Electromagnetic scattering by cylindrical orthotropic waveguide irises

L. P. Castro[1], R. Duduchava[2] and D. Kapanadze[2]

Abstract

The paper is devoted to the mathematical analysis of scattered time-harmonic electromagnetic waves by an infinitely long cylindrical orthotropic waveguide iris. This is modeled by an orthotropic Maxwell system in a cylindrical waveguide iris for plane waves propagating in the x_3-direction, imbedded in an isotropic infinite medium. The problem is equivalently reduced to a 2-dimensional boundary-contact problem with the operator $\operatorname{div} M \operatorname{grad} + k^2$ inside the domain and the (Helmholtz) operator $\Delta + k^2 = \operatorname{div}\operatorname{grad} + k^2$ outside the domain. Here M is a 2×2 positive definite, symmetric matrix with constant, real valued entries. The unique solvability of the appropriate boundary value problems is proved and the regularity of solutions is established in Bessel potential spaces.

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1 Introduction and formulation of the problem

We will consider the scattering of time-harmonic electromagnetic waves by an infinitely long cylindrical orthotropic waveguide iris with cross section Ω_+ with smooth boundary $\partial\Omega_+$. It is assumed that a thin sheet of a certain material is placed across some part of the boundary $S_1 \subset \partial\Omega_+$ to form what is known as an iris (cf. [DKM1], [Jo1, Section 4.10]); see Figure 1 below).

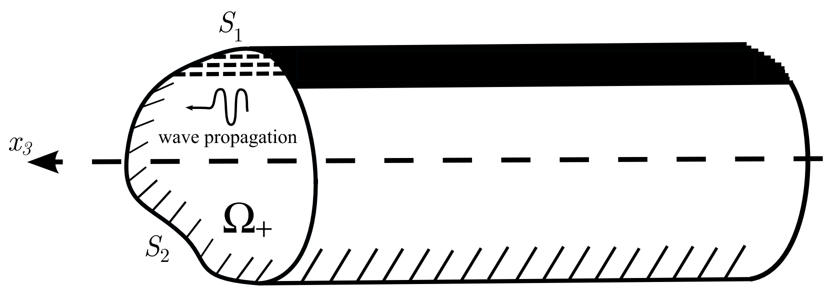

Figure 1: A cylindrical orthotropic waveguide iris

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It is well known that in anisotropic materials such as crystals, magneto-plasma, and ferrites which possess the anisotropic properties at optical or microwave frequencies, the direction of the electric field $\mathbf{E}$ (or the magnetic filed $\mathbf{H}$) usually does not coincide with that of the electric flux density (or the magnetic flux density).

To formulate the problem, we need some notation. The waveguide will be denoted by

$$\Omega_+^* := \Omega_+ \times \mathbb{R}_{x_3} = \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \Omega_+, x_3 \in \mathbb{R} \right\}.$$

Let $\Omega_-^* := \mathbb{R}^3 \setminus \overline{\Omega_+^*}$ be the complementary domain to the waveguide. According to the notation introduced above, we will also use $S_j^* := S_j \times \mathbb{R}_{x_3}$, $j = 1, 2$.

Denote by ϵ_0 and μ_0 the permittivity and permeability constants of the media outside the waveguide (i.e., in the exterior of the scatterer) and assume that the electric field $\mathbf{E}^-$ and the magnetic filed $\mathbf{H}^-$ satisfy the reduced isotropic Maxwell equations

$$\begin{cases} \operatorname{curl} \mathbf{E}^- - ik\mathbf{H}^- = 0, \\ \operatorname{curl} \mathbf{H}^- + ik\mathbf{E}^- = 0 \end{cases} \tag{1.1}$$

where the wave number $k > 0$ is defined by $k^2 \epsilon_0 \mu_0 \omega^2$, with frequency $\omega > 0$.

For the waveguide it is assumed that the electric field $\mathbf{E}^+$ and the magnetic field $\mathbf{H}^+$ satisfy the following reduced Maxwell equations

$$\begin{cases} \operatorname{curl} \mathbf{E}^+ - ik\mathbf{H}^+ = 0, \\ \operatorname{curl} \mathbf{H}^+ + ikN\mathbf{E}^+ = 0 \end{cases} \tag{1.2}$$

where the reactive index N is a matrix given by

$$N = \frac{1}{\epsilon_0} \epsilon$$

with ϵ being the electric permittivity. We assume, for simplicity, that the electric conductivity is equal to 0 and the electric permittivity ϵ is a constant symmetric matrix with real entries. From the energy conservation principle it follows that ϵ (and thus N) is positive definite, cf. [BDS1].

We consider the case where the permittivity tensor is represented by a 3×3 constant matrix. Similarly, it is possible to relate the magnetic flux density and the magnetic field intensity in terms of tensor permeability represented by a 3×3 constant matrix, e.g., as in the case of certain magnetic ferrites. This type of problems can be treated analogously without any significant changes in our approach, and therefore, we restrict ourselves to the case where the electric permittivity ϵ is a constant symmetric matrix with real entries.

Now let us assume that the waveguide is orthotropic, i.e., the matrix N is of the form

$$N = \begin{pmatrix} n_{11} & n_{12} & 0 \\ n_{21} & n_{22} & 0 \\ 0 & 0 & n_{33} \end{pmatrix},$$

which is positive definite $\langle N\xi,\xi\rangle \geq c|\xi|^2$ for all $\xi \in \mathbb{R}^3$ and some $c > 0$, and the electromagnetic fields are independent of the x_3-variable.

For a given bounded or unbounded domain $\mathscr{D} \subset \mathbb{R}^n$ and $s > 1/2$ denote by $H^s(\mathscr{D})$ the standard Bessel potential space of smoothness s on the domains $\mathscr{D}$. The Bessel potential space $TH^{s-1/2}(\partial\mathscr{D})^3$ of tangent vector fields of smoothness $s-1/2$ on the surface $\partial\mathscr{D}$ is defined as follows

$$TH^{s-1/2}(\partial\mathscr{D})^3 := \{\boldsymbol{\nu} \times \mathbf{A}|_{\partial\mathscr{D}} : \mathbf{A} \in H^s(\mathscr{D})^3\},$$

where $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)^\top$ denotes the unit normal vector field to $\partial\mathscr{D}$. Similarly we define the space $TH^{-s+1/2}(\partial\mathscr{D})^3$ which is the dual space to $TH^{s-1/2}(\partial\mathscr{D})^3$.

The notation $\widetilde{H}^s(S)$ is used for the closed subspace of $H^s(\partial\mathscr{D})$ comprising functions (distributions) supported inside a subsurface $S \subset \partial\mathscr{D}$ endowed with the subspace topology. The space $H^s(S)$ comprises the restrictions $r_S\varphi$ of functions (distributions) $\varphi \in H^s(\partial\mathscr{D})$ to the subsurface S and is endowed with the quotient norm of $H^s(\partial\mathscr{D})/\widetilde{H}^s(S^c)$, where S^c is the complementary surface (cf. [Tr1] for details).

Finally note that the space $H^s_{\mathrm{loc}(x_3)}(\Omega^*_\pm)$ consists of those vector-functions (didstributions) Φ for which $\omega\Phi \in H^s(\Omega^*_\pm)$ for arbitrary smooth cut-off function in the third variable $\omega = \omega(x_3) \in C^\infty_0(\Omega^*_\pm)$

3D Problem Formulation: Let an electromagnetic wave $\mathbf{E}^+_{\mathrm{o}}$, $\mathbf{H}^+_{\mathrm{o}}$ propagate through the cylindrical orthotropic waveguide iris Ω^*_+ and generate the fields $\mathbf{E}^-_{\mathrm{t}}$, $\mathbf{H}^-_{\mathrm{t}}$ which are transmitted outside the waveguide. The fields $\mathbf{E}^+_{\mathrm{o}}$, $\mathbf{H}^+_{\mathrm{o}}$ satisfy the anisotropic Maxwell equations (1.2) in Ω^*_+, and $\mathbf{E}^-_{\mathrm{t}}$, $\mathbf{H}^-_{\mathrm{t}}$ satisfy (1.1) in Ω^*_- and the Silver–Müller radiation condition at infinity [Mu1, Si1]

$$\lim_{|(x_1,x_2,0)^\top|\to\infty} ((x_1,x_2,0)^\top \times \mathbf{E}(x) - \mathbf{H}(x)) = 0 \quad \text{for all} \quad x = (x_1,x_2,x_3) \in \mathbb{R}^3$$

uniformly in all directions $(x_1,x_2,0)^\top/|(x_1,x_2,0)^\top|$.

