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Lions' Lemma, Korn's Inequalities and the Lamé Operator on Hypersurfaces

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Dedicated to my friend and colleague Nikolai Vasilevski on the occasion of his 60th birthday anniversary

Abstract. We investigate partial differential equations on hypersurfaces written in the Cartesian coordinates of the ambient space. In particular, we generalize essentially Lions' Lemma, prove Korn's inequality and establish the unique continuation property from the boundary for Killing's vector fields, which are analogues of rigid motions in the Euclidean space. The obtained results, the Lax-Milgram lemma and some other results are applied to the investigation of the basic Dirichlet and Neumann boundary value problems for the Lamé equation on a hypersurface.

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Introduction

Partial differential equations (PDEs) on hypersurfaces and corresponding boundary value problems (BVPs) appear rather often in applications: see [Ha1, §72] for the heat conduction by surfaces, [Ar1, §10] for the equations of surface flow, [Ci1], [Ci3],[Ci4], [Ko2], [Go1] for thin flexural shell problems in elasticity, [AC1] for the vacuum Einstein equations describing gravitational fields, [TZ1, TW1] for the Navier-Stokes equations on spherical domains and spheres, [MM1] for minimal surfaces, [AMM1] for diffusion by surfaces, as well as the references therein. Furthermore, such equations arise naturally while studying the asymptotic behavior of solutions to elliptic boundary value problems in a neighborhood of conical points (see the classical reference [Ko1]).

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By a classical approach differential equations on surfaces are written with the help of covariant and contravariant frames, metric tensors and Christoffel symbols. To demonstrate a difference between a classical and the present approaches, let us consider an example. A surface $\mathscr{S}$ can be given by a local immersion

$$\Theta : \omega \to \mathscr{S}, \quad \omega \subset \mathbb{R}^{n-1}, \tag{0.1}$$

which means that the derivatives $\{\boldsymbol{g}_k := \partial_k \Theta\}_{k=1}^{n-1}$, constituting the *covariant frame* in the space of tangent vector fields to the surface $\mathscr{V}(\mathscr{S})$, are linearly independent. In equivalent formulation that means the Gram matrix $G_{\mathscr{S}}(\mathscr{X}) = [g_{jk}(\mathscr{X})]_{n-1\times n-1}$, $g_{jk} := \langle \boldsymbol{g}_j, \boldsymbol{g}_k \rangle$ has the inverse $G_{\mathscr{S}}^{-1}(\mathscr{X}) = [g^{jk}(\mathscr{X})]_{n-1\times n-1}$, $g^{jk} := \langle \boldsymbol{g}^j, \boldsymbol{g}^k \rangle$. Hereafter

$$\langle \boldsymbol{U}, \boldsymbol{V} \rangle := \sum_{j=1}^{n} U_j^0 V_j^0, \qquad \boldsymbol{U} = (U_1^0, \dots, U_n^0)^\top \in \mathbb{R}^n, \quad \boldsymbol{V} = (V_1^0, \dots, V_n^0)^\top \in \mathbb{R}^n$$

denotes the scalar product. The Gram matrix $G_{\mathscr{S}}(\mathscr{X})$ is also called *covariant metric tensor* and is responsible for the *Riemannian metric* on $\mathscr{S}$. Remarkably, the generating system of vectors $\{\boldsymbol{g}^k\}_{k=1}^{n-1}$ called the *contravariant frame* in the space of tangent vector fields $\mathscr{V}(\mathscr{S})$, is biorthogonal to the covariant frame $\langle \boldsymbol{g}_j, \boldsymbol{g}^k \rangle = \delta_{jk}$, $j, k = 1, \dots, n-1$.

The surface divergence and gradients in classical differential geometry (in intrinsic parameters of the surface $\mathscr{S}$) read as follows:

$$\begin{aligned} \operatorname{div}_{\mathscr{S}} \boldsymbol{U} &:= \left[\det G_{\mathscr{S}}\right]^{-1/2} \sum_{j=1}^{n} \partial_j \left\{ \left[\det G_{\mathscr{S}}\right]^{1/2} U^j \right\}, \\ \nabla_{\mathscr{S}} f &= \sum_{j,k=1}^{n-1} (g^{jk} \partial_j f)\, \partial_k, \qquad \boldsymbol{U} = \sum_{j=1}^{n-1} U^j \boldsymbol{g}_j \end{aligned} \tag{0.2}$$

(see [Ta2, Ch. 2, § 3]). The intrinsic parameters enable generalization to arbitrary manifolds, not necessarily immersed in the Euclidean space $\mathbb{R}^n$.

A derivative $\partial_{\boldsymbol{U}}^{\mathscr{S}} : C^1(\mathscr{S}) \to C^1(\mathscr{S})$ along some tangential vector field $\boldsymbol{U} \in \mathscr{V}(\mathscr{S})$ is called *covariant* if it is a linear automorphism of the space of tangential vector fields

$$\partial_{\boldsymbol{U}}^{\mathscr{S}} : \mathscr{V}(\mathscr{S}) \longrightarrow \mathscr{V}(\mathscr{S}). \tag{0.3}$$

The covariant derivative of a tangential vector field $\boldsymbol{V} = \sum_{j=1}^{n-1} V^j \boldsymbol{g}_j \in \mathscr{V}(\mathscr{S})$ along a tangential vector field $\boldsymbol{U} = \sum_{j=1}^{n-1} U^j \boldsymbol{g}_j \in \mathscr{V}(\mathscr{S})$ is defined by the formula

$$\partial_{\boldsymbol{U}}^{\mathscr{S}} \boldsymbol{V} := \pi_{\mathscr{S}} \partial_{\boldsymbol{U}} \boldsymbol{V} := \sum_{j,k,m=1}^{n-1} \left[U^j V^k \Gamma_{jk}^m + \delta_{jk} U^j \partial_j V^m \right] \boldsymbol{g}_m, \tag{0.4}$$

where $\Gamma^m_{jk}(x)$ are the *Christoffel symbols*

$$\Gamma^m_{jk}(x) := \langle \partial_k \boldsymbol{g}_j(x), \boldsymbol{g}^m(x)\rangle = \sum_{q=1}^{n-1} \frac{g^{mq}}{2}\big[\partial_k g_{jq}(x) + \partial_j g_{kq}(x) - \partial_q g_{jk}(x)\big]$$
$$:= \Gamma^m_{kj}(x)\,. \tag{0.5}$$

The calculus of differential operators on hypersurfaces presented here is based on Günter's derivatives. The definition applies the natural basis

$$\boldsymbol{e}^1 = (1,0,\dots,0)^\top, \dots, \boldsymbol{e}^n = (0,\dots,0,1)^\top \tag{0.6}$$

in the ambient Euclidean space $\mathbb{R}^n$ and the field of unit normal vectors to the surface $\mathscr{S}$

$$\boldsymbol{\nu}(\mathscr{X}) := \pm \frac{\boldsymbol{g}_1(\Theta^{-1}(\mathscr{X})) \wedge \cdots \wedge \boldsymbol{g}_{n-1}(\Theta^{-1}(\mathscr{X}))}{\big|\boldsymbol{g}_1(\Theta^{-1}(\mathscr{X})) \wedge \cdots \wedge \boldsymbol{g}_{n-1}(\Theta^{-1}(\mathscr{X}))\big|}, \qquad \mathscr{X} \in \mathscr{S}, \tag{0.7}$$

where $\boldsymbol{U}^{(1)} \wedge \cdots \wedge \boldsymbol{U}^{(n-1)}$ (or also $\boldsymbol{U}^{(1)} \times \cdots \times \boldsymbol{U}^{(n-1)}$) denotes the vector product of vectors $\boldsymbol{U}^{(1)}, \dots, \boldsymbol{U}^{(n-1)} \in \mathbb{R}^n$. If a hypersurface $\mathscr{S}$ in $\mathbb{R}^n$ is defined implicitly

$$\mathscr{S} = \Big\{\mathscr{X} \in \omega \ : \ \Psi_{\mathscr{S}}(\mathscr{X}) = 0\Big\}, \tag{0.8}$$

where $\Psi_{\mathscr{S}} : \omega \to \mathbb{R}$ is a C^k-mapping (or is a Lipschitz mapping) which is regular $\nabla\,\Psi(\mathscr{X}) \neq 0$, then the normalized gradient

$$\boldsymbol{\nu}(\mathscr{X}) := \pm \frac{\nabla \Psi_{\mathscr{S}}(\mathscr{X})}{\big|\nabla \Psi_{\mathscr{S}}(\mathscr{X})\big|}, \qquad \mathscr{X} \in \mathscr{S} \tag{0.9}$$

coincides with the outer unit normal vector provided the sign $\pm$ is chosen appropriately.

The collection of the tangential *Günter's derivatives* are defined as follows (cf. [Gu1], [KGBB1], [Du1]);

$$\mathscr{D}_j := \partial_j - \nu_j(\mathscr{X})\partial_{\boldsymbol{\nu}} = \partial_{\boldsymbol{d}^j}\,. \tag{0.10}$$

Here $\partial_{\boldsymbol{\nu}} := \sum_{j=1}^n \nu_j \partial_j$ denotes the normal derivative. For each $1 \le j \le n$, the first-order differential operator $\mathscr{D}_j = \partial_{\boldsymbol{d}^j}$ is the directional derivative along the tangential vector $\boldsymbol{d}^j := \pi_{\mathscr{S}}\, \boldsymbol{e}^j$, the projection of $\boldsymbol{e}^j$ on the space of tangent vector fields to $\mathscr{S}$. Here

$$\pi_{\mathscr{S}} : \mathbb{R}^n \to \mathscr{V}(\mathscr{S}), \quad \pi_{\mathscr{S}}(t) = I - \boldsymbol{\nu}(t)\boldsymbol{\nu}^\top(t) = \big[\delta_{jk} - \nu_j(t)\nu_k(t)\big]_{n\times n}, \quad t \in \mathscr{S} \tag{0.11}$$

defines the canonical orthogonal projection $\pi^2_{\mathscr{S}} = \pi_{\mathscr{S}}$ onto the space of tangent vector fields $\mathscr{V}(\mathscr{S})$ and $(\boldsymbol{\nu}, \pi_{\mathscr{S}} v) = 0$ for all $v \in \mathbb{R}^n$.

For tangential vector fields $\boldsymbol{V} \in \mathscr{V}(\mathscr{S})$ and $\boldsymbol{U} \in \mathscr{V}(\mathscr{S})$ we have representations

$$\boldsymbol{V} = \sum_{j=1}^n V^0_j \boldsymbol{e}^j = \sum_{j=1}^n V^0_j \boldsymbol{d}^j, \qquad \boldsymbol{U} = \sum_{j=1}^n U^0_j \boldsymbol{e}^j = \sum_{j=1}^n U^0_j \boldsymbol{d}^j. \tag{0.12}$$

The *surface gradient* $\nabla_{\mathscr{S}}\boldsymbol{U}$ and the *surface divergence* $\mathrm{div}_{\mathscr{S}}\boldsymbol{U}$ are defined as follows

$$\nabla_{\mathscr{S}}\boldsymbol{U} := (\mathscr{D}_1 U^0, \dots, \mathscr{D}_n U^0)^\top\,, \qquad \mathrm{div}_{\mathscr{S}}\boldsymbol{U}; = \sum_{j=1}^{n} \mathscr{D}_j U_j^0 \tag{0.13}$$

(cf. (0.2)) while for the *derivative of a vector field* $\boldsymbol{V}$ along $\boldsymbol{U}$ and the corresponding *covariant derivative* we have the formulae

$$\partial_{\boldsymbol{U}}\boldsymbol{V} = \sum_{j=1}^{n} U_j^0 \mathscr{D}_j \boldsymbol{V}\,, \qquad \partial_{\boldsymbol{U}}^{\mathscr{S}}\boldsymbol{V} = \sum_{j=1}^{n} U_j^0 \mathscr{D}_j^{\mathscr{S}} \boldsymbol{V} \tag{0.14}$$

(cf. (0.4)). Here $\mathscr{D}_j^{\mathscr{S}} : \mathscr{V}(\mathscr{S}) \to \mathscr{V}(\mathscr{S})$ is the *covariant Günter's derivative*

$$\mathscr{D}_j^{\mathscr{S}}\boldsymbol{V} := \pi_{\mathscr{S}}\mathscr{D}_j\boldsymbol{V} = \mathscr{D}_j\boldsymbol{V} - \langle\boldsymbol{\nu}, \mathscr{D}_j\boldsymbol{V}\rangle\boldsymbol{\nu}, \qquad j = 1, \dots, n. \tag{0.15}$$

