On the uniqueness of a solution to anisotropic Maxwell's equations

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Abstract. In the present paper we consider Maxwell's equations in an anisotropic media, when the dielectric permittivity ε and the magnetic permeability μ are 3×3 matrices. We formulate relevant boundary value problems, investigate a fundamental solution and find a Silver-Müller type radiation condition at infinity which ensures the uniqueness of solutions when permittivity and permeability matrices are real valued, symmetric, positive definite and proportional $\varepsilon = \kappa\mu,\ \kappa > 0$.

Dedicated to Israel Gohberg, the outstanding teacher and scientist, on his 80-th birthday anniversary

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Introduction

In the paper we analyse the uniqueness of solutions to the time harmonic exterior three-dimensional boundary value problems (BVPs) for *anisotropic Maxwell's equations*. It is well known that in the electro-magnetic wave scattering theory the most important question is the formulation of appropriate radiation conditions at infinity, which are crucial in the study of uniqueness questions. In the case of *isotropic Maxwell's equations* such conditions are *the Silver-Müler radiation conditions* which are counterparts of the *Sommerfeld radiation conditions* for the Helmholtz equation. In view of the celebrated Rellich-Vekua lemma it follows that the Helmholtz equation and isotropic Maxwell's equations do not admit non-trivial solutions decaying at infinity as $\mathscr{O}(|x|^{-1-\delta})$ with $\delta > 0$. This property plays an essential role in the study of direct and inverse acoustic and electromagnetic wave scattering (see, e.g., [CK1, Eo1, HW1, Jo1, Le1, Ne1, Ve1] and the references therein).

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Investigation of the same type problems for the general anisotropic case proved to be much more difficult and only few results are worked out so far. The main problem here consists in finding the appropriate radiation conditions at infinity, which, in turn, is closely related to the asymptotic properties of the corresponding fundamental solutions (see, e.g., [Va1, Wi1, Na1, Ag1] for special classes of strongly elliptic partial differential equations). As we will see below anisotropic Maxwell's equations, as well as the isotropic one, is not strongly elliptic and its characteristic surface represents a self-intersecting two dimensional manifold, in general.

In the present paper, we consider a special case of anisotropy when the electric permittivity $\varepsilon = [\varepsilon_{kj}]_{3\times 3}$ and the magnetic permeability $\mu = [\mu_{kj}]_{3\times 3}$ are real valued, symmetric, positive definite and proportional matrices $\varepsilon = \kappa\mu,\ \kappa > 0$. For this particular case we explicitly construct fundamental matrices, formulate the corresponding Silver-Müler type radiation conditions and prove the uniqueness theorems for the exterior BVPs.

1. Basic boundary value problems for Maxwell's equations

Throughout the paper we denote by Ω a domain, which can be bounded or unbounded, while the notation Ω^+ stands for a bounded domain and $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$.

Maxwell's equations

$$\begin{cases} \mathbf{curl}\,\boldsymbol{H} + i\omega\varepsilon\boldsymbol{E} = 0\,, \\ \mathbf{curl}\,\boldsymbol{E} - i\omega\mu\boldsymbol{H} = 0\,, \end{cases} \qquad \text{in} \quad \Omega \subset \mathbb{R}^3 \tag{1}$$

for $\omega > 0$ govern the scattering of time-harmonic electromagnetic waves with frequency ω in a domain Ω. $\boldsymbol{E} = (E_1, E_2, E_3)^\top$ and $\boldsymbol{H} = (H_1, H_2, H_3)^\top$ are 3 vector-functions, representing the scattered electric and magnetic waves respectively. Here and in what follows the symbol $(\cdot)^\top$ denotes transposition and

$$\mathbf{curl} := \begin{bmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{bmatrix}.$$

System (1) can also be written in matrix form

$$\mathbf{M}(D)\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} = 0\,, \qquad \mathbf{M}(D) := \begin{bmatrix} i\omega\varepsilon I_3 & \mathbf{curl} \\ \mathbf{curl} & -i\omega\mu I_3 \end{bmatrix}. \tag{2}$$

$$D := -i(\partial_1, \partial_2, \partial_3)^\top, \quad \partial_j := \frac{\partial}{\partial x_j}, \quad j = 1, 2, 3.$$

The scope of the present investigation is to consider an anisotropic case when relative dielectric permittivity $\varepsilon = [\varepsilon_{jk}]_{3\times 3}$ and relative magnetic permeability $\mu = [\mu_{jk}]_{3\times 3}$ in (1) are real valued symmetric positive definite constant matrices, i.e.,

$$\langle \varepsilon\xi, \xi\rangle \geq c|\xi|^2\,, \qquad \langle \mu\xi, \xi\rangle \geq d|\xi|^2\,, \qquad \forall \xi \in \mathbb{C}^3 \tag{3}$$

with some positive constants $c > 0$, $d > 0$ and where

$$\langle \eta, \xi \rangle := \sum_{j=1}^{3} \eta_j \overline{\xi}_j, \quad \eta,\ \xi \in \mathbb{C}^3.$$

Consequently, these matrices admit the square roots $\varepsilon^{1/2}$, $\mu^{1/2}$. In some models of anisotropic media the positive definiteness (3) is a consequence of the energy conservation law (cf., e.g., [BDS1]).

By solving $\boldsymbol{E}$ from the first equation in (1) and introducing the result into the second one we obtain an equivalent system

$$\begin{cases} \mathbf{curl}\, \varepsilon^{-1} \mathbf{curl}\, \boldsymbol{H} - \omega^2 \mu \boldsymbol{H} = 0\,, \\ \boldsymbol{E} = i(\omega\varepsilon)^{-1} \mathbf{curl}\, \boldsymbol{H} \end{cases} \qquad \text{in} \quad \Omega\,, \tag{4}$$

or, by first solving $\boldsymbol{H}$ from the second equation and introducing the result into the first one we obtain another equivalent system

$$\begin{cases} \mathbf{curl}\, \mu^{-1} \mathbf{curl}\, \boldsymbol{E} - \omega^2 \varepsilon \boldsymbol{E} = 0\,, \\ \boldsymbol{H} = -i(\omega\mu)^{-1} \mathbf{curl}\, \boldsymbol{E} \end{cases} \qquad \text{in} \quad \Omega\,. \tag{5}$$

Since $\operatorname{div} \mathbf{curl} = 0$, after applying the divergence operator div to the first equations of the systems (4) and (5), we get

$$\operatorname{div}(\mu\, \boldsymbol{H}) = \operatorname{div}(\varepsilon\, \boldsymbol{E}) = 0\,. \tag{6}$$

Here we will only investigate the system (5). Results for the system (4) can be worked out analogously.

For a rigorous formulation of conditions providing the unique solvability of the formulated boundary value problems we use the Bessel potential $\mathbb{H}^r_p(\Omega)$, $\mathbb{H}^r_p(\mathscr{S})$, $\mathbb{H}^r_{p,\mathrm{loc}}(\Omega)$, $\mathbb{H}^r_{p,\mathrm{com}}(\Omega)$ and Besov $\mathbb{B}^r_{p,q}(\Omega)$, $\mathbb{B}^r_{p,p}(\mathscr{S})$ spaces, $-\infty < r < \infty$, $1 < p, q < \infty$, when $\Omega \subset \mathbb{R}^3$ is a domain and $\mathscr{S}$ is the sufficiently smooth boundary surface of Ω. Note that, for an unbounded domain Ω, the space $\mathbb{H}^r_{p,\mathrm{loc}}(\Omega)$ comprises all distributions u for which $\psi\, u \in \mathbb{H}^r_p(\Omega)$ where $\psi \in \mathbb{C}^\infty_0(\mathbb{R}^3)$ is arbitrary. As usual, for the spaces $\mathbb{H}^r_2(\Omega)$, $\mathbb{H}^r_2(\mathscr{S})$, $\mathbb{H}^r_{2,\mathrm{loc}}(\Omega)$, $\mathbb{H}^r_{2,\mathrm{com}}(\Omega)$ we use the notation $\mathbb{H}^r(\Omega)$, $\mathbb{H}^r(\mathscr{S})$, $\mathbb{H}^r_{\mathrm{loc}}(\Omega)$, $\mathbb{H}^r_{\mathrm{com}}(\Omega)$.

It is well known that $\mathbb{W}^{r-1/p}_p(\mathscr{S}) = \mathbb{B}^{r-1/p}_{p,p}(\mathscr{S})$ (Sobolev-Slobodetski space) is a trace space for $\mathbb{H}^r_p(\Omega)$, provided $r > 1/p$. If $\mathscr{C}$ is an open smooth subsurface of a hypersurface $\mathscr{S}$ in $\mathbb{R}^3$, we use the spaces $\mathbb{H}^r_p(\mathscr{C})$ and $\widetilde{\mathbb{H}}^r_p(\mathscr{C})$. The space $\mathbb{H}^r_p(\mathscr{C})$ comprises those functions φ which have extensions to functions $\phi \in \mathbb{H}^r_p(\mathscr{S})$. The space $\widetilde{\mathbb{H}}^r_p(\mathscr{C})$ comprises functions $\varphi \in \mathbb{H}^r_p(\mathscr{S})$ which are supported in $\overline{\mathscr{C}}$ (functions with "vanishing traces on the boundary $\partial\mathscr{C}$"). For detailed definitions and properties of these spaces we refer to, e. g., [Hr1, HW1, Tr1]).

Finally, as usual for the Maxwell's equations, we need the following special space

$$\mathbb{H}(\mathbf{curl}; \Omega) := \{ \boldsymbol{U} \in \mathbb{L}_2(\Omega) \ : \ \mathbf{curl}\, \boldsymbol{U} \in \mathbb{L}_2(\Omega) \}.$$

We also use the notation $\mathbb{H}_{\rm loc}(\mathbf{curl};\Omega)$, meaning the Frechet space of all locally integrable vector functions $\boldsymbol{U}$ and $\mathbf{curl}\, U$ instead of global integrability if the underlying domain Ω is unbounded, and the space $\mathbb{H}(\mathbf{curl};\Omega)$ if Ω is bounded.

Note that $\mathbb{H}^1(\Omega)$ is a proper subspace of $\mathbb{H}(\mathbf{curl};\Omega)$. Indeed, $\boldsymbol{U}+\operatorname{grad}\psi\in\mathbb{H}(\mathbf{curl};\Omega)$ for a vector function $\boldsymbol{U}\in\mathbb{H}^1(\Omega)$ and a scalar function $\psi\in\mathbb{H}^1(\Omega)$ but, in general, $\boldsymbol{U}+\operatorname{grad}\psi\notin\mathbb{H}^1(\Omega)$.

Next we recall basic boundary value problems for Maxwell's equations written for the electric field:

I. The "magnetic" BVP

$$\begin{cases} \mathbf{curl}\,\mu^{-1}\mathbf{curl}\,\boldsymbol{E}-\omega^2\varepsilon\boldsymbol{E}=0 & \text{in}\quad \Omega\subset\mathbb{R}^3\,,\\ \gamma_{\mathscr{S}}\left(\boldsymbol{\nu}\times\left(\mu^{-1}\mathbf{curl}\,\boldsymbol{E}\right)\right)=\mathbf{e} & \text{on}\quad \mathscr{S}:=\partial\Omega\,, \end{cases} \tag{7a}$$

$$\boldsymbol{E}\in\mathbb{H}_{\rm loc}(\mathbf{curl};\Omega),\qquad \mathbf{e}\in\mathbb{H}^{-1/2}(\mathscr{S}),$$

where $\gamma_{\mathscr{S}}$ is the trace operator on the boundary and the symbol $\times$ denotes the vector product of vectors;

II. The "electric" BVP

$$\begin{cases} \mathbf{curl}\,\mu^{-1}\mathbf{curl}\,\boldsymbol{E}-\omega^2\varepsilon\boldsymbol{E}=0 & \text{in}\quad \Omega\subset\mathbb{R}^3\,,\\ \gamma_{\mathscr{S}}\left(\boldsymbol{\nu}\times\boldsymbol{E}\right)=\mathbf{f} \qquad \text{on}\quad \mathscr{S}\,, \end{cases} \tag{7b}$$

$$\boldsymbol{E}\in\mathbb{H}_{\rm loc}(\mathbf{curl};\Omega),\qquad \mathbf{f}\in\mathbb{H}^{1/2}(\mathscr{S});$$

III. The "mixed" BVP

$$\begin{cases} \mathbf{curl}\,\mu^{-1}\mathbf{curl}\,\boldsymbol{E}-\omega^2\varepsilon\boldsymbol{E}=0 & \text{in}\quad \Omega\subset\mathbb{R}^3\,,\\ \gamma_{\mathscr{S}_N}\left(\boldsymbol{\nu}\times\left(\mu^{-1}\mathbf{curl}\,\boldsymbol{E}\right)\right)=\mathbf{e_N} & \text{on}\quad \mathscr{S}_N\,,\\ \gamma_{\mathscr{S}_D}\left(\boldsymbol{\nu}\times\boldsymbol{E}\right)=\mathbf{f_D} & \text{on}\quad \mathscr{S}_D\,, \end{cases} \tag{7c}$$

$$\boldsymbol{E}\in\mathbb{H}_{\rm loc}(\mathbf{curl};\Omega),\quad \mathbf{e_N}\in\mathbb{H}^{-1/2}(\mathscr{S}_N),\quad \mathbf{f_D}\in\mathbb{H}^{1/2}(\mathscr{S}_D),$$

where $\mathscr{S}_D$ and $\mathscr{S}_N$ are disjoint parts of the boundary surface $\mathscr{S}:=\overline{\mathscr{S}}_N\cup\overline{\mathscr{S}}_D$.