In addition, it is assumed that an incident plane wave $\mathbf{E}^-_{\mathrm{i}}$, $\mathbf{H}^-_{\mathrm{i}}$ hits the waveguide iris Ω^*_+ from the complementary domain Ω^*_-. As a result, the waveguide iris generates in Ω^*_- a scattered field $\mathbf{E}^-_{\mathrm{s}}$, $\mathbf{H}^-_{\mathrm{s}}$ which has to satisfy (1.1) and the radiation condition at infinity. The fields $\mathbf{E}^+_{\mathrm{t}}$, $\mathbf{H}^+_{\mathrm{t}}$ transmitted inside the waveguide iris Ω^*_+ satisfy the corresponding Maxwell equations (1.2) in Ω^*_+.

The tangential components of the total electromagnetic fields $\mathbf{E}^- = \mathbf{E}^-_{\mathrm{i}} + \mathbf{E}^-_{\mathrm{s}} + \mathbf{E}^-_{\mathrm{t}}$, $\mathbf{H}^- = \mathbf{H}^-_{\mathrm{i}} + \mathbf{H}^-_{\mathrm{s}} + \mathbf{H}^-_{\mathrm{t}}$ outside the waveguide and $\mathbf{E}^+ = \mathbf{E}^+_{\mathrm{o}} + \mathbf{E}^+_{\mathrm{t}}$, $\mathbf{H}^+ = \mathbf{H}^+_{\mathrm{o}} + \mathbf{H}^+_{\mathrm{t}}$ inside the waveguide are continuous across S^*_2, i.e.,

$$\left.\begin{aligned} \boldsymbol{\nu} \times (\mathbf{E}^-_{\mathrm{i}} + \mathbf{E}^-_{\mathrm{s}} + \mathbf{E}^-_{\mathrm{t}}) &= \boldsymbol{\nu} \times (\mathbf{E}^+_{\mathrm{o}} + \mathbf{E}^+_{\mathrm{t}}), \\ \boldsymbol{\nu} \times (\mathbf{H}^-_{\mathrm{i}} + \mathbf{H}^-_{\mathrm{s}} + \mathbf{H}^-_{\mathrm{t}}) &= \boldsymbol{\nu} \times (\mathbf{H}^+_{\mathrm{o}} + \mathbf{H}^+_{\mathrm{t}}) \end{aligned}\right. \qquad \text{on} \quad S^*_2.$$

Across the part S^*_1 of the boundary the solution satisfies the impedance boundary conditions:

$$\boldsymbol{\nu} \times (\mathbf{E}^-_{\mathrm{i}} + \mathbf{E}^-_{\mathrm{s}} + \mathbf{E}^-_{\mathrm{t}}) - \lambda_-\, \boldsymbol{\nu} \times (\boldsymbol{\nu} \times (\mathbf{H}^-_{\mathrm{i}} + \mathbf{H}^-_{\mathrm{s}} + \mathbf{H}^-_{\mathrm{t}})) = 0 \qquad \text{on} \quad S^*_1$$

and

$$\boldsymbol{\nu} \times (\mathbf{E}_{\rm o}^+ + \mathbf{E}_{\rm t}^+) - \lambda_+ \boldsymbol{\nu} \times (\mathbf{H}_{\rm o}^+ + \mathbf{H}_{\rm t}^+) = 0 \qquad \text{on} \quad S_1^*,$$

where $\lambda_\pm$ are the surface impedance parameters defined on the corresponding faces of S_1^*.

Note that in this paper we consider the polarized electromagnetic fields which are independent of the x_3-coordinate. For example, a plane wave is a constant-frequency wave whose wavefronts (surfaces of constant phase) are infinite parallel planes of constant amplitude normal to the phase velocity vector $\mathbf{k}^{\rm inc} \in \mathbb{R}^3$:

$$\mathbf{U}(\mathbf{x}, t) = \mathbf{A} e^{i(\mathbf{k}^{\rm inc} \cdot \mathbf{x} - \omega t)}, \qquad \mathbf{x} = x_1 \mathbf{e}^1 + x_2 \mathbf{e}^2 + x_3 \mathbf{e}^3 \in \mathbb{R}^3.$$

Here $\mathbf{A} = A_1 \mathbf{e}^1 + A_2 \mathbf{e}^2 + A_3 \mathbf{e}^3 \in \mathbb{R}^3$ is the (complex) amplitude vector. Note that if the unit phase velocity vector is independent of the x_3 variable $\mathbf{k}^{\rm inc} = \sin\theta \mathbf{e}^1 + \cos\theta \mathbf{e}^2$, then the scalar product $\mathbf{k}^{\rm inc} \cdot \mathbf{x}$ is also independent of x_3 and, consequently, all three components of the wave $\mathbf{U} = (U_1, U_2, U_3)^\top = U_1 \mathbf{e}^1 + U_2 \mathbf{e}^2 + U_3 \mathbf{e}^3$ are independent of x_3.

Polarization is a property of waves that describes the orientation of their oscillations. For transverse waves such as many electromagnetic waves, it describes the orientation of oscillations in the plane perpendicular to the wave's direction of travel. Oscillations may be oriented in a single direction (linear polarization), or the oscillation direction may rotate as the wave travels (circular or elliptical polarization). For longitudinal waves such as sound waves in fluids the direction of oscillation is, by definition, along the direction of travel. Guided modes in waveguides and optical fibers can carry waves with both transverse and longitudinal oscillations. Such waves do have polarization.

The above assumption about plane waves propagating in the $\mathbf{e}^3$-direction leads to two groups of scalar equations which are known as TM mode and TE mode, respectively, cf. (2.3) and (2.4). Note also that the electromagnetic fields $\mathbf{E}_{\rm o}^+, \mathbf{H}_{\rm o}^+ \in H^1_{{\rm loc}(x_3)}(\Omega_+^*)^3$, $\mathbf{E}_{\rm i}^-, \mathbf{H}_{\rm i}^- \in H^1_{{\rm loc}(x_3)}(\Omega_-^*)^3$ are given while the electromagnetic fields $\mathbf{E}_{\rm i}^-$, $\mathbf{H}_{\rm i}^-$, $\mathbf{E}_{\rm t}^-$, $\mathbf{H}_{\rm t}^- \in H^1_{{\rm loc}(x_3)}(\Omega_-^*)^3$, $\mathbf{E}_{\rm i}^+$, $\mathbf{H}_{\rm i}^+ \in H^1_{{\rm loc}(x_3)}(\Omega_+^*)^3$ should be found. Note also that $\mathbf{E}_{\rm i}^- = 0$, $\mathbf{H}_{\rm i}^- = 0$ imply that the scattered and the transmitted fields vanish identically, and $\mathbf{E}_{\rm o}^+ = 0$, $\mathbf{H}_{\rm o}^+ = 0$ imply $\mathbf{E}_{\rm t}^- = 0$, $\mathbf{H}_{\rm t}^- = 0$.

Theorem 1.1 *Let* $0 \le \varepsilon < \frac{1}{2}$. *If* $\mathbf{E}_{\rm o}^+, \mathbf{H}_{\rm o}^+ \in H^{1+\varepsilon}_{{\rm loc}(x_3)}(\Omega_+^*)^3$, $\mathbf{E}_{\rm i}^-, \mathbf{H}_{\rm i}^- \in H^{1+\varepsilon}_{{\rm loc}(x_3)}(\Omega_-^*)^3$, *and the above mentioned Silver–Müller radiation condition holds, then the 3D problem has a unique solution* $\mathbf{E}_{\rm i}^-$, $\mathbf{H}_{\rm i}^-$, $\mathbf{E}_{\rm t}^-$, $\mathbf{H}_{\rm t}^- \in H^{1+\varepsilon}_{{\rm loc}(x_3)}(\Omega_-^*)^3$, $\mathbf{E}_{\rm i}^+$, $\mathbf{H}_{\rm i}^+ \in H^{1+\varepsilon}_{{\rm loc}(x_3)}(\Omega_+^*)^3$ *representable by vector potentials.*

The proof of Theorem 1.1 is divided into several steps. we prove separately the uniqueness and the existence of a solution, its continuous dependence on the data and the representation formula for a solution. This will be done first for $\varepsilon = 0$; later we prove improved regularity of the solution for $0 < \varepsilon < 1/2$ in corresponding Bessel potential spaces.

