Compl. anal. oper. theory 2 (2008), 327–343
© 2008 Birkhäuser Verlag Basel/Switzerland
1661-8254/020327-17, *published online* March 28, 2008
DOI 10.1007/s11785-008-0055-0

Complex Analysis and Operator Theory

Solvability of Singular Integro-Differential Equations with Multiple Complex Shifts

Luís Pinheiro de Castro, Roland Duduchava, and Frank-Olme Speck

Dedicated to the memory of Prof. Dr. G. S. Litvinchuk

Abstract. We consider initial value problems for functional equations on the half-axis that contain Hilbert transforms, derivatives and complex shifts. The class of problems is motivated by various applications. Results are Fredholm and invertibility criteria as well as explicit analytical solution in cases where techniques for the constructive factorization of symbol matrices are available.

Mathematics Subject Classification (2000). Primary 45E10; Secondary 47B35, 47A53, 47A68.

Keywords. Singular integro-differential equation, initial value problems, Fredholm property, solvability.

1. Introduction

In the present work we investigate a class of convolution equations on the half-line with mixed real and complex shifts in Sobolev space settings. More precisely, for $n \in \mathbb{N} := \{1, 2, \ldots\}$, we will consider the initial value problem (IVP) for an integro-differential equation

$$\begin{aligned} \psi(x) + \sum_{j=0}^{n} \frac{c_j}{\pi i} \int_0^{+\infty} \frac{\psi^{(j)}(y)}{y - x + \alpha_j}\, dy = f(x)\,, \quad x \in]0, +\infty[\,, \\ \psi(0) = d_0\,, \ldots, \psi^{(n-1)}(0) = d_{n-1}\,, \end{aligned} \tag{1.1}$$

where the function f and the constants $d_0, \ldots, d_{n-1}$, $c_0, \ldots, c_n$, $\alpha_0, \ldots, \alpha_n$, are given data with $c_n \neq 0$. The elements $\psi^{(j)}$ denote the derivatives of order j of the

This work was supported in part by *Unidade de Investigação Matemática e Aplicações* of Universidade de Aveiro and *Centro de Matemática e Aplicações* of Instituto Superior Técnico, Universidade Técnica de Lisboa, through the Portuguese Science Foundation (*FCT–Fundação para a Ciência e a Tecnologia*), co-financed by the European Community.

unknown ψ, which is sought in Bessel potential or Sobolev–Slobodeckij spaces. We will reduce the IVP (1.1) to an equivalent Wiener–Hopf equation.

Singular integral operators with multiple complex shifts on the real line $\mathbb{R} = (-\infty, +\infty)$ are translation invariant and can be written in the form

$$W_a^0 \varphi = \mathcal{F}^{-1} a \cdot \mathcal{F}\varphi \,, \quad \varphi \in \mathbb{L}^2(\mathbb{R}) \,, \tag{1.2}$$

where $a(\xi) = \mathcal{F}k(\xi)$, $\xi \in \mathbb{R}$, is the Fourier transform of the integration kernel $k(x-y)$. The following three conditions for the operator $W_a^0 : \mathbb{L}^2(\mathbb{R}) \to \mathbb{L}^2(\mathbb{R})$ are equivalent: i) W_a^0 is Fredholm (i.e., has the Fredholm property); ii) W_a^0 is invertible; iii) the Fourier symbol a is elliptic, i.e., $\inf_{\xi\in\mathbb{R}} |a(\xi)| > 0$. The same holds for the operator $W_a^0 : \mathbb{L}^p(\mathbb{R}) \to \mathbb{L}^p(\mathbb{R})$ in other Lebesgue spaces with $1 < p < +\infty$, provided the Fourier symbol a belongs to a proper class of multipliers (cf. [9, 10, 12–15]). In the recent paper [23], based on holomorphic functions properties, the uniqueness of solution to a singular integral equation with multiple complex shifts on the real line in a Lebesgue space was proved again. It is clear that non-real shifts yield analytic symbol functions in the upper or lower complex half-planes with corresponding well-known consequences.

Singular integro-differential equations with continuous coefficients on closed contours and open arcs were investigated by many authors: I. Vekua, L. Magnaradze, G. Manjavidze, J. Krikunow etc. The famous airfoil Prandtl equation in aerodynamics [16] belongs to the same class. We refer to [15, §117] for a survey on this topic and cite here only a paper of N. Vekua in [21] (also cf. [15, §117]): using a representation by potentials I. Vekua reduced the problem to the investigation of the principal part of the equation. It was proved that elliptic equations have the Fredholm property and an index formula was found. A case of equation (1.1) with complex shifts, leading to equations with discontinuous symbols without oscillations, was treated in [17].

Let us start with some definitions. We denote by $\mathcal{S}(\mathbb{R})$ the Schwartz space of rapidly decreasing C^∞ functions on $\mathbb{R}$ and by $\mathcal{S}'(\mathbb{R})$ the space of tempered distributions. The Fourier transform of $\psi \in \mathcal{S}(\mathbb{R})$ is defined by $\mathcal{F}\psi(\xi) = \int_{\mathbb{R}} \psi(x) e^{i\xi x} dx$, $\xi \in \mathbb{R}$. It is a continuous operator in $\mathcal{S}(\mathbb{R})$ and is also defined for distributions, $\mathcal{F} : \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$, by duality. For $1 < p < +\infty$, $\mathbb{L}^p(\mathbb{R})$ denotes the well-known Banach space of complex-valued Lebesgue measurable functions ψ on $\mathbb{R}$, for which $|\psi|^p$ is integrable. The space $\mathbb{H}^{s,p}(\mathbb{R})$ of Bessel potentials, with $s \in \mathbb{R}$ and $p \in]1, +\infty[$, is defined as the space of distributions $\psi \in \mathcal{S}'(\mathbb{R})$ such that $\|\psi \,|\mathbb{H}^{s,p}(\mathbb{R})\| = \left\|\mathcal{F}^{-1}\langle\cdot\rangle^s \cdot \mathcal{F}\psi \,|\mathbb{L}^p(\mathbb{R})\right\| < +\infty$, for $\langle\xi\rangle = (1+\xi^2)^{1/2}$, $\xi \in \mathbb{R}$.

$\mathbb{W}^{s,p}(\mathbb{R})$ will denote the Sobolev–Slobodeckij spaces ($s \in \mathbb{R}, 1 < p < +\infty$). For definitions and basic properties of the spaces $\mathbb{W}^{s,p}(\mathbb{R})$ we refer to [20].

Moreover, we denote by $\mathbb{H}^{s,p}_{\pm}$ the closed subspace of $\mathbb{H}^{s,p}(\mathbb{R})$ consisting of those distributions which are supported in $\overline{\mathbb{R}}_{\pm}$. $\mathbb{H}^{s,p}(\mathbb{R}_+)$ denotes the space of distributions on $\overline{\mathbb{R}}_+$ which have extensions into $\mathbb{R}$ that belong to $\mathbb{H}^{s,p}(\mathbb{R})$. The space $\mathbb{H}^{s,p}(\mathbb{R}_+)$ is endowed with the norm of the quotient space $\mathbb{H}^{s,p}(\mathbb{R})/\mathbb{H}^{s,p}_-$.

Further let $\widetilde{\mathbb{H}}^{s,p}(\mathbb{R}_+)$ be the subspace of $\mathbb{H}^{s,p}(\mathbb{R}_+)$ functionals that are extendable by zero to the full axis within $\mathbb{H}^{s,p}(\mathbb{R})$.

