due to the conditions $\varphi^+_j - \varphi^-_j = 0$ on the complement surface $\mathscr{S}^c := \mathscr{C} \setminus \overline{\mathscr{S}}$ we prove that this particular extension is "continuous" across the complement surface $\mathscr{S}^c$. ∎

2 Perturbed poly-harmonic equations

The perturbed poly-harmonic (poly-Laplacian) operator

$$\boldsymbol{\Delta}^m_\omega\varphi := \boldsymbol{\Delta}^m\varphi + (-\omega^2)^m\varphi = (\mathrm{div}\nabla)^m\,\varphi + (-\omega^2)^m\varphi\,, \tag{5}$$

$$m = 1, 2, \ldots, \quad k > 0,$$

is elliptic, self-adjoint $(\boldsymbol{\Delta}^m_\omega)^* = \boldsymbol{\Delta}^m_\omega$ and is sign definite in the following sense for some $c > 0$. Then $\mathbf{\Delta}^m_\omega \ : \ \mathbb{H}^m(\mathbb{R}^n) \to \mathbb{H}^{-m}(\mathbb{R}^n)$ has the trivial kernel $\operatorname{Ker} \mathbf{\Delta}^m_\omega = \{0\}$ and cokernel $\operatorname{Coker} \mathbf{\Delta}^m_\omega = \{0\}$.

If $\mathscr{K}_{\mathbf{\Delta}^m_\omega}(x)$ denotes the fundamental solution of $\mathbf{\Delta}^m_\omega$, it decays at infinity fast

$$x^\beta \partial^\alpha \mathscr{K}_{\mathbf{\Delta}^m_\omega}(x) = \mathscr{O}\left(|x|^{|\alpha|+|\beta|} e^{-\omega|x|}\right) = \mathscr{o}(1) \qquad \text{as } |x| \to \infty, \ \forall\, \alpha, \beta \in \mathbb{N}^n. \quad (7)$$

Let

$$\boldsymbol{g}_j \in \mathbb{B}^{s-j-1/p}_{p,p}(\mathscr{C}), \quad j = 0, \dots, m-1, \quad f \in \widetilde{\mathbb{H}}^{s-2m}_p(\Omega), \qquad (8)$$
$$1 < p < \infty, \quad s \in \mathbb{R}$$

and look for a solution $\varphi \in \mathbb{H}^s_p(\Omega)$ of the boundary value problem

$$\begin{cases} (\mathbf{\Delta}^m_\omega \varphi)(x) = f(x), & x \in \Omega, \\ (\gamma_{\mathscr{C}} \boldsymbol{D}_j(x, D)\varphi)(\mathscr{x}) = \boldsymbol{g}_j(\mathscr{x}), & \mathscr{x} \in \mathscr{C}, \quad j = 0, \dots, m-1, \end{cases} \qquad (9)$$

where

$$\boldsymbol{D}_0(x, D) := I, \quad \boldsymbol{D}_j(x, D) := \begin{cases} -\mathbf{\Delta}^r & \text{if} \quad j = 2r, \\ \partial_{\boldsymbol{\nu}} \mathbf{\Delta}^r & \text{if} \quad j = 2r+1, \end{cases} \qquad (10)$$
$$j = 1, \dots, 2m-1$$

are the natural boundary operators from Green formula below (11) and (12).

The proof of the next Lemma 2.1 is standard, based on the Gauss formula.

Lemma 2.1 *For the operator $\mathbf{\Delta}^m_\omega$ in the domain $\Omega \subset \mathbb{R}^n$ with the smooth boundary $\mathscr{C} := \partial\Omega$ the following I and II Green formulae are valid:*

$$(\mathbf{\Delta}^m_\omega \varphi, \psi)_\Omega = \sum_{j=[m/2]}^{2m-1} ((\boldsymbol{D}_j(\mathscr{x}, D)\varphi)^+, (\boldsymbol{C}_j(\mathscr{x}, D)\psi)^+)_{\mathscr{C}} + \mathbb{A}(\varphi, \psi), \quad (11)$$