2 Reduction to the two-dimensional case

For polarized vector fields (for plane waves) it is possible to equivalently reduce 3D problems to 2D problems with respect to the third components of the electromagnetic field.

First note that under the assumptions made in the preceding section the original system (1.2) splits into two systems of scalar equations

$$\begin{aligned} \partial_{x_1} E_3 &= -ikH_2 \\ \partial_{x_2} E_3 &= ikH_1 \\ \partial_{x_1} H_2 - \partial_{x_2} H_1 &= -ikn_{33}E_3 \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} \partial_{x_1} H_3 &= ik(n_{21}E_1 + n_{22}E_2) \\ \partial_{x_2} H_3 &= -ik(n_{11}E_1 + n_{12}E_2) \\ \partial_{x_1} E_2 - \partial_{x_2} E_1 &= ikH_3\,. \end{aligned} \tag{2.4}$$

Equations (2.3) only involve H_1, H_2 and E_3 and describe the scattering problem for an electromagnetic wave polarized perpendicular to the x_3-axis. Writing now system (2.3) in the matrix form we get

$$\begin{aligned} &\operatorname{grad} E_3 = -ik \begin{pmatrix} H_2 \\ -H_1 \end{pmatrix}, \\ &\operatorname{div} \begin{pmatrix} H_2 \\ -H_1 \end{pmatrix} = -ikn_{33}E_3, \end{aligned} \tag{2.5}$$

where $\operatorname{grad} u := (\partial_{x_1} u, \partial_{x_2} u)^\top$ is the gradient and $\operatorname{div} U = \partial_{x_1} U_1 + \partial_{x_2} U_2$ is the divergence in two independent variables $(x_1, x_2)^\top \in \mathbb{R}^2$. By excluding $(H_2, -H_1)^\top$ from equations (2.5) we obtain the Helmholtz equation with respect to $v := E_3$

$$\operatorname{div}\operatorname{grad} v + k^2 n_{33} v = \Delta\, v + k^2 n_{33} v = 0, \tag{2.6}$$

which is well-investigated in the mathematical literature (cf. [CK1, Kr1]).

Writing now the system (2.4) in matrix form we get

$$\begin{aligned} &\operatorname{grad} H_3 = ik \begin{pmatrix} n_{22} & -n_{21} \\ -n_{12} & n_{11} \end{pmatrix} \begin{pmatrix} E_2 \\ -E_1 \end{pmatrix}, \\ &\operatorname{div} \begin{pmatrix} E_2 \\ -E_1 \end{pmatrix} = ikH_3. \end{aligned} \tag{2.7}$$

By excluding E_1 and E_2 from equations (2.7) we obtain the equation with respect to $u = H_3$:

$$\operatorname{div} M \operatorname{grad} u + k^2 u = 0, \tag{2.8}$$

where the matrix M is given by

$$M = \begin{pmatrix} n_{22} & -n_{21} \\ -n_{12} & n_{11} \end{pmatrix}^{-1} = \frac{1}{n_{11}n_{22} - n_{21}n_{12}} \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix}. \tag{2.9}$$

The matrix M in (2.9) is positive definite since N is also positive definite. Thus, there exists the unique symmetric and positive definite square root $M^{\frac{1}{2}}$ of M. Let v be a solution of the Helmholtz equation

$$\Delta v + k^2 v = 0. \tag{2.10}$$

It is quite simple to check that the function

$$u(x) := v(M^{-\frac{1}{2}}x) \tag{2.11}$$

is a solution of equation (2.8) (cf. [Kr1, § 4.1.8]). Moreover, since

$$\delta(M^{-\frac{1}{2}}x) = \det\, M^{\frac{1}{2}}\delta(x) = \sqrt{\det\, M}\delta(x)$$

and the fundamental solution of the Helmholtz equation in $\mathbb{R}^2$ is

$$\Gamma_0(x) = -\frac{i}{4}\, H_0^{(1)}(k|x|), \tag{2.12}$$

where $H_0^{(1)}(r)$ is the Hankel function of first kind of order 0, then the fundamental solution of (2.8) is

$$\Gamma(x) = -\frac{i}{4\sqrt{\det\, M}} H_0^{(1)}(k|M^{-\frac{1}{2}}x|)\,. \tag{2.13}$$

To avoid eventual confusion, we preserve the notation $\boldsymbol{\nu} = (\nu_1, \nu_2, 0)^\top$ for the 3-dimensional outer unit normal vector to Ω_+^* and denote by $\mathbf{n} = (\nu_1, \nu_2)^\top$ the 2-dimensional exterior unit normal vector to Ω_+ (which is the projection of $\boldsymbol{\nu}$ on the hyperplane spanned by the coordinate vectors $\mathbf{e}^1$, $\mathbf{e}^2$).

From the divergence theorem we obtain the Green formula

$$\int_{\Omega_+} A(D)u\overline{v}\, dx + \int_{\Omega_+} \left[\langle M\mathrm{grad}\, u, \overline{\mathrm{grad}\, v}\rangle - k^2 u\, \overline{v}\right] dx = \int_S T(D, \mathbf{n})u\, \overline{v}\, ds \tag{2.14}$$

for all $u \in H^1(\Omega_+)$, $v \in H^1(\Omega_+)$; here the operator

$$A(D)u := \mathrm{div} M\mathrm{grad}\, u + k^2 u \tag{2.15}$$

appears on the left-hand side of equation (2.8), $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^2$ and

$$\begin{gathered} T(D, \mathbf{n}(t))\varphi(t) := \langle \mathbf{n}(t), M\mathrm{grad}\, \varphi(t)\rangle\,, \\ t \in S := S_1 \cup S_2\,, \quad \varphi \in H^1(\Omega_+) \end{gathered} \tag{2.16}$$

is a “natural” boundary operator.

Remark 2.1 *If $u \in H^1(\Omega_+)$ is a solution of equation* (2.8)*, the left-hand side in* (2.14) *exists for arbitrary $v \in H^1(\Omega_+)$. Then the right-hand side in* (2.14) *is defined correctly and, by the classical trace theorem, $v^+ := v\big|_S \in H^{\frac{1}{2}}(S)$. From the duality argument it follows that the trace $[T(D, \mathbf{n})u]^+ := T(D, \mathbf{n})u\big|_S$ on the boundary exists and is an element of $H^{-\frac{1}{2}}(S)$.*

Lemma 2.2 *Let* $u \in H^1(\Omega_+)$ *be a solution to equation* (2.8) *in* Ω_+. *Then*

$$u(x) = \int_S \{u(t)T(D,\mathbf{n}(t))\Gamma(x-t) - T(D,\mathbf{n}(t))u(t)\,\Gamma(x-t)\}\,ds(t), \quad x \in \Omega_+. \tag{2.17}$$

Proof: The representation formula (2.17) is proved by a standard approach, similarly to the case of the Helmholtz equation.
■

Note that due to equation (1.1) the condition $\boldsymbol{\nu} \times \mathbf{E} = \mathbf{f}$, which is equivalent to the condition

$$\boldsymbol{\nu} \times N^{-1}\mathrm{curl}\,\mathbf{H} = -ik\mathbf{f}, \tag{2.18}$$

where $\boldsymbol{\nu} = (\nu_1,\nu_2,0)^\top$ and $\mathbf{f} = (f_1,f_2,f_3)^\top \in TH^{-\frac{1}{2}}(S^*)$, and relations (2.7), (2.9) imply the following boundary condition for H_3

$$\langle \mathbf{n}, M\mathrm{grad}\,H_3\rangle\langle(\nu_1,\nu_2)^\top, M\mathrm{grad}\,H_3\rangle = ikf_3.$$

Thus, the boundary condition (2.18) is equivalently transformed to a Neumann condition (cf. (2.16)) for equation (2.8). Arguing similarly, we find that the boundary condition

$$\boldsymbol{\nu} \times \mathbf{H} = \mathbf{f}, \qquad \mathbf{f} \in TH^{\frac{1}{2}}(S^*) \tag{2.19}$$

is equivalently transformed to the Dirichlet condition for H_3:

$$H_3 = \frac{f_1}{\nu_2} \quad \text{if} \quad \nu_2 \neq 0 \qquad \text{and} \qquad H_3 = -\frac{f_2}{\nu_1} \quad \text{if} \quad \nu_2 = 0.$$

These observations lead to the following reformulation of the problem.