The Lamé operator $\mathscr{L}_{\mathscr{S}}$ on $\mathscr{S}$ is the natural operator associated with the Euler-Lagrange equations for a variational integral. The starting point is the total free (elastic) energy

$$\mathscr{E}[\boldsymbol{U}] := \int_{\mathscr{S}} E(y, \mathscr{D}^{\mathscr{S}}\boldsymbol{U}(y))\,dS, \quad \mathscr{D}^{\mathscr{S}}\boldsymbol{U} := \big[(\mathscr{D}_j^{\mathscr{S}}\boldsymbol{U})_k^0\big]_{n\times n}\,, \quad \boldsymbol{U} \in \mathscr{V}(\mathscr{S}), \tag{0.16}$$

ignoring at the moment the displacement boundary conditions (Koiter's model). Equilibria states correspond to minimizers of the above variational integral (see [NH1, § 5.2]). The kernel $E = (\mathfrak{S}_{\mathscr{S}}, \mathrm{Def}_{\mathscr{S}})$ depends bi-linearly on the stress $\mathfrak{S}_{\mathscr{S}} = \big[\mathfrak{S}^{jk}\big]_{n\times n}$ and the deformation $\mathscr{D}^{\mathscr{S}}$ tensors. The following form of the important deformation (strain) tensor was identified in [DMM1]

$$\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}) = \big[\mathfrak{D}_{jk}(\boldsymbol{U})\big]_{n\times n}\,, \quad \boldsymbol{U} = \sum_{j=1}^{n} U_j^0 \boldsymbol{d}^j \in \mathscr{V}(\mathscr{S}), \quad j,k = 1, \dots, n, \tag{0.17}$$

$$\mathfrak{D}_{jk}(\boldsymbol{U}) := \frac{1}{2}\big[(\mathscr{D}_j^{\mathscr{S}}\boldsymbol{U})_k^0 + (\mathscr{D}_k^{\mathscr{S}}\boldsymbol{U})_j^0\big] = \frac{1}{2}\Big[\mathscr{D}_k U_j^0 + \mathscr{D}_j U_k^0 + \sum_{m=1}^{n} U_m^0 \mathscr{D}_m\big(\nu_j\nu_k\big)\Big]\,,$$

where $(\mathscr{D}_j^{\mathscr{S}}\boldsymbol{U})_k^0 := \langle\mathscr{D}_j^{\mathscr{S}}\boldsymbol{U}, \boldsymbol{e}^k\rangle$. Hooke's law states that $\mathfrak{S}_{\mathscr{S}} = \mathbb{T}\,\mathrm{Def}_{\mathscr{S}}$ for some linear fourth-order tensor $\mathbb{T} := \big[c_{jk\ell m}\big]_{n\times n\times n\times n}$, which is positive definite:

$$\langle\mathbb{T}\zeta, \zeta\rangle := \sum_{i,j,k,\ell=1}^{n} c_{ijk\ell}\zeta_{ij}\overline{\zeta}_{k\ell} \geq C_0 \sum_{i,j=1}^{n} |\zeta_{i,j}|^2 := C_0|\zeta|^2 \tag{0.18}$$

for all symmetric tensors $\zeta_{ij} = \zeta_{ji} \in \mathbb{C}$, $\zeta := \big[\zeta_{ij}\big]_{n\times n}$. Moreover, $\mathbb{T}$ has the following symmetry properties:

$$c_{ijk\ell} = c_{ij\ell k} = c_{k\ell ij} \qquad \forall\, i,j,k,\ell\,. \tag{0.19}$$

The following form of the Lamé operator for a linear anisotropic elastic medium was identified in [DMM1]:

$$\mathscr{L}_{\mathscr{S}} = \mathrm{Def}^*_{\mathscr{S}}\mathbb{T}\,\mathrm{Def}_{\mathscr{S}} = \Big[\sum_{\ell m=1}^{n} c_{jk\ell m}\mathscr{D}^{\mathscr{S}}_j\mathscr{D}^{\mathscr{S}}_\ell\Big]_{n\times n}, \qquad \boldsymbol{U}\in\mathscr{V}(\mathscr{S}), \tag{0.20}$$

The adjoint operator to the deformation tensor

$$\mathrm{Def}^*_{\mathscr{S}}\mathfrak{U} := \frac{1}{2}\sum_{j=1}^{n}\big\{(\mathscr{D}^{\mathscr{S}}_j)^*\big[\mathfrak{U}^{jk}+\mathfrak{U}^{kj}\big]\big\}_{k=1}^{n} \qquad \text{for} \quad \mathfrak{U}=\|\mathfrak{U}^{jk}\|_{n\times n} \tag{0.21}$$

maps tensor functions to vector functions.

For an isotropic medium

$$c_{jklm} = \lambda\delta_{jk}\delta_{lm} + \mu\big[\delta_{jl}\delta_{km}+\delta_{jm}\delta_{kl}\big] \tag{0.22}$$

and the Lamé operator acquires a simpler form

$$\begin{aligned}\mathscr{L}_{\mathscr{S}}\boldsymbol{U} &= -\lambda\nabla_{\mathscr{S}}\,\mathrm{div}_{\mathscr{S}}\boldsymbol{U} + 2\mu\,\mathrm{Def}^*_{\mathscr{S}}\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}\\ &= -\mu\pi_{\mathscr{S}}\Delta_{\mathscr{S}}\,\boldsymbol{U} - (\lambda+\mu)\,\nabla_{\mathscr{S}}\mathrm{div}_{\mathscr{S}}\boldsymbol{U} - \mu\,\mathscr{H}^0_{\mathscr{S}}\mathscr{W}_{\mathscr{S}}\boldsymbol{U}, \quad \boldsymbol{U}\in\mathscr{V}(\mathscr{S})\end{aligned} \tag{0.23}$$

(cf. (0.11) for the projection $\pi_{\mathscr{S}}$). $\lambda,\mu\in\mathbb{R}$ are the Lamé moduli, whereas

$$\mathscr{H}^0_{\mathscr{S}} = -\mathrm{div}_{\mathscr{S}}\boldsymbol{\nu} := -\sum_{j=1}^{n}\mathscr{D}_j\nu_j = \mathrm{Tr}\,\mathscr{W}_{\mathscr{S}}, \qquad \mathscr{W}_{\mathscr{S}} = -\big[\mathscr{D}_j\nu_k\big]_{n\times n}. \tag{0.24}$$

Note, that $\mathscr{H}_{\mathscr{S}} := (n-1)^{-1}\mathscr{H}^0_{\mathscr{S}}$ represents the *mean curvature* of the surface $\mathscr{S}$; $\mathscr{W}_{\mathscr{S}}$ is the *Weingarten curvature tensor* of $\mathscr{S}$; Eigenvalues of $\mathscr{W}_{\mathscr{S}}$, except one which is 0, represent all principal curvatures of the surface $\mathscr{S}$.

Note, that Günter's derivatives were already applied in [MM1] to minimal surfaces and in [Gu1], [KGBB1] to the problems of 3D elasticity.

We believe that our results should be useful in numerical and engineering applications (cf. [AN1], [Be1], [Ce1], [Co1], [DaL1], [BGS1], [Sm1]). Having in mind applications, equations in Cartesian coordinates are simpler for approximation and numerical treatment.

The paper is organized as follows. § 1 is auxiliary. In § 2 we prove generalized Lions' Lemma for the Bessel potential spaces $\mathbb{H}^s_p(\mathscr{S})$ on ~~closed and on open hypersurfaces.~~ The result is applied to the proof of important Korn's inequality for Killing's vector fields.

In § 3 we investigate Killing's vector fields, which constitute the kernel of the Lamé operator and represent analogues of rigid motions in $\mathbb{R}^n$. The most important result there states that the class of Killing's vector fields has the unique continuation property from the boundary: if such a field vanishes on a set of positive measure on the boundary of an ~~open hypersurface,~~ it vanishes on this hypersurface identically. The result is applied to prove further Korn's inequality "without boundary condition" and to the investigation of basic BVPs for the Lamé equation.

In § 4 we prove the ellipticity of the Lamé operator, which follows also from the Gårding's inequality

$$(\mathscr{L}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{U})_{\mathscr{S}} \geq C_1\|\boldsymbol{U}|\mathbb{H}^1(\mathscr{S})\|^2 - C_0\|\boldsymbol{U}|\mathbb{L}_2(\mathscr{S})\|^2.$$

For a closed hypersurface $\mathscr{S}$ the kernel Ker $\mathscr{L}_{\mathscr{S}}$ coincides with the space of Killing's vector fields. Moreover, the operator $\mathscr{L}_{\mathscr{S}} + \mathscr{B}I \,:\, \mathbb{H}^s_p(\mathscr{S}) \to \mathbb{H}^{s-2}_p(\mathscr{S})$ is invertible if $\mathscr{S}$ is closed and smooth, $1 < p < \infty$, $s \in \mathbb{R}$ and $\mathscr{B} \neq 0$ is a non-negative function.

In §§ 5–7 we investigate the Dirichlet and the Neumann boundary value problems for the Lamé operator on an open hypersurface $\mathscr{C}$ under a minimal requirements on the surface. Namely, we require that the immersion Θ in (0.1) (or the implicit function $\Psi_{\mathscr{S}}$ in (0.8)), representing the surface $\mathscr{C}$, has the bounded second derivative $\Theta \in (\mathbb{H}^2_\infty)^n$ ($\Psi_{\mathscr{S}} \in \mathbb{H}^2_\infty$, respectively). The Dirichlet problem

$$\begin{cases} \mathscr{L}_{\mathscr{S}}\boldsymbol{U} = \boldsymbol{F} \quad \text{in} \quad \mathscr{C}\,, \\ \boldsymbol{U}\big|_\Gamma = \boldsymbol{G} \quad \text{on} \quad \Gamma := \partial\mathscr{S}, \end{cases} \qquad \boldsymbol{F} \in \widetilde{\mathbb{H}}^{-1}(\mathscr{C}), \quad \boldsymbol{G} \in \mathbb{H}^{1/2}(\Gamma), \tag{0.25}$$

where $\boldsymbol{U} = \sum_{j=1}^n U_j^0 \boldsymbol{d}^j \in \mathscr{V}(\mathscr{C}) \cap \mathbb{H}^1(\mathscr{C})^n$ is the (tangential) generalized displacement vector field of the elastic hypersurface $\mathscr{S}$, is reduced to an equivalent Dirichlet BVP with vanishing boundary data $\boldsymbol{G} = 0$, which, in its turn, is equivalent to the invertibility of the operator

$$\mathscr{L}_{\mathscr{C}} \,:\, \widetilde{\mathbb{H}}^{-1}(\mathscr{C}) \to \mathbb{H}^{-1}(\mathscr{C}).$$

The invertibility derived from Gårding's inequality proved there. For the investigation of the Neumann BVP we apply the Lax-Milgramm Lemma, based on the coerciveness of the corresponding sesquilinear form.

1. Sobolev spaces and Bessel potential operators

Proposition 1.1. (*cf.* [DMM1]). *The surface divergence* $\operatorname{div}_{\mathscr{S}}$ *and the surface gradient* $\nabla_{\mathscr{S}}$ (*cf.* (0.13)) *are dual operators* $(\nabla_{\mathscr{S}}\,\varphi,\boldsymbol{U})_{\mathscr{S}} := (\varphi, \operatorname{div}_{\mathscr{S}}\boldsymbol{U})_{\mathscr{S}}$ *with respect to the usual scalar product of* (*square integrable*) *vector functions on the surface* $\mathscr{S}$

$$(\boldsymbol{U},\boldsymbol{V})_{\mathscr{S}} = \int_{\mathscr{S}} \langle \boldsymbol{U}(t), \overline{\boldsymbol{V}(t)}\rangle\, dS \qquad \forall\, \boldsymbol{U},\boldsymbol{V} \in \mathscr{V}(\mathscr{S}). \tag{1.1}$$

The Laplace-Beltrami operator $\Delta_{\mathscr{S}} := \operatorname{div}_{\mathscr{S}}\nabla_{\mathscr{S}}$ *on* $\mathscr{S}$ *writes*

$$\boldsymbol{\Delta}_{\mathscr{S}}\,\psi = -\operatorname{div}_{\mathscr{S}}\nabla_{\mathscr{S}}\psi = \sum_{j=1}^n \mathscr{D}_j^2\psi \qquad \forall\,\psi \in C^2(\mathscr{S})\,. \tag{1.2}$$

We remind that the surface gradient $\nabla_{\mathscr{S}}$ maps scalar functions to the tangential vector fields

$$\nabla_{\mathscr{S}} \,:\, C^1(\mathscr{S}) \to \mathscr{V}(\mathscr{S}) \subset C(\mathscr{S},\mathbb{C}^n) \tag{1.3}$$

and the scalar product with the normal vector vanishes $\langle \boldsymbol{\nu}(\mathscr{X}), \nabla_{\mathscr{S}}\varphi(\mathscr{X})\rangle \equiv 0$ for all $\varphi \in C^1(\mathscr{S})$ and all $\mathscr{X} \in \mathscr{S}$.

Tangential derivatives can be applied to a definition of Sobolev spaces $\mathbb{H}^m_p(\mathscr{S})$, $m \in \mathbb{N}^0$, $1 \le p < \infty$ on an ℓ-smooth surface $\mathscr{S}$ if $m \le \ell$:

$$\mathbb{H}^m_p(\mathscr{S}) := \{\varphi \in D'(\mathscr{S}) \ : \ \mathscr{D}^\alpha_{\mathscr{S}}\varphi \in \mathbb{L}_p(\mathscr{S})\,,\ \forall\, \alpha \in \mathbb{N}^n_0\,,\ |\alpha| \le m\}\,, \tag{1.4}$$

$$\mathscr{D}^\alpha_{\mathscr{S}} := \mathscr{D}^{\alpha_1}_1 \cdots \mathscr{D}^{\alpha_n}_n .$$

The derivative of $\varphi \in D'(\mathscr{S})$ in (1.4) is understood, as usual, in the distributional sense

$$(\mathscr{D}_j\varphi, \psi)_{\mathscr{S}} := (\varphi, \mathscr{D}^*_j\psi)_{\mathscr{S}},$$

where $\mathscr{D}^*_j$ is the formal dual operator to $\mathscr{D}_j$ (cf. [DMM1]):

$$\mathscr{D}^*_j\varphi = -\mathscr{D}_j\varphi - \nu_j\mathscr{H}^0_{\mathscr{S}}\varphi\,, \qquad \varphi \in C^1(\mathscr{S})\,. \tag{1.5}$$

The space $\mathbb{H}^1_p(\mathscr{S})$ is well defined if $\mathscr{S}$ is a Lipschitz hypersurface.