If $\mathscr{S}$ is an orientable, smooth, open surface in $\mathbb{R}^3$ with a boundary $\Gamma:=\partial\mathscr{S}$, it has two faces $\mathscr{S}^-$ and $\mathscr{S}^+$, which differ by the orientation of the normal vector field $\nu(x)$, which points from $\mathscr{S}^+$ to $\mathscr{S}^-$. The natural BVPs for **scattering of electromagnetic field by an open surface** $\mathscr{S}$ in $\mathbb{R}^3\setminus\overline{\mathscr{S}}$ are the following:

I. The crack type "magnetic-magnetic" BVP

$$\begin{cases} \mathbf{curl}\,\mu^{-1}\mathbf{curl}\,\boldsymbol{E}-\omega^2\varepsilon\boldsymbol{E}=0 & \text{in}\quad \mathbb{R}^3\setminus\overline{\mathscr{S}}\,,\\ \gamma_{\mathscr{S}^\pm}\left(\boldsymbol{\nu}\times\left(\mu^{-1}\mathbf{curl}\,\boldsymbol{E}\right)\right)=\mathbf{e}^\pm & \text{on}\quad \mathscr{S}\,, \end{cases} \tag{8a}$$

$$\boldsymbol{E}\in\mathbb{H}_{\rm loc}(\mathbf{curl};\mathbb{R}^3\setminus\overline{\mathscr{S}}),\quad \mathbf{e}^\pm\in\mathbb{H}^{-1/2}(\mathscr{S});$$

II. The screen type "electric-electric" BVP

$$\begin{cases} \mathbf{curl}\,\mu^{-1}\mathbf{curl}\,\boldsymbol{E} - \omega^2\varepsilon\boldsymbol{E} = 0 & \text{in} \quad \mathbb{R}^3 \setminus \overline{\mathscr{S}}\,, \\ \gamma_{\mathscr{S}^\pm}\,(\boldsymbol{\nu} \times \boldsymbol{E}) = \mathbf{f}^\pm & \text{on} \quad \mathscr{S}, \end{cases} \tag{8b}$$

$$\boldsymbol{E} \in \mathbb{H}_{\rm loc}(\mathbf{curl}; \mathbb{R}^3 \setminus \overline{\mathscr{S}}), \qquad \mathbf{f}^\pm \in \mathbb{H}^{1/2}(\mathscr{S});$$

III. The "magnetic-electric" BVP

$$\begin{cases} \mathbf{curl}\,\mu^{-1}\mathbf{curl}\,\boldsymbol{E} - \omega^2\varepsilon\boldsymbol{E} = 0 & \text{in} \quad \mathbb{R}^3 \setminus \overline{\mathscr{S}}\,, \\ \gamma_{\mathscr{S}^+}\left(\boldsymbol{\nu} \times \left(\mu^{-1}\mathbf{curl}\,\boldsymbol{E}\right)\right) = \mathbf{e}^+\,, \quad \gamma_{\mathscr{S}^-}\,(\boldsymbol{\nu} \times \boldsymbol{E}) = \mathbf{f}^- & \text{on} \quad \mathscr{S}\,, \end{cases} \tag{8c}$$

$$\boldsymbol{E} \in \mathbb{H}_{\rm loc}(\mathbf{curl}; \mathbb{R}^3 \setminus \overline{\mathscr{S}}), \qquad \mathbf{e}^+ \in \mathbb{H}^{-1/2}(\mathscr{S}), \quad \mathbf{f}^- \in \mathbb{H}^{1/2}(\mathscr{S});$$

IV. The "mixed-mixed" type BVP

$$\begin{cases} \mathbf{curl}\,\mu^{-1}\mathbf{curl}\,\boldsymbol{E} - \omega^2\varepsilon\boldsymbol{E} = 0 & \text{in} \quad \mathbb{R}^3 \setminus \overline{\mathscr{S}}, \\ \gamma_{\mathscr{S}_N^\pm}\left[\boldsymbol{\nu} \times \left(\mu^{-1}\mathbf{curl}\,\boldsymbol{E}\right)\right] = \mathbf{e}_{\mathbf{N}}^\pm, & \text{on} \quad \mathscr{S}_N^\pm, \\ \gamma_{\mathscr{S}_N^\pm}\left[\boldsymbol{\nu} \times \boldsymbol{E}\right] = \mathbf{f}_{\mathbf{D}}^\pm & \text{on} \quad \mathscr{S}_D^\pm, \end{cases} \tag{8d}$$

$$\boldsymbol{E} \in \mathbb{H}_{\rm loc}(\mathbf{curl}; \mathbb{R}^3 \setminus \overline{\mathscr{S}}), \qquad \mathbf{e}_{\mathbf{N}}^\pm \in \mathbb{H}^{-1/2}(\mathscr{S}_N^\pm), \quad \mathbf{f}_{\mathbf{D}}^\pm \in \mathbb{H}^{1/2}(\mathscr{S}_D^\pm),$$

where $\overline{\mathscr{S}_N^\pm} \cup \overline{\mathscr{S}_D^\pm} = \overline{\mathscr{S}}$ and $\mathscr{S}_N^+ \cap \mathscr{S}_D^+ = \varnothing$, $\mathscr{S}_N^- \cap \mathscr{S}_D^- = \varnothing$.

All BVPs (8a)-(8d) and BVPs (7a)-(7c) for an unbounded domain Ω should be endowed with a special condition at infinity. If the medium is isotropic, i. e., the permeability and the permittivity coefficients are scalar constants, the radiation conditions are well-known (cf., e.g., [CK1, Eo1, Jo1, Ne1] etc.). For example, the classical radiation condition imposed on the electric field reads

$$\frac{\partial \boldsymbol{E}(x)}{\partial R} - i\sigma k\boldsymbol{E} = \mathscr{O}\left(R^{-2}\right) \qquad \text{for} \quad R = |x| \to \infty, \tag{9}$$

where $k = \omega\sqrt{\varepsilon\mu}$ and either $\sigma = -1$ for incoming waves or $\sigma = +1$ for outgoing waves. Similar condition can also be imposed on the magnetic field $\boldsymbol{H}$. The Silver-Müller radiation condition is imposed on both fields either

$$\left|\sqrt{\varepsilon}\boldsymbol{E}(x) \times \hat{x} + \sqrt{\mu}\boldsymbol{H}(x)\right| = \mathscr{O}\left(R^{-2}\right) \qquad \text{for} \quad R = |x| \to \infty \tag{10}$$

or

$$\left|\sqrt{\varepsilon}\boldsymbol{E}(x) - \sqrt{\mu}\boldsymbol{H}(x) \times \hat{x}\right| = \mathscr{O}\left(R^{-2}\right) \qquad \text{for} \quad R = |x| \to \infty, \tag{11}$$

where $\hat{x} := \dfrac{x}{|x|}$.

The basic boundary value problems for the magnetic field $\boldsymbol{H}$ and the differential equation (4) are formulated similarly to (7a)-(7c) and (8a)-(8d).

Remark 1.1. *We can derive solutions to the screen type (the "electric") BVP for electric* $\boldsymbol{E}$ *field indirectly, provided we can solve the crack type (the "magnetic") BVP for the magnetic field* $\boldsymbol{H}$ *and vice versa.*

Indeed, let $\boldsymbol{H}$ *be a solution to the "magnetic" boundary value problem with a boundary data* $\mathbf{h}$ *for the magnetic field* $\boldsymbol{H}$*. Due to the second equations in* (4)*, we get*

$$\gamma_{\mathscr{S}}(\boldsymbol{\nu}\times\boldsymbol{E}) = \frac{i}{\omega}\gamma_{\mathscr{S}}(\nu\times\left(\varepsilon^{-1}\mathbf{curl}\,\boldsymbol{H}\right)) = \frac{i}{\omega}\mathbf{h}.$$

Therefore the vector field $\boldsymbol{E} = i(\omega\varepsilon)^{-1}\mathbf{curl}\,\boldsymbol{H}$ *is a solution to the "electric" BVP* (7b) *with the boundary data* $\mathbf{f} = \dfrac{i}{\omega}\mathbf{h}$.

The same is true, due to the second equations in (5) *and* (4)*, for the all three remaining BVPs for the magnetic* $\boldsymbol{H}$ *and the electric* $\boldsymbol{E}$ *vector fields.*

Radiation conditions for the matrix coefficients ε and μ are unknown so far. In §5 a radiation condition for anisotropic Maxwell's equations is derived when the permittivity and permeability matrices ε and μ are real valued, positive definite, symmetric and proportional $\varepsilon = \kappa\mu$. The radiation conditions ensure the uniqueness of a solution. As a first step to the investigation let us simplify the main object, namely, the system (1).

Let ε_1, ε_2, ε_3, μ_1, μ_2, μ_3 be the eigenvalues of the permittivity and permeability matrices. Due to (3) they are positive $\varepsilon_j > 0$, $\mu_j > 0$, $j = 1, 2, 3$. Consider following Maxwell's equations

$$\begin{cases} \mathbf{curl}\,\boldsymbol{H}^* + i\omega\varepsilon^*\boldsymbol{E}^* = 0\,, \\ \mathbf{curl}\,\boldsymbol{E}^* - i\omega\mu^*\boldsymbol{H}^* = 0\,, \end{cases} \qquad \text{in} \quad \Omega^*\subset\mathbb{R}^3, \tag{12}$$

with the diagonal permittivity and permeability matrices

$$\varepsilon^* = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}, \qquad \mu^* = \begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{bmatrix}.$$

Lemma 1.2. *Let the permittivity* ε *and the permeability* μ *be real valued, positive definite and proportional matrices*

$$\varepsilon = \kappa\mu, \qquad \kappa > 0. \tag{13}$$

Then there exists an orthogonal matrix

$$\mathscr{R}\,:\,\mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad |\mathscr{R}\,x| = |x|, \quad \mathscr{R}^{-1} = \mathscr{R}^\top,$$

which establishes the following equivalence between Maxwell's equations (1) *and* (12)*:* $\Omega^* := \mathscr{R}\,\Omega$ *and*

$$\boldsymbol{E}^*(x^*) := \mathscr{R}\,\boldsymbol{E}(\mathscr{R}^\top x^*) \quad \boldsymbol{H}^*(x) := \mathscr{R}\,\boldsymbol{H}(\mathscr{R}^\top x^*), \qquad \forall x^* := \mathscr{R}x \in \Omega^*. \tag{14}$$

Proof: The proof is based on the following well known result (see, e.g., [Me1, § 7.5] and [Ga1, § IX.10]): *a matrix* $\mathscr{A}\in\mathbb{C}^{n\times n}$ *is unitarily similar to a diagonal matrix* $\mathscr{D}$*, i.e.,* $\mathscr{A} = \mathscr{U}^\top\mathscr{D}\mathscr{U}$ *with* $\mathscr{U}^\top\mathscr{U} = I$*, if and only if the matrix* $\mathscr{A}$ *is normal, i.e., commutes with its adjoint* $\mathscr{A}^*\mathscr{A} = \mathscr{A}\mathscr{A}^*$.

Since the matrices ε and μ are real valued, positive definite and proportional matrices there exists an orthogonal, i.e., real valued and unitary, matrix $\mathscr{R}$ which reduces them to the diagonal (Jordan) form simultaneously

$$\varepsilon = \mathscr{R}^\top \varepsilon^* \mathscr{R}, \qquad \mu = \mathscr{R}^\top \mu^* \mathscr{R}. \tag{15}$$

By introducing the representations (15) into the system (1), applying the transformation $\mathscr{R}$ to both sides of equations and changing the variable to a new one $x^* = \mathscr{R}\, x$, we obtain the following:

$$\begin{cases} \mathbf{curl}^* \boldsymbol{H}^*(x^*) + i\omega\varepsilon^* \boldsymbol{E}^*(x^*) = 0\,, \\ \mathbf{curl}^* \boldsymbol{E}^*(x^*) - i\omega\mu^* \boldsymbol{H}^*(x^*) = 0\,, \end{cases} \qquad x^* \in \Omega^*, \tag{16}$$

where $\mathbf{curl}^*\boldsymbol{U}(x^*) := \mathscr{R}\,\mathbf{curl}\,\mathscr{R}^\top \boldsymbol{U}(x)$. Let $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, $\boldsymbol{R}_3$ be the vector columns of the transposed matrix $\mathscr{R}^\top$. Then

$$\mathscr{R}^\top = (\boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3), \quad \langle \boldsymbol{R}_j, \boldsymbol{R}_k \rangle = \delta_{jk}, \tag{17}$$

and we find

$$\begin{aligned} \mathbf{curl}^*\boldsymbol{U} = \mathscr{R}\mathbf{curl}\,\mathscr{R}^\top \boldsymbol{U} &= \begin{pmatrix} \boldsymbol{R}_1^\top \\ \boldsymbol{R}_2^\top \\ \boldsymbol{R}_3^\top \end{pmatrix} \nabla_x \times (\boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3)\boldsymbol{U} \\ &= [\langle \boldsymbol{R}_j, \nabla_x \times \boldsymbol{R}_k \rangle]_{3\times 3}\, \boldsymbol{U} = -\,[\langle \boldsymbol{R}_j \times \boldsymbol{R}_k, \nabla_x \rangle]_{3\times 3}\, \boldsymbol{U} \\ &= \begin{bmatrix} 0 & -\langle \boldsymbol{R}_3, \nabla_x \rangle & \langle \boldsymbol{R}_2, \nabla_x \rangle \\ \langle \boldsymbol{R}_3, \nabla_x \rangle & 0 & -\langle \boldsymbol{R}_1, \nabla_x \rangle \\ -\langle \boldsymbol{R}_2, \nabla_x \rangle & \langle \boldsymbol{R}_1, \nabla_x \rangle & 0 \end{bmatrix} \boldsymbol{U} \\ &= \begin{bmatrix} 0 & -\partial_{x_3^*} & \partial_{x_2^*} \\ \partial_{x_3^*} & 0 & -\partial_{x_1^*} \\ -\partial_{x_2^*} & \partial_{x_1^*} & 0 \end{bmatrix} \boldsymbol{U}, \end{aligned} \tag{18}$$

since the variables after transformation are $x_j^* = \langle \boldsymbol{R}_j, x \rangle,\ \ j = 1, 2, 3$. The last three equalities in (18) follow with the help of the formulae:

$$\langle \boldsymbol{R}_j, \nabla \times \boldsymbol{R}_k \rangle = -\langle \boldsymbol{R}_j \times \boldsymbol{R}_k, \nabla \rangle = -\varepsilon_{jkm} \langle \boldsymbol{R}_m, \nabla \rangle,$$

$$\boldsymbol{R}_1 \times \boldsymbol{R}_2 = \boldsymbol{R}_3\,, \qquad \boldsymbol{R}_2 \times \boldsymbol{R}_3 = \boldsymbol{R}_1\,, \qquad \boldsymbol{R}_3 \times \boldsymbol{R}_1 = \boldsymbol{R}_2\,,$$

where ε_{jkm} is the Levi-Civita symbol (the permutation sign), $j, k, m = 1, 2, 3$. The equality (18) acomplishes the proof. ■

Remark 1.3. *Hereafter, if not stated otherwise, we will assume that ε and μ are real valued, positive definite, proportional (cf.* (13)*) and diagonal matrices*

$$\varepsilon = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}, \qquad \mu = \begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{bmatrix}. \tag{19}$$

Remark 1.4. *Finally, let us note that for a complex valued wave frequency* $\operatorname{Im}\omega \neq 0$ *and arbitrary real valued, symmetric and positive definite matrices* μ *and* ε*, a fundamental solution to Maxwell's operator exists and decays at infinity exponentially.*

Moreover, each above formulated basic BVPs for Maxwell's equations has a unique solution in the class of polynomially bounded vector-functions, represented by layer potentials and actually these solutions decay exponentially at infinity.