2D Problem Formulation: Find elements $u^\pm \in H^1(\Omega_\pm)$ which satisfy

$$(\Delta + k^2)u^- = 0 \quad \text{in} \quad \Omega_- \tag{2.20}$$

$$(\mathrm{grad}\,M\,\mathrm{div} + k^2)u^+ = 0 \quad \text{in} \quad \Omega_+ \tag{2.21}$$

and the boundary conditions

$$u^- - u^+ = g_0, \quad \frac{\partial u^-}{\partial \mathbf{n}} - T(D,\mathbf{n})u^+ = g_1 \quad \text{on} \quad S_2, \tag{2.22}$$

$$\frac{\partial u^-}{\partial \mathbf{n}} + ip^-u^- = h_0, \quad T(D,\mathbf{n})u^+ - ip^+u^- = h_1 \quad \text{on} \quad S_1, \tag{2.23}$$

where $g_j \in H^{\frac{1}{2}-j}(S_2)$, $h_j \in H^{-\frac{1}{2}}(S_1)$, $j = 0,1$, and $p^\pm \in \mathbb{C}$.

We should add the compatibility condition on the surface

$$h_0 - h_1 \in \widetilde{H}^{-\frac{1}{2}}(S_1) \tag{2.24}$$

and the Sommerfeld radiation condition at infinity $u^- \in \mathrm{Som}(\Omega_-)$:

$$\frac{\partial}{\partial|x|}u^-(x) - i|k|u^-(x) = \mathscr{O}(|x|^{-\frac{3}{2}}), \quad |x| = |(x_1, x_2)| \to \infty \tag{2.25}$$

uniformly in all directions $x/|x|$ (which takes the role of the Silver-Müller radiation condition mentioned in the 3D formulation).

From now on we refer to the boundary value problem (2.20)-(2.24) endowed with the Sommerfeld radiation condition at infinity (2.25), as the Problem $\mathscr{P}$.

Lemma 2.3 *Consider the orthotropic Maxwell system* (1.2) *in the cylindrical waveguide iris* Ω_+^* *for plane waves propagating in the* e^3*-direction. Search for solutions which belong to the Sobolev spaces*

$$\mathbf{E} = (E_1, E_2, E_3)^\top \in H^1(\Omega_+)^3 \qquad \textit{and} \qquad \mathbf{H} = (H_1, H_2, H_3)^\top \in H^1(\Omega_+)^3$$

in the cross-sections

$$\Omega_+ := \left\{x = (x_1, x_2, x_3)^\top \in \Omega_+^* \;:\; x_3 = \mathrm{const}\right\} .$$

System (1.2) *splits into equivalent system of scalar equations* (2.6) *and* (2.8) *for components* $v = E_3 \in H^1(\Omega_+)$ *and* $u = H_3 \in H^1(\Omega_+)$*, the solutions are independent of the variable* x_3 *and the full solutions* $\mathbf{E}$ *and* $\mathbf{H}$ *are uniquely recovered by formulae* (2.5) *and* (2.7).

Proof: The algebraic equivalence of system (1.2) and of the system of scalar equations (2.6) and (2.8), including the full solution recovery formulae (2.5) and (2.7). has been proved above.

It is also clear that if $\mathbf{E}^+, \mathbf{H}^+ \in H^1_{\mathrm{loc}(x_3)}(\Omega_+^*)^3$ (independent of the variable x_3) are any solutions of the 3D problem, then their third components $E_3, H_3 \in H^1(\Omega_+)$ are solutions of the corresponding 2D problems, respectively.

To show the converse assertion, first note that with the help of the reduced 2D problem the third components of the electromagnetic fields $E_3, H_3 \in H^1(\Omega_+)$ are found, which help to recover the full solutions $\mathbf{E} = (E_1, E_2, E_3)^\top$ and $\mathbf{H} = (H_1, H_2, H_3)^\top$ of the 3D problem. Moreover, an easy calculation shows that the boundary conditions are then met automatically, since the data on the boundary S are tangent (they appear as the vector product with the normal vector).

Finally, note that since the boundary S is smooth the boundary condition

$$\boldsymbol{\nu} \times \mathbf{A} \in TH^{\frac{1}{2}}(S^*)^3 \subset H^{\frac{1}{2}}(S^*)^3$$

implies $\mathbf{A} \in H^1_{\mathrm{loc}(x_3)}(\Omega_+^*)^3$ (cf. [ABDG1, Remark 2.14], [GP1], [Co1]). In its turn this ensures that all components of the source electromagnetic fields fall into $H^1(\Omega_+)$ (cf. the boundary conditions in the 3D problem). ∎

Note also that for M being the identity matrix, which corresponds to the isotropic Maxwell system (1.1) in Ω_-^*, a similar result is a trivial consequence of Lemma 2.3.

These observations lead us to the main conclusion of this section: the 3D problem formulated above for polarized vector fields (for plane waves) is equivalently reduced to the corresponding 2D problem with respect to the third components of the electromagnetic field.

3 Uniqueness theorem

The following lemma is a special case of a more general result for partial differential equations known as Holmgren's theorem.

Lemma 3.1 *Let $u^+ \in H^1(\Omega_+)$ be a solution of* (2.21) *in Ω_+ and*

$$u^+ = \frac{\partial u^+}{\partial \mathbf{n}} = 0 \quad \textit{on } S_o$$

for some open subset $S_o \subset \partial\Omega_+$. Then u vanishes identically in Ω_+.

Remark 3.2 *If u is a solution to a second order elliptic equation $\mathbf{A}(x, D)u = 0$ with Lipschitz continuous top order coefficients on the domain Ω and vanishes to infinite order at one point of Ω, then it vanishes identically everywhere on Ω.*

The result was proved in [AKS1] for a multidimensional domain $\Omega \subset \mathbb{R}^n$. via the method of "Carleman estimates". Another proof, involving the monotonicity of a frequency function was given by N. Garofalo and F. Lin (see [GL1, GL2]). For equations with real analytic (with constant) coefficients the result follows from Holmgren's uniqueness theorem (cf. [Ho1, Theorem 8.6.5]).

Theorem 3.3 *If $\Re\mathrm{e}\, p^\pm \geq 0$, then the Problem $\mathscr{P}$ has a unique solution.*

Proof: To establish the announced uniqueness, we will show that the homogeneous boundary conditions ($g_j = h_j = 0$, $j = 1, 2$) imply that $u_\pm$ vanish identically. Let R be a sufficiently large positive number such that $\overline{\Omega_+} \subset G_R$, where G_R is the disk centered at the origin with radius R. Then the Green's formula yields

$$\int_{\Omega_+} \left[\langle M^{\frac{1}{2}} \mathrm{grad}\, u^+, M^{\frac{1}{2}} \mathrm{grad}\, \overline{u^+}\rangle - k^2 |u^+|^2\right] dx = \int_{\partial\Omega_+} T(D, \mathbf{n}) u^+\, \overline{u^+}\, ds \tag{3.26}$$

and

$$\int_{G_R \cap \Omega_-} \left[|\nabla u^-|^2 - k^2 |u^-|^2\right] dx = -\int_{\partial\Omega_+} \frac{\partial u^-}{\partial \mathbf{n}} \overline{u^-}\, ds + \int_{\partial G_R} \frac{\partial u^-}{\partial \mathbf{n}} \overline{u^-}\, ds. \tag{3.27}$$

Then, by summing (3.26) and (3.27) and using (2.22)–(2.23), we obtain

$$\int_{\Omega_+} \left[\langle M^{\frac{1}{2}} \mathrm{grad}\, u^+, M^{\frac{1}{2}} \mathrm{grad}\, \overline{u^+}\rangle - k^2 |u^+|^2\right] dx + \int_{G_R \cap \Omega_-} \left[|\nabla u^-|^2 - k^2 |u^-|^2\right] dx$$

$$= \int_{S_1} T(D,\mathbf{n})u^+\,\overline{u^+}\,ds - \int_{S_1} \frac{\partial u^-}{\partial \mathbf{n}}\,\overline{u^-}\,ds + \int_{\partial G_R} \frac{\partial u^-}{\partial \mathbf{n}}\,\overline{u^-}\,ds$$

$$= ip^+\langle u^+, u^+\rangle_{S_1} + ip^-\langle u^-, u^-\rangle_{S_1} + \int_{\partial G_R} \frac{\partial u^-}{\partial \mathbf{n}}\,\overline{u^-}\,ds \tag{3.28}$$