Analogous spaces are defined for the Sobolev–Slobodeckij case $\mathbb{W}^{s,p}$ [20]. For $1/p - 1 < s < 1/p$, the spaces $\widetilde{\mathbb{H}}^{s,p}(\mathbb{R}_+)$ and $\mathbb{H}^{s,p}(\mathbb{R}_+)$ coincide and $\mathbb{H}^{s,p}_+$ can be identified with the previous (by restriction to $\mathbb{R}_+$ or zero extension, respectively). The same holds true for $\widetilde{\mathbb{W}}^{s,p}(\mathbb{R}_+)$ and $\mathbb{W}^{s,p}(\mathbb{R}_+)$ (cf. [20]) and $\mathbb{W}^{s,p}_+$, respectively. For $s = 0$ these spaces coincide with the Lebesgue spaces and we use $\mathbb{L}^p(\mathbb{R}_+)$ instead of $\mathbb{H}^{0,p}(\mathbb{R}_+)$ and L^p_+ for $\mathbb{H}^{s,p}_+$.

Remark 1.1. The above notation is used in [8] and [11]. It is also common to denote the present space $\mathbb{H}^{s,p}_+$ by $\widetilde{\mathbb{H}}^{s,p}(\mathbb{R}_+)$, see [20] and some work of the authors [2]. The identification of both spaces is immediate (by restriction and zero extension) for positive s but not valid in general:

$$r_+\mathbb{H}^{s,p}_+ = \widetilde{\mathbb{H}}^{s,p}(\mathbb{R}_+) \quad \texttt{iff} \quad s \geq \frac{1}{p} - 1\,. \tag{1.3}$$

For $s < 1/p - 1$, the space $\mathbb{H}^{s,p}_+$ contains distributions F supported only at the origin $\operatorname{supp} F = \{0\}$ (e.g., the δ-functional and its derivatives), but the operator r_+ (in the distributional sense) annihilates such terms. The δ-functional does not belong to $\widetilde{\mathbb{H}}^{s,p}(\mathbb{R}_+)$ nor to $\widetilde{\mathbb{W}}^{s,p}(\mathbb{R}_+)$ for any s and p.

In the next Proposition 1.2 we expose interpolation properties of the spaces defined in the present section. For the proof and further details we refer to [20, §2.4.1, §2.4.2, §2.10.1, §2.10.4].

Proposition 1.2. *Let*

$$s_0, s_1 \in \mathbb{R}\,, \quad 0 < \theta < 1\,, \quad 1 \leq p_0, p_1, r \leq \infty\,,$$
$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}\,, \quad s = (1-\theta)s_0 + \theta s_1\,.$$

For the real $(\,\cdot\,,\,\cdot\,)_{\theta,p}$ *and the complex* $(\,\cdot\,,\,\cdot\,)_\theta$ *interpolation functors the following holds:*

i. $(\mathbb{H}^{s_0,p_0}(\mathbb{R}), \mathbb{H}^{s_1,p_1}(\mathbb{R}))_\theta = \mathbb{H}^{s,p}(\mathbb{R})$*;*
ii. $(\mathbb{H}^{s_0,r}(\mathbb{R}), \mathbb{H}^{s_1,r}(\mathbb{R}))_{\theta,r} = \mathbb{W}^{s,r}(\mathbb{R})$ *provided* $s_0 \neq s_1$*;*
iii. $(\mathbb{W}^{s_0,p_0}(\mathbb{R}), \mathbb{W}^{s_1,p_1}(\mathbb{R}))_\theta = \mathbb{W}^{s,p}(\mathbb{R})$.

The same interpolation results hold if the spaces $\mathbb{H}^{s_j,p_j}(\mathbb{R})$ *are replaced by the spaces* $\mathbb{H}^{s_j,p_j}_+$, $\widetilde{\mathbb{H}}^{s_j,p_j}(\mathbb{R}_+)$, $\mathbb{H}^{s_j,p_j}(\mathbb{R}_+)$ *and the space* $\mathbb{W}^{s_j,p_j}(\mathbb{R})$ *by* $\mathbb{W}^{s_j,p_j}_+$, $\widetilde{\mathbb{W}}^{s_j,p_j}(\mathbb{R}_+)$, $\mathbb{W}^{s_j,p_j}(\mathbb{R}_+)$*, respectively.*

Corollary 1.3. *Let for an operator*

$$A\,:\,\mathbb{H}^{s,p}(\mathbb{R}) \longrightarrow \mathbb{H}^{s-r,p}(\mathbb{R}) \tag{1.4}$$

hold one of the following properties: (i) *A is an invertible operator,* (ii) *A is a semi-invertible operator* (*i.e., A is invertible from the left or A is invertible from the right*), (iii) *A is a Fredholm operator,* (iv) *A is a semi-Fredholm operator* (*i.e.,* *A has a left or has a right regularizer), for some* $p \in (1, +\infty)$ *and all* $s \in (s_0, s_1)$, *with* $-\infty < s_0 < s_1 < +\infty$.

Then, for the same $p \in (1, +\infty)$ *and all* $s \in (s_0, s_1)$, *the operator*

$$A : \mathbb{W}^{s,p}(\mathbb{R}) \longrightarrow \mathbb{W}^{s-r,p}(\mathbb{R}) \tag{1.5}$$

is bounded or is (i) *invertible,* (ii) *semi-invertible,* (iii) *Fredholm,* (iv) *semi-Fredholm, respectively. Moreover, if* A (*in* (1.4) *and* (1.5)) *is a Fredholm operator then it has the same kernel and cokernel in all these spaces.*

The same invertibility, semi-invertibility, Fredholm and semi-Fredholm properties hold for arbitrary continuous operator mapping any couple of Bessel potential $\mathbb{H}^{s,p}_+$, $\widetilde{\mathbb{H}}^{s,p}(\mathbb{R}_+)$, $\mathbb{H}^{s,p}(\mathbb{R}_+)$ *and Sobolev–Slobodeckij* $\mathbb{W}^{s,p}_+$, $\widetilde{\mathbb{W}}^{s,p}(\mathbb{R}_+)$, $\mathbb{W}^{s,p}(\mathbb{R}_+)$ *spaces.*

Proof. The boundedness property follows due to the interpolation in Proposition 1.2.ii.

Concerning the invertibility: If A^{-1} is the inverse to A, it is the same for all s and, due to Proposition 1.2.ii, the operator $A^{-1} : \mathbb{W}^{s-r,p}(\mathbb{R}) \to \mathbb{W}^{s,p}(\mathbb{R})$ is then bounded for all $s \in (s_0, s_1)$ and, obviously, is the inverse to A in (1.5).

Similarly: if $A : \mathbb{H}^{s,p}(\mathbb{R}) \to \mathbb{H}^{s-r,p}(\mathbb{R})$ is Fredholm, then it has a regularizer R fulfilling $RA = I + K_1$ and $AR = I + K_2$, where $R : \mathbb{H}^{s-r,p}(\mathbb{R}) \to \mathbb{H}^{s,p}(\mathbb{R})$ is bounded, and $K_1 : \mathbb{H}^{s,p}(\mathbb{R}) \to \mathbb{H}^{s,p}(\mathbb{R})$ and $K_2 : \mathbb{H}^{s-r,p}(\mathbb{R}) \to \mathbb{H}^{s-r,p}(\mathbb{R})$ are finite rank operators (i.e., have finite dimensional images). Using the stability results, we can choose a regularizer R so that the finite rank operators K_1 and K_2 will be bounded for all $s \in (s_0, s_1)$. Then, obviously, R is a regularizer for all $s \in (s_0, s_1)$ and, due to Proposition 1.2.ii, $R : \mathbb{W}^{s-r,p}(\mathbb{R}) \to \mathbb{W}^{s,p}(\mathbb{R})$ is bounded for all $s \in (s_0, s_1)$ and is a regularizer to A in (1.5).

Moreover, it is known that if A in (1.5) is Fredholm and has the same regularizer for $p \in (p_0, p_1)$ and all $s \in (s_0, s_1)$, then its kernel and cokernel is the same in these spaces (cf., e.g., [6]).