$$(\mathbf{\Delta}^m_\omega \varphi, \psi)_\Omega - (\varphi, \mathbf{\Delta}^m_\omega \psi)_\omega = \sum_{j=0}^{2m-1} ((\boldsymbol{D}_j(\mathscr{x}, D)\varphi)^+, (\boldsymbol{C}_j(\mathscr{x}, D)\psi)^+)_{\mathscr{C}} \quad (12)$$

for arbitrary $\varphi, \psi \in \mathbb{X}^{2m}(\Omega)$, where $\mathbb{A}(\varphi, \psi)$ is the bilinear form:

$$\mathbb{A}(\varphi, \psi) := \begin{cases} (\mathbf{\Delta}^\ell \varphi, \mathbf{\Delta}^\ell \psi)_\Omega + \omega^{2m}(\varphi, \psi)_\Omega & m = 2\ell, \\ -(\nabla \mathbf{\Delta}^\ell \varphi, \nabla \mathbf{\Delta}^\ell \psi)_\Omega - \omega^{2m}(\varphi, \psi)_\Omega & m = 2\ell + 1. \end{cases} \qquad (13)$$

For a solution $\varphi \in \mathbb{X}^m_p(\Omega)$ of the equation $\mathbf{\Delta}^m_\omega \varphi = f$ with $f \in \widetilde{\mathbb{X}}^{-m}_p(\Omega)$ the traces $\gamma_{\mathscr{C}} \mathbf{\Delta}^{\ell+j}\varphi$ and $\gamma_{\mathscr{C}} \partial_{\boldsymbol{\nu}} \mathbf{\Delta}^{\ell+j}\varphi$, $j = 0, 1, \dots, [(m+1)/2]$ exist and

$$\gamma_{\mathscr{C}} \mathbf{\Delta}^j \varphi \in \mathbb{B}^{m-2j-\frac{1}{p}}_{p,p}(\mathscr{C}), \qquad \gamma_{\mathscr{C}} \partial_{\boldsymbol{\nu}} \mathbf{\Delta}^j \varphi \in \mathbb{B}^{m-2j-1-\frac{1}{p}-1}_{p,p}(\mathscr{C}). \qquad (14)$$

Corollary 2.2 *A solution $\varphi \in \mathbb{H}^m_p(\Omega)$ to boundary value problem* (8)-(9) *is represented as follows:*

$$\varphi(x) = (\mathbf{N}_\Omega f)(x) + \sum_{j=0}^{2m-1} \boldsymbol{V}_j\big(\boldsymbol{D}_j(x, D)\varphi\big)^+(x), \qquad x \in \Omega, \qquad (15)$$

where

$$(\mathbf{N}_\Omega\varphi)(x) := \int\limits_\Omega \mathscr{K}_{\mathbf{\Delta}^m_\omega}(x-y)\varphi(y)\,dy \tag{16}$$

$$(\boldsymbol{V}_j\psi)(x) := \oint\limits_{\mathscr{C}} (\boldsymbol{C}_j(\mathscr{x},D)\mathscr{K}_{\mathbf{\Delta}^m_\omega})(\mathscr{x}-x)\psi(\mathscr{x})\,d\sigma \tag{17}$$

are, respectively, the volume (Newton) potential and the layer potentials (cf. [Du1]).

Theorem 2.3 *If solution $\varphi \in \mathbb{H}^m(\Omega)$ to BVP* (9)-(8) *for $p = 2$ and $s = m$ exists, it is unique.*

Dirichlet problem (9)-(8) *for $p = 2$ and $s = m$ has a unique solution $\varphi \in \mathbb{H}^m(\Omega)$ for arbitrary right-hand side $\boldsymbol{g}_j \in \mathbb{H}^{m-j-\frac{1}{2}}$, $j = 0, \dots, m-1$ and for $f \in \widetilde{\mathbb{H}}^{-m}(\Omega)$.*

Lemma 2.4 *$\mathbf{N}_\Omega$ is a ΨDO of order $-2m$ with a proper symbol, while $\boldsymbol{V}_j$ is the potential operator of order $-j-1/p$ and map the spaces*

$$\begin{aligned} \mathbf{N}_\Omega &: \widetilde{\mathbb{H}}^\theta_p(\Omega) \longrightarrow \mathbb{H}^{\theta-2m}_p(\Omega), \\ \boldsymbol{V}_j &: \mathbb{B}^\theta_{p,p}(\mathscr{C}) \longrightarrow \mathbb{H}^{\theta+j+\frac{1}{p}}_p(\Omega) \bigcap \mathbb{S}(\Omega) \qquad j = 0, 1, \dots, 2m-1 \end{aligned} \tag{18}$$

continuously for arbitrary $1 < p < \infty$ and $\theta \in \mathbb{R}$.