For real valued frequencies matters are different and we consider the case in the next section.

2. A fundamental solution to Maxwell's operator

The equation

$$\boldsymbol{M}_\mu(D)F(x) = \delta(x)I_3\,, \quad \boldsymbol{M}_\mu(D) := \mathbf{curl}\,\mu^{-1}\mathbf{curl}\,, \tag{20}$$

$$\mathbf{F} = (F_1, F_2, F_3)^\top, \quad x \in \mathbb{R}^3,$$

(cf. (5)), where I_3 is the identity matrix, has no fundamental solution. In fact, the determinant of the symbol (the characteristic polynomial) of this operator vanishes identically,

$$\det\, \boldsymbol{M}_\mu(\xi) = \det\, \sigma_{\mathbf{curl}}(\xi)\, \det\, \mu^{-1} \det\, \sigma_{\mathbf{curl}}(\xi) \equiv 0, \tag{21}$$

where $\sigma_{\mathbf{curl}}(\xi)$ is the symbol of the operator $\mathbf{curl}$:

$$\sigma_{\mathbf{curl}}(\xi) := \begin{bmatrix} 0 & i\xi_3 & -i\xi_2 \\ -i\xi_3 & 0 & i\xi_1 \\ i\xi_2 & -i\xi_1 & 0 \end{bmatrix}. \tag{22}$$

The absence of the fundamental solution is a consequence of the following theorem.

Theorem 2.1. *A partial differential operator* $\mathbf{P}(D) = \sum\limits_{|\alpha|\le m} p_\alpha \partial^\alpha$ *with constant matrix coefficients* $p_\alpha \in \mathbb{C}^{N\times N}$ *has a fundamental solution* $\mathbf{F}_\mathbf{P} \in \mathbb{S}'(\mathbb{R}^n)$ *if and only if the determinant of the symbol*

$$P(\xi) = \sigma_\mathbf{P}(\xi) := \sum_{|\alpha|\le m} p_\alpha(-i\xi)^\alpha, \qquad \xi \in \mathbb{R}^n,$$

does not vanish identically.

Proof: The proof is based on the Malgrange-Ehrenpreis theorem on the existence of the fundamental solution for the scalar equation (cf., e.g., [Hr1]).

Let $\det\, P(\xi) \not\equiv 0$ and consider the formal co-factor matrix of $\mathbf{P}(D)$

$$\mathbf{A}_\mathbf{P}(D) := \big[\mathbf{A}_{jk}(D)\big]_{N\times N}\,, \quad \mathbf{A}_{jk}(D) = (-1)^{j+k}\mathbf{M}_{kj}(D)\,, \tag{23}$$

where $\mathbf{M}_{kj}(D)$ are the $(N-1)$-dimensional minors of $\mathbf{P}(D)$. Then

$$\mathbf{A}_P(D)\mathbf{P}(D) = \mathbf{P}(D)\mathbf{A}_P(D) = \operatorname{diag}\{\det\, \mathbf{P}(D), \ldots, \det\, \mathbf{P}(D)\}\,.$$

The distribution

$$\mathbf{F}_\mathbf{P} := \mathbf{A}_\mathbf{P}(D)\operatorname{diag}\{\mathbf{F}_{\det\,\mathbf{P}}, \ldots, \mathbf{F}_{\det\,\mathbf{P}}\},$$

where $\mathbf{F}_{\det\,\mathbf{P}}$ is the fundamental solution of the scalar equation $\det\,\mathbf{P}(D)F(x)=\delta(x)$ (cf. Malgrange-Ehrenpreis theorem; cf. [Hr1]) is the claimed fundamental solution of $\mathbf{P}(D)$.

Next we assume that the determinant vanishes identically, i.e., $\det\,P(\xi)\equiv 0$. Then $\det\,\mathbf{P}(D)=0$ and the rows of the operator matrix are linearly dependent. There exists a non-singular permutation $N\times N$ matrix $\mathscr{H}$ with constant entries, such that the first row of the matrix-operator $\widetilde{\mathbf{P}}(D)=\mathscr{H}\mathbf{P}(D)$ is identically 0. If we assume that a fundamental solution exists, i.e., $\mathbf{P}(D)\mathbf{F}_{\mathbf{P}}=\delta I_N$, we get the following equality

$$(0,c_2,\dots,c_N)^\top=\big(\mathscr{H}\mathbf{P}(D)\big)\mathbf{F}_{\mathbf{P}}u=\mathscr{H}\big(\mathbf{P}(D)\mathbf{F}_{\mathbf{P}}u\big)=\mathscr{H}\delta u=\mathscr{H}u(0)$$

for all $u\in\mathbb{S}(\mathbb{R}^n)$. Since the test vector-function u is arbitrary and the matrix $\mathscr{H}$ is invertible, the latter equality is a contradiction. ■

In contrast to equations (20) the corresponding spectral equation

$$\boldsymbol{M}_e(D)\Phi_e=\delta I_3\,,\qquad \boldsymbol{M}_e(D):=\boldsymbol{M}_\mu(D)-\omega^2\mu I \tag{24}$$

has a fundamental solution.

Theorem 2.2. *The fundamental solution of the equation in* (24) *is given by*

$$\Phi_e=\boldsymbol{M}_e^{\#}(D)\,\mathbf{F}_{\det\,\boldsymbol{M}_e}I_3 \tag{25}$$

where $\boldsymbol{M}_e^{\#}(D)$ denotes the formal co-factor matrix operator of $\boldsymbol{M}_e(D)$ and $\mathbf{F}_{\det\,\boldsymbol{M}_e}$ is a fundamental solution of the equation

$$\det\,\boldsymbol{M}_e(D)\,\mathbf{F}_{\det\,\boldsymbol{M}_e}=\delta\,.$$

Proof: Due to Theorem 2.1 the fundamental solution $\mathbf{F}_{\det\,\boldsymbol{M}_e}$ exists and implies the existence of the fundamental solution Φ_e for $\boldsymbol{M}_e(D)$:

$$\boldsymbol{M}_e(D)\Phi_e=\boldsymbol{M}_e(D)\boldsymbol{M}_e^{\#}(D)\,\mathbf{F}_{\det\,\boldsymbol{M}_e}I_3=\det\,\boldsymbol{M}_e(D)\,\mathbf{F}_{\det\,\boldsymbol{M}_e}I_3=\delta I_3.$$

■

Remark 2.3. *The symbol $M_e(\xi)$ of the operators $\boldsymbol{M}_e(D)$ in* (24) *is not elliptic and even not hypoelliptic. To be* hypoelliptic *(of the class $\mathbb{HL}^{m,m_1}_{\rho,0}(\mathbb{R}^n\times\mathbb{R}^n)$ for $m_1,m\in\mathbb{N}_0$, $m_1\le m$), the principal symbol $\sigma_{\boldsymbol{A}}(x,\xi)$ of a matrix differential (or a pseudodifferential) operator $\boldsymbol{A}(x,D)$ needs, by definition, to meet the following two conditions* [Hr1]*:*

i. *there exist positive constants C_1 and C_2, such that the inequalities*

$$C_1|\xi|^{m_1}\le|\det\,\sigma_{\boldsymbol{A}}(x,\xi)|\le C_2|\xi|^m\qquad\forall\,x,\xi\in\mathbb{R}^n \tag{26}$$

hold;

ii. *for arbitrary $\alpha,\beta\in\mathbb{R}^n$, $|\alpha|+|\beta|\neq 0$, there exist positive constants $C_{\alpha,\beta}$ and $\rho>0$, such that*

$$\Big|\det\,[\sigma^{-1}_{\boldsymbol{A}}(x,\xi)(\sigma_{\boldsymbol{A}})^{(\alpha)}_{(\beta)}(x,\xi)]\Big|\le C_{\alpha,\beta}|\xi|^{-\rho|\alpha|}\qquad\forall\,x,\xi\in\mathbb{R}^n\,, \tag{27}$$

where $(\sigma_{\boldsymbol{A}})^{(\alpha)}_{(\beta)}(x,\xi):=\partial^\beta_x\partial^\alpha_\xi\sigma_{\boldsymbol{A}}(x,\xi)$.

If the indices coincide $m_1=m$, the symbol $\sigma_{\boldsymbol{A}}(x,D)$ is elliptic *from the Hörmander class $\mathbb{HL}^m_{\rho,0}(\mathbb{R}^n\times\mathbb{R}^n)$.*

To show that the symbol $M_e(\xi)$ is not hypoelliptic we will check that the second condition (27) fails for it. In fact:

$$\begin{aligned}\det M_e(\xi) &= \det\left[\sigma_{\mathbf{curl}}(\xi)\mu^{-1}\sigma_{\mathbf{curl}}(\xi)-\omega^2\varepsilon\right]\\ &= \omega^2\mathscr{P}_4(\xi)+\omega^4\mathscr{P}_2(\xi)-\omega^6\det\varepsilon\,. \end{aligned}\tag{28}$$

Here $\mathscr{P}_k(\xi)$ is a homogeneous polynomial of order $k=2,4$. Then

$$\begin{gathered}\mathrm{ord}\left[\,M_e(\xi)-\omega^2\varepsilon\right]^{-1}=0\,,\\ \mathrm{ord}\left[\,M_e(\xi)-\omega^2\varepsilon\right]^{-1}\partial_j\left[M_e(\xi)-\omega^2\varepsilon\right]=+1,\end{gathered}\tag{29}$$

and the condition (27) fails.

The next proposition is well known (cf. [Ne1], [CK1]).

Proposition 2.4. *Either of the following functions*

$$\Phi^{\pm}_{\mathfrak{M}}(x)=\frac{e^{\pm ik|x|}}{4\pi|x|}I_3+\frac{1}{4\pi k^2}\nabla\nabla^\top\frac{e^{\pm ik|x|}}{|x|}\,.\tag{30}$$

is a fundamental solution of the equation

$$\mathfrak{M}\Phi_{\mathfrak{M}}:=\mathbf{curl}^{\,2}\,\Phi_{\mathfrak{M}}-k^2\Phi_{\mathfrak{M}}=\delta I_3.\tag{31}$$

Proof: The fundamental solution is equal to the inverse Fourier transform of the inverse symbol

$$\Phi_{\mathfrak{M}}(x)=\mathscr{F}^{-1}_{\xi\to x}\left[\mathfrak{M}^{-1}(\xi)\right].\tag{32}$$

Since the symbol equals (cf. (22))

$$\mathfrak{M}(\xi)=\begin{bmatrix}0 & i\xi_3 & -i\xi_2\\ -i\xi_3 & 0 & i\xi_1\\ i\xi_2 & -i\xi_1 & 0\end{bmatrix}^2-k^2I_3=(|\xi|^2-k^2)I_3-\xi\xi^\top,\quad \xi\in\mathbb{R}^3,\tag{33}$$

let us look for the inverse in the form

$$\mathfrak{M}^{-1}(\xi):=\frac{1}{|\xi|^2-k^2}I_3-\alpha\xi\xi^\top,$$

where α is an unknown scalar function. Since $\xi^\top\xi=|\xi|^2$, the condition $\mathfrak{M}^{-1}(\xi)\mathfrak{M}(\xi)\equiv I_3$ provides the equality

$$\alpha(\xi)=\frac{1}{k^2(|\xi|^2-k^2)}\,,$$

which is well-defined outside the sphere $|\xi|^2=k^2$. Then,

$$\mathfrak{M}^{-1}(\xi):=\frac{1}{|\xi|^2-k^2}\left[I_3-\frac{1}{k^2}\xi\xi^\top\right],\qquad |\xi|\neq k.\tag{34}$$

To regularize the singular integral, let us temporarily replace k by a complex valued parameter $k\pm i\theta$, where $\theta>0$ is small. By inserting (34) (with $k\pm i\theta$) into (32) and by applying the identity

$$\nabla_x\nabla_x^\top\left[e^{-i\langle x,\xi\rangle}\right]=-\xi\xi^\top e^{-i\langle x,\xi\rangle}\,,\qquad x,\xi\in\mathbb{R}^3,$$

we proceed as follows:

$$\Phi^{\pm}_{\mathfrak{M}}(x) = \frac{1}{2\pi^3}\lim_{\theta\to 0+}\left[\int_{\mathbb{R}^3}\frac{e^{-i\langle x,\xi\rangle}d\xi}{|\xi|^2-(k\pm i\theta)^2}I_3 - \int_{\mathbb{R}^3}\frac{e^{-i\langle x,\xi\rangle}\xi\xi^\top d\xi}{(k\pm i\theta)^2\big[|\xi|^2-(k\pm i\theta)^2\big]}\right] \quad (35)$$

$$= \lim_{\theta\to 0+}\frac{1}{2\pi^3}\left[\int_{\mathbb{R}^3}\frac{e^{-i\langle x,\xi\rangle}d\xi}{|\xi|^2-(k\pm i\theta)^2}I_3 + \frac{1}{(k\pm i\theta)^2}\nabla\nabla^\top\int_{\mathbb{R}^3}\frac{e^{-i\langle x,\xi\rangle}d\xi}{|\xi|^2-(k\pm i\theta)^2}\right].$$