The remaining cases (when A is semi-invertible or semi-Fredholm, and when the spaces are different $\mathbb{H}^{s,p}_+$, $\widetilde{\mathbb{H}}^{s,p}(\mathbb{R}_+)$ etc.) are treated similarly. □

Due to the foregoing Proposition 1.2 and Corollary 1.3, the results of the present article hold for both scales of spaces ($\mathbb{H}$ and $\mathbb{W}$) simultaneously. Therefore, $\mathbb{X}^{s,p}(\mathbb{R})$, $\mathbb{X}^{s,p}_+$, $\mathbb{X}^{s,p}(\mathbb{R}_+)$ and $\widetilde{\mathbb{X}}^{s,p}(\mathbb{R}_+)$ will stand for the corresponding spaces either in the $\mathbb{H}$ or in the $\mathbb{W}$ scales. Anyway, some techniques are valid only for the $\mathbb{H}$ scales. In these cases, the $\mathbb{W}$ spaces are treated separately.

2. Wiener–Hopf operators associated with the initial value problem

The initial value problem (1.1) will be considered in the following setting

$$f \in \mathbb{X}^{s-n,p}(\mathbb{R}_+)\,, \quad \psi \in \mathbb{X}^{s,p}(\mathbb{R}_+)\,, \quad 1 < p < +\infty\,, \quad n+\frac{1}{p}-1 < s < n+\frac{1}{p}\,. \tag{2.1}$$

The conditions on s and p ensure the existence of the values $\psi(0),\dots,\psi^{n-1}(0)$, because, due to Sobolev's lemma (cf. [20]), the embedding $\mathbb{X}^{s,p}(\mathbb{R}_+)\subset C^{n-1}(\mathbb{R}_+)$ is continuous.

By introducing a new unknown function

$$\varphi(t) := \psi(t) - e^{-t}\sum_{k=0}^{n-1}\frac{(-t)^k}{k!}d_k\,, \tag{2.2}$$

we find easily that the initial conditions for $\psi\in\mathbb{X}^{s,p}(\mathbb{R}_+)$ in (1.1) are transformed into

$$\varphi\in\mathbb{X}^{s,p}(\mathbb{R}_+)\,,\quad \varphi(0)=\cdots=\varphi^{(n-1)}(0)=0\,. \tag{2.3}$$

Since (2.3) can also be written in a compact form $\varphi\in\mathbb{X}^{s,p}_+$, we have proved the following.

Theorem 2.1. *The IVP* (1.1) *is equivalent to the following equation:*

$$\varphi(x)+\sum_{j=0}^{n}\frac{c_j}{\pi i}\int_0^{+\infty}\frac{\varphi^{(j)}(y)}{y-x+\alpha_j}\,dy = g(x)\,,\quad x\in\mathbb{R}_+\,, \tag{2.4}$$

$$g\in\mathbb{X}^{s-n,p}(\mathbb{R}_+)\,,\quad \varphi\in\widetilde{\mathbb{X}}^{s,p}(\mathbb{R}_+)\,,$$
$$1<p<+\infty\,,\quad n+\frac{1}{p}-1<s<n+\frac{1}{p}\,,$$

where the solution φ is related to the solution ψ of (1.1) *by formulae* (2.2).

Equation (2.4) may be viewed as a convolution integral equation with shifts on the half-line (if interpreting now the above φ in the role of its extension by zero to the full real line):

$$r_+\varphi + r_+\sum_{j=0}^{n}c_jT_{\alpha_j}K*\varphi^{(j)} = r_+(\delta*\varphi)+r_+\sum_{j=0}^{n}c_jT_{\alpha_j}K*\varphi^{(j)} = g\,,$$
$$g\in\mathbb{X}^{s-n,p}(\mathbb{R}_+)\,,\quad \varphi\in\mathbb{X}^{s,p}_+ \tag{2.5}$$

where r_+ denotes the restriction from $\mathbb{R}$ to $\mathbb{R}_+$, and δ is the Dirac delta function. In addition, $(T_{\alpha_j}\varphi)(\xi)=\varphi(\xi-\alpha_j)$ is the α_j-shift operator, $K(\xi)=1/\pi i\xi$ is the singular kernel, and $(k*\varphi)(x)=\int_{-\infty}^{+\infty}k(x-y)\varphi(y)\,dy$, $x\in\mathbb{R}$, denotes, as usual, the convolution.

The Hilbert transformation

$$S_{\mathbb{R}}\varphi(x)=(K*\varphi)(x)=\frac{1}{\pi i}\int_{\mathbb{R}}\frac{\varphi(y)}{y-x}\,dy\,, \tag{2.6}$$

is understood in the Cauchy principal value sense (cf., e.g., [15], [14, Chapter II, Section 1]). In the operators sense $S_{\mathbb{R}}$ and T_{α_j} are convolutions

$$S_{\mathbb{R}}=\mathcal{F}^{-1}(-\operatorname{sign}\xi)\cdot\mathcal{F} \tag{2.7}$$

(cf. [5]) and

$$T_{\alpha_j} = \mathcal{F}^{-1} \exp(i\alpha_j\xi)\chi_j(\xi) \cdot \mathcal{F}\,, \tag{2.8}$$

$$\chi_j(\xi) = \begin{cases} 1 & \text{if} \quad \alpha_j \in \mathbb{R}\,, \\ \chi_+(\xi) & \text{if} \quad \Im m\, \alpha_j > 0\,, \\ \chi_-(\xi) & \text{if} \quad \Im m\, \alpha_j < 0\,, \end{cases} \tag{2.9}$$

where $\chi_\pm(\xi)$ denote the characteristic (indicator) functions of the positive or negative half-line, respectively.

Since $\mathcal{F}\varphi^{(j)} = (-i\xi)^j \mathcal{F}\varphi$, (2.4) can be interpreted as a convolution equation $W_a\varphi = g$ where

$$W_a = r_+\mathcal{F}^{-1} a \cdot \mathcal{F}\,, \tag{2.10}$$

$$a(\xi) = 1 - \sum_{j=0}^{n} c_j(\operatorname{sign}\xi)(-i\xi)^j \exp(i\alpha_j\xi)\chi_j(\xi)\,.$$

Therefore, we are interested to study the convolution type operator W_a and the solvability properties of the corresponding equation

$$W_a\varphi = g\,, \tag{2.11}$$

$$W_a : \mathbb{X}^{s,p}_+ \to \mathbb{X}^{s-n,p}(\mathbb{R}_+)$$

given by (2.10).

Corollary 2.2. *The IVP* (1.1) *and equation* (2.4) *are equivalent to the equation* (2.11) *with the operator* W_a *in* (2.11), *provided the symbol* a *in* (2.11) *is defined by* (2.10) *and the solution* φ *is related to the solution* ψ *of* (1.1) *by formulae* (2.2).

From the above-mentioned reduction and from the structure of the spaces in the domain of (2.11) it is clear that the IVP (1.1) is equivalent to the equation $W_a\varphi = g$ provided the solutions are related by formula (2.2).

Theorem 2.3 (Lifting of Wiener–Hopf operators in spaces of Bessel potentials). *The operator* W_a *in* (2.10) *and* (2.11) *is well-defined and continuous. In the* $\mathbb{H}$ *scale it is toplinear* (*i.e., algebraically and topologically*) *equivalent to a Wiener–Hopf operator acting on Lebesgue spaces*

$$W_{a_{s,n}} = r_+\Lambda_-^{s-n}\ell^{(s-n)} W_a \Lambda_+^{-s} \ : \ \mathbb{L}^p_+ \to \mathbb{L}^p(\mathbb{R}_+)\,,$$

$$a_{s,n}(\xi) = \zeta^s(\xi)\frac{a(\xi)}{(\xi - i)^n}\,, \quad \zeta^s(\xi) := \left(\frac{\xi - i}{\xi + i}\right)^s\,, \tag{2.12}$$

where

$$\Lambda_+^{-s} = \mathcal{F}^{-1}(\xi + i)^{-s} \cdot \mathcal{F} : \mathbb{L}^p_+ \to \widetilde{\mathbb{H}}^{s,p}\,,$$

$$r_+\Lambda_-^{s-n}\ell^{(s-n)} = W_{(\xi-i)^{s-n}}\ell^{(s-n)} : \mathbb{H}^{s-n,p}(\mathbb{R}_+) \to \mathbb{L}^p(\mathbb{R}_+) \tag{2.13}$$

are isomorphisms between the corresponding spaces. Here $\ell^{(s-n)} : \mathbb{H}^{s-n,p}(\mathbb{R}_+) \to \mathbb{H}^{s-n,p}(\mathbb{R})$ *is a bounded extension operator and can be replaced by any other extension into the space* $\mathbb{H}^{s,p}(\mathbb{R})$*, i.e., the particular choice does not change the definition of the last mentioned operator in* (2.13).