Equivalently, $\mathbb{H}^m_p(\mathscr{S})$ is the closure of the space $C^\ell(\mathscr{S})$ (or of $C^\infty(\mathscr{S})$ if $\mathscr{S}$ is infinitely smooth $\ell = \infty$) with respect to the norm

$$\|\varphi \,|\, \mathbb{H}^m_p(\mathscr{S})\,\| := \left[\sum_{|\alpha|\le m} \|\mathscr{D}^\alpha_{\mathscr{S}}\varphi \,|\, \mathbb{L}_p(\mathscr{S})\|_p\right]^{1/p}. \tag{1.6}$$

Moreover, $\mathbb{H}^m_2(\mathscr{S})$ is a Hilbert space with the scalar product

$$(\varphi, \psi)^{(m)}_{\mathscr{S}} := \sum_{|\alpha|\le m} \oint_{\mathscr{S}} \mathscr{D}^\alpha_{\mathscr{S}}\varphi)(\mathscr{X})\overline{\mathscr{D}^\alpha_{\mathscr{S}}\psi(\mathscr{X})}dS\,. \tag{1.7}$$

As usual, $\mathbb{H}^{-m}_2(\mathscr{S})$ with an integer $m \in \mathbb{N}$ denotes the space of distributions of the negative order $-m$ which is dual to the Sobolev space $\mathbb{H}^m_2(\mathscr{S})$.

We write, as customary, $\mathbb{H}^m(\mathscr{S})$ instead of $\mathbb{H}^m_2(\mathscr{S})$.

To accomplish the definition of the Banach spaces $\mathbb{H}^m_p(\mathscr{S})$ we need to prove the following.

Lemma 1.2. *For $\varphi \in C^1(\mathscr{S})$ the surface gradient vanishes $\nabla_{\mathscr{S}}\varphi \equiv 0$ if and only if $\varphi(\mathscr{X}) \equiv$ const.*

Proof. We only have to show that $\nabla_{\mathscr{S}}\,\varphi \equiv 0$ implies $\varphi(\mathscr{X}) \equiv$ const. The inverse implication is trivial. Let

$$\Omega^\varepsilon_{\mathscr{S}} := \mathscr{S} \times [-\varepsilon, \varepsilon] := \{\mathscr{X} + t\boldsymbol{\nu} \ : \ \mathscr{X} \in \mathscr{S}, \quad -\varepsilon < t < \varepsilon\}$$

be the *tubular neighborhood* of the surface of the thickness 2ε, with the middle surface $\mathscr{S}$. Taking ε sufficiently small, we can assume that the domain $\Omega^\varepsilon_{\mathscr{S}}$ has no self-intersections. Any function $\varphi \in C^1(\mathscr{S})$ is extended as a constant along the normal vector: $\widetilde{\varphi}(x,t) := \varphi(x)$, $x, t \in \Omega^\varepsilon_{\mathscr{S}}$. Then the normal derivatives are applicable and vanish: $\partial_{\boldsymbol{\nu}}\widetilde{\varphi} = 0$. Therefore the coordinate derivatives also are applicable and $\partial_j\widetilde{\varphi}(\mathscr{X}, t) = \mathscr{D}_j\varphi(\mathscr{X}) \equiv 0$ for all $j = 1, \ldots, n$ and all $(\mathscr{X}, t) \in \Omega^\varepsilon_{\mathscr{S}}$. But this implies $\widetilde{\varphi}(\mathscr{X}) \equiv$ const and, restricted to the surface, $\varphi(\mathscr{X}) = \gamma_{\mathscr{S}}\widetilde{\varphi}(\mathscr{X}) =$ const. $\square$

To the equivalence of the norm in (1.6) with the usual one defined by a partition of unity we only remark that among the "pull back" operators of n covariant derivatives there always can be selected locally $n-1$ linearly independent linear differential operators of order 1 of the variable $x \in \mathbb{R}^{n-1}$, which can equivalently be replaced by the coordinate derivatives $\partial_1, \dots, \partial_{n-1}$.

Lemma 1.3. *Let $\varphi \in \mathbb{H}^2(\Omega^\varepsilon_{\mathscr{S}}) \cap C^1(\mathscr{S})$ and $\gamma_{\mathscr{S}}\nabla\varphi$, $\gamma_{\mathscr{S}}\partial_{\boldsymbol{\nu}}\varphi$ denote the traces on $\mathscr{S}$ of the spatial gradient and of the normal derivative, while $\nabla_{\mathscr{S}}\varphi$ denote the surface gradient. Then*

$$\|\gamma_{\mathscr{S}}\nabla\varphi \,|\, \mathbb{L}_2(\mathscr{S})\|_2^2 = \|\nabla_{\mathscr{S}}\varphi \,|\, \mathbb{L}_2(\mathscr{S})\|_2^2 + \|\gamma_{\mathscr{S}}\partial_{\boldsymbol{\nu}}\varphi \,|\, \mathbb{L}_2(\mathscr{S})\|_2^2 \,. \tag{1.8}$$

Proof. Indeed,

$$\begin{aligned}
\|\nabla_{\mathscr{S}}\varphi \,|\, \mathbb{L}_2(\mathscr{S})\|^2 &= \sum_{j=1}^n \oint_{\mathscr{S}} \mathscr{D}_j\varphi(\mathscr{X})\overline{\mathscr{D}_j\varphi(\mathscr{X})}\,dS \\
&= \sum_{j=1}^n \oint_{\mathscr{S}} (\partial_j\varphi(\mathscr{X}) - \nu_j(\mathscr{X})\partial_{\boldsymbol{\nu}(x)}\varphi(\mathscr{X}))\overline{(\partial_j\varphi(\mathscr{X}) - \nu_j(\mathscr{X})\partial_{\boldsymbol{\nu}(x)}\varphi(\mathscr{X}))}\,dS \\
&= \sum_{j=1}^n \Bigg[\oint_{\mathscr{S}} (\partial_j\varphi(\mathscr{X})\overline{\partial_j\varphi(\mathscr{X})}\,dS - \oint_{\mathscr{S}} \nu_j(\mathscr{X})\partial_j\varphi(\mathscr{X})\overline{\partial_{\boldsymbol{\nu}(x)}\varphi(\mathscr{X})}\,dS \\
&\quad - \oint_{\mathscr{S}} \partial_{\boldsymbol{\nu}(x)}\varphi(\mathscr{X})\overline{\nu_j(\mathscr{X})\partial_j\varphi(\mathscr{X})}\,dS + \nu_j^2(\mathscr{X})\oint_{\mathscr{S}} \partial_{\boldsymbol{\nu}(x)}\varphi(\mathscr{X})\overline{\partial_{\boldsymbol{\nu}(x)}\varphi(\mathscr{X})}\,dS\Bigg] \\
&= \|\gamma_{\mathscr{S}}\nabla\varphi \,|\, \mathbb{L}_2(\mathscr{S})\|^2 - \|\gamma_{\mathscr{S}}\partial_{\boldsymbol{\nu}}\varphi \,|\, \mathbb{L}_2(\mathscr{S})\|^2
\end{aligned}$$

and (1.8) follows. □

Lemma 1.4. *The operator*

$$\boldsymbol{\Delta}_{\mathscr{S},\mu} := \mu I - \boldsymbol{\Delta}_{\mathscr{S}} \;:\; \mathbb{H}^1(\mathscr{S}) \to \mathbb{H}^{-1}(\mathscr{S})\,, \qquad \mu = \text{const} > 0 \tag{1.9}$$

is positive definite, elliptic and invertible. For arbitrary $s \in \mathbb{R}$ the power $\boldsymbol{\Delta}^s_{\mathscr{S},\mu}$ is a self-adjoint positive definite pseudodifferential operator with a trivial kernel $\operatorname{Ker}\boldsymbol{\Delta}^r_{\mathscr{S},\mu} = \{0\}$ *in the Sobolev space $\mathbb{W}^m_p(\mathscr{S}) = \mathbb{H}^m_p(\mathscr{S})$ for all $m = 1, \dots, m$ and all $1 < p < \infty$.*

Proof. The positive definiteness (also implying self-adjointness, ellipticity and invertibility) of $\boldsymbol{\Delta}_{\mathscr{S},\mu}$ follows from Proposition 1.1

$$((\mu I - \boldsymbol{\Delta}_{\mathscr{S}})\varphi, \varphi)_{\mathscr{S}} = \mu\|\varphi\big|\mathbb{L}_2(\mathscr{S})\|^2 + \|\nabla_{\mathscr{S}}\varphi\big|\mathbb{L}_2(\mathscr{S})\| \geq C\|\varphi\big|\mathbb{H}^1(\mathscr{S})\|^2$$

with $C := \min\{1, \mu\} > 0$. Then the powers $\boldsymbol{\Delta}^s_{\mathscr{S},\mu}$, $s \in \mathbb{R}$ exist and are pseudodifferential operators (cf., e.g., [Sh1]). We quote [DNS1] (also see [Ag1, Du2, Ka1] and [DNS2] for a most general result) that an elliptic pseudodifferential operators on a ~~closed manifold~~ has the same kernel and cokernel in the spaces $\mathbb{H}^m_p(\mathscr{S})$ for all $m = 1, \dots, \ell$ and all $1 < p < \infty$. □

Now we are able to define the Bessel potential space $\mathbb{H}^s_p(\mathscr{S})$ for arbitrary $s \in \mathbb{R}$ and $1 < p < \infty$:

$$\mathbb{H}^s_p(\mathscr{S}) := \left\{\varphi \,:\, \|\varphi \,|\, \mathbb{H}^s_p(\mathscr{S})\,\| := \|\boldsymbol{\Delta}^{s/2}_{\mathscr{S},1}\varphi \,|\, \mathbb{L}_p(\mathscr{S})\,\| < \infty\right\}. \tag{1.10}$$

The Sobolev spaces with negative indices $\mathbb{H}^{-s}_p(\mathscr{S})$, $s < 0$, $1 < p < \infty$ are dual to $\mathbb{H}^s_{p'}(\mathscr{S})$, $p' := \dfrac{p}{p-1}$, with respect to the sesquilinear form $(\varphi, \psi)_{\mathscr{S}}$ (cf. (1.1)) extended by continuity to duality between pairs $\varphi \in \mathbb{H}^s_{p'}(\mathscr{S})$ and $\psi \in \mathbb{H}^{-s}_p(\mathscr{S})$.

The embeddings $\mathbb{H}^s_p(\mathscr{S}) \subset \mathbb{L}_p(\mathscr{S}) \subset \mathbb{H}^{-s}_p(\mathscr{S})$, for $s > 0$, are continuous, even compact, and for integer-valued parameter $s = m$ the space $\mathbb{H}^{-m}_p(\mathscr{S})$ is the convex linear hull of distributional derivatives of $\mathbb{L}_p(\mathscr{S})$-functions:

$$\mathbb{H}^{-m}_p(\mathscr{S}) := \mathscr{L}\left\{\mathscr{D}^\alpha\varphi \,:\, \varphi \in \mathbb{L}_p(\mathscr{S}) \text{ for all } \mathscr{D}^\alpha = \mathscr{D}^{\alpha_1}_1 \cdots \mathscr{D}^{\alpha_n}_n\,, \quad |\alpha| \le m\right\}.$$

If $\mathscr{C}$ is an ~~open subsurface~~ with the Lipschitz boundary $\Gamma = \partial\mathscr{C} \neq \emptyset$, $\widetilde{\mathbb{H}}^s_p(\mathscr{C})$ denotes the space of functions ~~obtained by closing the space $C^\infty_0(\mathscr{C})$ of smooth functions with compact support in the norm of $\mathbb{H}^s_p(\mathscr{S})$,~~ where $\mathscr{S}$ is a smooth ~~closed surface~~ which extends the surface $\mathscr{C}$. Let $\mathscr{C}^+ := \mathscr{C}$ and $\mathscr{C}^- := \mathscr{C}^c = \mathscr{S} \setminus \overline{\mathscr{C}}$ denote the complemented ~~open surface~~ $\mathscr{S} = \mathscr{C}^+ \cup \overline{\mathscr{C}^-}$; the notation $\mathbb{H}^s_p(\mathscr{C})$ is used for the factor space $\mathbb{H}^s_p(\mathscr{S})/\widetilde{\mathbb{H}}^s_p(\mathscr{C}^-)$; the space $\mathbb{H}^s_p(\mathscr{C})$ can also be viewed as the space of restrictions $r_{\mathscr{C}}\varphi := \varphi\big|_{\mathscr{C}}$ of all functions $\varphi \in \mathbb{H}^s_p(\mathscr{S})$ to the subsurface $\mathscr{C} = \mathscr{C}^+$.

We refer to [Tr1] and [DS1] for details about similar spaces.

2. Lions' Lemma and Korn's inequalities

The following generalizes essentially J.L. Lions' Lemma (cf. [DaL1, p.111], [Ta1], [AG1, Proposition 2.10], [Ci3, § 1.7], [Mc1]).

Lemma 2.1. *Let $\mathscr{S}$ be a 2-smooth ~~closed hypersurface~~ in $\mathbb{R}^n$. Then the inclusions $\varphi \in \mathbb{H}^{-1}_p(\mathscr{S})$, $\mathscr{D}_j\varphi \in \mathbb{H}^{-1}_p(\mathscr{S})$, for all $j = 1, \ldots, n$ imply $\varphi \in \mathbb{L}_p(\mathscr{S})$.*

Moreover, the assertion remains valid for a hypersurface $\mathscr{C}$ with the Lipschitz boundary $\Gamma := \partial\mathscr{C}$ and the spaces $\mathbb{H}^{-1}_p(\mathscr{C})$ and $\widetilde{\mathbb{H}}^{-1}_p(\mathscr{C})$.