To calculate the integral in (35) it is convenient to introduce the spherical coordinates $\xi = \rho\eta$, $\eta \in \mathbb{S}^2 \subset \mathbb{R}^3$ and apply the residue theorem. After a standard manipulation we get the following:

$$G^{\pm}(x) := \frac{1}{(2\pi)^3}\lim_{\theta\to 0+}\int_{\mathbb{R}^3}\frac{e^{-i\langle x,\xi\rangle}d\xi}{|\xi|^2-(k\pm i\theta)^2} = \frac{1}{2\pi^2|x|}\lim_{\theta\to 0+}\int_0^\infty\frac{\rho\sin(\rho|x|)}{\rho^2-(k\pm i\theta)^2}d\rho$$

$$= \frac{e^{\pm ik|x|}}{4\pi|x|}, \quad x\in\mathbb{R}^3. \quad (36)$$

By inserting the obtained integral in (35) we arrive at (30). ■

Theorem 2.5. *Let coefficients ε and μ be diagonal and proportional (see* (13) *and Remark* 1.3*). Then the fundamental solution Φ_e in* (24) *is written in explicit form*

$$\Phi^{\pm}_e(x) = \mathscr{F}^{-1}_{\xi\to x}\left[M_e^{-1}(\xi,\omega)\right] \quad (37)$$

$$= \frac{1}{4\pi\omega^2\kappa(\det\mu)^{3/2}}\begin{bmatrix}\partial_1^2+\omega^2\kappa\mu_2\mu_3 & -\partial_1\partial_2 & -\partial_1\partial_3\\ -\partial_1\partial_2 & \partial_2^2+\omega^2\kappa\mu_1\mu_3 & -\partial_2\partial_3\\ -\partial_1\partial_3 & -\partial_2\partial_3 & \partial_3^2+\omega^2\kappa\mu_1\mu_2\end{bmatrix}$$

$$\times\frac{e^{\pm i\omega\sqrt{\kappa\ \det\ \mu}\,|\widetilde{x}|}}{|\widetilde{x}|} = \frac{e^{\pm i\omega\sqrt{\kappa\ \det\ \mu}|\widetilde{x}|}}{4\pi|\widetilde{x}|}\Phi_{e,\infty}(\widetilde{x}) + \mathscr{O}\left(|x|^{-2}\right) \quad as \quad |x|\to\infty,$$

where $\widetilde{x} := (\widetilde{x}_1,\widetilde{x}_2,\widetilde{x}_3)^\top$, $\widetilde{x}_j := \dfrac{x_j}{\sqrt{\mu_j}}$, $j=1,2,3$, *and the matrix*

$$\Phi_{e,\infty}(\widetilde{x}) := \begin{bmatrix}\dfrac{|\widetilde{x}|^2-\widetilde{x}_1^2}{\mu_1|\widetilde{x}|^2\sqrt{\det\mu}} & \dfrac{\sqrt{\mu_3}\,\widetilde{x}_1\widetilde{x}_2}{|\widetilde{x}|^2\det\mu} & \dfrac{\sqrt{\mu_2}\,\widetilde{x}_1\widetilde{x}_3}{|\widetilde{x}|^2\det\mu}\\ \dfrac{\sqrt{\mu_3}\,\widetilde{x}_1\widetilde{x}_2}{|\widetilde{x}|^2\det\mu} & \dfrac{|\widetilde{x}|^2-\widetilde{x}_2^2}{\mu_2|\widetilde{x}|^2\sqrt{\det\mu}} & \dfrac{\sqrt{\mu_1}\,\widetilde{x}_2\widetilde{x}_3}{|\widetilde{x}|^2\det\mu}\\ \dfrac{\sqrt{\mu_2}\,\widetilde{x}_1\widetilde{x}_3}{|\widetilde{x}|^2\det\mu} & \dfrac{\sqrt{\mu_1}\,\widetilde{x}_2\widetilde{x}_3}{|\widetilde{x}|^2\det\mu} & \dfrac{|\widetilde{x}|^2-\widetilde{x}_3^2}{\mu_3|\widetilde{x}|^2\sqrt{\det\mu}}\end{bmatrix} \quad (38)$$

is known as the far field pattern.

Proof: If ε and μ are diagonal (cf. (19)), the operator $\boldsymbol{M}_e(D)$ in (24) acquires the form:

$$M_e(\xi,\omega) = \sigma_{\mathbf{curl}}(\xi)\,\mu^{-1}\sigma_{\mathbf{curl}}(\xi) - \omega^2\varepsilon I$$

$$= \begin{bmatrix} 0 & i\xi_3 & -i\xi_2 \\ -i\xi_3 & 0 & i\xi_1 \\ i\xi_2 & -i\xi_1 & 0 \end{bmatrix} \begin{bmatrix} \mu_1^{-1} & 0 & 0 \\ 0 & \mu_2^{-1} & 0 \\ 0 & 0 & \mu_3^{-1} \end{bmatrix} \begin{bmatrix} 0 & i\xi_3 & -i\xi_2 \\ -i\xi_3 & 0 & i\xi_1 \\ i\xi_2 & -i\xi_1 & 0 \end{bmatrix} - \omega^2\varepsilon I$$

$$= \begin{bmatrix} \mu_3^{-1}\xi_2^2 + \mu_2^{-1}\xi_3^2 - \omega^2\varepsilon_1 & -\mu_3^{-1}\xi_1\xi_2 & -\mu_2^{-1}\xi_1\xi_3 \\ -\mu_3^{-1}\xi_1\xi_2 & \mu_3^{-1}\xi_1^2 + \mu_1^{-1}\xi_3^2 - \omega^2\varepsilon_2 & -\mu_1^{-1}\xi_2\xi_3 \\ -\mu_2^{-1}\xi_1\xi_3 & -\mu_1^{-1}\xi_2\xi_3 & \mu_2^{-1}\xi_1^2 + \mu_1^{-1}\xi_2^2 - \omega^2\varepsilon_3 \end{bmatrix}.$$

We have:

$$\begin{aligned}
&\det\, M_e(\xi,\omega) \\
&= \big[\mu_3^{-1}\xi_2^2 + \mu_2^{-1}\xi_3^2 - \omega^2\varepsilon_1\big]\big[\mu_3^{-1}\xi_1^2 + \mu_1^{-1}\xi_3^2 - \omega^2\varepsilon_2\big]\big[\mu_2^{-1}\xi_1^2 + \mu_1^{-1}\xi_2^2 - \omega^2\varepsilon_3\big] \\
&\quad -2\mu_1^{-1}\mu_2^{-1}\mu_3^{-1}\xi_1^2\xi_2^2\xi_3^2 - \mu_1^{-2}\big[\mu_3^{-1}\xi_2^2 + \mu_2^{-1}\xi_3^2 - \omega^2\varepsilon_1\big]\xi_2^2\xi_3^2 \\
&\quad -\mu_2^{-2}\big[\mu_3^{-1}\xi_1^2 + \mu_1^{-1}\xi_3^2 - \omega^2\varepsilon_2\big]\xi_1^2\xi_3^2 - \mu_3^{-2}\big[\mu_2^{-1}\xi_1^2 + \mu_1^{-1}\xi_2^2 - \omega^2\varepsilon_3\big]\xi_1^2\xi_2^2 \\
&= -\omega^2\left[\varepsilon_1\xi_1^2 + \varepsilon_2\xi_2^2 + \varepsilon_3\xi_3^2 - \omega^2\varepsilon_1\varepsilon_3\mu_2\right] \\
&\quad \times\left[\mu_2^{-1}\mu_3^{-1}\xi_1^2 + \mu_1^{-1}\mu_3^{-1}\xi_2^2 + \mu_1^{-1}\mu_2^{-1}\xi_3^2 - \omega^2\varepsilon_2\mu_2^{-1}\right] \\
&\quad +\omega^4[\varepsilon_2^2\mu_2^{-1} + \varepsilon_1\varepsilon_3\mu_1^{-1}\mu_2\mu_3^{-1} - \varepsilon_1\varepsilon_2\mu_1^{-1} - \varepsilon_2\varepsilon_3\mu_3^{-1}]\xi_2^2 \\
&= -\omega^2\det\,\varepsilon\Big[\frac{\xi_1^2}{\mu_2\varepsilon_3} + \frac{\xi_2^2}{\varepsilon_1\mu_2\varepsilon_3\varepsilon_2^{-1}} + \frac{\xi_3^2}{\varepsilon_1\mu_2} - \omega^2\Big]\Big[\frac{\xi_1^2}{\mu_3\varepsilon_2} + \frac{\xi_2^2}{\mu_1\varepsilon_2\mu_3\mu_2^{-1}} + \frac{\xi_3^2}{\mu_1\varepsilon_2} - \omega^2\Big] \\
&\quad +\omega^4\Big[\varepsilon_2^2\mu_2^{-1} + \varepsilon_1\varepsilon_3\mu_1^{-1}\mu_2\mu_3^{-1} - \varepsilon_1\varepsilon_2\mu_1^{-1} - \varepsilon_2\varepsilon_3\mu_3^{-1}\Big]\xi_2^2.
\end{aligned}$$

For diagonal and proportional matrices (see (13) and (19)), we get the following simplification

$$M_e(\xi,\omega) = \sigma_{\mathbf{curl}}(\xi)\,\mu^{-1}\sigma_{\mathbf{curl}}(\xi) - \omega^2\kappa\mu I \tag{39}$$

$$= \begin{bmatrix} \mu_3^{-1}\xi_2^2 + \mu_2^{-1}\xi_3^2 - \omega_1^2\mu_1 & -\mu_3^{-1}\xi_1\xi_2 & -\mu_2^{-1}\xi_1\xi_3 \\ -\mu_3^{-1}\xi_1\xi_2 & \mu_3^{-1}\xi_1^2 + \mu_1^{-1}\xi_3^2 - \omega_1^2\mu_2 & -\mu_1^{-1}\xi_2\xi_3 \\ -\mu_2^{-1}\xi_1\xi_3 & -\mu_1^{-1}\xi_2\xi_3 & \mu_2^{-1}\xi_1^2 + \mu_1^{-1}\xi_2^2 - \omega_1^2\mu_3 \end{bmatrix},$$

$$\det\, M_e(\xi,\omega) = -\frac{\omega^2\det\,\varepsilon}{\kappa^2}\left[\frac{\xi_1^2}{\mu_2\mu_3} + \frac{\xi_2^2}{\mu_1\mu_3} + \frac{\xi_3^2}{\mu_1\mu_2} - \kappa\omega^2\right]^2$$

$$= -\frac{\omega^2\det\,\varepsilon}{\kappa^2\det\,\mu}\left[\mu_1\xi_1^2 + \mu_2\xi_2^2 + \mu_3\xi_3^2 - \kappa\omega^2\det\,\mu\right]^2 \tag{40}$$

$$= -\omega_1^2 \left[\mu_1\xi_1^2 + \mu_2\xi_2^2 + \mu_3\xi_3^2 - \omega_1^2 \det\,\mu\right]^2, \tag{41}$$

where $\omega_1^2 := \omega^2\kappa$. It is easy to see that all minors of the matrix (39) have the factor $\mu_1\xi_1^2 + \mu_2\xi_2^2 + \mu_3\xi_3^2 - \omega_1^2 \det\,\mu$. Indeed, we have

$$\begin{aligned}
\big(M_e(\xi,\omega)\big)_{11} &= [\mu_3^{-1}\xi_1^2 + \mu_1^{-1}\xi_3^2 - \omega_1^2\mu_2][\mu_2^{-1}\xi_1^2 + \mu_1^{-1}\xi_2^2 - \omega_1^2\mu_3] - \mu_1^{-2}\xi_2^2\xi_3^2 \\
&= (\xi_1^2 - \omega^2\kappa\mu_2\mu_3)\left[\frac{\xi_1^2}{\mu_2\mu_3} + \frac{\xi_2^2}{\mu_1\mu_3} + \frac{\xi_3^2}{\mu_1\mu_2} - \omega^2\kappa\right] \\
&= \frac{\xi_1^2 - \omega^2\kappa\mu_2\mu_3}{\det\,\mu}\left[\mu_1\xi_1^2 + \mu_2\xi_2^2 + \mu_3\xi_3^2 - \omega_1^2 \det\,\mu\right],
\end{aligned}$$

$$\begin{aligned}
\big(M_e(\xi,\omega)\big)_{22} &= (\xi_2^2 - \omega^2\kappa\mu_1\mu_3)\left[\frac{\xi_1^2}{\mu_2\mu_3} + \frac{\xi_2^2}{\mu_1\mu_3} + \frac{\xi_3^2}{\mu_1\mu_2} - \omega^2\kappa\right] \\
&= \frac{\xi_2^2 - \omega^2\kappa\mu_1\mu_3}{\det\,\mu}\left[\mu_1\xi_1^2 + \mu_2\xi_2^2 + \mu_3\xi_3^2 - \omega_1^2 \det\,\mu\right],
\end{aligned}$$

$$\begin{aligned}
\big(M_e(\xi,\omega)\big)_{33} &= (\xi_3^2 - \omega^2\kappa\mu_1\mu_2)\left[\frac{\xi_1^2}{\mu_2\mu_3} + \frac{\xi_2^2}{\mu_1\mu_3} + \frac{\xi_3^2}{\mu_1\mu_2} - \omega^2\kappa\right] \\
&= \frac{\xi_3^2 - \omega^2\kappa\mu_1\mu_2}{\det\,\mu}\left[\mu_1\xi_1^2 + \mu_2\xi_2^2 + \mu_3\xi_3^2 - \omega_1^2 \det\,\mu\right], \\
\big(M_e(\xi,\omega)\big)_{12} &= \big(M_e(\xi,\omega)\big)_{21} = \mu_3^{-1}\xi_1\xi_2[\mu_2^{-1}\xi_1^2 + \mu_1^{-1}\xi_2^2 - \omega_1^2\mu_3] - \mu_1^{-1}\mu_2^{-1}\xi_1\xi_2\xi_3^2 \\
&= -\xi_1\xi_2\left[\frac{\xi_1^2}{\mu_2\mu_3} + \frac{\xi_2^2}{\mu_1\mu_3} + \frac{\xi_3^2}{\mu_1\mu_2} - \omega^2\kappa\right] \\
&= -\frac{\xi_1\xi_2}{\det\,\mu}\left[\mu_1\xi_1^2 + \mu_2\xi_2^2 + \mu_3\xi_3^2 - \omega_1^2 \det\,\mu\right], \\
\big(M_e(\xi,\omega)\big)_{13} &= \big(M_e(\xi,\omega)\big)_{31} = -\xi_1\xi_3\left[\frac{\xi_1^2}{\mu_2\mu_3} + \frac{\xi_2^2}{\mu_1\mu_3} + \frac{\xi_3^2}{\mu_1\mu_2} - \omega^2\kappa\right] \\
&= -\frac{\xi_1\xi_3}{\det\,\mu}\left[\mu_1\xi_1^2 + \mu_2\xi_2^2 + \mu_3\xi_3^2 - \omega_1^2 \det\,\mu\right], \\
\big(M_e(\xi,\omega)\big)_{23} &= \big(M_e(\xi,\omega)\big)_{32} = -\xi_2\xi_3\left[\frac{\xi_1^2}{\mu_2\mu_3} + \frac{\xi_2^2}{\mu_1\mu_3} + \frac{\xi_3^2}{\mu_1\mu_2} - \omega^2\kappa\right] \\
&= -\frac{\xi_2\xi_3}{\det\,\mu}\left[\mu_1\xi_1^2 + \mu_2\xi_2^2 + \mu_3\xi_3^2 - \omega_1^2 \det\,\mu\right].
\end{aligned}$$