Proof. First of all, we realize that due to the particular form of (2.9), and also due to the remaining factors of a, the function $a_{s,n}$ belongs to the algebra $\mathcal{M}^p(\mathbb{R})$ of Fourier multipliers in $\mathbb{L}^p$. In particular, this ensures that $W_{a_{s,n}} : \mathbb{L}^p_+ \to \mathbb{L}^p(\mathbb{R}_+)$ is a well-defined bounded operator.

Secondly, with the help of the isomorphisms (2.13) (cf. [8, Lemma 4.4 and Theorem 4.6] or [20, §2.3.4 and §2.10.3] and [5]), the operator W_a in (2.11) is lifted to the toplinear equivalent operator between the Lebesgue spaces (2.12). Due to the analytic continuation properties of the symbols $\lambda^{s-n}_-(\xi) := (\xi - i)^{s-n}$ and $\lambda^{-s}_+(\xi) := (\xi + i)^{-s}$ we get the identity $r_+\Lambda^{s-n}_-\ell^{(s-n)}W_a\Lambda^{-s}_+ = r_+\mathcal{F}^{-1}\lambda^{s-n}_- a \cdot \lambda^{-s}_+\mathcal{F}W_{a_{s,n}}$, which completes the proof. □

Remark 2.4. For $\mathbb{W}$ spaces, lifting works in a different way: the spaces $\mathbb{W}^{s,p}_+$ can be lifted to $\mathbb{L}^p$-spaces for integer orders $s = 1, 2, \ldots$ only (i.e., for the case of pure Sobolev spaces). But we can apply Corollary 1.3 to extend invertibility, semi-invertibility, Fredholm and semi-Fredholm properties to these spaces, provided they are valid for the Bessel potential spaces.

2.1. Case $\alpha_n = 0$

The Fourier symbol $a_{s,n}$ of the operator $W_{a_{s,n}}$ in (2.12) might have discontinuities at $\xi = 0$ and at $\xi = \infty$:

$$\begin{aligned} a_{s,n}(0 \pm 0) &= \frac{(-1)^s}{(-i)^n}\big[1 - c_0 \mathrm{sign}(\pm 1)\chi_0(0 \pm 0)\big] \\ &= e^{\pi i(s+n/2)}\big[1 \mp c_0\chi_0(0 \pm 0)\big] \qquad (2.14) \\ \chi_0(0 \pm 0) &= \begin{cases} 1 & \text{if} \quad \alpha_0 \in \mathbb{R}\,, \\ \frac{1\pm 1}{2} & \text{if} \quad \Im m\, \alpha_0 > 0\,, \\ \frac{1\mp 1}{2} & \text{if} \quad \Im m\, \alpha_0 < 0\,, \end{cases} \\ a_{s,n}(-\infty) &= e^{-\pi i n/2}c_n\,, \quad a_{s,n}(+\infty) - e^{\pi i(2s-n/2)}c_n\,. \qquad (2.15) \end{aligned}$$

To describe Fredholm properties and to obtain the index of the operator W_a we introduce a full symbol, filling the gaps between the possible discontinuities in the image of the piecewise continuous symbol $a_{s,n}$ (cf. [5]):

$$a_{s,n,p}(\xi,\eta) = \begin{cases} a_{s,n}(\xi) \quad \text{if} \quad \xi \in \mathbb{R}\,, \quad \xi \neq 0\,, \\ \frac{e^{\pi i(s+n/2)}}{2}\left[a_1 + a_2 \coth \pi\Big(\frac{i}{p} + \eta\Big)\right] \\ \qquad \text{if} \quad \xi = 0\,, \quad \eta \in \ddot{\mathbb{R}}\,, \\ \frac{e^{-\pi i n/2}c_n}{2}\left[\big[1 - e^{2\pi i s}\big] + \big[1 + e^{2\pi i s}\big]\coth \pi\Big(\frac{i}{p} + \eta\Big)\right] \\ \qquad \text{if} \quad \xi = \infty\,, \quad \eta \in \ddot{\mathbb{R}}\,, \end{cases} \qquad (2.16)$$

$$a_1 = \begin{cases} 2 & \text{if} \quad \alpha_0 \in \mathbb{R}\,, \\ 2 - c_0 & \text{if} \quad \Im m\, \alpha_0 > 0\,, \\ 2 + c_0 & \text{if} \quad \Im m\, \alpha_0 < 0\,, \end{cases} \qquad a_2 = \begin{cases} 2c_0 & \text{if} \quad \alpha_0 \in \mathbb{R}\,, \\ c_0 & \text{if} \quad \Im m\, \alpha_0 \neq 0\,, \end{cases}$$

where $\ddot{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ and $\dot{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ (cf. below) are the two point and the one point compactification of the real axis with the standard topologies.

It is easy to ascertain that the full image of the symbol $a_{s,n,p}$ in the complex plane is a closed curve. If the full symbol is elliptic

$$\inf_{\xi,\eta\in\mathbb{R}} |a_{s,n,p}(\xi,\eta)| > 0\,, \tag{2.17}$$

then the winding number $\operatorname{ind} a_{s,n,p}$ of the oriented image $\big\{a_{s,n,p}(\xi,\eta) : \xi \in \dot{\mathbb{R}}\,,\ \eta \in \ddot{\mathbb{R}}\big\}$ (the image curve of $a_{s,n,p}(\xi,\eta)$ when the variables ξ and η range through $\dot{\mathbb{R}}$ and $\ddot{\mathbb{R}}$, respectively) is called the index of $a_{s,n,p}$.

Theorem 2.5. *Let $\alpha_n = 0$. The operator W_a in* (2.10)–(2.11) *is Fredholm* (*in $\mathbb{X}$ spaces*) *if and only if the symbol $a_{s,n,p}$ in* (2.16) *is elliptic in the sense of* (2.17).

If the operator is Fredholm, then the Fredholm index is given by the formula

$$\operatorname{Ind} W_a = \dim \operatorname{Ker} W_a - \operatorname{codim} \operatorname{Im} W_a = -\operatorname{ind} a_{s,n,p}\,. \tag{2.18}$$

Moreover:

(a) *If $\operatorname{Ind} W_a = 0$, then the operator W_a has trivial kernel and cokernel.*
(b) *If $m = -\operatorname{Ind} W_a > 0$, then the operator W_a has trivial kernel and an m-dimensional cokernel.*
(c) *If $m = \operatorname{Ind} W_a > 0$, then the operator W_a has trivial cokernel and an m-dimensional kernel.*

Proof. By virtue of Corollary 1.3 it suffices to prove the theorem for the $\mathbb{H}$ scale only.

Due to the equivalence relation explicitly given in (2.12) between operators W_a and $W_{a_{s,n}}$, we conclude that they have the same Fredholm characteristics. Thus, we can proceed by studying the operator $W_{a_{s,n}}$ in (2.12).