Proof. First we assume that $\mathscr{S}$ is a ~~closed surface.~~ The proof is based on the following facts (cf. [Hr1, Sh1, Ta2, Tr1]):

A. The "lifting operators" (the Bessel potential operator) $\Lambda^{\pm 1}_{\mathscr{S}}(\mathscr{x}, D) := \boldsymbol{\Delta}^{\pm 1/2}_{\mathscr{S},1}$ (cf. Lemma 1.4 and (1.10)), are invertible $\Lambda^{\pm 1}_{\mathscr{S}}(\mathscr{x}, D)\Lambda^{\mp 1}_{\mathscr{S}}(\mathscr{x}, D) = I$, mapping isometrically the spaces

$$\begin{aligned} \Lambda^{-1}_{\mathscr{S}}(\mathscr{x}, D) \,&:\, \mathbb{H}^{m-1}_p(\mathscr{S}) \to \mathbb{H}^m_p(\mathscr{S}), \\ \Lambda_{\mathscr{S}}(\mathscr{x}, D) \,&:\, \mathbb{H}^m_p(\mathscr{S}) \to \mathbb{H}^{m-1}_p(\mathscr{S}) \end{aligned} \tag{2.1}$$

for arbitrary $m = 0, \pm 1, \ldots$ and are pseudodifferential operators of order ± 1, respectively.

B. The commutant

$$[\mathscr{D}_j, \Lambda^{-1}_{\mathscr{S}}(\mathscr{X}, D)] := \mathscr{D}_j \Lambda^{-1}_{\mathscr{S}}(\mathscr{X}, D) - \Lambda^{-1}_{\mathscr{S}}(\mathscr{X}, D)\mathscr{D}_j \tag{2.2}$$

with the pseudodifferential operator $\mathscr{D}_j$ has order -1 and maps continuously the spaces

$$[\mathscr{D}_j, \Lambda^{-1}_{\mathscr{S}}(\mathscr{X}, D)] \;:\; \mathbb{H}^{-1}_p(\mathscr{S}) \to \mathbb{L}_p(\mathscr{S}).$$

The assertion (**B**) is a well-known property of pseudodifferential operators and can be retrieved from many sources [Hr1, Sh1, Ta1, Tr1].

Let $\varphi \in \mathbb{H}^{-1}_p(\mathscr{S})$, $\mathscr{D}_j\varphi \in \mathbb{H}^{-1}_p(\mathscr{S})$, for all $j = 1, \dots, n$. Then, due to (2.1), $\psi := \Lambda^{-1}_{\mathscr{S}}(\mathscr{X}, D)\varphi \in \mathbb{L}_p(\mathscr{S})$ and, due to (2.2), $\mathscr{D}_j\psi = [\mathscr{D}_j, \Lambda^{-1}_{\mathscr{S}}(\mathscr{X}, D)]\varphi + \Lambda^{-1}_{\mathscr{S}}(\mathscr{X}, D)\mathscr{D}_j\varphi \in \mathbb{L}_p(\mathscr{S})$ for all $j = 1, \dots, n$. By the definition of the space $\mathbb{H}^1_p(\mathscr{S}) = \mathbb{H}^1_p(\mathscr{S})$ in (1.6) we conclude that $\psi \in \mathbb{H}^1_p(\mathscr{S})$. Due to (2.1) we get finally $\varphi = \Lambda_{\mathscr{S}}(\mathscr{X}, D)\psi \in \mathbb{L}_p(\mathscr{S})$.

If $\mathscr{C}$ has non-empty Lipschitz boundary $\Gamma \neq \emptyset$, there exist pseudodifferential operators

$$\begin{aligned} &\Lambda^{-1}_{-}(\mathscr{X}, D) \;:\; \mathbb{H}^{-1}_p(\mathscr{C}) \to \mathbb{L}_p(\mathscr{C})\,, \\ &\Lambda^{-1}_{+}(\mathscr{X}, D) \;:\; \widetilde{\mathbb{H}}^{-1}_p(\mathscr{C}) \to \widetilde{\mathbb{L}}_p(\mathscr{C})\,, \end{aligned} \tag{2.3}$$

of order -1, arranging isomorphisms between the indicated spaces, and their inverses are $\Lambda_{\pm}(\mathscr{X}, D)$, respectively (cf. [DS1]).

Moreover, the commutants $[\mathscr{D}_j, \Lambda^{-1}_{\pm}(\mathscr{X}, D)] := \mathscr{D}_j\Lambda^{-1}_{\pm}(\mathscr{X}, D) - \Lambda^{-1}_{\pm}(\mathscr{X}, D)\mathscr{D}_j$ have order -1, i.e., mapping continuously the spaces $[\mathscr{D}_j, \Lambda^{-1}_{-}(\mathscr{X}, D)] \,:\, \mathbb{H}^{-1}_p(\mathscr{C}) \to \mathbb{L}_p(\mathscr{C})$ and $[\mathscr{D}_j, \Lambda^{-1}_{+}(\mathscr{X}, D)] \,:\, \widetilde{\mathbb{H}}^{-1}_p(\mathscr{C}) \to \mathbb{L}_p(\mathscr{C})$.

By using the formulated assertions the proof is completed as in the case of a closed surface $\mathscr{S}$. □

The foregoing Lemma 2.1 has the following generalization for the Bessel potential spaces $\widetilde{\mathbb{H}}^s_p(\mathscr{S})$.

Lemma 2.2. *If $\mathscr{S}$ is closed, sufficiently smooth, $1 < p < \infty$, $s \in \mathbb{R}$, $m = 1, 2, \dots$ and*

$$\varphi \in \mathbb{H}^{s-m}_p(\mathscr{S})\,, \quad \mathscr{D}^\alpha\varphi = \mathscr{D}^{\alpha_1}_1 \cdots \mathscr{D}^{\alpha_n}_n\varphi \in \mathbb{H}^{s-m}_p(\mathscr{S}) \quad \textit{for all} \quad |\alpha| \leq m,$$

then $\varphi \in \mathbb{H}^s_p(\mathscr{S})$.

Moreover, the assertion remains valid for a hypersurface $\mathscr{C}$ with the Lipschitz boundary $\Gamma := \partial\mathscr{C}$ and the spaces $\mathbb{H}^s_p(\mathscr{C})$ and $\widetilde{\mathbb{H}}^s_p(\mathscr{C})$.

Proof. Assume first $\mathscr{S}$ has no boundary. The proof is based on similar facts as in the foregoing case:

A. The "lifting operator" (the Bessel potential operator) $\Lambda^r_{\mathscr{S}}(\mathscr{X}, D) := \mathbf{\Delta}^{r/2}_{\mathscr{S},1}$ (cf. Lemma 1.4 and (1.10)) maps isometrically the spaces

$$\Lambda^r_{\mathscr{S}}(\mathscr{X}, D) \;:\; \mathbb{H}^s_p(\mathscr{S}) \to \mathbb{H}^{s-r}_p(\mathscr{S})\,, \quad r \in \mathbb{R} \tag{2.4}$$

has the inverse $\Lambda^{-r}_{\mathscr{S}}(\mathscr{X}, D)$ and is a pseudodifferential operator of order r.

B. The commutant

$$[\mathscr{D}^\alpha, \Lambda^r_{\mathscr{S}}(\mathscr{x}, D)] := \mathscr{D}^\alpha \Lambda^r_{\mathscr{S}}(\mathscr{x}, D) - \Lambda^r_{\mathscr{S}}(\mathscr{x}, D)\mathscr{D}^\alpha \tag{2.5}$$

is a pseudodifferential operator of order $|\alpha| + r - 1$ and maps continuously the spaces

$$[\mathscr{D}^\alpha, \Lambda^r_{\mathscr{S}}(\mathscr{x}, D)] \,:\, \mathbb{H}^\gamma_p(\mathscr{S}) \to \mathbb{H}^{\gamma-|\alpha|-r+1}_p(\mathscr{S}), \qquad \forall\, \gamma \in \mathbb{R}.$$

Assume that $m = 1$. Then $\varphi \in \mathbb{H}^{s-1}_p(\mathscr{S})$ and, due to (2.4), (2.5), it follows that $\psi := \Lambda^{s-1}_{\mathscr{S}}(\mathscr{x}, D)\varphi \in \mathbb{L}_p(\mathscr{S})$, $\mathscr{D}_j\psi = [\mathscr{D}_j, \Lambda^{s-1}_{\mathscr{S}}(\mathscr{x}, D)]\varphi + \Lambda^{s-1}_{\mathscr{S}}(\mathscr{x}, D)\mathscr{D}_j\varphi \in \mathbb{L}_p(\mathscr{S})$ for all $j = 1, \ldots, n$. By the definition of the space $\mathbb{H}^1_p(\mathscr{S}) = \mathbb{H}^1_p(\mathscr{S})$ in (1.6) the inclusion $\psi \in \mathbb{H}^1_p(\mathscr{S})$ follows. Due to (2.4) we get finally $\varphi = \Lambda^{1-s}_{\mathscr{S}}(\mathscr{x}, D)\psi \in \mathbb{H}^s_p(\mathscr{S})$.

Now assume: $m = 2, 3, \ldots$ and the assertion is valid for $m - 1$. Then, due to the hypothesis, $\psi_j := \mathscr{D}_j\varphi \in \mathbb{H}^{s-m}_p(\mathscr{S})$ for $j = 1, \ldots, n$. Moreover, due to the same hypothesis,

$$\mathscr{D}^\alpha\psi_j := \mathscr{D}^\alpha\mathscr{D}_j\varphi \in \mathbb{H}^{s-m}_p(\mathscr{S}) \quad \text{for all} \quad |\alpha| \le m-1 \quad \text{and all} \quad j = 1, \ldots, n.$$

Hence the induction hypothesis implies that $\psi_j := \mathscr{D}_j\varphi \in \mathbb{H}^{s-1}_p(\mathscr{S})$ for $j = 1, \ldots, n$. Now it follows from the already considered case $m = 1$ that $\varphi \in \mathbb{H}^s_p(\mathscr{S})$.

If $\mathscr{C}$ has the non-empty Lipschitz boundary $\Gamma \neq \emptyset$, there exist pseudodifferential operators

$$\Lambda^r_-(\mathscr{x}, D) \;:\; \mathbb{H}^s_p(\mathscr{C}) \to \mathbb{H}^{s-r}_p(\mathscr{C})\,, \qquad \Lambda^r_+(\mathscr{x}, D) \;:\; \widetilde{\mathbb{H}}^s_p(\mathscr{C}) \to \widetilde{\mathbb{H}}^{s-r}_p(\mathscr{C})\,, \tag{2.6}$$

arranging isomorphisms between the indicated spaces, and their inverses are $\Lambda^{-r}_-(\mathscr{x}, D)$, $\Lambda^{-r}_+(\mathscr{x}, D)$ (cf. [DS1]).

Moreover, the pseudodifferential operators $\Lambda^{-r}_\pm(\mathscr{x}, D)$ have order $-r$ and the commutants $[\mathscr{D}^\alpha, \Lambda^{-r}_\pm(\mathscr{x}, D)] := \mathscr{D}^\alpha\Lambda^{-1}_\pm(\mathscr{x}, D) - \Lambda^{-r}_\pm(\mathscr{x}, D)\mathscr{D}^\alpha$ have order $|\alpha| - r - 1$, i.e., mapping continuously the spaces $[\mathscr{D}^\alpha, \Lambda^{-r}_-(\mathscr{x}, D)] \,:\, \mathbb{H}^\gamma_p(\mathscr{C}) \to \mathbb{H}^{\gamma+r+1-|\alpha|}_p(\mathscr{C})$ and $[\mathscr{D}^\alpha, \Lambda^{-r}_+(\mathscr{x}, D)] \,:\, \widetilde{\mathbb{H}}^\gamma_p(\mathscr{C}) \to \widetilde{\mathbb{H}}^{\gamma+r+1-|\alpha|}_p(\mathscr{C})$.

By using the formulated assertions the proof is completed as in the foregoing cases. □

Theorem 2.3. (Korn's I inequality "without boundary condition"). *Let* $\mathscr{S} \subset \mathbb{R}^n$ *be a Lipshitz hypersurface without boundary,* $\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}) := [\mathfrak{D}_{jk}(\boldsymbol{U})]_{n\times n}$ *be the deformation tensor* (*cf.* (0.17)) *and*

$$\big\|\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\big|\mathbb{L}_p(\mathscr{S})\big\| := \left[\sum_{j,k=1}^n \big\|\mathfrak{D}_{jk}\boldsymbol{U}\big|\mathbb{L}_p(\mathscr{S})\big\|^p\right]^{1/p}, \qquad \boldsymbol{U} \in \mathbb{H}^1_p(\mathscr{S}) \tag{2.7}$$

for $1 < p < \infty$*. Then*

$$\big\|\boldsymbol{U}\big|\mathbb{H}^1_p(\mathscr{S})\big\| \le M\left[\big\|\boldsymbol{U}\big|\mathbb{L}_p(\mathscr{S})\big\|^p + \big\|\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\big|\mathbb{L}_p(\mathscr{S})\big\|^p\right]^{1/p} \tag{2.8}$$

for some constant $M > 0$ or, equivalently, the mapping

$$\boldsymbol{U} \mapsto \left[\left\|\boldsymbol{U}\middle|\mathbb{L}_p(\mathscr{S})\right\|^p + \left\|\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\middle|\mathbb{L}_p(\mathscr{S})\right\|^p\right]^{1/p}$$

is an equivalent norm on the space $\mathbb{H}^1_p(\mathscr{S})$.

Proof. Consider the space

$$\widehat{\mathbb{H}}^1_p(\mathscr{S}) := \left\{\boldsymbol{U} = \left(U_1, \dots, U_n\right)^\top \,:\, U_j\,,\, \mathfrak{D}_{jk}(\boldsymbol{U}) \in \mathbb{L}_p(\mathscr{S}) \ \text{ for all } j,k = 1,\dots,n\right\} \tag{2.9}$$

endowed with the norm (cf. (2.8)):

$$\left\|\boldsymbol{U}\middle|\widehat{\mathbb{H}}^1_p(\mathscr{S})\right\| := \left[\left\|\boldsymbol{U}\middle|\mathbb{L}_p(\mathscr{S})\right\|^p + \left\|\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\middle|\mathbb{L}_p(\mathscr{S})\right\|^p\right]^{1/p}. \tag{2.10}$$

The derivatives here are understood in the distributional sense

$$(\mathfrak{D}_{jk}(\boldsymbol{U}), \psi)_{\mathscr{S}} := \frac{1}{2}(U_k, \mathscr{D}^*_j\psi)_{\mathscr{S}} + \frac{1}{2}(U_j, \mathscr{D}^*_k\psi)_{\mathscr{S}} \qquad \forall\,\psi \in C^1(\mathscr{S})$$

(cf. (1.5) for the formal dual operator $\mathscr{D}^*_j$).