The following Plemelji formulae are valid

$$\begin{aligned} &[(\boldsymbol{D}_j\boldsymbol{V}_j\psi)]^\pm(\mathscr{x}) = \pm\frac{1}{2}\psi(\mathscr{x}) + (\boldsymbol{V}_{jj}\psi)(\mathscr{x}), \\ &[(\boldsymbol{D}_j\boldsymbol{V}_k\psi)]^-(\mathscr{x}) = (\boldsymbol{V}_{jk}\psi)(\mathscr{x}) = [(\boldsymbol{D}_j\boldsymbol{V}_k\psi)]^+(\mathscr{x}), \\ &\mathscr{x} \in \mathscr{C}, \quad j,k = 0,1,\dots,2m-1, \quad j \neq k, \quad \psi \in C(\mathscr{C}), \end{aligned} \tag{19}$$

where $\boldsymbol{V}_{jk}(\mathscr{x},D)$ is the direct value of the potential operator $\boldsymbol{D}_j(t,\partial)\boldsymbol{V}_k$ on the boundary $\mathscr{C}$. It represents a ΨDO of order $\operatorname{ord}\boldsymbol{V}_{jk}(\mathscr{x},D) = -2m+1+j+2m-1-k = j-k$, $j,k = 0,1,\dots,2m-1$.

Proof: For the standard proofs we refer to [Du1, DNS1]. QED

Theorem 2.5 *BVP* (9) *has a solution for arbitrary right-hand side f and data $\boldsymbol{g}_0, \dots, \boldsymbol{g}_{m-1}$ which satisfy conditions* (8)*, represented by formula*

$$\varphi(x) = (\mathbf{N}_\Omega f)(x) + \sum_{j=0}^{m-1}(\boldsymbol{V}_j g_j)(x) + + \sum_{j=m}^{2m-1}(\boldsymbol{V}_j\psi_j)(x), \qquad x \in \Omega \tag{20}$$

where the vector-function $\Psi := (\psi_0, \dots, \psi_{m-1})^\top \in \bigotimes_{j=0}^{m-1} \mathbb{B}^{s-m-j-1/p}_{p,p}(\mathscr{C})$ is a unique solution to the following boundary pseudodifferential system

$$\boldsymbol{W}(\mathscr{x},D)\Psi(\mathscr{x}) = \frac{1}{2}\boldsymbol{G}(\mathscr{x}) - \vec{\boldsymbol{D}}^{(m)}(x,D)N_\Omega\, f(\mathscr{x}) - \boldsymbol{W}_0\boldsymbol{G}(\mathscr{x}), \quad \mathscr{x} \in \mathscr{C}, \tag{21}$$

$$\begin{aligned} \boldsymbol{G} &:= (\boldsymbol{g}_0, \dots, \boldsymbol{g}_{m-1})^\top \in \bigotimes_{j=0}^{m-1} \mathbb{B}^{s-j-1/p}_{p,p}(\mathscr{C}), \\ \vec{\boldsymbol{D}}^{(m)}(x,D) &:= (\boldsymbol{D}_0(x,D), \dots, \boldsymbol{D}_{m-1}(x,D))^\top \end{aligned} \tag{22}$$

and W, W_0 are matrix operators

$$\boldsymbol{W} := \begin{bmatrix} \boldsymbol{V}_{0m} & \cdots & \boldsymbol{V}_{0(2m-1)} \\ \boldsymbol{V}_{1m} & \cdots & \boldsymbol{V}_{1(2m-1)} \\ \cdots & \cdots & \cdots \\ \boldsymbol{V}_{(m-1)m} & \cdots & \boldsymbol{V}_{(m-1)(2m-1)} \end{bmatrix}, \boldsymbol{W}_0 := \begin{bmatrix} \boldsymbol{V}_{00} & \cdots & \boldsymbol{V}_{0(m-1)} \\ \boldsymbol{V}_{10} & \cdots & \boldsymbol{V}_{1(m-1)} \\ \cdots & \cdots & \cdots \\ \boldsymbol{V}_{(m-1)0} & \cdots & \boldsymbol{V}_{(m-1)(m-1)} \end{bmatrix}.$$