The Fourier symbol $a_{s,n}$ of $W_{a_{s,n}}$ (cf. (2.12) and (2.10))

$$a_{s,n}(\xi) = \left(\frac{\xi - i}{\xi + i}\right)^s \frac{1}{(\xi - i)^n} \left(1 - \sum_{j=0}^{n-1} c_j(-i)^j |\xi| \xi^{j-1} \exp(i\alpha_j\xi)\chi_j(\xi) - c_n(-i)^n |\xi| \xi^{n-1}\right) \tag{2.19}$$

is a piecewise continuous function having at most two discontinuities (at zero and at infinity). Therefore, the well-known criterion for Wiener–Hopf operators with piecewise continuous Fourier symbols to have the Fredholm property (cf. [5, Theorem 4.2]) can be applied to $W_{a_{s,n}} : \mathbb{L}^p_+ \to \mathbb{L}^p(\mathbb{R}_+)$. Namely, $W_{a_{s,n}}$ is a

Fredholm operator if and only if $\inf_{\xi\,,\,\eta}|a_{s,n,p}(\xi,\eta)|>0$, where

$$a_{s,n,p}(\xi,\eta)=\begin{cases} a_{s,n}(\xi) & \text{if} \quad \xi\in\mathbb{R}\setminus\{0\} \\ \frac{1}{2}\big[a_{s,n}(0-0)+a_{s,n}(0+0)\big] & \\ +\frac{1}{2}\big[a_{s,n}(0-0)-a_{s,n}(0+0)\big]\coth\pi\Big(\frac{i}{p}+\eta\Big) & \text{if} \quad \xi=0\,,\eta\in\ddot{\mathbb{R}} \\ \frac{1}{2}\big[a_{s,n}(-\infty)+a_{s,n}(+\infty)\big] & \\ +\frac{1}{2}\big[a_{s,n}(-\infty)-a_{s,n}(+\infty)\big]\coth\pi\Big(\frac{i}{p}+\eta\Big) & \text{if} \quad \xi=\infty\,,\eta\in\ddot{\mathbb{R}}\,. \end{cases} \tag{2.20}$$

By inserting here the values (2.14) and (2.15), the full symbol (2.20) acquires the form (2.16).

The proof is completed by a reference to [5, Theorem 4.2]. □

2.1.1. Case $\alpha_n = c_0 = \cdots = c_{n-1} = 0$ and $c_n \neq 0$. Let us use the following standard notation: a real number $s\in\mathbb{R}$ decomposes into the sum

$$s=[s]+\{s\}\,,\quad [s]=0,\pm1,\pm2,\dots\,,\quad 0\le\{s\}<1\,,$$

where $[s]$ denotes the integer part and $\{s\}$ the fractional part of s.

Corollary 2.6. *Let $\alpha_n = 0$, and $c_0 = \cdots = c_{n-1} = 0$ with $c_n \neq 0$. The operator $W_a : \mathbb{X}^{s,p}_{+} \to \mathbb{X}^{s-n,p}(\mathbb{R}_+)$, defined in (2.10), is Fredholm if and only if*

$$\begin{cases} c_n\notin\mathbb{R} & \textit{if} \quad n \textit{ is even} \\ c_n(-i)^n\notin\,]0,+\infty[& \textit{if} \quad n \textit{ is odd} \end{cases} \quad \textit{and} \quad \left|\{s\}-\frac{1}{p}\right|\neq\frac{1}{2}\,. \tag{2.21}$$

Moreover, when the operator is Fredholm the index is given by

$$\operatorname{Ind}W_a=\begin{cases} N-[s] & \textit{if} \quad \left|\{s\}-\frac{1}{p}\right|<\frac{1}{2} \\ N-[s]-1 & \textit{if} \quad \{s\}-\frac{1}{p}>\frac{1}{2} \\ N-[s]+1 & \textit{if} \quad \{s\}-\frac{1}{p}<-\frac{1}{2} \end{cases}. \tag{2.22}$$

Proof. Once again, due to the equivalence relation of Theorem 2.3, we only need to consider the operator $W_{a_{s,n}}$. By noting that the Fourier symbol of $W_{a_{s,n}}$ has a special form

$$\left(a_{s,n}\right)_0(\xi)=\zeta^s(\xi)\,\lambda_-^{-n}(\xi)\left(1-c_n(-i)^n|\xi|\xi^{n-1}\right) \tag{2.23}$$

(cf. (2.12) for $\zeta^s(\xi)$) we only have to consider the simplified full symbol $a_{s,n}$ in (2.23) instead of that one in (2.12) and apply Theorem 2.5. □

Since a (well-defined) scalar Wiener–Hopf operator in Lebesgue spaces with a non-identically zero Fourier symbol have a trivial kernel or a trivial cokernel (see the *Coburn–Simonenko Theorem* [5]), from (2.22) we directly obtain the following.

Corollary 2.7. *Let $\alpha_n = c_0 = \cdots = c_{n-1} = 0$ and $c_n \notin \mathbb{R}$ if n is even or alternatively $c_n(-i)^n \notin\,]0,+\infty[$ if n is odd.*

(i) *If ($[s] = N$ and $|s - N - 1/p| < 1/2$) or ($[s] = N - 1$ and $\{s\} - 1/p > 1/2$) or ($[s] = N + 1$ and $\{s\} - 1/p < -1/2$), then W_a is an invertible operator.*

(ii) *If* $([s] > N$ *and* $|\{s\} - 1/p| < 1/2)$ *or* $([s] > N - 1$ *and* $\{s\} - 1/p > 1/2)$ *or* $([s] > N + 1$ *and* $\{s\} - 1/p < -1/2)$ *, then operator* W_a *is only left-invertible.*

(iii) *If* $([s] < N$ *and* $|\{s\} - 1/p| < 1/2)$ *or* $([s] < N - 1$ *and* $\{s\} - 1/p > 1/2)$ *or* $([s] < N+1$ *and* $\{s\}-1/p < -1/2)$ *, then operator* W_a *is only right-invertible.*

2.2. Case $\Im m\, \alpha_n \neq 0$

Theorem 2.8. *If* $\Im m\, \alpha_n \neq 0$*, then the operator* $W_a : \mathbb{X}_+^{s,p} \to \mathbb{X}^{s-n,p}(\mathbb{R}_+)$*, defined in* (2.10)–(2.11)*, is not a Fredholm operator.*

Proof. In the present case, the Fourier symbol of $W_{a_{s,n}}$,

$$a_{s,n}(\xi) = \zeta^s(\xi)\, \lambda_-^{-n}(\xi) \left(1 - \sum_{j=0}^{n} c_j (-i\xi)^j (\operatorname{sign} \xi) \chi_j(\xi) \exp(i\alpha_j \xi) \right), \tag{2.24}$$

turns out to be zero at infinity. This occurs essentially because of the combination of the value of

$$\chi_n(\xi) = \begin{cases} \chi_+(\xi) & \text{if} \quad \Im m\, \alpha_n > 0\,, \\ \chi_-(\xi) & \text{if} \quad \Im m\, \alpha_n < 0\,, \end{cases} \tag{2.25}$$

with the element $\exp(i\alpha_n\xi) = \exp(i(\Re e\, \alpha_n)\xi) \exp(-(\Im m\, \alpha_n)\xi)$, where $\Im m\, \alpha_n \neq 0$. As a consequence, we obtain in this case

$$\inf_{\xi,\eta} \left| a_{s,n,p}(\xi,\eta) \right| = 0\,. \tag{2.26}$$

Therefore, since the Fredholm criterion used in the proof of Theorem 2.5 also applies to the present case (i.e., $W_{a_{s,n}}$ is a Fredholm operator if and only if $\inf_{\xi,\eta} |a_{s,n,p}(\xi,\eta)| > 0$), the result is proved. □

2.3. Case $\Re e\, \alpha_n \neq 0$ and $\Im m\, \alpha_n = 0$

In the present case we shall see that the corresponding Fourier symbol of W_a oscillates at infinity (is almost periodic). Thus, we will start by defining some known characteristics of such functions.