It is sufficient to prove that the spaces $\mathbb{H}^1_p(\mathscr{S})$ and $\widehat{\mathbb{H}}^1_p(\mathscr{S})$ are identical. The inclusion $\mathbb{H}^1_p(\mathscr{S}) \subset \widehat{\mathbb{H}}^1_p(\mathscr{S})$ is trivial and we only check the inverse inclusion $\widehat{\mathbb{H}}^1_p(\mathscr{S}) \subset \mathbb{H}^1_p(\mathscr{S})$.

To this end take $\boldsymbol{U} \in \widehat{\mathbb{H}}^1_p(\mathscr{S})$ and note that the inclusions $\boldsymbol{U} \in \mathbb{L}_p(\mathscr{S})$, $\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}) \in \mathbb{L}_p(\mathscr{S})$ (i.e., $\mathfrak{D}_{jk}\boldsymbol{U} \in \mathbb{L}_p(\mathscr{S})$ for all $j,k = 1,\dots,n$) imply

$$\widetilde{\mathfrak{D}}_{jk}(\boldsymbol{U}) = \frac{1}{2}\Big[\mathscr{D}_k U_j + \mathscr{D}_j U_k\Big] = \mathfrak{D}_{jk}(\boldsymbol{U}) - \frac{1}{2}\sum_{r=1}^n \partial_r\big(\nu_j\nu_k\big)U_r \in \mathbb{L}_p(\mathscr{S}) \tag{2.11}$$

for all $j,k = 1,\dots,n$. Then (cf. [DMM1, Proposition 4.4.iv] for the commutator $[\mathscr{D}_j, \mathscr{D}_k]$):

$$\mathscr{D}_j U_k \in \mathbb{H}^{-1}_p(\mathscr{S}), \quad \big[\mathscr{D}_j, \mathscr{D}_k\big]U_m = \sum_{r=1}^n \big[\nu_j\mathscr{D}_k\nu_r - \nu_k\mathscr{D}_j\nu_r\big]\mathscr{D}_r U_m \in \mathbb{H}^{-1}_p(\mathscr{S}),$$

$$\mathscr{D}_k\mathscr{D}_j U_m = \mathscr{D}_j\widetilde{\mathfrak{D}}_{km}(\boldsymbol{U}) + \mathscr{D}_k\widetilde{\mathfrak{D}}_{jm}(\boldsymbol{U}) - \mathscr{D}_m\widetilde{\mathfrak{D}}_{jk}(\boldsymbol{U}) - \frac{1}{2}\big[\mathscr{D}_j, \mathscr{D}_k\big]U_m$$

$$-\frac{1}{2}\big[\mathscr{D}_j, \mathscr{D}_m\big]U_k - \frac{1}{2}\big[\mathscr{D}_k, \mathscr{D}_m\big]U_j \in \mathbb{H}^{-1}_p(\mathscr{S}) \quad \text{for} \quad j,k,m = 1,\dots,n\,,$$

Due to Lemma 2.1 of J.L. Lions this implies $\mathscr{D}_j U_m \in \mathbb{L}_p(\mathscr{S})$ for all $j,m = 1,\dots,n$ and the claimed result $\boldsymbol{U} \in \mathbb{H}^1_p(\mathscr{S})$ follows. □

Remark 2.4. The foregoing Theorem 2.3 is proved by P. Ciarlet in [Ci3] for the case $p = 2$, $m = 1$, ~~closed manifold (without boundary),~~ for curvilinear coordinates and covariant derivatives.

A remarkable consequence of Korn's inequality (2.8) is that the space

$$\mathbb{H}^1_p(\mathscr{S}) := \left\{\boldsymbol{U} = \left(U_1, \dots, U_n\right)^\top \,:\, U_j\,,\, \mathscr{D}_k U_j \in \mathbb{L}_p(\mathscr{S}) \ \text{ for all } j,k = 1,\dots,n\right\}$$

(cf. (1.6)) and the space $\widehat{\mathbb{H}}^1_p(\mathscr{S})$ (cf. (2.9)) are isomorphic (i.e., can be identified), although only $\dfrac{n(n+1)}{2} < n^2$ linear combinations of the n^2 derivatives $\mathscr{D}_j U_k$, $j,k = 1,\dots,n$ participate in the definition of the space $\widehat{\mathbb{H}}^1_p(\mathscr{S})$.

3. Killing's vector fields and the unique continuation from the boundary

Definition 3.1. Let $\mathscr{S}$ be a hypersurface in the Euclidean space $\mathbb{R}^n$. The space $\mathscr{R}(\mathscr{S})$ of solutions to the deformation equations

$$\begin{aligned}\mathfrak{D}_{jk}(\boldsymbol{U}) &:= \frac{1}{2}\big[(\mathscr{D}^{\mathscr{S}}_j \boldsymbol{U})^0_k + (\mathscr{D}^{\mathscr{S}}_k \boldsymbol{U})^0_j\big]\\ &= \frac{1}{2}\Big[\mathscr{D}_k U^0_j + \mathscr{D}_j U^0_k + \sum_{m=1}^n U^0_m \mathscr{D}_m\big(\nu_j\nu_k\big)\Big] = 0\,, \qquad (3.1)\\ \boldsymbol{U} &= \sum_{j=1}^n U^0_j \boldsymbol{d}^j \in \mathscr{V}(\mathscr{S}), \quad j,k = 1,\dots,n\end{aligned}$$

(cf. (0.17)) is called the space of *Killing's vector fields.*

Killing's vector fields on a domain in the Euclidean space $\Omega \subset \mathbb{R}^n$ are known as the *rigid motions* and we start with this simplest class.

The space of rigid motions $\mathscr{R}(\Omega)$ extends naturally to the entire $\mathbb{R}^n$ and consists of linear vector functions

$$\boldsymbol{V}(x) = a + \mathscr{B}x\,, \qquad \mathscr{B} = \big[b_{jk}\big]_{n\times n}\,, \qquad a \in \mathbb{R}^n\,, \quad x \in \mathbb{R}^n\,, \qquad (3.2)$$

where the matrix $\mathscr{B}$ is skew symmetric

$$\mathscr{B} := \begin{bmatrix} 0 & b_{12} & b_{13} & \cdots & b_{1(n-2)} & b_{1(n-1)}\\ -b_{12} & 0 & b_{21} & \cdots & b_{1(n-3)} & b_{2(n-2)}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ -b_{1(n-2)} & -b_{2(n-3)} & -b_{3(n-4)} & \cdots & 0 & b_{(n-1)1}\\ -b_{1(n-1)} & -b_{2(n-2)} & -b_{3(n-3)} & \cdots & -b_{(n-1)1} & 0 \end{bmatrix} = -\mathscr{B}^\top \qquad (3.3)$$

with real-valued entries $b_{jk} \in \mathbb{R}$. For $n = 3, 4, \dots$ the space $\mathscr{R}(\mathbb{R}^n)$ is finite-dimensional and $\dim \mathscr{R}(\mathbb{R}^n) = n + \dfrac{n(n-1)}{2} = \dfrac{n(n+1)}{2}$.

Note that for $n = 3$ the vector field $\boldsymbol{V} \in \mathscr{R}(\Omega)$, $\Omega \subset \mathbb{R}^3$, is the classical rigid displacement

$$\begin{aligned}&\boldsymbol{V}(x) = a + \mathscr{B}x = a + b \wedge x\,,\\ &b := (b_1, b_2, b_3)^\top \in \mathbb{R}^3\,, \quad x \in \Omega\,,\end{aligned} \qquad \mathscr{B} := \begin{bmatrix} 0 & -b_3 & b_2\\ b_3 & 0 & -b_1\\ -b_2 & b_1 & 0\end{bmatrix}. \qquad (3.4)$$

Definition 3.2. We call a subset $\mathscr{M} \subset \mathbb{R}^n$ *essentially m-dimensional* and write $\operatorname{ess\,dim} \mathscr{M} = m$, if there exist $m+1$ points $\mathscr{X}^0, \mathscr{X}^1, \cdots, \mathscr{X}^m \in \mathscr{M}$ such that the vectors $\big\{\mathscr{X}^j - \mathscr{X}^0\big\}_{j=1}^m$ are linearly independent.

Note, that any m-dimensional subset $\mathscr{M} \subset \mathbb{R}^m$ is essentially m-dimensional, because contains m linearly independent vectors. Moreover, any collection of $m+1$ points in $\mathbb{R}^m$ (a 0-dimensional subset!) is essentially m-dimensional, provided these points does not belong to any $m-1$-dimensional hyperplane.

Proposition 3.3. *Let*

$$\operatorname{Def}(\boldsymbol{U}) := \left[\mathfrak{D}^0_{jk}(\boldsymbol{U})\right]_{n\times n}, \tag{3.5}$$

$$\mathfrak{D}^0_{jk}(\boldsymbol{U}) = \frac{1}{2}\left[\partial_k U^0_j + \partial_j U^0_k\right], \qquad \boldsymbol{U} = \sum_{j=1}^n U^0_j \boldsymbol{e}^j$$

be the deformation tensor in Cartesian coordinates.

The linear space $\mathscr{R}(\mathbb{R}^n)$ of rigid motions (of Killing's vector fields) in $\mathbb{R}^n$ consists of vector fields $\boldsymbol{K} = (K^0_1, \dots, K^0_n)^\top$ which are solutions to the system

$$2\mathfrak{D}^0_{jk}(\boldsymbol{K})(x) = \partial_k K^0_j(x) + \partial_j K^0_k(x) = 0 \qquad x \in \mathscr{S} \quad \text{for all} \quad j,k = 1,\dots,n. \tag{3.6}$$

If a rigid motion vanishes on an essentially $(n-1)$-dimensional subset $\boldsymbol{K}(\mathscr{x}) = 0$ for all $\mathscr{x} \in \mathscr{M}$, $\operatorname{ess\,dim}\mathscr{M} = n-1$, or at infinity $\boldsymbol{K}(x) = \mathscr{o}(1)$ as $|x| \to \infty$, then $\boldsymbol{K}$ vanishes identically $\boldsymbol{K}(x) \equiv 0$ on $\mathbb{R}^n$.

Proof. The proof can be retrieved from many sources. We quote only two of them [Ci2, KGBB1]. □

Remark 3.4. For the deformation tensor in Cartesian coordinates $\operatorname{Def}(\boldsymbol{U})$ (cf. (3.5)) in a domain $\Omega \subset \mathbb{R}^n$ Korn's inequality

$$\left\|\boldsymbol{U}\middle|\mathbb{H}^1_p(\Omega)\right\| \le M\left[\left\|\boldsymbol{U}\middle|\mathbb{L}_p(\Omega)\right\|^P + \left\|\operatorname{Def}(\boldsymbol{U})\middle|\mathbb{L}_p(\Omega)\right\|^p\right]^{1/p}, \quad 1 < p < \infty \tag{3.7}$$

with some constant $M > 0$ is well known and is proved, e.g., in [Ci2] (cf. (2.7) for a similar norm).

In contrast to the rigid motions in $\mathbb{R}^n$ nobody can describe Killing's vector fields on hypersurfaces explicitly so far. The next Theorem 3.5 underlines importance of Killing's vector fields for the Lamé equation on hypersurfaces. Later we investigate properties of Killing's vector fields to prepare tools for investigations of boundary value problems for the Lamé equation.

Theorem 3.5. *Let $\mathscr{S}$ be an ℓ-smooth ~~closed hypersurface~~ in $\mathbb{R}^n$ and $\ell \ge 2$. The Lamé operator $\mathscr{L}_\mathscr{S}$ for an isotropic media (cf. (0.23))*

$$\mathscr{L}_\mathscr{S} \;:\; \mathbb{H}^{s+1}_p(\mathscr{S}) \to \mathbb{H}^{s-1}_p(\mathscr{S}) \tag{3.8}$$

is self-adjoint $\mathscr{L}^*_\mathscr{S} = \mathscr{L}_\mathscr{S}$, elliptic, Fredholm and $\operatorname{Ind}\mathscr{L}_\mathscr{S} = 0$ for all $1 < p < \infty$ and all $s \in \mathbb{R}$, provided that $|s| \le \ell$.

The kernel of the operator $\operatorname{Ker}\mathscr{L}_\mathscr{S} \subset \mathbb{H}^s_p(\mathscr{S})$ is independent of the parameters p and s, coincides with the space of Killing's vector fields

$$\operatorname{Ker}\mathscr{L}_\mathscr{S} = \{\boldsymbol{U} \in \mathscr{V}(\mathscr{S}) \;:\; \mathscr{L}_\mathscr{S}\boldsymbol{U} = 0\} = \mathscr{R}(\mathscr{S}), \tag{3.9}$$

is finite-dimensional ~~and~~ $\dim\mathscr{R}(\mathscr{S}) = \dim\operatorname{Ker}\mathscr{L}_\mathscr{S} < \infty$.

If $\mathscr{S}$ *is* C^∞ *smooth, then the Killing's vector fields are smooth as well* $\mathscr{R}(\mathscr{S}) \subset C^\infty(\mathscr{S})$.