If $s \geqslant m$, any solution φ to BVP (9) *is represented by formula* (20) *and the solution is unique.*

Proof: With the help of the Green formulae (11) and (12), invoking Lax-Milgram lemma, the result is proved first for $p = 2$ (see similar proofs in [DNS1, DS1]).

Then, using the representation formulae (18), Plemelji formulae (19) we derive equivalent boundary ΨDOs and prove their solvability for a general p and s.

■

3 Extension with potentials from a hypersurface

Based on Theorem 2.5 we can prove similar result for the BVP with general boundary conditions

$$\begin{cases} (\boldsymbol{\Delta}^m_\omega \varphi)(x) = f(x), & x \in \Omega, \\ (\gamma_{\mathscr{C}} \vec{\boldsymbol{B}}^{(m)}(x, D)\varphi)(\mathscr{x}) = \boldsymbol{G}(\mathscr{x}), & \mathscr{x} \in \mathscr{C}, \end{cases} \tag{23}$$

$$\boldsymbol{G} := (g_0, \ldots, g_{m-1})^\top \in \bigotimes_{j=0}^{m-1} \mathbb{B}^{s-j-1/p}_{p,p}(\mathscr{C}),$$

where $\vec{\boldsymbol{B}}^{(m)}\,(\boldsymbol{B}_0(x, D), \ldots, \boldsymbol{B}_{m-1}(x, D))^\top$ is a Dirichlet system of order m, ord $\boldsymbol{B}_j = j$, $j = 0, 1, \ldots, m-1$ and conditions (8) hold. Let

$$\vec{\boldsymbol{B}}^{(m)}(x, D) = \boldsymbol{M}^{(m\times m)}_{\boldsymbol{B},\boldsymbol{D}}(x, \mathscr{D})\vec{\boldsymbol{D}}^{(m)}(x, D)\,, \tag{24}$$

where $\vec{\boldsymbol{D}}^{(m)}(x, D)$ is defined in (21) and $\boldsymbol{M}^{(m\times m)}_{\boldsymbol{B},\boldsymbol{D}}(x, \mathscr{D})$ is a lower triangular invertible matrix operator (called **admissible**).

Let us explain what is meant under the *generalized traces* $\gamma^\pm_{\mathscr{C}}\psi = \psi^\pm \in \mathbb{B}^{\theta-\frac{1}{p}}_{p,p}(\mathscr{C})$ for distributions $\psi \in \mathbb{H}^\theta_p(\Omega^\pm)$ when $-\infty < \theta \leqslant \dfrac{1}{p}$. For this in a vicinity $\Omega^\pm_{\mathscr{C}} \subset \Omega^\pm$ of the surface $\mathscr{C}$ consider a local coordinate system $x = (\mathscr{x}, t) \in \Omega^\pm_{\mathscr{C}}$, where $\mathscr{x} \in \mathscr{C}$ and $0 \leqslant \pm t < \varepsilon$ is the transverse variable, equal to the signed distance from the reference point x to the surface $\mathscr{C}$. Then we define

$$\psi^\pm := \gamma_{\mathscr{S}^\pm}\psi \quad \text{if} \quad \lim_{0<\pm t\to 0} \big\|\psi(\cdot, t) - \psi^\pm(\cdot)\big|\mathbb{B}^{\theta-\frac{1}{p}}_{p,p}(\mathscr{C})\big\|. \tag{25}$$

Lemma 3.1 *The Plemelji formulae* (19) *are valid for $\psi \in \mathbb{B}^\theta_{p,p}(\mathscr{C})$ when $1 < p < \infty$ and $\theta \in \mathbb{R}$ are arbitrary.*