First of all, by AP^p we will denote the smallest closed subalgebra of $\mathcal{M}^p(\mathbb{R})$ that contains all the functions $\exp(ic\xi)$ (with variable $\xi \in \mathbb{R}$ and real fixed constants c):

$$AP^p := \operatorname{alg}_{\mathcal{M}^p(\mathbb{R})} \big\{ \exp(ic\xi) : c \in \mathbb{R} \big\}\,. \tag{2.27}$$

Every $\phi \in AP^p$ has a finite mean value

$$M(\phi) := \lim_{\omega\to\infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} \phi(\xi)\, d\xi\,. \tag{2.28}$$

We will use the notation $\mathcal{G}Z$ for the collection of all invertible elements of an algebra Z. If $\phi \in \mathcal{G}AP^p$, Bohr proved that ϕ admits a multiplicative decomposition in the form

$$\phi(\xi) = \exp\big(i\kappa(\phi)\xi\big) \exp\big(\psi(\xi)\big) \quad \text{for all} \quad \xi \in \mathbb{R}\,, \tag{2.29}$$

where $\psi \in AP^p$, and $\kappa(\phi)$ is a real number uniquely determined by ϕ and usually called the *mean motion of* ϕ. It is also known that (for $\phi \in \mathcal{G}AP^p$)

$$\kappa(\phi) = \lim_{\omega\to+\infty} \frac{(\arg\phi)(\omega) - (\arg\phi)(-\omega)}{2\omega}\,. \tag{2.30}$$

Finally, for $\phi \in \mathcal{G}AP^p$, we will also need the so-called *geometric mean value of* ϕ:

$$d(\phi) := \exp\big(M(\psi)\big)\,, \tag{2.31}$$

where $\psi \in AP^p$ is the element that appears in the factorization (2.29).

We will subdivide the present case into two significant situations: When $c_0 = 0$, and when $c_0 \neq 0$.

2.3.1. Subcase $c_0 = 0$.

Theorem 2.9. *Let $\alpha_n \neq 0$, $\Im m\,\alpha_n = 0$ and $c_0 = 0$. The operator $W_a : \mathbb{X}_+^{s,p} \to \mathbb{X}^{s-n,p}(\mathbb{R}_+)$, defined in* (2.10)*, is normally solvable if and only if the symbol is elliptic:*

$$\inf_{\xi\in\mathbb{R}} \big|a_{s,n}(\xi)\big| > 0\,. \tag{2.32}$$

Under condition (2.32)*, we have only two possibilities:*

(i) *If $\Re e\,\alpha_n > 0$, then W_a is a left invertible operator and* $\dim \operatorname{coker} W_a = \infty$*.*
(ii) *If $\Re e\,\alpha_n < 0$, then W_a is a right invertible operator and* $\dim \ker W_a = \infty$*.*

Proof. Similarly as before, it is enough to consider here the $\mathbb{H}$ spaces case (then, by the use of Corollary 1.3, the same conclusions hold for the $\mathbb{W}$ spaces case).

Under the present conditions the Fourier symbol $a_{s,n}$ takes the form

$$\begin{aligned} a_{s,n}(\xi) = \zeta^s(\xi)\,\lambda_-^{-n}(\xi)\Bigg(1 - \sum_{j=1}^{n-1} b_j(-i)^j|\xi|\xi^{j-1}\chi_j\exp(i\alpha_j\xi) \\ - c_n(-i)^n|\xi|\xi^{n-1}\exp\Big(i\big(\Re e\,\alpha_n\big)\xi\Big)\Bigg)\,. \end{aligned} \tag{2.33}$$

Thus, $a_{s,n}$ is a semi-almost periodic function, i.e., it belongs to the smallest closed subalgebra of $\mathcal{M}^p(\mathbb{R})$ that contains the almost periodic functions and the continuous ones on the real line with a possible jump at infinity:

$$a_{s,n} \in SAP^p := \operatorname{alg}_{\mathcal{M}^p(\mathbb{R})}\big\{AP^p, C(\ddot{\mathbb{R}})\big\}\,. \tag{2.34}$$

In the remaining part of the proof we will work with $a_{s,n}/(-c_n(-i)^n)$ instead of $a_{s,n}$, for simplicity. Since the corresponding Wiener–Hopf operators are toplinear equivalent, they posses the same Fredholm characteristics.

By a well-known criterion of Sarason [18], it is possible to present a different representation of SAP^p elements that is useful in our particular case. Namely,

$$a_{s,n}/\big(-c_n(-i)^n\big) = (1-u)\phi_l + u\phi_r + \phi_0\,, \tag{2.35}$$

where $u \in C(\ddot{\mathbb{R}})$ with $u(-\infty) = 0$ and $u(+\infty) = 1$, and $\phi_0 \in C(\dot{\mathbb{R}})$ with $\phi(\infty) = 0$ and with $\phi_{l,r}\exp(i(\Re e\,\alpha_n)\xi) \in AP^p$.

Thus, noticing that $\kappa(\phi_l) = \kappa(\phi_r) = \Re e\, \alpha_n$ and $d(\phi_l) = d(\phi_r)$, theorems of Duduchava–Saginashvili about Wiener–Hopf operators with semi-almost periodic Fourier symbols in $\mathbb{L}^p$ spaces apply (cf. [7, Theorems 2.1 and 2.2]) and lead to the above statement. ☐

2.3.2. Subcase $c_0 \neq 0$. For $c_0 \neq 0$ the Fourier symbol $a_{s,n}(\xi)$ of the operator $W_{a_{s,n}}$ in (2.12) has a jump discontinuity at $\xi = 0$:

$$a_{s,n}(0 \pm 0) = e^{\pi i(s+n/2)}\big[1 \mp c_0\chi_0(0 \pm 0)\big]\,, \tag{2.36}$$

where

$$\chi_0(0 \pm 0) = \begin{cases} 1 & \text{if} \quad \alpha_0 \in \mathbb{R}\,, \\ \frac{1\pm 1}{2} & \text{if} \quad \Im m\, \alpha_0 > 0\,, \\ \frac{1\mp 1}{2} & \text{if} \quad \Im m\, \alpha_0 < 0\,. \end{cases}$$

Therefore the symbol $a_{s,n}$ belongs to the algebra of piecewise almost periodic functions, generated by the almost periodic functions and the piecewise continuous functions. In fact, $a_{s,n}$ has the jump discontinuity at zero (see above) and almost periodic oscillation at infinity (see the foregoing subsection).

Theorem 2.10. *Let $\alpha_n \neq 0$, $\Im m\, \alpha_n = 0$ and $c_0 \neq 0$. The operator $W_a : \mathbb{X}^{s,p}_+ \to \mathbb{X}^{s-n,p}(\mathbb{R}_+)$, defined in* (2.10)*, is normally solvable if and only if the numbers c_j and α_j are such that*

$$\inf_{\xi \in \mathbb{R}\setminus\{0\}} \big|a_{s,n}(\xi)\big| > 0 \tag{2.37}$$

and

(a) $c_0 \coth \pi(i/p + \eta) \neq -1$ *for all* $\eta \in \ddot{\mathbb{R}}$*, in the case of* $\alpha_0 \in \mathbb{R}$*;*
(b) $c_0 \coth \pi(i/p + \eta) \neq c_0 - 2$ *for all* $\eta \in \ddot{\mathbb{R}}$*, in the case of* $\Im m\, \alpha_0 > 0$*;*
(c) $c_0 \coth \pi(i/p + \eta) \neq -c_0 - 2$ *for all* $\eta \in \ddot{\mathbb{R}}$*, in the case of* $\Im m\, \alpha_0 < 0$*.*

Moreover, when W_a is normally solvable, the following alternative holds:

(i) *If $\Re e\, \alpha_n > 0$, then W_a is a left invertible operator and* $\dim \operatorname{coker} W_a = \infty$*;*
(ii) *If $\Re e\, \alpha_n < 0$, then W_a is a right invertible operator and* $\dim \ker W_a = \infty$*.*

Proof. Observing the above identification of the Fourier symbol of the operator (within the piecewise almost periodic class), the result follows by combining the techniques of the last subsection with the Fredholm theory for Wiener–Hopf operators with discontinuous symbols (and acting between L^p spaces; cf. [5]). In this way, the condition prompted by the discontinuity of the Fourier symbol at zero

$$\begin{aligned} a_{s,n,p}(0,\eta) = {}& \frac{1}{2}\big[a_{s,n}(0-0) + a_{s,n}(0+0)\big] \\ & + \frac{1}{2}\big[a_{s,n}(0-0) - a_{s,n}(0+0)\big] \coth \pi\Big(\frac{i}{p} + \eta\Big) \neq 0\,, \quad \text{for all} \quad \eta \in \ddot{\mathbb{R}} \end{aligned}$$

turns out to be equivalent to (a), (b) or (c) when $\alpha_0 \in \mathbb{R}$, $\Im m\, \alpha_0 > 0$ or $\Im m\, \alpha_0 < 0$, respectively.

As about propositions (i) and (ii), they follow exactly by the same reasoning as in the previous subsection. ☐

As a concretization of the previous result to the particular Hilbert spaces case ($p = 2$) and invertible Fourier symbols a (within the present case), we remark that the operator $W_a : \mathbb{X}_+^{s,2} \to \mathbb{X}^{s-n,2}(\mathbb{R}_+)$ is normally solvable if and only if:

(a') $1 + c_0 - 2c_0\mu \neq 0$ for all $\mu \in [0, 1]$, in the case of $\alpha_0 \in \mathbb{R}$;
(b') $1 - c_0\mu \neq 0$ for all $\mu \in [0, 1]$, in the case of $\Im m\, \alpha_0 > 0$;
(c') $1 + c_0 - c_0\mu \neq 0$ for all $\mu \in [0, 1]$, in the case of $\Im m\, \alpha_0 < 0$.

3. Explicit solution of the initial value problem for $\alpha_n = 0$ and $c_0 = \cdots = c_{n-1} = 0$

In the present section we suppose that the conditions (2.21) hold and hence that W_a is a Fredholm operator.

We will apply a generalized factorization of the symbol $a_{s,n}$ (see (2.23)) to describe invertibility properties and the index of the operators $W_{a_{s,n}}$.

Let us start by observing that

$$a_{s,n} = \zeta^{\{s\}-1/2}\, \widetilde{a_{s,n}}\,, \tag{3.1}$$

where

$$\widetilde{a_{s,n}}(\xi) = \zeta^{[s]-(n-1)/2}(\xi)\ \frac{1 - c_n(-i)^n |\xi| \xi^{n-1}}{(\xi^2+1)^{1/2}(\xi^2+1)^{(n-1)/2}} \tag{3.2}$$

is a non-vanishing continuous function and $\widetilde{a_{s,n}}(\pm\infty) = -c_n(-i)^n$.

We will make use of the Cauchy projections:

$$P_{\mathbb{R}}^{\pm} = \frac{1}{2}(I \pm S_{\mathbb{R}})\,, \tag{3.3}$$

where I is denoting the identity operator in $\mathbb{L}^p(\mathbb{R})$.

Definition 3.1 ([5,14]). A function $\Phi \in \mathcal{G}\mathbb{L}^\infty(\mathbb{R})$ admits *a generalized factorization relative to* $\mathbb{L}^p(\mathbb{R})$, $1 < p < +\infty$, if Φ has a representation

$$\Phi(\xi) = \Phi_-(\xi)\, \zeta^\varkappa(\xi)\, \Phi_+(\xi)\,, \quad \text{for almost all} \quad \xi \in \mathbb{R}\,, \tag{3.4}$$

where $\varkappa \in \mathbb{Z}$, $\lambda_+^{-1}\Phi_+ \in P_{\mathbb{R}}^+\mathbb{L}^q(\mathbb{R})$, $\lambda_+^{-1}\Phi_+^{-1} \in P_{\mathbb{R}}^+\mathbb{L}^p(\mathbb{R})$, $\lambda_-^{-1}\Phi_- \in P_{\mathbb{R}}^-\mathbb{L}^p(\mathbb{R})$, $\lambda_-^{-1}\Phi_-^{-1} \in P_{\mathbb{R}}^-\mathbb{L}^q(\mathbb{R})$, for $q = p/(p-1)$, and $\Phi_- S_{\mathbb{R}} \Phi_-^{-1} I$ is an operator defined on a dense subset of $\mathbb{L}^p(\mathbb{R})$ possessing a bounded extension to $\mathbb{L}^p(\mathbb{R})$.

Due to (3.1)–(3.2), and bearing in mind the last definition, we derive the following.

Theorem 3.2. *The function* $a_{s,n}$ *in* (3.1) *admits a generalized factorization with respect to* $\mathbb{L}^p(\mathbb{R})$

$$a_{s,n} = a_{s,n_-}\, \zeta^\varkappa\, a_{s,n_+} \tag{3.5}$$

where

$$\varkappa = \begin{cases} [s]-N\,, & \text{if} \quad \left|\{s\}-\frac{1}{p}\right| < \frac{1}{2} \\ [s]-N+1\,, & \text{if} \quad \{s\}-\frac{1}{p} > \frac{1}{2} \\ [s]-N-1\,, & \text{if} \quad \{s\}-\frac{1}{p} < -\frac{1}{2} \end{cases} \tag{3.6}$$

$$a_{s,n_\pm} = \begin{cases} \Psi_\pm \left(\frac{-2i}{\lambda_\mp}\right)^{\mp(\{s\}-1/2)} & \text{if} \quad \left|\{s\}-\frac{1}{p}\right| < \frac{1}{2} \\ \Psi_\pm \left(\frac{-2i}{\lambda_\mp}\right)^{\mp(\{s\}-3/2)} & \text{if} \quad \{s\}-\frac{1}{p} > \frac{1}{2} \\ \Psi_\pm \left(\frac{-2i}{\lambda_\mp}\right)^{\mp(\{s\}+1/2)} & \text{if} \quad \{s\}-\frac{1}{p} < -\frac{1}{2} \end{cases} \tag{3.7}$$

and $\Psi_\pm = \exp P^\pm_{\mathbb{R}} \ln\left(\zeta^{N-[s]}\, \widetilde{a_{s,n}}\right)$, *with* $\widetilde{a_{s,n}}$ *defined in* (3.2).

Theorem 3.3. (a) *For a given* $g \in \mathbb{X}^{s-n,p}(\mathbb{R}_+)$, *and under the conditions of proposition* (i) *in Corollary* 2.7, *the IVP* (2.4) *admits a unique solution* $\varphi \in \widetilde{\mathbb{X}}^{s,p}(\mathbb{R}_+)$ *in the form*

$$\varphi = r_+ \mathcal{F}^{-1} \lambda_+^{-s} (a_{s,n_+})^{-1} P^+_{\mathbb{R}} \lambda_-^{s-n} (a_{s,n_-})^{-1} \cdot \mathcal{F}\ell^{(s-n)} g \tag{3.8}$$

where $a_{s,n_\pm}$ *are given by* (3.7), *with* $\varkappa = 0$.