$\mathscr{L}_\mathscr{S}$ *is non-negative on the space* $\mathbb{H}^1(\mathscr{S})$ *and positive definite on the orthogonal complement* $\mathbb{H}^1_\mathscr{R}(\mathscr{S})$ *to the kernel*

$$(\mathscr{L}_\mathscr{S}\boldsymbol{U}, \boldsymbol{U})_\mathscr{S} \geq 0 \quad \text{for all} \quad \boldsymbol{U} \in \mathbb{H}^1(\mathscr{S}), \tag{3.10}$$

$$(\mathscr{L}_\mathscr{S}\boldsymbol{U}, \boldsymbol{U})_\mathscr{S} \geq C\big\|\boldsymbol{U}\big|\mathbb{H}^1(\mathscr{S})\big\|^2 \quad \text{for all} \quad \boldsymbol{U} \in \mathbb{H}^1_\mathscr{R}(\mathscr{S}), \quad C > 0, \tag{3.11}$$

~~*where* $\mathbb{H}^1_\mathscr{R}(\mathscr{S})$ *is the orthogonally complemented subspace to* $\mathscr{R}(\mathscr{S})$ *in* $\mathbb{H}^1(\mathscr{S})$.~~

Moreover, the following Gårding's inequality

$$(\mathscr{L}_\mathscr{S}\boldsymbol{U}, \boldsymbol{U})_\mathscr{S} \geq C_1\|\boldsymbol{U}|\mathbb{H}^1(\mathscr{S})\|^2 - C_0\|\boldsymbol{U}|\mathbb{H}^{-r}(\mathscr{S})\|^2 \tag{3.12}$$

holds for all $\boldsymbol{U} \in \mathbb{H}^1(\mathscr{S})$, *with arbitrary* $0 < r \leq \ell$ *and some positive constants* $C_0 > 0$, $C_1 > 0$.

The *proof* will be given later, in § 4. Here we draw the following consequence.

Corollary 3.6. *Let* $\mathscr{S} \subset \mathbb{R}^n$ *be a Lipschitz hypersurface without boundary,*

$$\mathrm{Def}_\mathscr{S}(\boldsymbol{U}) := [\mathfrak{D}_{jk}(\boldsymbol{U})]_{n\times n}$$

be the deformation tensor (*cf.* (0.17)) *and the norm*

$$\big\|\mathrm{Def}_\mathscr{S}(\boldsymbol{U})\big|\mathbb{L}_2(\mathscr{S})\big\|$$

be defined by (2.7).

Then the following Korn's inequality

$$\big\|\mathrm{Def}_\mathscr{S}(\boldsymbol{U})\big|\mathbb{L}_2(\mathscr{S})\big\| \geq c\big\|\boldsymbol{U}\big|\mathbb{H}^1(\mathscr{S})\big\| \qquad \forall\, \boldsymbol{U} \in \mathbb{H}^1_\mathscr{R}(\mathscr{S}) \tag{3.13}$$

holds for some constant $c > 0$ *or, equivalently, the mapping*

$$\boldsymbol{U} \mapsto \big\|\mathrm{Def}_\mathscr{S}(\boldsymbol{U})\big|\mathbb{L}_2(\mathscr{S})\big\|$$

is an equivalent norm on the orthogonal complement $\mathbb{H}^1_\mathscr{R}(\mathscr{S})$ *to the space of Killing's vector fields.*

Proof. Due to Korn's inequality (2.8) for $p = 2$

$$\big\|\boldsymbol{U}\big|\mathbb{L}_2(\mathscr{S})\big\|^2 \geq M_1\Big[\big\|\boldsymbol{U}\big|\mathbb{H}^1(\mathscr{S})\big\|^2 - \big\|\boldsymbol{U}\big|\mathbb{H}^1(\mathscr{S})\big\|\big\|\mathrm{Def}_\mathscr{S}(\boldsymbol{U})\big|\mathbb{L}_2(\mathscr{S})\big\|^2\Big]$$

the mapping $\mathrm{Def}_\mathscr{S} : \mathbb{H}^1_\mathscr{R}(\mathscr{S}) \to \mathbb{L}_2(\mathscr{S})$ is Fredholm and has index 0. The inequality (3.13) follows since the mapping is injective (has an empty kernel). □

Let us recall some results related to the uniqueness of solutions to arbitrary elliptic equation.

Definition 3.7. Let Ω be an open subset with the Lipschitz boundary $\partial\Omega \neq \emptyset$ either on a Lipschitz hypersurface $\mathscr{S} \subset \mathbb{R}^n$ or in the Euclidean space $\mathbb{R}^{n-1}$.

A class of functions $\mathscr{U}(\Omega)$ defined in a domain Ω in $\mathbb{R}^n$, is said to have the *strong unique continuation property*, if every $u \in \mathscr{U}(\Omega)$ in this class which vanishes to infinite order at one point must vanish identically.

If a surface $\mathscr{S}$ is C^∞-smooth, any elliptic operator on $\mathscr{S}$ has the strong unique continuation property due to Holmgren's theorem. But we can have more.

Lemma 3.8. *Let $\mathscr{S}$ be a C^2-smooth hypersurface in $\mathbb{R}^n$. The class of solutions to a second-order elliptic equation $\mathbb{A}(\mathscr{X},\mathscr{D})u = 0$, with Lipschitz continuous top-order coefficients on a surface $\mathscr{S}$ has the strong unique continuation property.*

In particular, if the solution $u(\mathscr{X}) = 0$ vanishes in any open subset of $\mathscr{S}$ it vanishes identically on entire $\mathscr{S}$.

Proof. The result was proved in [AKS1] for a domain $\Omega \subset \mathbb{R}^n$ by the method of "Carleman estimates" (also see [Hr1, Volume 3, Theorem 17.2.6]). Another proof, involving monotonicity of the frequency function was discovered by N. Garofalo and F. Lin (see [GL1, GL2]). A differential equation $\mathbb{A}(\mathscr{X},\mathscr{D})u(\mathscr{X}) = 0$ with Lipschitz continuous top-order coefficients on a C^2-smooth surface $\mathscr{S}$ is locally equivalent to a differential equation with Lipschitz continuous top-order coefficients on a domain $\Omega \subset \mathbb{R}^{n-1}$. Therefore a solution $u(\mathscr{X})$ has the strong unique continuation property locally (on each coordinate chart) on $\mathscr{S}$.

Since $\mathscr{S}$ is covered by a finite number of local coordinate charts which intersect on open neighborhoods, a solution $u(\mathscr{X})$ has the strong unique continuation property globally on $\mathscr{S}$. □

Remark 3.9. If the top-order coefficients of a second-order elliptic equation $\mathbb{A}(\mathscr{X},\mathscr{D})u = 0$ in open subsets $\Omega \subset \mathbb{R}^n$, $n \geq 3$, are merely Hölder continuous, with exponent less than 1, examples due to A. Plis [Pl1] and K. Miller [Mi1] show that a solution $u(x)$ does not have the strong unique continuation property.

Lemma 3.10. *Let $\mathscr{C}$ be a C^2-smooth hypersurface in $\mathbb{R}^n$ with the Lipschitz boundary $\Gamma := \partial\mathscr{C}$ and $\gamma \subset \Gamma$ be an open part of the boundary Γ. Let $\mathbb{A}(\mathscr{X},\mathscr{D})$ be a second-order elliptic system with Lipschitz continuous top-order matrix coefficients on a surface $\mathscr{S}$.*

The Cauchy problem

$$\begin{cases} \mathbb{A}(\mathscr{X},\mathscr{D})u = 0 & \text{on} \quad \mathscr{C}, \qquad u \in \mathbb{H}^1(\Omega), \\ u(\mathfrak{s}) = 0 & \text{for all} \quad \mathfrak{s} \in \gamma, \\ (\partial_{\boldsymbol{V}} u)(\mathfrak{s}) = 0 & \text{for all} \quad \mathfrak{s} \in \gamma, \end{cases} \tag{3.14}$$

where $\boldsymbol{V}$ is a non-tangent vector to Γ, but tangent to $\mathscr{S}$, has only a trivial solution $u(\mathscr{X}) = 0$ on entire $\mathscr{S}$.

Proof. With a local diffeomorphism the Cauchy problem (3.14) is transformed into a similar problem on a domain $\Omega \subset \mathbb{R}^{n-1}$ with the Cauchy data vanishing on some open subset of the boundary $\gamma \subset \Gamma := \partial\Omega$.

Let us, for simplicity, use the same notation $\gamma \subset \Gamma = \partial\Omega$, the non-tangent vector $\boldsymbol{V}$ to γ, the function u and the differential operator $\mathbb{A}(x,\mathscr{D})$ for the transformed Cauchy problem in the transformed domain Ω. Moreover, we will suppose that γ is a part of the hypersurface $x_1 = 0$ (otherwise we can transform the domain Ω again). We also use new variables $t = x_1$ and $x := (x_2, \dots, x_{n-1})$. Then $(0, x) \in \gamma$ while $(t, x) \in \Omega$ for all small $0 < t < \varepsilon$ and some $x \in \Omega'$.

Thus, the natural basis element $\boldsymbol{e}^1$ (cf. (0.6)) is orthogonal to γ and, therefore, $\boldsymbol{e}^1 = c_1(x)\boldsymbol{V}(0, x) + c_2(x)\boldsymbol{g}^j(x)$ for some unit tangential vector $\boldsymbol{g}^j(x)$ to γ for all $x \in \Omega'$ and some scalar functions $c_1(x)$, $c_2(x)$. Then, due to the third line in (3.14),

$$(\partial_t u)(0, x) = \partial_{\boldsymbol{e}^j} u(0, x) = c_1(x)\partial_{\boldsymbol{V}} u(0, x) + c_2(x)\partial\boldsymbol{g}^j u(0, x) = 0$$

because any derivative along tangential vector to γ vanishes $\partial\boldsymbol{g}^j u(0, x) = 0$ due to the second line in (3.14).

The second-order equation $\mathbb{A}(t, x; \mathscr{D})$ can be written in the form

$$\mathbb{A}(t, x, D)u = \mathbb{A}(t, x; \boldsymbol{e}^1)\partial_t^2 u + \mathbb{A}_1(t, x; D)\partial_t u + \mathbb{A}_2(t, x; D), \qquad D := -i\partial_x,$$

where $\mathbb{A}_1(t, x; \boldsymbol{e}^1)$ is the (invertible) matrix function, $\mathbb{A}_1(t, x; D)$ and $\mathbb{A}_2(t, x; D)$ are differential operators of orders 1 and 2 respectively, compiled of derivatives ∂_x, $x \in \Omega'$. Therefore, if $\mathbb{A}_j^0(t, x; D) := \mathbb{A}^{-1}(t, x; \boldsymbol{e}^1)\mathbb{A}_j(t, x; D)$, $j = 1, 2$, the Cauchy problem (3.14) transforms into

$$\begin{cases} \partial_t^2 u(t, x) + \mathbb{A}_1^0(t, x; D)\partial_t u(t, x) + \mathbb{A}_2^0(t, x; D)u(t, x) = 0 & \text{on } (t, x) \in \Omega_\varepsilon, \\ u(0, x) = 0 & \text{for all } x \in \Omega', \\ (\partial_t u)(0, x) = 0 & \text{for all } x \in \Omega', \end{cases} \tag{3.15}$$

where $\Omega_\varepsilon := (0, \varepsilon) \times \Omega' \subset \Omega$, $u \in \mathbb{H}^1(\Omega_\varepsilon)$ and $\gamma := \big\{(0, x) \,:\, x \in \Omega'\big\}$.

Now let us recall the inequality (see [Miz1, § 4.3, Theorem 4.3, § 6.14], [Sch1, § 4-7, Lemma 4-21]): There is a constant C which depends on ε and $\mathbb{A}(t, x; D)$ only and such that the inequality

$$\int_{\Omega_\varepsilon} e^{-\lambda t}|v(t, x)|^2 dt\, dx \le C \int_{\Omega_\varepsilon} e^{-\lambda t}|(\mathbb{A}(t, x; D)v)(t, x)|^2 dt\, dx, \tag{3.16}$$

holds for $\mathbb{A}(t, x; D)v \in \mathbb{L}_2(\Omega_\varepsilon)$, $v \in C^\infty(\Omega_\varepsilon)$; moreover, $v(t, x)$ should vanish near $t = \varepsilon$ and should have vanishing Cauchy data $v(0, x) = (\partial_t v)(0, x) = 0$ for all $x \in \Omega'$.