Since R is assumed to be sufficiently large, we can apply the radiation condition on the circle ∂G_R. Let us now detach the imaginary part of the equation (3.28) and use the fact that $u(x) = \mathscr{O}(|x|^{-\frac{1}{2}})$ as $|x| \to \infty$. Then we obtain

$$\Re\mathrm{e}\, p^+\langle u^+, u^+\rangle_{S_1} + \Re\mathrm{e}\, p^-\langle u^-, u^-\rangle_{S_1} + |k| \int_{x\in\partial G_R} |u(x)|^2 ds = \mathscr{O}(R^{-1}) \quad \text{as } R \to \infty, \tag{3.29}$$

which yields

$$\lim_{R\to\infty} \int_{\partial G_R} |u|^2\, dS = 0\ .$$

due to the conditions $\Re\mathrm{e}\, p^{\pm} \geq 0$. Therefore, from the Rellich-Vekua theorem it follows that $u^- = 0$ in Ω_- [Ve1]. This together with (3.29) and the homogeneous boundary conditions imply that $u^+ = 0$ and $T(D,\mathbf{n})u^+ = 0$ on $\partial\Omega_+$. Then Lemma 3.1 gives $u^+ = 0$ in Ω_+. ∎

Remark 3.4 *It is easy to verify that the uniqueness result remains valid even if $S_1 = \emptyset$ or $S_2 = \emptyset$, i.e., if we have either a pure transmission or a pure impedance boundary condition on the entire boundary S.*

4 Layer potentials

Let us start with the case of the Helmholtz equation (2.10). Analogous results for equation (2.8) can be easily derived by a similar approach (cf. (2.11)).

Recall that $\Gamma_0(x)$ denotes a standard fundamental solution to the Helmholtz equation in two dimensions (2.8), satisfying the Sommerfeld radiation condition (2.25) at infinity.

The corresponding single and double layer potentials are of the form

$$V(\psi)(x) = \int_S \Gamma_0(x-y)\psi(y)dS\ , \quad x \notin S\ ,$$

$$W(\varphi)(x) = \int_S [\partial_{\mathbf{n}(y)}\Gamma_0(x-y)]\varphi(y)dS\ , \quad x \notin S\ ,$$

where ψ and φ are density functions.

Note that by standard arguments for Green identities we obtain the following integral representation of a solution to the homogeneous Helmholtz equation

$$\pm \int_{\partial\Omega_\pm} \left\{ [\partial_{\mathbf{n}(y)}\Gamma_0(x-y)][u(y)]^{\pm} - \Gamma_0(x-y)[\partial_{\mathbf{n}(y)}u(y)]^{\pm} \right\} ds = \begin{cases} u(x) & \text{for } x \in \Omega_\pm \\ 0 & \text{for } x \in \Omega_\mp, \end{cases} \tag{4.30}$$

which is integrable $u \in H^1(\Omega_+)$ for a bounded domain and is a radiating solution $u \in H^1_{\rm loc}(\Omega_-) \cap {\rm Som}(\Omega_-)$ for an unbounded domain (cf. [Ve1]).

Let us now recall some properties of the potentials introduced above. First of all they have the following mapping properties (cf., e.g., [DNS1])

$$\begin{aligned} V \ &: \ H^s(S) \to H^{s+1+\frac{1}{2}}_{\rm loc}(\Omega_-) \cap {\rm Som}(\Omega_-), \\ &: \ H^s(S) \to H^{s+1+\frac{1}{2}}(\Omega_+)\ , \\ W \ &: \ H^s(S) \to H^{s+\frac{1}{2}}_{\rm loc}(\Omega_-) \cap {\rm Som}(\Omega_-), \\ &: \ H^s(S) \to H^{s+\frac{1}{2}}(\Omega_+)\ , \end{aligned} \tag{4.31}$$

where $s \in \mathbb{R}$ (see e.g. [Ne1, p. 102] for a topology in the Sommerfeld space ${\rm Som}(\Omega_-)$). The following jump relations (the Plemelji formulae) are well known

$$\begin{aligned} &[V(\psi)]^+_S = [V(\psi)]^-_S =: V_{-1}(\psi), \quad [\partial_{\mathbf{n}} V(\psi)]^{\pm}_S =: [\mp\tfrac{1}{2}I + W_0](\psi) \\ &[W(\varphi)]^{\pm}_S =: [\pm\tfrac{1}{2}I + W_0^*](\varphi), \qquad [\partial_{\mathbf{n}} W(\varphi)]^+_S = [\partial_{\mathbf{n}} W(\varphi)]^-_S =: W_{+1}(\varphi) \end{aligned} \tag{4.32}$$

where I denotes the identity operator, and

$$V_{-1}(\psi)(z) \ := \ \int_S \Gamma_0(z-y)\psi(y)dS\ , \quad z \in S \tag{4.33}$$

$$W_0(\psi)(z) \ := \ \int_S [\partial_{\mathbf{n}(z)}\Gamma_0(z-y)]\psi(y)dS\ , \quad z \in S \tag{4.34}$$

$$W_0^*(\varphi)(z) \ := \ \int_S [\partial_{\mathbf{n}(y)}\Gamma_0(y-z)]\varphi(y)dS\ , \quad z \in S \tag{4.35}$$

$$W_{+1}(\varphi)(z) \ := \ \lim_{x \to z \in S} \partial_{\mathbf{n}(x)} \int_S [\partial_{\mathbf{n}(y)}\Gamma_0(y-x)]^\top \varphi(y)dS\ , \quad z \in S \tag{4.36}$$

are the direct values of the above potentials on the boundary.

Theorem 4.1 [DNS1] *Operators* (4.33)–(4.36) *can be extended to the following bounded mappings*

$$r_{S_j} V_{-1} \ : \ \widetilde{H}^s(S_j) \to H^{s+1}(S_j)\ , \tag{4.37}$$

$$r_{S_j} W_0,\ r_{S_j} W_0^* \ : \ \widetilde{H}^s(S_j) \to H^s(S_j)\ , \tag{4.38}$$

$$r_{S_j} W_{+1} \ : \ \widetilde{H}^{s+1}(S_j) \to H^s(S_j)\ , \tag{4.39}$$

where $s \in \mathbb{R}$ *is arbitrary and*

$$r_{S_j} : H^s(S) \to H^s(S_j), \qquad j = 1, 2$$

is the restriction operator. Moreover, these operators are pseudodifferential of order -1, 0, 0, *and* 1, *respectively. The operators* (4.37) *and* (4.39) *are invertible provided* $-1 < s < 0$.

Inserting in the single and double layer potentials (4.30), and then applying formulas (4.32), we obtain the identities

$$\begin{aligned} W_0^* V_{-1} &= V_{-1} W_0 , & W_{+1} W_0^* &= W_0 W_{+1} , \\ V_{-1} W_{+1} &= -\frac{1}{4} I + (W_0^*)^2 , & W_{+1} V_{-1} &= -\frac{1}{4} I + (W_0)^2 . \end{aligned} \tag{4.40}$$

It can be proved (cf. [DNS1]) that the homogeneous principal symbols $\sigma(V_{-1})(x,\xi)$, $\sigma(W_0)(x,\xi)$, $\sigma(W_0^*)(x,\xi)$, $\sigma(W_{+1})(x,\xi)$ have the following properties

$$\begin{gathered} \sigma(W_0)(x,\xi) =: iK(x,\xi), \quad \sigma(W_0^*)(x,\xi) =: -iK(x,\xi), \\ \sigma(W_0)(x,-\xi) = \sigma(W_0)(x,\xi) , \end{gathered} \tag{4.41}$$

and

$$\begin{gathered} \sigma(V_{-1})(x,\xi) = \sigma(V_{-1})(x,-\xi) =: H(x,\xi), \\ \sigma(W_{+1})(x,\xi) = \sigma(W_{+1})(x,-\xi) =: L(x,\xi), \end{gathered} \tag{4.42}$$

where K, H, L are real valued functions. Recall that $-H$ and L are positive definite on S_j, i.e., for all $\xi \in \mathbb{R}\backslash\{0\}$, $x \in S_j$ and $\eta \in \mathbb{C}$ we have

$$-H(x,\xi)\eta \cdot \eta \geq c_1|\xi|^{-1}|\eta|^2 , \quad L(x,\xi)\eta \cdot \eta \geq c_2|\xi||\eta|^2 , \tag{4.43}$$

for $c_j = \text{const} > 0$, $j = 1, 2$. Moreover, from (4.40) we easily derive

$$\sigma(W_0^*) = \sigma(W_0) = 0 , \ -\sigma(V_{-1})\sigma(W_{+1}) = -\sigma(W_{+1})\sigma(V_{-1}) = \frac{1}{4}. \tag{4.44}$$

Remark 4.2 *From (4.44) it follows that the operators*

$$r_{S_j} W_0 \ , \ r_{S_j} W_0^* : \widetilde{H}^s(S_j) \to H^s(S_j)$$

are compact for all $s \in \mathbb{R}$ and $j = 1, 2$.