(b) *Under the conditions of proposition* (ii) (*or, respectively,* (iii)) *in Corollary* 2.7 *and corresponding situation in Theorem* 3.2, *the operator* W_a *is left-invertible* (*respectively, right-invertible*) *by*

$$W_a^- = \mathcal{F}^{-1} \lambda_+^{-s} (a_{s,n_+})^{-1} P^+_{\mathbb{R}} \zeta^{-\varkappa} P^+_{\mathbb{R}} \lambda_-^{s-n} (a_{s,n_-})^{-1} \cdot \mathcal{F}\ell^{(s-n)}\,. \tag{3.9}$$

Proof. In all considered situations in (a) and (b), the result is a consequence of the construction of the inverse or lateral inverse of W_a, from the generalized factorization of $a_{s,n}$ (presented in Theorem 3.2), and from the explicit equivalence relation between W_a and $W_{a_{s,n}}$ (cf. Theorem 2.3) in the case of $\mathbb{H}$ spaces and from Corollary 1.3 in the case of $\mathbb{W}$ spaces.

In detail, for the $\mathbb{H}$ spaces case (with Λ_-^{s-n} and Λ_+^{-s} in (2.13)), and $\varkappa = 0$, or $\varkappa > 0$, or $\varkappa < 0$, a direct computation shows that

$$\begin{aligned} W_a^- &= \Lambda_+^{-s} W^-_{a_{s,n}} \ell_0 r_+ \Lambda_-^{s-n} \ell^{(s-n)} \\ &= \Lambda_+^{-s} \ell_0 r_+ \mathcal{F}^{-1} (a_{s,n_+})^{-1} \cdot \mathcal{F} \ell_0 r_+ \mathcal{F}^{-1} \zeta^{-\varkappa} \cdot \mathcal{F} \ell_0 r_+ \mathcal{F}^{-1} (a_{s,n_-})^{-1} \cdot \mathcal{F} \Lambda_-^{s-n} \ell^{(s-n)} \\ &= \mathcal{F}^{-1} \lambda_+^{-s} (a_{s,n_+})^{-1} \cdot \mathcal{F} \ell_0 r_+ \mathcal{F}^{-1} \zeta^{-\varkappa} \cdot \mathcal{F} \ell_0 r_+ \mathcal{F}^{-1} \lambda_-^{s-n} (a_{s,n_-})^{-1} \cdot \mathcal{F} \ell^{(s-n)} \end{aligned}$$

is the inverse, left-inverse, or right-inverse of W_a, respectively and $\ell_0 : \mathbb{L}^p(\mathbb{R}_+) \to \mathbb{L}^p_+$ denoting the extension by zero. □

We end up by reminding that for piecewise almost periodic Fourier symbols the *Factorization Theory* [1] is not well developed in contrast to the continuous and even piecewise continuous cases. Therefore, we can not proceed further and obtain explicit lateral inverses to W_a for the Fredholm operators considered in the Subsection 2.3.

4. Concluding remarks

There are various possible generalizations which can be treated straightforwardly by a similar approach. For instance, multiple shifts within the same derivative are admissible, however leading to more complicated formulas.

In Case 3.2 one can find a criterion for the semi-Fredholm property, which then is equivalent to one-sided regularizability and generalized invertibility. Moreover, the operator can be factorized, in that case, into a shift and a Fredholm operator.

The reduction of the IVP (1.1) by substitution to an equivalent problem with homogeneous initial data can be interpreted in terms of a so-called (toplinear) equivalence after extension relation (see [3]) between the corresponding operators. This relation is characteristic for the operators having isomorphic kernels and co-kernels and, therefore, shows the (logical) equivalence of invertibility, of the Fredholm property, etc.

References

[1] A. Böttcher, Yu. I. Karlovich, and I. M. Spitkovsky, *Convolution Operators and Factorization of Almost Periodic Matrix Functions*, Birkhäuser, Basel, 2002.

[2] L. P. Castro, R. Duduchava, and F.-O. Speck, Finite interval convolution operators with transmission property, *Integral Equations Operator Theory* **52** (2005), 165–179.

[3] L. P. Castro and F.-O. Speck, Relations between convolution type operators on intervals and on the half-line, *Integral Equations Operator Theory* **37** (2000), 169–207.

[4] A. Devinatz and M. Shinbrot, General Wiener–Hopf Operators, *Trans. Am. Math. Soc.* **145** (1969), 467–494.

[5] R. Duduchava, *Integral Equations with Fixed Singularities*, Teubner, Leipzig, 1979.

[6] R. Duduchava, D. Natroshvili, and E. Shargorodsky, Boundary value problems of the mathematical theory of cracks, *Proceedings of I. Vekua Institute of Applied Mathematics,* Tbilisi State University **39**, 1990, 68–84.

[7] R. Duduchava and A. Saginashvili, Convolution integral equations on a half-line with semi-almost-periodic presymbols, *Differ. Equations* **17** (1981), 207–216.

[8] G. I. Èskin, *Boundary Value Problems for Elliptic Pseudodifferential Equations*, Translations of Mathematical Monographs 52, AMS, Providence, R.I., 1981.

[9] F. D. Gakhov, *Boundary Value Problems* (reprint of the 1966 English translation), Dover Publications, New York, 1990.

[10] I. C. Gohberg and M. G. Kreĭn, Systems of integral equations on a half line with kernels depending on the difference of arguments, *Amer. Math. Soc. Transl.* **14** (1960), 217–287.

[11] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman, London, 1985.

[12] N. Karapetiants and S. Samko, *Equations with Involutive Operators*, Birkhäuser, Boston, 2001.

[13] G. S. Litvinchuk, *Solvability Theory of Boundary Value Problems and Singular Integral Equations with Shift*, Mathematics and its Applications **523**, Kluwer Academic Publishers, Dordrecht, 2000.

[14] S. G. Mikhlin and S. Prössdorf, *Singular Integral Operators*, Springer-Verlag, Berlin, 1986.

[15] N. I. Muskhelishvili, *Singular Integral Equations*, Nauka, Moscow 1968; 3-rd edition in Russian (English translation of the 1-st edition: P. Noordhoff, Groningen, 1953; German translation of the 2-nd edition: Akademie Verlag, Berlin, 1965).

[16] L. Prandtl, *Prandtl's Essentials of Fluid Mechanics* (second edition), Applied Mathematical Sciences, vol. 158, Springer-Verlag, New York, 2004.

[17] A. Saginashvili, Singular integral operators on the axes with complex shifts (Russian, English summary), *Soobsh. Akademii nauk Gruz. SSR* **126**, (1987), 249–252.

[18] D. Sarason, Toeplitz operators with semi-almost periodic symbols, *Duke Math. J.* **44** (1977), 357–364.

[19] F.-O. Speck, *General Wiener–Hopf Factorization Methods*, Research Notes in Mathematics **119**, Pitman (Advanced Publishing Program), Boston, 1985.

[20] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, Barth, Leipzig, 1995.

[21] N. Vekua, On one differential boundary value problem of linear conjugation for several unknown functions in the case of open contours (Georgian), *Soobshch. AN GSSR* **21** (1958), No. 5, 513–518.

[22] N. Wiener and E. Hopf, Über eine Klasse Singulärer Integralgleichungen, *Sitzungsber. Preuss. Akad. Wiss., Phys.-Math. Kl.* **30/32** (1931), 696–706.

[23] Y. J. Xu, On the uniqueness of solution for a kind of singular integral equations, *Appl. Anal.* **82** (2003), 741–746.

Luís Pinheiro de Castro
Department of Mathematics
Research Unit "Matemática e Aplicações"
University of Aveiro
P-3810–193 Aveiro
Portugal
e-mail: castro@ua.pt

Roland Duduchava
Andrea Razmadze Mathematical Institute
M. Alexidze str. 1
Tbilisi 0193
Georgia
e-mail: dudu@rmi.acnet.ge

Frank-Olme Speck
Department of Mathematics
Research Unit "Center for Functional Analysis and Applications"
I.S.T., Technical University of Lisbon
Avenida Rovisco Pais
P-1049–001 Lisbon
Portugal
e-mail: fspeck@math.ist.utl.pt

Submitted: March 30, 2007.
Accepted: November 5, 2007.