Applying the variable transformation $\sqrt{\mu_j}\xi_j = \eta_j$ and the Fourier transformation formula (36), we easily obtain

$$\mathscr{F}^{-1}_{\xi\to x}\left[\frac{1}{\mu_1\xi_1^2 + \mu_2\xi_2^2 + \mu_3\xi_3^2 - \kappa\omega^2 \det\,\mu \pm i0}\right] = \frac{e^{\pm i\omega|\widetilde{x}|\sqrt{\kappa\ \det\,\mu}}}{4\pi|\widetilde{x}|\sqrt{\det\,\mu}},$$

where $\widetilde{x}$ is defined in (38). From the obtained expressions for the determinant, minors and the latter formula for Fourier transformation we easily derive formula (38)

$$\Phi_e^{\pm}(x) = \mathscr{F}^{-1}_{\xi\to x}\left[M_e^{-1}(\xi,\omega)\right]$$

$$= -\frac{1}{\kappa\omega^2 \det\,\mu}\mathscr{F}^{-1}_{\xi\to x}\left[\frac{1}{\mu_1\xi_1^2+\mu_2\xi_2^2+\mu_3\xi_3^2-\kappa\omega^2\det\,\mu \pm i0}\times\right.$$

$$\left.\times\begin{bmatrix} \xi_1^2-\omega^2\kappa\mu_2\mu_3 & -\xi_1\xi_2 & -\xi_1\xi_3 \\ -\xi_1\xi_2 & \xi_2^2-\omega^2\kappa\mu_1\mu_3 & -\xi_2\xi_3 \\ -\xi_1\xi_3 & -\xi_2\xi_3 & \xi_3^2-\omega^2\kappa\mu_1\mu_2 \end{bmatrix}\right]$$

$$= \frac{1}{\kappa\omega^2\det\,\mu}\begin{bmatrix} \partial_1^2+\omega^2\kappa\mu_2\mu_3 & -\partial_1\partial_2 & -\partial_1\partial_3 \\ -\partial_1\partial_2 & \partial_2^2+\omega^2\kappa\mu_1\mu_3 & -\partial_2\partial_3 \\ -\partial_1\partial_3 & -\partial_2\partial_3 & \partial_3^2+\omega^2\kappa\mu_1\mu_2 \end{bmatrix}$$

$$\times\mathscr{F}^{-1}_{\xi\to x}\left[\frac{1}{\mu_1\xi_1^2+\mu_2\xi_2^2+\mu_3\xi_3^2-\kappa\omega^2\det\,\mu\pm i0}\right]$$

$$= \frac{1}{\kappa\omega^2\det\,\mu}\begin{bmatrix} \partial_1^2+\omega^2\kappa\mu_2\mu_3 & -\partial_1\partial_2 & -\partial_1\partial_3 \\ -\partial_1\partial_2 & \partial_2^2+\omega^2\kappa\mu_1\mu_3 & -\partial_2\partial_3 \\ -\partial_1\partial_3 & -\partial_2\partial_3 & \partial_3^2+\omega^2\kappa\mu_1\mu_2 \end{bmatrix}\frac{e^{\pm i\omega|\widetilde{x}|\sqrt{\kappa\,\det\,\mu}}}{4\pi|\widetilde{x}|\sqrt{\det\,\mu}},$$

where the variable $\widetilde{x}$ is defined in (38). ■

Remark 2.6. *It can be checked that the necessary and sufficient condition for the polynomial* $\det\,M_e(\xi,\omega)$ *to be factored into two second degree polynomials,* $\det\,M_e(\xi,\omega) = P_1(\xi)P_2(\xi)$*, is the condition that one of the following equalities hold:*

$$\frac{\varepsilon_1}{\mu_1}=\frac{\varepsilon_2}{\mu_2},\quad \frac{\varepsilon_1}{\mu_1}=\frac{\varepsilon_3}{\mu_3},\quad \frac{\varepsilon_2}{\mu_2}=\frac{\varepsilon_3}{\mu_3}$$

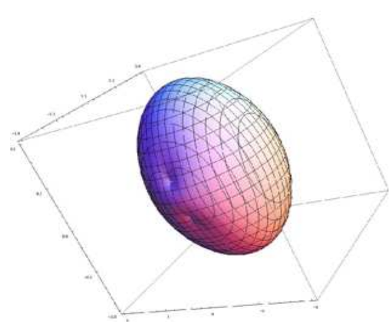

Fig. 1: Outer characteristic ellipsoid

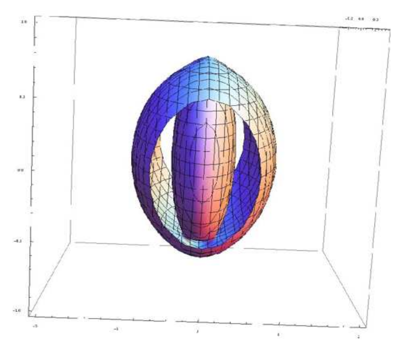

Fig. 2: Section of the characteristic surface

If (13) *is not fulfilled, then the equations* $P_i(\xi) = 0$, $i = 1, 2$, *determine two different ellipsoidal surfaces with two touching points at the endpoints of common axes (see Fig.1 and Fig.2). If conditions* (13) *and* (19) *hold, the ellipsoids coincide.*

3. Green's formulae

Here we apply the results of [Du1] and derive Green's formulae for Maxwell's equations (5), needed for our analysis. For convenience we also use the notation $\gamma^{\pm}_{\mathscr{S}}\boldsymbol{U} = \boldsymbol{U}^{\pm}$

Lemma 3.1. *For a domain $\Omega^+ \subset \mathbb{R}^3$ with a smooth boundary $\mathscr{S} := \partial\Omega^+$ the following Green's formula holds*

$$(\mathbf{curl}\,\boldsymbol{U}, \boldsymbol{V})_{\Omega^+} - (\boldsymbol{U}, \mathbf{curl}\,\boldsymbol{V})_{\Omega^+} = (\boldsymbol{\nu}\times\boldsymbol{U}^+, \boldsymbol{V}^+)_{\mathscr{S}} = -(\boldsymbol{U}^+, \boldsymbol{\nu}\times\boldsymbol{V}^+)_{\mathscr{S}}, \qquad \boldsymbol{U}, \boldsymbol{V} \in \mathbb{H}^1(\Omega^+)\,, \tag{42}$$

where $\boldsymbol{U}^+ = (U_1^+, U_2^+, U_3^+)^\top$ denotes the trace on the boundary $\mathscr{S}$,

$$(\boldsymbol{U}, \boldsymbol{V})_{\boldsymbol{G}} := \int\limits_G \langle \boldsymbol{U}(x), \boldsymbol{V}(x)\rangle\, dx.$$

In particular,

$$(\mathbf{curl}\,\boldsymbol{U}, \nabla\, v)_{\Omega^+} = -(\boldsymbol{U}^+, \mathscr{M}_{\mathscr{S}} v^+)_{\mathscr{S}}\,, \qquad \boldsymbol{U} \in \mathbb{H}^1(\Omega^+), \quad v \in \mathbb{H}^1(\Omega^+)\,, \tag{43}$$

where the brackets $(\cdot,\cdot)_{\mathscr{S}}$ denotes the duality between adjoint spaces $\mathbb{H}^s(\mathscr{S})$ and $\mathbb{H}^{-s}(\mathscr{S})$,

$$\mathscr{M}_{\mathscr{S}} := \boldsymbol{\nu}\times\nabla = (\mathscr{M}_{23}, \mathscr{M}_{31}, \mathscr{M}_{12})^\top, \tag{44}$$

and $\mathscr{M}_{jk} = \nu_j\partial_k - \nu_k\partial_j$ are Stoke's tangential differentiation operators on the boundary surface $\mathscr{S}$.

Proof: Formula (42) is a simple consequence of the Gauss integration by parts formula

$$(\partial_j\, u, \psi)_{\Omega^+} = (\nu_j u^+, \psi^+)_{\mathscr{S}} - (u, \partial_j\,\psi)_{\Omega^+}\,, \qquad u, \psi \in \mathbb{H}^1(\Omega^+) \tag{45}$$

In fact,

$$\begin{aligned}
&(\mathbf{curl}\,\boldsymbol{U}, \boldsymbol{V})_{\Omega^+} \\
&\quad = \int_{\Omega^+} \big[(\partial_2 U_3 - \partial_3\, U_2)V^1 + (\partial_3 U_1 - \partial_1\, U_3)V^2 + (\partial_1 U_2 - \partial_2\, U_1)V^3\big]\, dx \\
&\quad = \int_{\mathscr{S}} \big[(\nu_2 U_3^+ - \nu_3 U_2^+)V_1^+ + (\nu_3 U_1^+ - \nu_1\, U_3^+)V_2^+ + (\nu_1 U_2^+ - \nu_2\, U_1^+)V_3^+\big]\, dS \\
&\qquad + \int_{\Omega^+} \big[(\partial_2 V_3 - \partial_3\, V_2)U_1 + (\partial_3 V_1 - \partial_1\, V_3)U_2 + (\partial_1 V_2 - \partial_2\, V_1)U_3\big]\, dx \\
&\quad = -(\boldsymbol{\nu}\times\boldsymbol{U}^+, \boldsymbol{V}^+)_{\mathscr{S}} + (\boldsymbol{U}, \mathbf{curl}\boldsymbol{V})_{\Omega^+}.
\end{aligned} \tag{46}$$

Since $\boldsymbol{\nu}\times\boldsymbol{U}$ can be interpreted as the application of the skew symmetric matrix

$$\boldsymbol{\nu}\times\boldsymbol{U} = \mathscr{N}\boldsymbol{U}\,, \qquad \mathscr{N} := \begin{bmatrix} 0 & -\nu_3 & \nu_2 \\ \nu_3 & 0 & -\nu_1 \\ -\nu_2 & \nu_1 & 0 \end{bmatrix} = -\mathscr{N}^\top,$$

we get

$$(\boldsymbol{\nu}\times\boldsymbol{U}^+, \boldsymbol{V}^+) = (\mathscr{N}\boldsymbol{U}^+, \boldsymbol{V}^+) = -(\boldsymbol{U}^+, \mathscr{N}\boldsymbol{V}^+) = -(\boldsymbol{U}^+, \boldsymbol{\nu}\times\boldsymbol{V}^+)\,,$$

and this accomplishes the proof of (42).

To prove (43) first note that

$$\boldsymbol{\nu}\times(\nabla v)^{+}=(\boldsymbol{\nu}\times\nabla v)^{+}=(\mathscr{M}_{\mathscr{S}}v)^{+}=\mathscr{M}_{\mathscr{S}}v^{+}\qquad\forall\, v\in\mathbb{H}^{2}(\Omega^{+}),\tag{47}$$

because $\mathscr{M}_{jk}$ are tangential derivatives (cf. (44)) and therefore it commutes with the trace operator

$$\big(\mathscr{M}_{\mathscr{S}}v\big)^{+}=\mathscr{M}_{\mathscr{S}}v^{+}.\tag{48}$$

Moreover, due to equality (48) it is sufficient to suppose $v\in\mathbb{H}^{1}(\Omega^{+})$ in (47): if $v\in\mathbb{H}^{1}(\Omega^{+})$ then $v^{+}\in\mathbb{H}^{1/2}(\mathscr{S})$ by the classical trace theorem and therefore $\big(\mathscr{M}_{\mathscr{S}}v\big)^{+}:=\mathscr{M}_{\mathscr{S}}v^{+}\in\mathbb{H}^{-1/2}(\mathscr{S})$.

Equation (43) is a consequence of (42). In fact,

$$\begin{aligned}(\mathbf{curl}\,\boldsymbol{U},\nabla v)_{\Omega^{+}}&=-(\boldsymbol{U}^{+},\boldsymbol{\nu}\times(\nabla v)^{+})_{\mathscr{S}}+(\boldsymbol{U},\mathbf{curl}\nabla v)_{\Omega^{+}}=-(U^{+},\big(\mathscr{M}_{\mathscr{S}}v\big)^{+})_{\mathscr{S}}\\&=-(U^{+},\mathscr{M}_{\mathscr{S}}v^{+})_{\mathscr{S}}\end{aligned}$$

since $\mathbf{curl}\nabla=0$. ■

For anisotropic Maxwell's equations we have the following.

Theorem 3.2. *The operator*

$$\boldsymbol{M}_{e}=\mathbf{curl}\,\mu^{-1}\mathbf{curl}\,-\omega^{2}\varepsilon I\tag{49}$$

(cf. (6)*) is formally self adjoint $\boldsymbol{M}_{e}^{*}=\boldsymbol{M}_{e}$ and the following Green's formulae hold*

$$(\boldsymbol{M}_{e}\boldsymbol{U},\boldsymbol{V})_{\Omega^{+}}$$

$$=(\boldsymbol{\nu}\times(\mu^{-1}\mathbf{curl}\,\boldsymbol{U})^{+},\boldsymbol{V}^{+})_{\mathscr{S}}+(\mu^{-1}\mathbf{curl}\,\boldsymbol{U},\mathbf{curl}\,\boldsymbol{V})_{\Omega^{+}}-\omega^{2}(\varepsilon\,\boldsymbol{U},\boldsymbol{V})_{\Omega^{+}}\tag{50a}$$

$$=-((\mu^{-1}\mathbf{curl}\,\boldsymbol{U})^{+},\boldsymbol{\nu}\times\boldsymbol{V}^{+})_{\mathscr{S}}+(\mu^{-1}\mathbf{curl}\,\boldsymbol{U},\mathbf{curl}\,\boldsymbol{V})_{\Omega^{+}}-\omega^{2}(\varepsilon\,\boldsymbol{U},\boldsymbol{V})_{\Omega^{+}},\tag{50b}$$

$$(\boldsymbol{M}_{e}\boldsymbol{U},\boldsymbol{V})_{\Omega^{+}}-(\boldsymbol{U},\boldsymbol{M}_{e}\boldsymbol{V})_{\Omega^{+}}$$

$$=(\boldsymbol{\nu}\times(\mu^{-1}\mathbf{curl}\,\boldsymbol{U})^{+},\boldsymbol{V}^{+})_{\mathscr{S}}-(\boldsymbol{U}^{+},\boldsymbol{\nu}\times(\mu^{-1}\mathbf{curl}\,\boldsymbol{V})^{+})_{\mathscr{S}}\tag{50c}$$

$$=-((\mu^{-1}\mathbf{curl}\,\boldsymbol{U})^{+},\boldsymbol{\nu}\times\boldsymbol{V}^{+})_{\mathscr{S}}+(\boldsymbol{\nu}\times\boldsymbol{U}^{+},(\mu^{-1}\mathbf{curl}\,\boldsymbol{V})^{+})_{\mathscr{S}}\tag{50d}$$

provided $\boldsymbol{U},\boldsymbol{V}\in\mathbb{H}^{1}(\Omega^{+})$, and additionally, $\boldsymbol{M}_{e}\boldsymbol{U}\in\widetilde{\mathbb{H}}^{-1}(\Omega^{+})$ in (50a) *and* (50b)*, while $\boldsymbol{M}_{e}\boldsymbol{U}$, $\boldsymbol{M}_{e}\boldsymbol{V}\in\widetilde{\mathbb{H}}^{-1}(\Omega^{+})$ in* (50c) *and* (50d)*.*

Proof: The claimed formulae follow from Lemma (3.1). ■

4. Representation of solutions and layer potentials

In the present section we continue to apply the results of [Du1] to Maxwell's equations (also see [CK1]). For simplicity we suppose that the boundary $\mathscr{S}=\partial\Omega$ is a C^{∞} smooth surface.