For our purposes below, we need to distinguish between the layer potentials inside and outside Ω_+. To this end, let us introduce the following notation

$$V_-(\psi)(x) = \int_S \Gamma_0(x-y)\psi(y)dS , \quad x \in \Omega_- , \tag{4.45}$$

$$W_-(\varphi)(x) = \int_S [\partial_{\mathbf{n}(y)}\Gamma_0(x-y)]\varphi(y)dS , \quad x \in \Omega_- \tag{4.46}$$

for the single and double layer potentials related to (2.10), and

$$V_+(\psi)(x) = \int_S \Gamma(x-y)\psi(y)dS , \quad x \in \Omega_+ , \tag{4.47}$$

$$W_+(\varphi)(x) = \int_S [T(D,\mathbf{n}(y))\Gamma(x-y)]\varphi(y)dS , \quad x \in \Omega_+ \tag{4.48}$$

for the single and double layer potentials related to (2.8).

Note that analogous results are valid for the operators in (4.47) and (4.48). Indeed, besides the observation made at the end of Section 2, from (2.11) it also follows that the Dirichlet type condition transforms to a Dirichlet type condition, i.e., we have $[v]^+(M^{-\frac{1}{2}}x) = [u]^+(x)$, $x \in S$. For the Neumann type condition we have that the unit normal vectors on S are

$$\boldsymbol{\nu} := \frac{M^{-\frac{1}{2}}\mathbf{n}}{|M^{-\frac{1}{2}}\mathbf{n}|}. \tag{4.49}$$

Then

$$\partial_{\mathbf{n}} = \mathbf{n} \cdot \mathrm{grad}_y = \mathbf{n} \cdot M^{\frac{1}{2}}\mathrm{grad}_x = M^{-\frac{1}{2}}\mathbf{n} \cdot M\mathrm{grad}_x = |M^{-\frac{1}{2}}\mathbf{n}|\, T(D, \boldsymbol{\nu}).$$

Therefore, for example, the relations for the single and double layer potentials are as follows:

$$\begin{aligned} &[V_+(\psi)]_S^+ =: V_{-1}^+(\psi), && [T(D,\boldsymbol{\nu})V_+(\psi)]_S^+ =: [-\tfrac{1}{2}I + W_0^+](\psi) \\ &[W_+(\varphi)]_S^+ =: [\tfrac{1}{2}I + (W_0^+)^*](\varphi), && [T(D,\boldsymbol{\nu})W_+(\varphi)]_S^+ =: W_{+1}^+(\varphi). \end{aligned} \tag{4.50}$$

The potential operators involved here are similar to (4.33)-(4.36) with the only difference: the kernels $\Gamma_0(x-y)$ and $\partial_{\mathbf{n}(y)}\Gamma_0(x-y)$ in (4.33)-(4.36) are replaced by $\Gamma(x-y)$ and $T(D, \mathbf{n}(y))\Gamma(x-y)$, respectively. For more details concerning the properties of layer potentials cf. [Du1].

5 Existence of a solution

Since the scope of the paper is to show, along with the existence and uniqueness, the regularity of solutions, we take from the beginning $0 \le \varepsilon < \frac{1}{2}$. Modifications for the $\varepsilon = 0$ are obvious.

Consider the analytic functions

$$\Lambda_\pm^s(\xi) := (\xi \pm i)^s = (1+\xi^2)^{\frac{s}{2}} \exp\left\{ s\, i \arg(\xi \pm i) \right\},$$

in the complex plane cutted along the negative real axis and with branches chosen so that $\arg(\xi \pm i) \to 0$ as $\xi \to +\infty$ (cf. Example 1.7 in [Es81] for further details).

Lemma 5.1 [Es81, §4] *Let $s, r \in \mathbb{R}$, and consider the (Bessel potential) operators*

$$\begin{aligned} \Lambda_+^s(D) &= (D+i)^s \\ \Lambda_-^s(D) &= r_{\mathbb{R}_+}(D-i)^s \ell^{(r)}, \end{aligned}$$

where $(D \pm i)^{\pm s} = \mathscr{F}^{-1}(\xi \pm i)^{\pm s} \cdot \mathscr{F}$, for the Fourier transformation $\mathscr{F}$ and $\ell^{(r)} : H^r(\mathbb{R}_+) \to H^r(\mathbb{R})$ being any bounded extension operator in these spaces (the result is independent of a particular choice of $\Lambda_-^s(D)$).

The Bessel potential operators arrange isomorphisms of the spaces

$$\begin{aligned} \Lambda_+^s(D) &: \widetilde{H}^r(\mathbb{R}_+) \to \widetilde{H}^{r-s}(\mathbb{R}_+), \\ \Lambda_-^s(D) &: H^r(\mathbb{R}_+) \to H^{r-s}(\mathbb{R}_+) \end{aligned}$$

for arbitrary $s, r \in \mathbb{R}$.

Set

$$\mathscr{E}_+(\xi) := \begin{pmatrix} \Lambda_+^{-1/2-\varepsilon}(\xi) & 0 \\ 0 & \Lambda_+^{1/2-\varepsilon}(\xi) \end{pmatrix}, \; \mathscr{E}_-(\xi) := \begin{pmatrix} \Lambda_-^{-1/2+\varepsilon}(\xi) & 0 \\ 0 & \Lambda_+^{-1/2+\varepsilon}(\xi) \end{pmatrix}.$$

For definiteness, we fix the arguments of the analytic functions as follows

$$\arg(\xi+i) \in (0,\pi)\;, \quad \arg(\xi-i) \in (-\pi,0) \qquad \text{for all} \quad \xi \in \mathbb{R}.$$

Then for the principal symbols of the above introduced Bessel potential operators we have

$$\begin{aligned} &\sigma(\Lambda^s_\pm)(1) = 1, \quad \sigma(\Lambda^s_+)(-1) = \exp\big\{si\pi\big\}\;, \\ &\sigma(\Lambda^s_-)(-1) = \exp\big\{-si\pi\big\} \qquad \text{for all} \quad s \in \mathbb{R}. \end{aligned} \tag{5.51}$$

We look for a solution of Problem $\mathscr{P}$ in the form

$$u^\pm(x) = V_\pm\psi(x) + W_\pm\varphi(x), \quad x \in \Omega_\pm. \tag{5.52}$$

Let $\ell_j g_j$, $j = 0,1$ be arbitrary (but fixed) extensions of the right-hand sides $g_j \in H^{\frac{1}{2}+\varepsilon-j}(S_2)$ in (2.22) such that $\ell_j g_j \in H^{\frac{1}{2}+\varepsilon-j}(S)$ and $\tilde g_j \in \widetilde H^{\frac{1}{2}+\varepsilon-j}(S_1)$ be certain functions, to be determined later on (cf. (5.60) and Theorem 5.4 below). Then, the representation formula (5.52) together with the jump relations (4.32) and (4.50) and the boundary conditions (2.22) lead to the following system of pseudo-differential equations on S with unknowns φ and ψ:

$$\left\{ \begin{array}{rcl} (V^+_{-1} - V^-_{-1})\psi + (1 + (W^+_0)^* - (W^-_0)^*)\varphi &=& \ell_0 g_0 + \tilde g_0 \\ (-1 + W^+_0 - W^-_0)\psi + (W^+_{+1} - W^-_{+1})\varphi &=& \ell_1 g_1 + \tilde g_1 \end{array} \right. . \tag{5.53}$$

We would like to point out that, in general, $V^+_{-1} \neq V^-_{-1}$ and $W^+_{+1} \neq W^-_{+1}$ because they have different kernels (cf. (4.45), (4.47) (4.46) and (4.48)).