Theorem 3.2 *BVP* (23) *has a solution for arbitrary f and data $\boldsymbol{G}$, represented by the formula*

$$\varphi(x) = (\mathbf{N}_\Omega f)(x) + \sum_{j=0}^{m-1} (\boldsymbol{V}_j \widetilde{g}_j)(x) + \sum_{j=m}^{2m-1} (\boldsymbol{V}_j \psi_{j-m})(x)\,, \qquad x \in \Omega\,. \tag{26}$$

Here the vector $\Psi := (\psi_0, \ldots, \psi_{m-1})^\top \in \bigotimes_{j=0}^{m-1} \mathbb{B}^{s-m-j-1/p}_{p,p}(\mathscr{C})$ *is a unique solution to the following boundary pseudodifferential system*

$$\boldsymbol{W}(\mathscr{X}, D)\Psi(\mathscr{X}) = \frac{1}{2}\widetilde{\boldsymbol{G}}(\mathscr{X}) - \vec{\boldsymbol{D}}^{(m)}(x, D)N_\Omega f(\mathscr{X}) - \boldsymbol{W}_0\widetilde{\boldsymbol{G}}(\mathscr{X}), \qquad \mathscr{X} \in \mathscr{C}, \tag{27}$$

with the Ψ*DO* $\boldsymbol{W}(\mathscr{X}, D)$ *from* (21)

$$\widetilde{\boldsymbol{G}} := (\widetilde{g}_0, \ldots, \widetilde{g}_{m-1})^\top = \left[\boldsymbol{M}^{m\times m}_{\boldsymbol{B},\boldsymbol{D}}(\mathscr{X}, \mathscr{D})\right]^{-1}\boldsymbol{G} \in \bigotimes_{j=0}^{m-1} \mathbb{B}^{s-j-1/p}_{p,p}(\mathscr{C}). \tag{28}$$

If $s \geqslant m$*, any solution* φ *to BVP* (23) *is represented by formula* (26) *and the solution is unique.*

As we see, the space setting for the BVP (23) is also general and arbitrary $s \in \mathbb{R}$ can be taken, involving the Bessel potential spaces of distriburions for $s < 0$.

Applying this result, we can prove the following result on (explicit) continuation of the set of functions from a hypersurface.

Theorem 3.3 *Let* $m \in \mathbb{N}_0$, $\left\{\boldsymbol{\varphi}^\pm_j\right\}_{j=0}^{m-1}$ *be two* m*-tuples of functions (distributions, if* $s < 0$*)*

$$\Phi_j := (\boldsymbol{\varphi}^+_j + \boldsymbol{\varphi}^-_j, \boldsymbol{\varphi}^+_j - \boldsymbol{\varphi}^-_j) \in \overset{2}{\mathbb{B}}{}^{s-j}_{p,p}(\mathscr{S}), \qquad j = 0, 1, \ldots, m-1, \qquad 1 < p < \infty, \quad s \in \mathbb{R} \tag{29}$$

and and denote $p_s := p$ *if* $s \neq 0, \pm 1, \ldots$, $p_s < p$ *if* $s = 0, \pm 1, \ldots$.

For a given Dirichlet system of boundary operators

$$\vec{\boldsymbol{B}}^{(m)}(x, D) := \{\boldsymbol{B}_0(x, D), \ldots, \boldsymbol{B}_{m-1}(x, D)\}^\top \tag{30}$$

and the preassigned boundary data (29) *there exists an operator*

$$\mathscr{P} \ : \ \bigotimes_{j=0}^{m-1} \overset{2}{\mathbb{B}}{}^{s-j}_{p,p}(\mathscr{S}) \to \mathbb{H}^{s+1/p_s}_{p_s}(\mathbb{R}^n_{\mathscr{S}}) \cap \mathbb{S}(\mathbb{R}^n_{\mathscr{S}}) \tag{31}$$

which is a co-retraction $\gamma_{\mathscr{S}^\pm}\boldsymbol{B}_j\mathscr{P}\Phi = \boldsymbol{\varphi}^\pm_j$, $j = 0, 1, \ldots, m-1$ *to the trace operator* $\gamma_{\mathscr{C}}(\vec{\boldsymbol{B}}^{(m)}) := \{\gamma_{\mathscr{C}}\boldsymbol{B}_0(x, D), \ldots, \gamma_{\mathscr{C}}\boldsymbol{B}_{m-1}(x, D)\}^\top$.

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