Let us consider the following operators, related to the Maxwell systems (4) and (5): *Newton's potential*

$$\mathbf{N}^e_\Omega \boldsymbol{U}(x) := \int_\Omega \Phi_e(x-y)\boldsymbol{U}(y)\,dy, \qquad x\in\mathbb{R}^3, \tag{51}$$

the *single layer potential*

$$\mathbf{V}^e\boldsymbol{U}(x) := \oint_{\mathscr{S}} \Phi_e(x-\tau)\boldsymbol{U}(\tau)\,dS, \qquad x\in\mathbb{R}^3\setminus\mathscr{S}, \tag{52}$$

and the *double layer potential*

$$\mathbf{W}^e\boldsymbol{U}(x) := \oint_{\mathscr{S}} [(\gamma^e_N\Phi_e)(x-\tau)]^\top\boldsymbol{U}(\tau)\,dS, \qquad x\in\mathbb{R}^3\setminus\mathscr{S}, \tag{53}$$

where Φ_e denotes one of the fundamental solutions Φ^-_e or Φ^+_e and

$$\gamma^e_N\boldsymbol{V}(\tau) := \boldsymbol{\nu}(\tau)\times\mu^{-1}\mathbf{curl}\,\boldsymbol{V}(\tau), \qquad \tau\in\mathscr{S}, \tag{54}$$

denotes the *"magnetic" trace operator*.

Theorem 4.1. *Let Ω^+ be a bounded domain with infinitely smooth boundary $\mathscr{S}=\partial\Omega^+$ and $\Omega^- := \mathbb{R}^3\setminus\overline{\Omega}+$. The potential operators*

$$\begin{aligned}
\mathbf{N}^e_{\Omega^+} &: \mathbb{H}^s_p(\Omega^+)\to\mathbb{H}^s_p(\Omega^+),\\
\mathbf{V}^e &: \mathbb{H}^s_p(\mathscr{S})\to\mathbb{H}^{s+\frac{1}{p}-1}_{p,\mathrm{loc}}(\Omega^-),\\
&: \mathbb{H}^s_p(\mathscr{S})\to\mathbb{H}^{s+\frac{1}{p}-1}_{p}(\Omega^+),\\
\mathbf{W}^e &: \mathbb{H}^s_p(\mathscr{S})\to\mathbb{H}^{s+\frac{1}{p}-2}_{p,\mathrm{loc}}(\Omega^-),\\
&: \mathbb{H}^s_p(\mathscr{S})\to\mathbb{H}^{s+\frac{1}{p}-2}_{p}(\Omega^+),\\
\gamma_{\mathscr{S}}\mathbf{V}^e &: \mathbb{H}^s_p(\mathscr{S})\to\mathbb{H}^{s-1}_p(\mathscr{S}),\\
\gamma_{\mathscr{S}}\mathbf{W}^e &: \mathbb{H}^s_p(\mathscr{S})\to\mathbb{H}^{s-2}_p(\mathscr{S}),
\end{aligned} \tag{55}$$

are continuous for all $1<p<\infty$, $s\in\mathbb{R}$. Here $(\gamma_{\mathscr{S}}\Psi)(x) := \Psi(x)$ is the Dirichlet trace operator *on the boundary $\mathscr{S}=\partial\Omega^+$.*

Proof: The operators $\mathbf{N}^e_{\Omega^+}$, $\gamma_{\mathscr{S}}\mathbf{V}^e$, and $\gamma_{\mathscr{S}}\mathbf{W}^e$ are all pseudodifferential (abbreviated as ΨDO; cf. [DNS1, DNS2]). The symbol $\mathbf{N}^e_{\Omega^+}(\xi) = \mathscr{F}_{x\to\xi}[\Phi_e(x)]$ of the pseudodifferential operator $\mathbf{N}^e_{\Omega^+}$ coincides with the inverse symbol $M^{-1}_e(\xi)$ of the initial operator $\boldsymbol{M}_e$, which is a rational function uniformly bounded at infinity (cf. Theorem 2.2). Therefore the ΨDO $\mathbf{N}^e_{\Omega^+}$ has order 0, has the transmission property (as a ΨDO with a rational symbol), which implies the mapping property (55) for $\mathbf{N}^e_{\Omega^+}$.

For the potential operators $\mathbf{V}^e$ and $\mathbf{W}^e$ the proof is based on the above proved property of ΨDOs $\mathbf{N}^e_{\Omega^+}$ and the trace theorem and follows the proof of [Du1, Theorem 3.2].

Let us consider the following surface δ-function

$$(g \otimes \delta_{\mathscr{S}}, v)_{\mathbb{R}^3} := \oint\limits_{\mathscr{S}} g(\tau)\gamma_{\mathscr{S}} v(\tau) dS, \quad g \in C^\infty(\mathscr{S}), \quad v \in C_0^\infty(\mathbb{R}^3). \tag{56}$$

Obviously, $\operatorname{supp}(g \otimes \delta_{\mathscr{S}}) = \operatorname{supp} g \subset \mathscr{S}$.

The definition (56) is extendible to less regular functions. More precisely, the following holds: *Let* $1 < p < \infty$, $s < 0$, $g \in \mathbb{W}_p^s(\mathscr{S})$. *Then*

$$g \otimes \delta_{\mathscr{S}} \in \mathbb{H}_p^{s-\frac{1}{p'}}(\mathscr{S}) \subset \mathbb{H}_{p,\mathrm{com}}^{s-\frac{1}{p'}}(\mathbb{R}^3)\,, \tag{57}$$

where $p' = p/(p-1)$ (cf. [Du1, Lemma 4.9]).

The layer potential $\boldsymbol{V}^e$ can be written in the form

$$\begin{aligned} \boldsymbol{V}^e \boldsymbol{U}(x) := \oint_{\mathscr{S}} \Phi_e(x-\tau)\boldsymbol{U}(\tau)\, dS = \int_\Omega \Phi_e(x-y)(\boldsymbol{U} \otimes \delta_{\mathscr{S}})(y)\, dy, \\ = \mathbf{N}_\Omega^e(\boldsymbol{U} \otimes \delta_{\mathscr{S}})(x), \end{aligned} \tag{58}$$

where Ω is compact and $\mathscr{S} \subset \Omega$, and can be interpreted as a pseudodifferential operator. Assume, for simplicity, Ω is compact. From the inclusion (57) and the mapping property of the pseudodifferential operator $\mathbf{N}_\Omega^e$ in (55) we derive the mapping property of $\boldsymbol{V}^e$ in (55):

$$\begin{aligned} \left\|\boldsymbol{V}^e \boldsymbol{U} \big| \mathbb{H}_p^{s+\frac{1}{p}-1}(\Omega)\right\| \leq \left\|\mathbf{N}_\Omega^e(\boldsymbol{U} \otimes \delta_{\mathscr{S}}) \big| \mathbb{H}_p^{s-\frac{1}{p'}}(\Omega)\right\| \leq C_1 \left\|(\boldsymbol{U} \otimes \delta_{\mathscr{S}}) \big| \mathbb{H}_p^{s-\frac{1}{p'}}(\Omega)\right\| \\ \leq C_2 \left\|\boldsymbol{U} \big| \mathbb{H}_p^s(\mathscr{S})\right\|, \end{aligned}$$

provided $s < 0$.

The layer potential $\mathbf{W}^e$ is written in the form

$$\begin{aligned} \boldsymbol{W}^e \boldsymbol{U}(x) &= \oint_{\mathscr{S}} \left[\boldsymbol{T}(D_y, \boldsymbol{\nu})\Phi_e(x-\tau)\right]^\top \boldsymbol{U}(\tau)\, dS \\ &= \int_\Omega \left[\boldsymbol{T}(D_y, \mathscr{N}(y))\Phi_e(x-y)\right]^\top (\boldsymbol{U} \otimes \delta_{\mathscr{S}})(y)\, dy \\ &= \mathbf{D}_\Omega^e(\boldsymbol{U} \otimes \delta_{\mathscr{S}})(x), \quad \mathscr{S} \subset \Omega \end{aligned} \tag{59}$$

and the principal symbol of the ΨDO $\mathbf{D}_\Omega^e$ is

$$D_\Omega^e(x,\xi) := \mathscr{N}(x)\mu^{-1}\sigma_{\mathbf{curl}}(\xi)N_\Omega^e(\xi) = \mathscr{N}(x)\mu^{-1}\sigma_{\mathbf{curl}}(\xi)\mathscr{F}_{x\to\xi}\left[\Phi_e(x)\right], \tag{60}$$

$$\mathscr{N}(x) := \begin{bmatrix} 0 & -\mathscr{N}_3(x) & \mathscr{N}_2(x) \\ \mathscr{N}_3(x) & 0 & -\mathscr{N}_1(x) \\ -\mathscr{N}_2(x) & \mathscr{N}_1(x) & 0 \end{bmatrix},$$

where $(\mathscr{N}_1(x), \mathscr{N}_2(x), \mathscr{N}_3(x))^\top$ is some smooth extension of the normal vector field $\boldsymbol{\nu}(x)$ from $\mathscr{S}$ onto the domain Ω. Therefore, $\operatorname{ord} \mathbf{D}_\Omega^e = +1$ and this pseudodifferential operator has the following mapping property

$$\mathbf{D}^e_\Omega \quad : \quad \widetilde{\mathbb{H}}^s_p(\Omega) \to \mathbb{H}^{s-1}_{p,\mathrm{loc}}(\mathbb{R}^3). \tag{61}$$

From the inclusion (57) and the mapping property (61) we derive, as above, the mapping property of $\mathbf{W}^e$ in (55) provided $s < 0$.

For the case $s \geq 0$ we quote a similar proof in [Du1, Theorem 3.2] and drop the details since it needs some auxiliary assertions, proved in [Du1].

The mapping properties of ΨDOs $\gamma_{\mathscr{S}}\mathbf{V}^e$ and $\gamma_{\mathscr{S}}\mathbf{W}^e$, which are the traces of the potential operators $\mathbf{V}^e$ and $\mathbf{W}^e$, follow immediately due to the generalized trace theorem (see, e.g. [Se1]). ∎

Theorem 4.2. *Solutions of Maxwell's equations* (1) *in a compact domain* Ω^+ *with diagonal and proportional coefficients* ε *and* μ *(see* (13) *and Remark* 1.3*) are represented as follows*

$$\boldsymbol{E}(x) = \boldsymbol{W}^e(\gamma_D \boldsymbol{E})(x) - \boldsymbol{V}^e(\gamma^e_N \boldsymbol{E})(x), \qquad x \in \Omega^+. \tag{62}$$

Here $\gamma^e_N \boldsymbol{E}$ *is the "magnetic" trace operators (cf.* (54)*) and* $(\gamma_D \boldsymbol{E})(x) := \boldsymbol{E}^+(x)$ *is the "electric" trace operator on the boundary* $\mathscr{S} = \partial\Omega^+$.

Proof: By introducing the substitution

$$(\boldsymbol{M}_e\boldsymbol{U}, \boldsymbol{V})_{\Omega^+} - (\boldsymbol{U}, \boldsymbol{M}_e\boldsymbol{V})_{\Omega^+} = (\boldsymbol{\nu}\times(\mu^{-1}\mathbf{curl}\,\boldsymbol{U})^+, \boldsymbol{V}^+)_{\mathscr{S}} - (\boldsymbol{U}^+, \boldsymbol{\nu}\times(\mu^{-1}\mathbf{curl}\,\boldsymbol{V})^+)_{\mathscr{S}}$$

in the Green formula, where $\boldsymbol{U}$ is the fundamental solution $\boldsymbol{U} = \Phi_e$ and $\boldsymbol{V}$ is the electric field $\boldsymbol{V} = \boldsymbol{E}$, we obtain the representation of the solution $\boldsymbol{E}$ of the system (5). If we take into account that the Newton potential eliminates since we deal with a homogeneous system $(\boldsymbol{U}, \boldsymbol{M}_e\boldsymbol{V})_{\Omega^+} = (\Phi_e, \boldsymbol{M}_e\boldsymbol{E})_{\Omega^+} = 0$, then we see that the system (5) is equivalent to original Maxwell's equations (1) (see similar arguments in [Du1]). ∎

Remark 4.3. *The case of an unbounded domain* Ω^- *will be treated in Theorem* 5.1 *after we establish asymptotic properties of fundamental solutions.*

For non-homogeneous Maxwell's equations

$$\begin{cases} \mathbf{curl}\,\boldsymbol{H} + i\omega\varepsilon\boldsymbol{E} = \mathbf{f}\,, \\ \mathbf{curl}\,\boldsymbol{E} - i\omega\mu\boldsymbol{H} = \mathbf{g} \end{cases} \quad \text{in} \quad \Omega \tag{63}$$

the equivalent systems are

$$\begin{cases} \mathbf{curl}\,\varepsilon^{-1}\mathbf{curl}\,\boldsymbol{H} - \omega^2\mu\boldsymbol{H} = \omega^{-1}\mathbf{curl}(\varepsilon^{-1}\mathbf{f}) + \mathbf{g}\,, \\ \boldsymbol{E} = i(\omega\varepsilon)^{-1}\mathbf{curl}\,\boldsymbol{H} - i(\omega\varepsilon)^{-1}\mathbf{f} \end{cases} \quad \text{in} \quad \Omega \tag{64}$$

and

$$\begin{cases} \mathbf{curl}\,\mu^{-1}\mathbf{curl}\,\boldsymbol{E} - \omega^2\varepsilon\boldsymbol{E} = \mathbf{f} - \omega^{-1}\mathbf{curl}(\mu^{-1}\mathbf{g})\,, \\ \boldsymbol{H} = -i(\omega\mu)^{-1}\mathbf{curl}\,\boldsymbol{E} + i(\omega\mu)^{-1}\mathbf{g} \end{cases} \quad \text{in} \quad \Omega\,. \tag{65}$$