With the notation

$$\mathscr{A} := \begin{pmatrix} V^+_{-1} - V^-_{-1} & 1 + (W^+_0)^* - (W^-_0)^* \\ -1 + W^+_0 - W^-_0 & W^+_{+1} - W^-_{+1} \end{pmatrix}$$

and

$$\Phi := (\psi,\varphi)^\top\;, \quad G := (\ell_0 g_0, \ell_1 g_1)^\top\;, \quad \widetilde G := (\tilde g_0, \tilde g_1)^\top.$$

system (5.53) is equivalently written in a compact form

$$\mathscr{A}\Phi = G + \widetilde G \quad \text{on} \quad S\;, \tag{5.54}$$

where $\Phi \in H^{-\frac{1}{2}+\varepsilon}(S) \times H^{\frac{1}{2}+\varepsilon}(S)$, $G \in H^{\frac{1}{2}+\varepsilon}(S) \times H^{-\frac{1}{2}+\varepsilon}(S)$, and $\widetilde G \in \widetilde H^{\frac{1}{2}+\varepsilon}(S_1) \times \widetilde H^{-\frac{1}{2}+\varepsilon}(S_1)$.

Theorem 5.2 *The operator*

$$\mathscr{A} : H^{-\frac{1}{2}+\varepsilon}(S) \times H^{\frac{1}{2}+\varepsilon}(S) \to H^{\frac{1}{2}+\varepsilon}(S) \times H^{-\frac{1}{2}+\varepsilon}(S) \tag{5.55}$$

is invertible for all $\varepsilon \in \mathbb{R}$.

Proof: First note that we can ignore compact operators since they do not affect the Fredholm property and the Fredholm index. Thus, keeping in mind Remark 4.2, we start by analyzing the operator

$$\mathscr{A}_0 := \begin{pmatrix} V_{-1}^{+} - V_{-1}^{-} & 1 \\ -1 & W_{+1}^{+} - W_{+1}^{-} \end{pmatrix}.$$

For this we introduce the parameter-dependent operator

$$\mathscr{A}_\lambda := \begin{pmatrix} (1-\lambda)[V_{-1}^{+} - V_{-1}^{-}] & 1 \\ -1 & (1-\lambda)[W_{+1}^{+} - W_{+1}^{-}] \end{pmatrix}, \qquad 0 \le \lambda \le 1.$$

Note that $\mathscr{A}_\lambda$ is elliptic in the sense of Douglis-Nirenberg

$$\inf\{|\det \sigma(\mathscr{A}_\lambda)(x,\xi)| : x \in S\ ,\ |\xi| = 1\} > 0\,, \qquad 0 \le \lambda \le 1, \tag{5.56}$$

including the case $\lambda = 0$, i.e. the operator $\mathscr{A}_0$. Indeed, from (4.43) and (4.44) follows that

$$\begin{aligned} \det \sigma(\mathscr{A}_\lambda) &= 1 + (1-\lambda)^2[V_{-1}^{+}W_{+1}^{+} - V_{-1}^{+}W_{+1}^{-} - V_{-1}^{-}W_{+1}^{+} + V_{-1}^{-}W_{+1}^{-}] \\ &= 1 - \frac{1}{2}(1-\lambda)^2 - (1-\lambda)^2[V_{-1}^{+}W_{+1}^{-} + V_{-1}^{-}W_{+1}^{+}], \\ \det \sigma(\mathscr{A}_\lambda)(x,\xi) &\ge 1 - \frac{1}{2}(1-\lambda)^2 + 2c_1c_2(1-\lambda)^2|\xi|^2 \ge \frac{1}{2} \qquad \forall\, \lambda \in [0,1]. \end{aligned}$$

Thus, the operator $\mathscr{A}_0$ and therefore $\mathscr{A}$ in (5.55) are Fredholm.

Moreover, since the operator $\mathscr{A}_1$ is invertible

$$\mathscr{A}_1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathscr{A}_1^{-1} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

we conclude by the homotopy argument (see [MP1]) that $\operatorname{Ind}\mathscr{A}_0 = \operatorname{Ind}\mathscr{A}_1 = 0$. Then $\operatorname{Ind}\mathscr{A} = \operatorname{Ind}\mathscr{A}_0 = 0$. Now noting that the operator $\mathscr{A}$ corresponds to the boundary contact problem with the pure transmission conditions on the entire boundary, then applying Remark 3.4, we conclude that $\operatorname{Ker}\mathscr{A} = \{0\}$, which means that the operator (5.55) is invertible. ■

Due to Theorem 5.2 we then obtain

$$\Phi = \mathscr{A}^{-1}G + \mathscr{A}^{-1}\widetilde{G} \quad \text{on} \quad S\,. \tag{5.57}$$

Now from the representation formula (5.52) together with the jump relations (4.32) and (4.50) and the boundary conditions (2.23) we obtain the following system of pseudo-differential equations on S_1

$$\left\{ \begin{array}{lcl} r_{S_1}\big[(-\frac{1}{2}+(W_0^+)^*)\psi + W_{+1}^+\varphi - ip^+\big(V_{-1}^+\psi + (\frac{1}{2}+(W_0^+)^*)\varphi\big)\big] & = & h_0 \\ r_{S_1}\big[(\frac{1}{2}+(W_0^-)^*)\psi + W_{+1}^-\varphi + ip^-\big(V_{-1}^-\psi + (-\frac{1}{2}+(W_0^-)^*)\varphi\big)\big] & = & h_1 \end{array} \right. \tag{5.58}$$

System (5.58) is equivalently written in the form

$$r_{S_1}\mathscr{B}\Phi = H \quad \text{on} \quad S_1 \,, \tag{5.59}$$

where $H := (h_0, h_1 - h_0)^\top \in H^{-\frac{1}{2}+\varepsilon}(S_1) \times \widetilde{H}^{-\frac{1}{2}+\varepsilon}(S_1)$ and

$$\mathscr{B} := \begin{pmatrix} \mathscr{B}_{11} & \mathscr{B}_{12} \\ \mathscr{B}_{21} & \mathscr{B}_{22} \end{pmatrix},$$

$$\mathscr{B}_{11} := -\frac{1}{2} + (W_0^+)^* - ip^+V_{-1}^+, \quad \mathscr{B}_{12} := W_{+1}^+ - ip^+\left(\frac{1}{2} + (W_0^+)^*\right),$$
$$\mathscr{B}_{21} := 1 + (W_0^-)^* - (W_0^+)^* + ip^-V_{-1}^- + ip^+V_{-1}^+,$$
$$\mathscr{B}_{22} := W_{+1}^- - W_{+1}^+ + ip^-\left(-\frac{1}{2} + (W_0^-)^*\right) + ip^+\left(\frac{1}{2} + (W_0^+)^*\right).$$

Putting (5.57) into (5.59) we obtain a system of pseudo-differential equations on S_1 with respect to $\widetilde{G} = (\tilde{g}_0, \tilde{g}_1)^\top$.

$$r_{S_1}\mathscr{B}\mathscr{A}^{-1}\widetilde{G} = H - r_{S_1}\mathscr{B}\mathscr{A}^{-1}G \quad \text{on} \quad S_1 \,. \tag{5.60}$$

Theorem 5.3 *If* $\Re\mathrm{e}\, p^\pm \geq 0$*, then the operator*

$$\mathscr{B} : H^{-\frac{1}{2}+\varepsilon}(S) \times H^{\frac{1}{2}+\varepsilon}(S) \to H^{-\frac{1}{2}+\varepsilon}(S) \times H^{-\frac{1}{2}+\varepsilon}(S) \tag{5.61}$$

is invertible for all $\varepsilon \in \mathbb{R}$*.*

Proof: As in the proof of Theorem 5.2, we consider the operator

$$\mathscr{B}_0 := \begin{pmatrix} -\frac{1}{2} & W_{+1}^+ \\ 1 & W_{+1}^- - W_{+1}^+ \end{pmatrix}$$

by ignoring the corresponding compact operators in the structure of $\mathscr{B}$ (cf. Remark 4.2 and take into account the compact embeddings $H^{\frac{1}{2}+\varepsilon}(S_1) \hookrightarrow H^{-\frac{1}{2}+\varepsilon}(S_1)$).