Let $\rho \in C^2(0, \varepsilon)$ be a cut-off function: $\rho(t) = 1$ for $0 \le t < \varepsilon/2$ and $\rho(t) = 0$ for $3\varepsilon/4 \le t < \varepsilon$. Then $v := \rho u \in \mathbb{H}^1(\Omega_\varepsilon)$ and since $\mathbb{A}(t, x; D)u = 0$ on Ω_ε, we get

$$\begin{aligned} \mathbb{A}(t, x; D)(\rho u) &= \rho\mathbb{A}(t, x; D)u + (\partial_t^2 \rho)u + (\partial_t \rho)\partial_t u + (\partial_t \rho)\mathbb{A}_1^0(t, x; D)u \\ &= (\partial_t^2 \rho)u + (\partial_t \rho)\partial_t u + (\partial_t \rho)\mathbb{A}_1^0(t, x; D)u. \end{aligned}$$

We have asserted $u \in \mathbb{H}^1(\Omega_\varepsilon)$, $\rho \in C^2$ and this implies $(\partial_t^2 \rho)u \in \mathbb{H}^1(\Omega_\varepsilon) \subset \mathbb{L}_2(\Omega_\varepsilon)$, $(\partial_t \rho)\partial_t u \in \mathbb{L}_2(\Omega_\varepsilon)$. Note, that $(\partial_t \rho(t)$ vanishes for $0 < t < \varepsilon/2$. Therefore $(\partial_t \rho)\mathbb{A}_1^0(t, x; D)u$ vanishes in a neighborhood of the boundary $\gamma \subset \Gamma$. Due to a priori regularity result (cf. [LM1, Ch. 2, § 3.2, § 3.3]), a solution to an elliptic equation in (3.15) has additional regularity $u \in \mathbb{H}^2(\Omega_\varepsilon^0)$ for arbitrary Ω_ε^0 properly imbedded into Ω_ε. This implies $(\partial_t \rho)\mathbb{A}_1^0(t, x; D)u \in \mathbb{L}_2(\Omega_\varepsilon)$ and we conclude

$$\mathbb{A}(t, x; D)(\rho u) \in \mathbb{L}_2(\Omega_\varepsilon). \tag{3.17}$$

Introducing $v = \rho u$ into the inequality (3.16) we get

$$\int_{\Omega'}\int_0^{\varepsilon/4} e^{-\lambda t}|\rho(t)u(t,x)|^2 dt\,dx \le \int_{\Omega_\varepsilon} e^{-\lambda t}|\rho(t)u(t,x)|^2 dt\,dx$$
$$\le C\int_{\Omega'}\int_{\varepsilon/2}^{3\varepsilon/4} e^{-\lambda t}|(\mathbb{A}(t,x;D))\rho(t)u(t,x)|^2 dt\,dx.$$

This implies for $\lambda > 0$

$$\int_{\Omega'}\int_0^{\varepsilon/4} |\rho(t)u(t,x)|^2 dt\,dx \le e^{-\lambda\varepsilon/4}\int_{\Omega_\varepsilon} |(\mathbb{A}(t,x;D))\rho(t)u(t,x)|^2\,dt\,dx \le C_1 e^{-\lambda\varepsilon/4}.$$

where, due to (3.14), $C_1 > 0$ is a finite constant. By sending $\lambda \to \infty$ we get the desired result $u(t,x) = 0$ for all $0 \le t \le \varepsilon/4$ and all $x \in \Omega'$. Since $u(x)$ vanishes in a subset of the domain Ω, bordering γ, due to Lemma 3.8 the solution vanishes on entire Ω (on entire $\mathscr{C}$). □

Due to our specific interest (see ~~the~~ next Lemma 3.12) and many applications, for example to control theory, the following boundary unique continuation property is of special interest.

Definition 3.11. Let $\mathscr{S}$ be a Lipschitz hypersurface in $\mathbb{R}^n$ and $\mathscr{C} \subset \mathscr{S}$ be ~~an open subsurface~~ with the Lipschitz boundary $\Gamma = \partial\mathscr{C}$.

We say that a class of functions $\mathscr{U}(\Omega)$ has the *strong unique continuation property from the boundary* if a vector function $\boldsymbol{U} \in \mathscr{U}(\Omega)$ which vanishes $\boldsymbol{U}(\mathfrak{s}) = 0$, $\forall\,\mathfrak{s} \in \gamma$ on an open subset of the boundary $\gamma \subset \Gamma$, vanishes on the entire $\mathscr{C}$.

Lemma 3.12. *Let $\mathscr{S}$ be a C^2-smooth hypersurface in $\mathbb{R}^n$ and $\mathscr{C} \subset \mathscr{S}$ be ~~an open C^2-smooth subsurface~~.*

The set of Killing's vector fields $\mathscr{R}(\mathscr{S})$ on the open surface $\mathscr{C}$ has the strong unique continuation property from the boundary.

Proof. Let $\gamma \subset \Gamma := \partial\mathscr{C}$, $\operatorname{mes}\gamma > 0$ and $\boldsymbol{U}(\mathfrak{s}) = 0$ for all $\mathfrak{s} \in \gamma \subset \Gamma := \partial\mathscr{C}$. Then (cf. (3.1))

$$\begin{cases} (\mathscr{D}_j U_k^0)(\mathfrak{s}) + (\mathscr{D}_k U_j^0)(\mathfrak{s}) = -\sum\limits_{m=1}^n U_m^0(\mathfrak{s})\mathscr{D}_m\big(\nu_j(\mathfrak{s})\nu_k(\mathfrak{s})\big) = 0, \\ U_k^0(\mathfrak{s}) = 0 \qquad \forall\,\mathfrak{s} \in \gamma, \quad j,k = 1,\dots,n. \end{cases} \tag{3.18}$$

Among tangent vector fields generating the Günter's derivatives $\{\boldsymbol{d}^j(\mathfrak{s})\}_{j=1}^{n-1}$ only $n-1$ are linearly independent. One of vectors might collapse at a point $\boldsymbol{d}^j(\mathfrak{s}) = 0$ if the corresponding basis vector $\boldsymbol{e}^j$ is orthogonal to the surface at $\mathfrak{s} \in \mathscr{S}$, while others might be tangential to the subsurface Γ, except at least one, say $\boldsymbol{d}^n(\mathfrak{s})$, which is non-tangential to γ. Then from (3.18) follows

$$2(\mathscr{D}_n U_n^0)(\mathfrak{s}) = 0 \quad \text{and implies} \quad (\mathscr{D}_j U_n^0)(\mathfrak{s}) = 0 \tag{3.19}$$
$$\text{for all} \quad \mathfrak{s} \in \gamma \quad \text{and all} \quad j = 1,\dots,n.$$

Indeed, the vector $\boldsymbol{d}^j$, $1 \le j = 1 \le n-1$ is a linear combination $\boldsymbol{d}^j(\mathfrak{s}) = c_1(\mathfrak{s})\boldsymbol{d}^n(\mathfrak{s}) + c_2(\mathfrak{s})\boldsymbol{\tau}^j(\mathfrak{s})$ of the non-tangential vector $\boldsymbol{d}^n(\mathfrak{s})$ and of the projection $\boldsymbol{\tau}^j(\mathfrak{s}) := \pi_\gamma \boldsymbol{d}^j(\mathfrak{s})$ of $\boldsymbol{d}^j(\mathfrak{s})$ to the subsurface γ at the point $\mathfrak{s} \in \gamma$. Since U^n vanishes identically on γ, the derivative $(\partial_{\boldsymbol{\tau}^j} U^0_n)(\mathfrak{s}) = 0$ vanishes as well and (3.19) follows:

$$(\mathscr{D}_j U^0_n)(\mathfrak{s}) = c_1(\mathfrak{s})(\partial_{\boldsymbol{d}^n} U^0_n)(\mathfrak{s}) + c_2(\mathfrak{s})(\partial_{\boldsymbol{\tau}^j} U^0_n)(\mathfrak{s}) = c_1(\mathfrak{s})(\mathscr{D}_n U^0_n)(\mathfrak{s}) = 0 \quad \forall\, \mathfrak{s} \in \gamma.$$

Equalities (3.18) and (3.19) imply

$$(\mathscr{D}_n U^0_j(\mathfrak{s}) = -(\mathscr{D}_j U^0_n)(\mathfrak{s}) = 0 \qquad \forall\, \mathfrak{s} \in \gamma \quad \text{and all} \quad j = 1, \dots, n. \tag{3.20}$$

Thus, we have the following Cauchy problem

$$\begin{cases} \mathscr{L}_{\mathscr{C}}(\mathscr{X}, \mathscr{D})\boldsymbol{U}(\mathscr{X}) = 0 & \text{on} \quad \mathscr{C}, \\ \boldsymbol{U}(\mathfrak{s}) = 0 & \text{for all} \quad \mathfrak{s} \in \gamma, \\ (\mathscr{D}_n \boldsymbol{U})(\mathfrak{s}) = (\partial_{\boldsymbol{d}^n} \boldsymbol{U})(\mathfrak{s}) = 0 & \text{for all} \quad \mathfrak{s} \in \gamma, \end{cases} \tag{3.21}$$

where $\boldsymbol{d}^n$ is a vector filed non-tangential to Γ. Due to Lemma 3.10, $\boldsymbol{U}(\mathscr{X}) = 0$ for all $\mathscr{X} \in \mathscr{C}$. □

Corollary 3.13. (Korn's I inequality "with boundary condition"). *Let $\mathscr{C} \subset \mathbb{R}^n$ be a C^ℓ-smooth hypersurface with the Lipschitz boundary $\Gamma := \partial\mathscr{C} \neq \emptyset$ and $\ell \ge 2$, $|s| \le \ell$. Then*

$$\left\| \boldsymbol{U} \middle| \mathbb{H}^s_p(\mathscr{C}) \right\| \le M \left\| \mathrm{Def}_{\mathscr{C}}(\boldsymbol{U}) \middle| \mathbb{H}^{s-1}_p(\mathscr{C}) \right\| \qquad \forall\, \boldsymbol{U} \in \widetilde{\mathbb{H}}^s_p(\mathscr{C})$$

for some constant $M > 0$. In other words: the mapping

$$\boldsymbol{U} \mapsto \left\| \mathrm{Def}_{\mathscr{C}}(\boldsymbol{U}) \middle| \mathbb{H}^{s-1}_p(\mathscr{C}) \right\| \tag{3.22}$$

is an equivalent norm on the space $\widetilde{\mathbb{H}}^s_p(\mathscr{C})$.

Proof. If the claimed inequality (3.22) is false, there exists a sequence $\boldsymbol{U}^j \in \widetilde{\mathbb{H}}^s_p(\mathscr{C})$, $j = 1, 2, \dots$ such that

$$\left\| \boldsymbol{U}^j \middle| \mathbb{H}^s_p(\mathscr{C}) \right\| = 1 \quad \forall j = 1, 2, \dots \quad \lim_{j\to\infty} \left\| \mathrm{Def}_{\mathscr{C}}(\boldsymbol{U}^j) \middle| \mathbb{H}^{s-1}_p(\mathscr{C}) \right\| = 0.$$

Due to the compact embedding $\widetilde{\mathbb{H}}^s_p(\mathscr{C}) \subset \mathbb{H}^s_p(\mathscr{C}) \subset \mathbb{H}^{s-1}_p(\mathscr{C})$, a convergent subsequence $\boldsymbol{U}^{j_1}, \boldsymbol{U}^{j_2}, \dots$ in $\mathbb{H}^{s-1}_p(\mathscr{C})$ can be selected. Let $\boldsymbol{U}^0 = \lim_{k\to\infty} \boldsymbol{U}^{j_k}$. Then

$$\left\| \mathrm{Def}_{\mathscr{C}}(\boldsymbol{U}^0) \middle| \mathbb{H}^{s-1}_p(\mathscr{C}) \right\| = \lim_{k\to\infty} \left\| \mathrm{Def}_{\mathscr{C}}(\boldsymbol{U}^{j_k}) \middle| \mathbb{H}^{s-1}_p(\mathscr{C}) \right\| = 0$$

and $\boldsymbol{U}^0$ is a Killing's vector field. Since $\boldsymbol{U}(x) = 0$ on Γ, due to Lemma 3.12 $\boldsymbol{U}^0(x) = 0$ for all $x \in \mathscr{C}$ which contradicts to $\left\| \boldsymbol{U}^0 \middle| \mathbb{H}^s_p(\mathscr{C}) \right\| = \lim_{k\to\infty} \left\| \boldsymbol{U}^{j_k} \middle| \mathbb{H}^s_p(\mathscr{C}) \right\| = 1$. □

4. A local fundamental solution to the Lamé equation

Proof of Theorem 3.5. Let us check the ellipticity of $\mathscr{L}_\mathscr{S}$. The operator $\mathscr{L}_\mathscr{S}$ maps the tangential spaces and the principal symbol is defined on the cotangent space. The cotangent space is orthogonal to the normal vector and, therefore,

$$\mathscr{L}_\mathscr{S}(\mathscr{X},\xi)\eta=\mu|\xi|^2(1-\boldsymbol{\nu}\boldsymbol{\nu}^\top)\eta+(\lambda+\mu)\xi\xi^\top\eta=\mu|\xi|^2\eta+(\lambda+\mu)\xi\xi^\top\eta\,,\quad \forall\,\xi,\,\eta\perp\boldsymbol{\nu}.$$

Thus, while considering the principal symbol $\mathscr{L}_\mathscr{S}(\mathscr{X},\xi)$ we can ignore the projection $\pi_\mathscr{S}$. With this assumption, the principal symbol of $\mathscr{L}_\mathscr{S}$ reads

$$\mathscr{L}_\mathscr{S}(\mathscr{X},\xi)=\mu|\xi|^2+(\lambda+\mu)\xi\xi^\top\qquad\text{for}\quad(\mathscr{X},\xi)\in\mathbb{T}^*(\mathscr{S})\,.\tag{4.1}$$

The matrix $\mathscr{L}_\mathscr{S}(\mathscr{X},\xi)$ has eigenvalue $(\lambda+2\mu)|\xi|^2$ (the corresponding eigenvector is ξ) and $\mu|\xi|^2$ which has multiplicity $n-1$ (the corresponding eigenvectors θ^j are orthogonal to ξ: $\xi^\top\theta^j=\langle\xi,\theta^j\rangle=0$, $j=1,\dots,n-1$). Then

$$\det\mathscr{L}_\mathscr{S}(\mathscr{X},\xi)=(\lambda+2\mu)|\xi|^2\big[\mu|\xi|^2\big]^{n-1}=\mu^{n-1}(\lambda+2\mu)>0$$
$$\text{for}\quad(\mathscr{X},\xi)\in\mathbb{T}^*(\mathscr{S})\,,\quad|\xi|=1$$

and the ellipticity is proved.