Theorem 4.4. *Solutions of Maxwell's equations* (63) *in a domain* Ω^+ *with diagonal and proportional coefficients* ε *and* μ *(see* (13) *and Remark* 1.3*) are represented as follows*

$$\boldsymbol{H}(x) = \mathbf{N}^m_{\Omega^+}\left[\omega^{-1}\mathbf{curl}(\varepsilon^{-1}\mathbf{f}) + \mathbf{g}\right](x) + \mathbf{W}^m(\gamma_D\boldsymbol{H})(x) - \mathbf{V}^m(\gamma^m_N\boldsymbol{H})(x),$$
$$\boldsymbol{E}(x) = \mathbf{N}^e_{\Omega^+}\left[\mathbf{f} - \omega^{-1}\mathbf{curl}(\mu^{-1}\mathbf{g})\right](x) + W^e(\gamma_D\boldsymbol{E})(x) - V^e(\gamma^e_N\boldsymbol{E})(x), \quad x \in \Omega^+.$$

Proof: The proof is analogous to the proof of the foregoing Theorem 4.2 with a single difference: Newton's potential does not disappear

$$(\boldsymbol{U}, \boldsymbol{M}_e\boldsymbol{V})_{\Omega^+} = (\Phi_m, \boldsymbol{M}_e\boldsymbol{E})_{\Omega^+} = \mathbf{N}^e_{\Omega^+}\left[\mathbf{f} - \omega^{-1}\mathbf{curl}(\mu^{-1}\mathbf{g})\right]$$

(cf. equation (65)). ■

5. The uniqueness of a solution

A solution $\boldsymbol{E}$ of the system (1) is called *radiating* in an unbounded domain Ω^- if the asymptotic condition

$$\partial_j\boldsymbol{E}(x) - i\,\varkappa_e\,\frac{x_j}{\mu_j\,|\widetilde{x}|}\,\boldsymbol{E}(x) = \mathscr{O}(|x|^{-2}) \;\text{ as }\; |x| \to \infty\,, \quad j = 1, 2, 3, \qquad (66)$$

$$\varkappa_e := \omega\sqrt{\kappa\,\det\,\mu}, \quad \widetilde{x} := \left(\frac{x_1}{\sqrt{\mu_1}}, \frac{x_2}{\sqrt{\mu_2}}, \frac{x_3}{\sqrt{\mu_3}}\right), \qquad (67)$$

holds uniformly in all directions $x^*/|x^*|$, where

$$x^* = (x^*_1, x^*_2, x^*_3) := \Big(\frac{x_1}{\mu_1}\,, \frac{x_2}{\mu_2}\,, \frac{x_3}{\mu_3}\Big). \qquad (68)$$

A radiating solution $\boldsymbol{H}$ of the system (1) is defined similarly.

Without loss of generality we assume that the origin of the co-ordinate system belong to the bounded domain Ω^+ and R be a sufficiently large positive number, such that the domain Ω^+ lies inside the ellipsoid

$$\Psi(x) := \frac{|\widetilde{x}|^2}{R^2} = \frac{x_1^2}{\mu_1\,R^2} + \frac{x_2^2}{\mu_2\,R^2} + \frac{x_3^2}{\mu_3\,R^2} = 1\,. \qquad (69)$$

Further, let B_R denote the interior of the ellipsoid and $\Omega^-_R := \Omega^- \cap B_R$. Note that the exterior unit normal vector to the ellipsoidal surface $\Sigma_R := \partial B_R$ defined by equation (69) at the point $x \in \Sigma_R$ reads as

$$\boldsymbol{\nu}(x) = (\nu_1(x), \nu_2(x), \nu_3(x)) := \frac{\nabla\Psi(x)}{|\nabla\Psi(x)|} = \frac{1}{|x^*|}\Big(\frac{x_1}{\mu_1}\,, \frac{x_2}{\mu_2}\,, \frac{x_3}{\mu_3}\Big), \qquad (70)$$

(cf. (68) for x^*), where,

$$\nu_j(x) = \nu_j(\widehat{x}) = \frac{x_j}{\mu_j\,|x^*|} = \frac{x^*_j}{|x^*|}\,, \quad j = 1, 2, 3. \qquad (71)$$

Theorem 5.1. *Let* $\boldsymbol{E}, \boldsymbol{H} \in \mathbb{H}^1_{\rm loc}(\Omega^-)$ *be radiating solutions to Maxwell's equations* (1) *with diagonal and proportional anisotropic coefficients* ε *and* μ *(cf.* (13) *and* (19)*) in an exterior domain* Ω^-*. Then*

$$\begin{aligned}\boldsymbol{H}(x) &= \boldsymbol{W}^m(\gamma_D\boldsymbol{H})(x) - \boldsymbol{V}^m(\gamma_N^m\boldsymbol{H})(x),\\ \boldsymbol{E}(x) &= \boldsymbol{W}^e(\gamma_D\boldsymbol{E})(x) - \boldsymbol{V}^e(\gamma_N^e\boldsymbol{E})(x), \qquad x\in\Omega^-.\end{aligned} \tag{72}$$

Proof: We prove this proposition for the electric field $\boldsymbol{E}$ and fundamental solution Φ_e^+; the proof for other cases (for Φ_e^-, for the field $\boldsymbol{H}$ and fundamental solutions $\Phi_m^\pm$) are similar.

First note that the radiation condition (66) implies

$$\mathbf{curl}\,\boldsymbol{E} - i\varkappa_e\frac{|x^*|}{|\widetilde{x}|}[\boldsymbol{\nu}(\hat{x})\times\boldsymbol{E}] = \mathscr{O}\left(|x|^{-2}\right) \quad \text{as} \quad |x|\to\infty, \tag{73}$$

and further

$$\begin{aligned}\int_{\partial B_R}\Big\{\langle\mu^{-1}\mathbf{curl}\,\boldsymbol{E},\mathbf{curl}\,\boldsymbol{E}\rangle + \frac{\varkappa_e^2|x^*|^2}{|\widetilde{x}|^2}\langle\mu^{-1}(\boldsymbol{\nu}\times\boldsymbol{E}),\boldsymbol{\nu}\times\boldsymbol{E}\rangle\\ +2\frac{\varkappa_e|x^*|}{|\widetilde{x}|}\,{\rm Im}\langle\mu^{-1}\mathbf{curl}\boldsymbol{E},\boldsymbol{\nu}\times\boldsymbol{E}\rangle\Big\}\,dS\\ = \int_{\partial B_R}\left|\mu^{-\frac12}\mathbf{curl}\,\boldsymbol{E} - i\mu^{-\frac12}\varkappa_e\frac{|x^*|}{|\widetilde{x}|}[\boldsymbol{\nu}\times\boldsymbol{E}]\right|^2 dS\to 0 \quad \text{as} \quad R\to\infty,\end{aligned}$$

where $\mu^{-\frac12}$ is a square root of μ^{-1}.

Using the fact that $c_1 \le \dfrac{|x^*|}{|\widetilde{x}|} \le c_2$ with some positive constants c_1 and c_2 for all $x\in\mathbb{R}^3$ and since the integrand in the first integral is positive for all $x\in\partial B_R$, we obtain

$$\begin{aligned}\int_{\partial B_R}\Big\{\frac{|\widetilde{x}|}{|x^*|}\langle\mu^{-1}\mathbf{curl}\,\boldsymbol{E},\mathbf{curl}\,\boldsymbol{E}\rangle + \frac{\varkappa_e^2|x^*|}{|\widetilde{x}|}\langle\mu^{-1}(\boldsymbol{\nu}\times\boldsymbol{E}),\boldsymbol{\nu}\times\boldsymbol{E}\rangle\\ +2\varkappa_e\,{\rm Im}\langle\mu^{-1}\mathbf{curl}\boldsymbol{E},\boldsymbol{\nu}\times\boldsymbol{E}\rangle\Big\}\,dS\to 0 \quad \text{as} \quad R\to\infty.\end{aligned} \tag{74}$$

Green's formula in the domain Ω_R^- gives us

$$\begin{aligned}(\mu^{-1}\mathbf{curl}\,\boldsymbol{E},\mathbf{curl}\,\boldsymbol{E})_{\Omega_R^-} - \omega^2(\varepsilon\boldsymbol{E},\boldsymbol{E})_{\Omega_R^-} + (\mu^{-1}\mathbf{curl}\,\boldsymbol{E},\boldsymbol{\nu}\times\boldsymbol{E})_{\mathscr{S}}\\ = (\mu^{-1}\mathbf{curl}\,\boldsymbol{E},\boldsymbol{\nu}\times\boldsymbol{E})_{\partial B_R}.\end{aligned}$$

Now taking the imaginary part of the last equation and applying (74) we find that

$$\begin{aligned}\int_{\partial B_R}\Big\{\frac{|\widetilde{x}|}{|x^*|}\langle\mu^{-1}\mathbf{curl}\,\boldsymbol{E},\mathbf{curl}\,\boldsymbol{E}\rangle + \frac{\varkappa_e^2|x^*|}{|\widetilde{x}|}\langle\mu^{-1}(\boldsymbol{\nu}\times\boldsymbol{E}),\boldsymbol{\nu}\times\boldsymbol{E}\rangle\Big\}\,dS\\ = -2\varkappa_e\,{\rm Im}\int_{\partial\mathscr{S}}\langle\mu^{-1}\mathbf{curl}\boldsymbol{E},\boldsymbol{\nu}\times\boldsymbol{E}\rangle dS.\end{aligned} \tag{75}$$

Since both summands in the left hand side of (75) are nonnegative, they are bounded at infinity:

$$\int_{\partial B_R} |\boldsymbol{\nu}\times\boldsymbol{E}|^2 ds = \mathscr{O}(1) \quad \text{as} \quad R\to\infty. \tag{76}$$

Write the representation formula (62) in the bounded domain Ω^-_R:

$$\begin{aligned}\boldsymbol{E}(x) &= \oint_{\partial B_R\cup\mathscr{S}} [(\gamma^e_N \Phi^+_e)(x-\tau)]^\top(\gamma_D\boldsymbol{E})(\tau)\,dS - \oint_{\partial B_R\cup\mathscr{S}} \Phi^+_e(x-\tau)(\gamma^e_N\boldsymbol{E})(\tau)\,dS,\\ &= \boldsymbol{W}^e(\gamma_D\boldsymbol{E})(x) - \boldsymbol{V}^e(\gamma^e_N\boldsymbol{E})(x) + \mathscr{I}_R,\end{aligned} \tag{77}$$

where

$$\begin{aligned}\mathscr{I}_R &= -\oint_{\partial B_R} \Phi^+_e(x-\tau)(\gamma^e_N\boldsymbol{E})(\tau)\,dS + \oint_{\partial B_R} \big[(\gamma^e_N\Phi^+_e)(x-\tau)\big]^\top(\gamma_D\boldsymbol{E})(\tau)\,dS\\ &= \oint_{\partial B_R}\big[\boldsymbol{\nu}\times\Phi^+_e(x-\tau)\big]^\top(\mu^{-1}\mathbf{curl}\,\boldsymbol{E})(\tau)\,dS\\ &\quad - \oint_{\partial B_R}\big[(\mu^{-1}\mathbf{curl}\,\Phi^+_e)(x-\tau)\big]^\top(\boldsymbol{\nu}\times\boldsymbol{E})(\tau)\,dS\\ &= \oint_{\partial B_R}\big[\boldsymbol{\nu}(\tau)\times\Phi^+_e(x-\tau)\big]^\top\left[\mu^{-1}\mathbf{curl}\,\boldsymbol{E}(\tau) - i\varkappa_e\frac{|x^*|}{|\widetilde{x}|}\mu^{-1}(\boldsymbol{\nu}(\tau)\times\boldsymbol{E}(\tau))\right]dS\\ &\quad - \oint_{\partial B_R}\left[\big[\mathbf{curl}\,\Phi^+_e(x-\tau)\big]^\top - i\varkappa_e\frac{|x^*|}{|\widetilde{x}|}\big[\boldsymbol{\nu}(\tau)\times\Phi^+_e(x-\tau)\big]^\top\right]\mu^{-1}(\boldsymbol{\nu}\times\boldsymbol{E})(\tau)\,dS.\end{aligned}$$

Since $\Phi^+_e(x) = \mathscr{O}(|x|^{-1})$ at infinity, due to (79), (73), (76) and Schwartz inequality both integrals on the right-hand side vanish as $R\to\infty$ and the claimed representation for $\boldsymbol{E}$ in (72) follows from (77). ∎

Corollary 5.2. *Radiating solutions to Maxwell's equations* (1) *with anisotropic coefficients ε and μ as in* (13) *and* (19) *in an exterior domain Ω^- have the following asymptotic behaviour:*

$$\boldsymbol{H}(x) = \mathscr{O}\left(|x|^{-1}\right), \quad \boldsymbol{E}(x) = \mathscr{O}\left(|x|^{-1}\right) \qquad \text{as} \quad |x|\to\infty, \tag{78}$$

Proof: The proof follows immediately from the representation formulae (72) since the potential operators have the indicated asymptotic behaviour automatically. ∎

Clearly, each column of the fundamental matrix $\Phi^+_e(x)$ is a radiating vector due to the asymptotic formulae (38). Moreover, we have the following asymptotic relations for sufficiently large $|x|$

$$\Phi^+_e(x) = \frac{1}{4\pi\,|\widetilde{x}|}\, e^{i\,\varkappa_e\,|\widetilde{x}|}\,\Phi_{e,\infty}(\widehat{x}) + \mathscr{O}(|x|^{-2}),$$

$$\partial_j\Phi^+_e(x) = \frac{1}{4\pi\,|\widetilde{x}|}\,\frac{i\,\varkappa_e\,x_j}{\mu_j\,|\widetilde{x}|}\, e^{i\,\varkappa_e\,|\widetilde{x}|}\,\Phi_{e,\infty}(\widehat{x}) + \mathscr{O}(|x|^{-2}),$$

$$\partial_j\Phi^+_e(x) - i\,\varkappa_e\,\frac{x_j}{\mu_j\,|\widetilde{x}|}\,\Phi^+_e(x) = \mathscr{O}(|x|^{-2}), \quad j=1,2,3, \tag{79}$$

where $\varkappa_e$ and $\widetilde{x}$ are given by (67), $\widehat{x} = x/|x|$ and $\Phi_{e,\infty}(\widehat{x})$ is defined by (38).