Let us introduce the parameter-dependent operator

$$\mathscr{B}_\lambda := \begin{pmatrix} -\frac{1}{2}(1-\lambda) & W_{+1}^+ \\ 1 & (1-\lambda)[W_{+1}^- - W_{+1}^+] \end{pmatrix}$$

Note that $\mathscr{A}_\lambda$ is elliptic in the sense of Douglis-Nirenberg

$$\inf\{|\det\sigma(\mathscr{B}_\lambda)(x,\xi)| : x \in S \,,\ |\xi| = 1\} > 0 \,, \qquad 0 \leq \lambda \leq 1, \tag{5.62}$$

including the case $\lambda = 0$, i.e. the operator $\mathscr{B}_0$.

Indeed, from (4.43) it follows that

$$\det \sigma(\mathscr{B}_\lambda) = -\frac{1}{2}(1-\lambda)^2 W_{+1}^- - \left[1 - \frac{1}{2}(1-\lambda)^2\right] W_{+1}^+,$$
$$\det \sigma(\mathscr{B}_\lambda)(x,\xi) \geq -\frac{1}{2}c_2(1-\lambda)^2|\xi| - c_2\left[1 - \frac{1}{2}(1-\lambda)^2\right]|\xi| = \frac{c_2}{2}$$

for all $\lambda \in [0,1]$. Thus, the operator $\mathscr{B}_0$ and therefore $\mathscr{B}$ in (5.55) are Fredholm.

Moreover, since the operator $\mathscr{B}_1$ is invertible

$$\mathscr{B}_1 := \begin{pmatrix} 0 & W_{+1}^+ \\ 1 & 0 \end{pmatrix}, \qquad \mathscr{B}_1^{-1} := \begin{pmatrix} 0 & 1 \\ (W_{+1}^-)^{-1} & 0 \end{pmatrix},$$

we conclude by the homotopy argument that $\operatorname{Ind} \mathscr{B}_0 = \operatorname{Ind} \mathscr{B}_1 = 0$. Then $\operatorname{Ind} \mathscr{B}_0 = \operatorname{Ind} \mathscr{B} = 0$. Now, observing that the operator $\mathscr{B}$ corresponds to the boundary value problem with the pure impedance conditions on the entire boundary, applying Remark 3.4 and Theorem 3.3, we have $\operatorname{Ker} \mathscr{B} = \{0\}$, i.e., the operator (5.61) is invertible. ■

Theorem 5.4 *If* $\Re e\, p^\pm \geq 0$*, then the operator*

$$r_{S_1}\mathscr{B}\mathscr{A}^{-1} : \widetilde{H}^{\frac{1}{2}+\varepsilon}(S_1) \times \widetilde{H}^{-\frac{1}{2}+\varepsilon}(S_1) \to H^{-\frac{1}{2}+\varepsilon}(S_1) \times \widetilde{H}^{-\frac{1}{2}+\varepsilon}(S_1) \tag{5.63}$$

is invertible for all $0 \leq \varepsilon < \frac{1}{2}$.

Proof: As in the foregoing theorems, we again apply Remark 4.2 and the compactness of embedding $H^{\frac{1}{2}+\varepsilon}(S_1) \hookrightarrow H^{-\frac{1}{2}+\varepsilon}(S_1)$, ignore compact operators in the structure of $r_{S_1}\mathscr{B}\mathscr{A}^{-1}(x,D)$ and denote the remainder by $r_{S_1}(\mathscr{B}\mathscr{A}^{-1})_0(x,D)$.

If we now apply the local principle (see, e.g. [Sim1], [DS1]) and "freezing coefficients". Then the operator

$$r_{S_1}(\mathscr{B}\mathscr{A}^{-1})_0(x,D) : \widetilde{H}^{s+1}(S_1) \times \widetilde{H}^{s}(S_1) \to H^{s}(S_1) \times \widetilde{H}^{s}(S_1) \tag{5.64}$$
$$s := -\frac{1}{2} + \varepsilon$$

is Fredholm if and only if its local representatives

$$\mathscr{B}\mathscr{A}_0^{-1}(x_0,D) : H^{s+1}(\mathbb{R}) \times H^{s}(\mathbb{R}) \to H^{s}(\mathbb{R}) \times H^{s}(\mathbb{R}) \quad \text{if } x_0 \in S_1, \tag{5.65}$$
$$r_{\mathbb{R}_+}(\mathscr{B}\mathscr{A}^{-1})_0(x_0,D) : \widetilde{H}^{s+1}(\mathbb{R}_+) \times \widetilde{H}^{s}(\mathbb{R}_+) \to H^{s}(\mathbb{R}_+) \times \widetilde{H}^{s}(\mathbb{R}_+) \tag{5.66}$$
$$\text{if } x_0 \in \partial S_1.$$

are locally invertible for all $x_0 \in \overline{S}_1$.

In the case $x_0 \in S_1$, the invertibility of (5.65) is a direct consequence of Theorems 5.2 and 5.3.

For $x_0 \in \partial S_1$, the operator $r_{\mathbb{R}_+}(\mathscr{B}\mathscr{A}^{-1})_0$ is an upper triangular matrix operator

$$r_{\mathbb{R}_+}(\mathscr{B}\mathscr{A}^{-1})_0 := \begin{pmatrix} \mathscr{M}_{11} & \mathscr{M}_{12} \\ 0 & -I \end{pmatrix}$$

and the entry $\mathscr{M}_{11}$ is a self-adjoint operator $\mathscr{M}_{11}^* = \mathscr{M}_{11}$. Now we lift the operator (5.66) to an equivalent pseudo-differential (matrix) operator of order zero

$$\mathscr{C}_0 := \mathscr{E}_-(D) r_{\mathbb{R}_+}(\mathscr{B}\mathscr{A}^{-1})_0 \mathscr{E}_+(D) : L_2(\mathbb{R}_+) \times L_2(\mathbb{R}_+) \to L_2(\mathbb{R}_+) \times L_2(\mathbb{R}_+) \ . \tag{5.67}$$

From (4.42)-(4.44) we obtain the interior ellipticity. Moreover, the corresponding principal symbol

$$\sigma(\mathscr{C}_0)(x_0, \xi) = \sigma(\mathscr{E}_-)(\xi)\sigma(\mathscr{B}_0)(x_0, \xi)[\sigma(\mathscr{A}_0)(x_0, \xi)]^{-1}\sigma(\mathscr{E}_+)(\xi)$$

is an even matrix with respect to ξ. Since $\sigma(\mathscr{B}_0)(x_0, \xi)[\sigma(\mathscr{A}_0)(x_0, \xi)]^{-1} < 0$, for all $\xi \in \mathbb{R}$, we obtain that $\Im\mathrm{m}\det\sigma(\mathscr{C})(x_0, \xi) > 0$ for all $\xi \in \mathbb{R}$, provided $0 \leq \varepsilon < \frac{1}{2}$. Moreover,

$$\Im\mathrm{m}\det\mathscr{C}_\lambda(x_0) = (\lambda + (1-\lambda)e^{-2\varepsilon\pi i})\det\sigma(\mathscr{C})(x_0, +\infty) > 0$$

for all $0 \leq \lambda \leq 1$ and $0 \leq \varepsilon < \frac{1}{2}$; here

$$\mathscr{C}_\lambda(x_0) := \lambda\,\sigma(\mathscr{C})(x_0, +\infty) + (1-\lambda)\,\sigma(\mathscr{C})(x_0, -\infty).$$

Thus operator (5.66) is Fredholm with index 0. Since $-I$ is the $(2, 2)$-entry of the triangular matrix operator $r_+\mathscr{C}_0$, we obtain that the 11-entry is a Fredholm operator with index zero. Noting that after "freezing the coefficients" we have convolution operators, we conclude that the 11-entry and therefore the matrix operator $r_+\mathscr{C}_0$ are invertible. Thus (5.64) and therefore (5.63) are Fredholm operators with index 0. Then, due to Theorem 3.3, operator (5.63) is invertible. ■

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