Further, if y belongs to a compact set and $|x|$ is sufficiently large then we have

$$|\widetilde{x}-\widetilde{y}| = |\widetilde{x}| - |\widetilde{x}|^{-1}\,\langle\widetilde{x},\widetilde{y}\rangle + \mathscr{O}(|x|^{-1})\,,$$

$$|\widetilde{x}-\widetilde{y}|^{-1} = |\widetilde{x}|^{-1} + \mathscr{O}(|x|^{-2})\,,$$

$$e^{i\,\varkappa_e\,|\widetilde{x}-\widetilde{y}|} = e^{i\,\varkappa_e\,|\widetilde{x}|}\,e^{-i\,\varkappa_e\,|\widetilde{x}|^{-1}\langle\widetilde{x},\widetilde{y}\rangle} + \mathscr{O}(|x|^{-1})\,,$$

whence it follows that

$$\Phi^+_e(x-y) = \frac{1}{4\pi\,|\widetilde{x}|}\, e^{i\,\varkappa_e\,|\widetilde{x}|}\; e^{-i\,\varkappa_e\,|\widetilde{x}|^{-1}\,\langle\widetilde{x},\widetilde{y}\rangle}\,\Phi_{e,\infty}(\widehat{x}) + \mathscr{O}(|x|^{-2}),$$

$$\partial_j\Phi^+_e(x-y) = \frac{1}{4\pi\,|\widetilde{x}|}\,\frac{i\,\varkappa_e\,x_j}{\mu_j\,|\widetilde{x}|}\, e^{i\,\varkappa_e\,|\widetilde{x}|}\,e^{-i\,\varkappa_e\,|\widetilde{x}|^{-1}\langle\widetilde{x},\widetilde{y}\rangle}\,\Phi_{e,\infty}(\widehat{x}) + \mathscr{O}(|x|^{-2}),$$

$$\partial_j\Phi^+_e(x-y) - i\,\varkappa_e\,\frac{x_j}{\mu_j\,|\widetilde{x}|}\,\Phi^+_e(x-y) = \mathscr{O}(|x|^{-2}), \quad j=1,2,3.$$

These formulae can be differentiated arbitrarily many times

$$\partial^\alpha_x\partial^\beta_y\Phi^+_e(x-y) - (-1)^{|\beta|}\left(\frac{i\,\varkappa_e}{|\widetilde{x}|}\right)^{|\alpha+\beta|}\frac{x^{\alpha+\beta}}{\tilde{\mu}^{\alpha+\beta}}\Phi^+_e(x-y) = \mathscr{O}(|x|^{-2}) \tag{80}$$
$$\forall\,\alpha\,,\,\beta\in\mathbb{N}^3_0 \quad \text{as} \quad |x|\to\infty, \quad |y|\le M<\infty,$$

where besides standard notation x^α and ∂^α_x we use $\tilde{\mu} := (\mu_1,\mu_2,\mu_3)$, $\tilde{\mu}^\alpha := \mu_1^{\alpha_1}\mu_2^{\alpha_2}\mu_3^{\alpha_3}$.

Applying the above asymptotic relations and taking into account that radiating solutions to the homogeneous equation $\mathbf{M}_e(D)\boldsymbol{E}(x) = 0$ in the outer domain Ω^- are representable by linear combination of the single and double layer potentials (see Theorem 5.1) we easily derive

$$\boldsymbol{E}(x) = \frac{e^{i\,\varkappa_e\,|\widetilde{x}|}}{|\widetilde{x}|}\,\boldsymbol{E}_\infty(\widehat{x}) + \mathscr{O}(|x|^{-2}), \quad \widehat{x} = \frac{x}{|x|}\,, \tag{81}$$

$$\partial_j\boldsymbol{E}(x) = \frac{i\,\varkappa_e\,x_j}{\mu_j\,|\widetilde{x}|}\,\frac{e^{i\,\varkappa_e\,|\widetilde{x}|}}{|\widetilde{x}|}\,\boldsymbol{E}_\infty(\widehat{x}) + \mathscr{O}(|x|^{-2}), \quad j=1,2,3, \tag{82}$$

where $\boldsymbol{E}_\infty(\widehat{x}) = (E_{1,\infty}(\widehat{x}), E_{2,\infty}(\widehat{x}), E_{3,\infty}(\widehat{x}))^\top$ is the *far field pattern* of the radiating vector $\boldsymbol{E}$, cf. (38). Note that these asymptotic relations can be differentiated arbitrarily many times as well (cf. (80)):

$$\partial^\alpha \boldsymbol{E}(x) - \left(\frac{i\,\varkappa_e}{|\widetilde{x}|}\right)^{|\alpha|} \frac{x^\alpha}{\widetilde{\mu}^\alpha} \boldsymbol{E}_\infty(\widehat{x}) = \mathscr{O}(|x|^{-2}) \qquad \forall\,\alpha \in \mathbb{N}_0^3. \tag{83}$$

Now we prove the uniqueness theorems for the above formulated exterior boundary value problems.

Theorem 5.3. *Let $\boldsymbol{E}$ be a radiating solution to the homogeneous equation*

$$\mathbf{M}_e(D)\,\boldsymbol{E} = \mathbf{curl}\,\mu^{-1}\mathbf{curl}\,\boldsymbol{E} - \omega^2\,\varepsilon\,\boldsymbol{E} = 0 \tag{84}$$

in Ω^- satisfying the homogeneous boundary conditions for the "electric", "magnetic" or "mixed" problems on $\partial\Omega^-$, cf. (8a)–(8d)*. Then $\boldsymbol{E}$ vanishes identically in Ω^-.*

Proof: Let $\boldsymbol{U}$ be a solution of the homogenous exterior "electric", "magnetic" or "mixed" problem. By Green's formula (50b) for the domain Ω_R^- with vectors $\boldsymbol{U} = \boldsymbol{E}$ and $\boldsymbol{V} = \overline{\boldsymbol{E}}$, we obtain

$$-\int\limits_{\Sigma_R} \langle \mu^{-1}\,\mathbf{curl}\,\boldsymbol{E}, [\,\boldsymbol{\nu}\times\boldsymbol{E}\,]\rangle\, d\Sigma_R + \int\limits_{\Omega_R^-} \langle \mu^{-1}\,\mathbf{curl}\,\boldsymbol{E}, \mathbf{curl}\,\boldsymbol{E}\rangle\, dx - \omega^2 \int\limits_{\Omega_R^-} \langle \varepsilon\,\boldsymbol{E}, \boldsymbol{E}\rangle\, dx = 0\,, \tag{85}$$

where $\boldsymbol{\nu}$ is the exterior unit normal vector to Σ_R . Note that the surface integral over $\mathscr{S}$ expires due to the homogenous boundary conditions. Since the matrices μ and ε are positive definite the second and third summands in the left hand side expression of (85) are real and we conclude

$$\mathrm{Im} \int\limits_{\Sigma_R} \langle \mu^{-1}\,\mathbf{curl}\,\boldsymbol{E}, [\,\boldsymbol{\nu}\times\boldsymbol{E}\,]\rangle\, d\Sigma_R = 0\,. \tag{86}$$

In view of (68) the radiation condition (82) can be rewritten as

$$\partial_j \boldsymbol{E}(x) = \frac{i\,\varkappa_e}{|\widetilde{x}|}\,\frac{e^{i\,\varkappa_e\,|\widetilde{x}|}}{|\widetilde{x}|}\, x_j^*\,\boldsymbol{E}_\infty(\widehat{x}) + \mathscr{O}(|x|^{-2}), \quad j = 1,2,3. \tag{87}$$

Therefore for sufficiently large R and for $x \in \Sigma_R$ by (71) we have

$$\begin{aligned} \mathbf{curl}\,\boldsymbol{E}(x) &= \nabla\times\boldsymbol{E}(x) = \frac{i\,\varkappa_e}{|\widetilde{x}|^2}\, e^{i\,\varkappa_e\,|\widetilde{x}|}\,[\,x^*\times\boldsymbol{E}_\infty(\widehat{x})\,] + \mathscr{O}(|x|^{-2}) \\ &= \frac{i\,\varkappa_e\,|x^*|}{|\widetilde{x}|^2}\, e^{i\,\varkappa_e\,|\widetilde{x}|}\,[\,\boldsymbol{\nu}(\widehat{x})\times\boldsymbol{E}_\infty(\widehat{x})\,] + \mathscr{O}(|x|^{-2}), \quad j = 1,2,3. \end{aligned} \tag{88}$$

Take into account the asymptotic formulae (81) and (88) and transform equation (86)

$$\mathrm{Im} \int\limits_{\Sigma_R} \frac{i\,\varkappa_e\,|x^*|}{|\widetilde{x}|^3}\,\langle \mu^{-1}\,[\,\boldsymbol{\nu}(\widehat{x})\times\boldsymbol{E}_\infty(\widehat{x})\,], [\,\boldsymbol{\nu}(\widehat{x})\times\boldsymbol{E}_\infty(\widehat{x})\,]\rangle\, d\Sigma_R + \mathscr{O}(R^{-1}) = 0\,. \tag{89}$$

It can be easily verified that the integrand in (89) does not depend on R. Furthermore, since μ^{-1} is positive definite, $|\widetilde{x}| = R$ for $x \in \Sigma_R$ and $d\Sigma_R = R^2 d\Sigma_1$, by passing to the limit in (89) as $R \to \infty$ we finally arrive at the relation

$$\int\limits_{\Sigma_1} |x^*| \, \langle \mu^{-1} \, [\, \boldsymbol{\nu}(\widehat{x}) \times \boldsymbol{E}_\infty(\widehat{x}) \,], [\, \boldsymbol{\nu}(\widehat{x}) \times \boldsymbol{E}_\infty(\widehat{x}) \,] \rangle \, d\Sigma_1 = 0 \,, \tag{90}$$

where $\Sigma_1 = \partial B_1$ is the ellipsoidal surface defined by (69) with $R = 1$ and the integrand is non-negative. Note that $|x^*| \geq \min\{\mu_1^{-1/2}, \mu_2^{-1/2}, \mu_3^{-1/2}\} > 0$ for $x \in \Sigma_1$ in view of (68). Therefore from (90) it follows that

$$\langle \mu^{-1} \, [\, \boldsymbol{\nu}(\widehat{x}) \times \boldsymbol{E}_\infty(\widehat{x}) \,], [\, \boldsymbol{\nu}(\widehat{x}) \times \boldsymbol{E}_\infty(\widehat{x}) \,] \rangle = 0$$

which implies

$$\boldsymbol{\nu}(\widehat{x}) \times \boldsymbol{E}_\infty(\widehat{x}) = 0,$$

i.e.,

$$x^* \times \boldsymbol{E}_\infty(\widehat{x}) = 0,$$

where x^* is given by (68). Now from (88) we get

$$\mathbf{curl}\, \boldsymbol{E}(x) = \mathscr{O}(|x|^{-2}), \tag{91}$$

which leads to the asymptotic relation

$$\partial^\alpha \boldsymbol{E}(x) = \mathscr{O}(|x|^{-2}) \ \text{ for arbitrary multi-index } \ \alpha = (\alpha_1, \alpha_2, \alpha_3), \tag{92}$$

due to equation (84) and since we can differentiate (91) any times with respect to the variables x_j, $j = 1, 2, 3$.

To show that $\boldsymbol{E}$ vanishes identically in Ω^- we proceed as follows. From (41) and (84) it is clear that

$$\det \, \mathbf{M}_e(D) := \kappa \, \omega^2 \left[\mu_1 \partial_1^2 + \mu_2 \partial_2^2 + \mu_3 \partial_3^2 + \varkappa_e^2 \right]^2$$

and

$$\det \, \mathbf{M}_e(D) \boldsymbol{E}(x) = 0 \quad \text{in} \quad \Omega^-.$$

Therefore

$$\mathbf{\Lambda}^2(D) \, \boldsymbol{E}(x) = 0, \qquad \mathbf{\Lambda}(D) := \mu_1 \partial_1^2 + \mu_2 \partial_2^2 + \mu_3 \partial_3^2 + \varkappa_e^2 \,.$$

Let us introduce new variables z_k,

$$x_k = \sqrt{\mu_k} \, z_k, \quad k = 1, 2, 3,$$

and set

$$\boldsymbol{E}(x) = \boldsymbol{E}(\sqrt{\mu_1} \, z_1, \sqrt{\mu_2} \, z_2, \sqrt{\mu_3} \, z_3) =: \boldsymbol{V}(z).$$

It can be easily shown that the components of the vector function $\boldsymbol{V}$ solves the homogeneous equation

$$[\, \Delta + \varkappa_e^2 \,]^2 \, \boldsymbol{V}(z) = 0 \quad \text{for} \quad |z| > R_1,$$

where R_1 is some positive number and Δ is the Laplace operator. Moreover, in view of (92) we have

$$\partial^\alpha \boldsymbol{V}(z) = \mathscr{O}(|z|^{-2}) \tag{93}$$

for arbitrary multi-index α. Thus, $\boldsymbol{W}(z) := [\,\Delta + \varkappa_e^2\,]\,\boldsymbol{V}(z)$ solves the Helmholtz equation and for sufficiently large $|z|$

$$\boldsymbol{W}(z) = [\,\Delta + \varkappa_e^2\,]\,\boldsymbol{V}(z) = \mathscr{O}(|z|^{-2}),$$

i.e., there holds the equality

$$\lim_{A\to\infty} \int\limits_{|z|=A} |\boldsymbol{W}(z)|^2\,dS = 0.$$

Therefore, due to the well known Rellich-Vekua theorem $\boldsymbol{W}(z)$ vanishes identically for $|z| > R_1$, cf. [Ve1], [CK1],

$$\boldsymbol{W}(z) = [\,\Delta + \varkappa_e^2\,]\,\boldsymbol{V}(z) = 0 \quad \text{for} \quad |z| > R_1.$$

Again with the help of the asymptotic behavior (93) and the Rellich-Vekua theorem we conclude that $\boldsymbol{V}(z)$ vanishes for $|z| > R_1$. In turn this yields that $\boldsymbol{E}(x)$ vanishes for $|x| > R_2$ with some positive number R_2. Since $\boldsymbol{E}(x)$ is real analytic vector function with respect to the real variable $x \in \Omega^-$, we finally conclude that $\boldsymbol{E} = 0$ in Ω^-. ∎

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