The ellipticity of the differential operator $\mathscr{L}_\mathscr{S}=\mathscr{L}_\mathscr{S}(\mathscr{X},\mathscr{D})$ in (3.8) on a ~~closed manifold~~ $\mathscr{S}$, proved above, implies Fredholm property for all $1<p<\infty$ and all $s\in\mathbb{R}$. Indeed, $\mathscr{L}_\mathscr{S}(\mathscr{X},\mathscr{D})$ has a parametrix $\boldsymbol{R}_\mathscr{S}(\mathscr{X},\mathscr{D})$, which is a pseudodifferential operator (ΨDO) with the symbol $R_\mathscr{S}(\mathscr{X},\xi):=\chi(\xi)\mathscr{L}_\mathscr{S}^{-1}(\mathscr{X},\xi)$, where $\mathscr{L}_\mathscr{S}^{-1}(\mathscr{X},\xi)$ is the inverse symbol and $\chi\in C^\infty(\mathbb{R}^n)$ is a smooth function, $\chi(\xi)=1$ for $|\xi|>2$ and $\chi(\xi)=0$ for $|\xi|<1$. ΨDO $\boldsymbol{R}_\mathscr{S}(\mathscr{X},\mathscr{D})$ is a bounded operator between the spaces

$$\boldsymbol{R}_\mathscr{S}(\mathscr{X},\mathscr{D})\;:\;\mathbb{H}^{s-2}_p(\mathscr{S})\to\mathbb{H}^s_p(\mathscr{S}),\qquad\text{for all}\quad1<p<\infty,\quad s\in\mathbb{R},$$

because the symbol $R_\mathscr{S}(\mathscr{X},\xi)=\mathscr{L}_\mathscr{S}^{-1}(\mathscr{X},\xi)$ belongs to the Hörmander class $S^{-2}(\mathscr{S},\mathbb{R}^n)$

$$\left|\mathscr{D}^\alpha\partial^\beta_\xi R_\mathscr{S}(\mathscr{X},\xi)\right|\le C_{\alpha,\beta}|\xi|^{-2-|\beta|}$$

for all multi-indices $\alpha,\beta\in\mathbb{Z}^n_+$ (cf. [Hr1, Sh1, Ta2] for details).

The Fredholm property for the case $p=2$ and $s=1$ follows from Gårding's inequality (3.12) as well (cf. [HW1, Thorem 5.3.10] and [Mc1, Thorem 2.33]).

The Fredholm property implies the finite-dimensional kernel

$$\dim\operatorname{Ker}\mathscr{L}_\mathscr{S}(\mathscr{X},\mathscr{D})<\infty.$$

To prove that the index is trivial $\operatorname{Ind}\mathscr{L}_\mathscr{S}(\mathscr{X},\mathscr{D})=0$ for all $1<p<\infty$, $s\in\mathbb{R}$ we apply Gårding's inequality (3.12) and homotopy. For this purpose first note that the symbol $\mathscr{L}_\mathscr{S}(\mathscr{X},\xi)$ is positive definite (cf. (4.1))

$$\langle\mathscr{L}_\mathscr{S}(\mathscr{X},\xi)\eta,\eta\rangle=\mu|\xi|^2|\eta|^2+(\lambda+\mu)\langle\xi\xi^\top\eta,\eta\rangle=\mu|\xi|^2|\eta|^2+(\lambda+\mu)\sum_{j=1^n}(\xi_j\eta_j)^2$$
$$\ge\mu|\xi|^2|\eta|^2\qquad\forall\,\mathscr{X}\in\mathscr{S},\quad\forall\,\xi,\,\eta\in\mathbb{R}^n.\tag{4.2}$$

Further recall that the Bessel potential operator $\Lambda^2_{\mathscr{S}}(\mathscr{X}, D) \ : \ \mathbb{H}^s_p(\mathscr{S}) \to \mathbb{H}^{s-2}_p(\mathscr{S})$ (cf. (2.4)) lifting the Bessel potential spaces, has positive definite symbol

$$\langle \Lambda^2_{\mathscr{S}}(\mathscr{X}, \xi)\eta, \eta\rangle \geq C|\xi|^2|\eta|^2 \qquad \forall\, \mathscr{X} \in \mathscr{S}, \quad \forall\, \xi,\, \eta \in \mathbb{R}^n \tag{4.3}$$

(cf. [DS1]). Now consider the symbols $\boldsymbol{B}_\tau(\mathscr{X}, \xi) = (1-\tau)\mathscr{L}_{\mathscr{S}}(\mathscr{X}, \xi) + \tau\Lambda^2_{\mathscr{S}}(\mathscr{X}, \xi)$ and the corresponding ΨDO

$$\boldsymbol{B}_\tau(\mathscr{X}, \mathscr{D}) = (1-\tau)\mathscr{L}_{\mathscr{S}}(\mathscr{X}, \mathscr{D}) + \tau\Lambda^2_{\mathscr{S}}(\mathscr{X}, \mathscr{D}) \ : \ \mathbb{H}^s_p(\mathscr{S}) \to \mathbb{H}^{s-2}_p(\mathscr{S}). \tag{4.4}$$

Obviously, $\boldsymbol{B}_\tau(\mathscr{X}, \mathscr{D})$ is a continuous (with respect to $0 \leq \tau \leq 1$) homotopy connecting the operator $\boldsymbol{B}_0(\mathscr{X}, \mathscr{D}) = \mathscr{L}_{\mathscr{S}}(\mathscr{X}, \mathscr{D})$ with $\boldsymbol{B}_1(\mathscr{X}, \mathscr{D}) = \Lambda^2_{\mathscr{S}}(\mathscr{X}, \mathscr{D})$. Since the symbol $B_\tau(\mathscr{X}, \xi)$ is positive definite

$$\langle B_\tau(\mathscr{X}, \xi)\eta, \eta\rangle \geq [(1-\tau)\mu + \tau C]|\xi|^2|\eta|^2 \quad \forall\, \xi,\, \eta \in \mathbb{R}^n$$

(cf. (4.2) and (4.3)), it is elliptic and the operator $\boldsymbol{B}_\tau(\mathscr{X}, \mathscr{D})$ is then Fredholm for all $0 \leq \tau \leq 1$. Then $\operatorname{Ind}\mathscr{L}_{\mathscr{S}}(\mathscr{X}, \mathscr{D}) = \operatorname{Ind}\boldsymbol{B}_0(\mathscr{X}, \mathscr{D}) = \operatorname{Ind}\boldsymbol{B}_1(\mathscr{X}, \mathscr{D}) = \operatorname{Ind}\Lambda^2_{\mathscr{S}}(\mathscr{X}, \mathscr{D}) = 0$, since the operator $\Lambda^2_{\mathscr{S}}(\mathscr{X}, \mathscr{D})$ is invertible.

From the representation (0.23) follows that the bilinear form $\big(\mathscr{L}_{\mathscr{S}}\boldsymbol{U}, \boldsymbol{U}\big)_{\mathscr{S}}$ is non-negative

$$\begin{aligned}\big(\mathscr{L}_{\mathscr{S}}\boldsymbol{U}, \boldsymbol{U}\big)_{\mathscr{S}} &= \lambda\big(\operatorname{div}^*_{\mathscr{S}}\operatorname{div}_{\mathscr{S}}\boldsymbol{U}, \boldsymbol{U}\big)_{\mathscr{S}} + 2\mu\big(\operatorname{Def}^*_{\mathscr{S}}\operatorname{Def}_{\mathscr{S}}\boldsymbol{U}, \boldsymbol{U}\big)_{\mathscr{S}} \\ &= \lambda\big\|\operatorname{div}_{\mathscr{S}}\boldsymbol{U}\big|\mathbb{L}_2(\mathscr{S})\big\|^2 + 2\mu\big\|\operatorname{Def}_{\mathscr{S}}\boldsymbol{U}\big|\mathbb{L}_2(\mathscr{S})\big\|^2 \geq 0\end{aligned} \tag{4.5}$$

(cf. (3.10)) and only vanishes if $\boldsymbol{U}$ is a Killing's vector field $\operatorname{Def}_{\mathscr{S}}\boldsymbol{U} = 0$. Indeed, $\mathfrak{D}_{jj}\boldsymbol{U} = (\mathscr{D}^{\mathscr{S}}_j\boldsymbol{U})^0_j = 0$, $j = 1, \dots, n$, if $\operatorname{Def}_{\mathscr{S}}\boldsymbol{U} = 0$ and, due to (0.13),

$$\operatorname{div}_{\mathscr{S}}\boldsymbol{U} = \sum_{j=1}^n \mathscr{D}_j U^0_j = \sum_{j=1}^n \mathscr{D}_j U^0_j + \frac{1}{2}\sum_{j=1}^n \partial_{\boldsymbol{U}}(\nu_j)^2 = \sum_{j=1}^n (\mathscr{D}^{\mathscr{S}}_j\boldsymbol{U})^0_j = 0 \qquad \forall\, \boldsymbol{U} \in \mathscr{R}(\mathscr{S}) \tag{4.6}$$

since $|\boldsymbol{\nu}(\mathscr{X})| \equiv 1$. Thence, due to (4.5), $\mathscr{R}(\mathscr{S}) \subset \operatorname{Ker}\mathscr{L}_{\mathscr{S}}$. The inverse inclusion follows also from (4.5) because $\operatorname{Def}_{\mathscr{S}}(\boldsymbol{U}) = 0$ if $\mathscr{L}_{\mathscr{S}}(\mathscr{X}, \mathscr{D})\boldsymbol{U} = 0$. This accomplishes the proof of (3.9).

The estimate (3.11) is a direct consequence of (3.10) and of (3.9): Since the operator $\mathscr{L}_{\mathscr{S}}$ is Fredholm, self-adjoint and $\operatorname{Ker}\mathscr{L}_{\mathscr{S}} = \mathscr{R}(\mathscr{S})$, then also $\operatorname{Coker}\mathscr{L}_{\mathscr{S}} = \mathscr{R}(\mathscr{S})$ and, therefore, the mapping

$$\mathscr{L}_{\mathscr{S}} \ : \ \mathbb{H}^1_{\mathscr{R}}(\mathscr{S}) \longrightarrow \mathbb{H}^{-1}_{\mathscr{R}}(\mathscr{S})$$

is one-to-one, i.e., is invertible. The established invertibility implies the claimed inequality (3.11).

A priori regularity property of solutions to partial differential equations (cf. [Ta2, Hr1]) states that the ellipticity of $\mathscr{L}_{\mathscr{S}}(\mathscr{X}, \mathscr{D})$ provides $C^\ell(\mathscr{S})$-smoothness of any solution $\boldsymbol{K}$ to the homogeneous equation $\mathscr{L}_{\mathscr{S}}(\mathscr{X}, \mathscr{D})\boldsymbol{K} = 0$ (the hypersurface $\mathscr{S}$ is C^ℓ-smooth). Due to the embeddings $\mathbb{H}^r_q(\mathscr{S}) \subset \mathbb{H}^s_p(\mathscr{S})$, $s \leq r$, $p \leq q$, then the kernel $\operatorname{Ker}\mathscr{L}_{\mathscr{S}}(\mathscr{X}, \mathscr{D})$ is independent of the space $\mathbb{H}^s_p(\mathscr{S})$ provided that the spaces are well defined, which is the case if $|s| \leq \ell$ (cf. [Ag1, Du2, DNS2, Ka1] for similar assertions).

In particular, the Killing's vector fields $\mathscr{R}(\mathscr{S}) = \operatorname{Ker} \mathscr{L}_{\mathscr{S}}(\mathscr{X}, \mathscr{D})$ are smooth $\mathscr{R}(\mathscr{S}) \subset C^\infty(\mathscr{S})$ provided that the hypersurface $\mathscr{S}$ is C^∞-smooth.

Let $\{\boldsymbol{K}_j\}_{j=1}^m$ be an orthogonal basis $(\boldsymbol{K}_j, \boldsymbol{K}_k)_{\mathscr{S}} = \delta_{jk}$ in the finite-dimensional space of Killing's vector fields $\mathscr{R}(\mathscr{S})$. Let

$$\boldsymbol{T}\boldsymbol{U}(\mathscr{X}) := \sum_{j=1}^m (\boldsymbol{K}_j, \boldsymbol{U})_{\mathscr{S}} \, \boldsymbol{K}_j(\mathscr{X}), \qquad \mathscr{X} \in \mathscr{S}. \tag{4.7}$$

Due to the proved part $\{\boldsymbol{K}_j\}_{j=1}^m \subset C^\ell(\mathscr{S})$ and the operator $\boldsymbol{T}$ is smoothing $\boldsymbol{T} : \mathbb{H}^{-r}(\mathscr{S}) \to \mathbb{H}^r(\mathscr{S})$ (is infinitely smoothing if $\ell = \infty$). Then, the operator

$$\mathscr{L}_{\mathscr{S}} + \boldsymbol{T} \; : \; \mathbb{H}^1(\mathscr{S}) \to \mathbb{H}^{-1}(\mathscr{S})$$

is invertible and non-negative

$$(\mathscr{L}_{\mathscr{S}} + \boldsymbol{T})\boldsymbol{U}, \boldsymbol{U})_{\mathscr{S}} = (\mathscr{L}_{\mathscr{S}}\boldsymbol{U}, \boldsymbol{U})_{\mathscr{S}} + \sum_{j=1}^m (\boldsymbol{K}_j, \boldsymbol{U})^2_{\mathscr{S}} \geq 0$$

(cf. (4.5)). This implies that $\mathscr{L}_{\mathscr{S}} + \boldsymbol{T}$ is positive definite

$$(\mathscr{L}_{\mathscr{S}}\boldsymbol{U}, \boldsymbol{U})_{\mathscr{S}} + \boldsymbol{T})\boldsymbol{U}, \boldsymbol{U})_\Gamma \geq C_1 \big\|\boldsymbol{U}\big|\mathbb{H}^1(\mathscr{S})\big\|^2$$

and we write

$$\begin{aligned}(\mathscr{L}_{\mathscr{S}}\boldsymbol{U}, \boldsymbol{U})_{\mathscr{S}} &:= ((\mathscr{L}_{\mathscr{S}} + \boldsymbol{T})\boldsymbol{U}, \boldsymbol{U})_{\mathscr{S}} + (\boldsymbol{T}\boldsymbol{U}, \boldsymbol{U})_{\mathscr{S}} \\ &\geq C_1 \big\|\boldsymbol{U}\big|\mathbb{H}^1(\mathscr{S})\big\|^2 + (\boldsymbol{T}\boldsymbol{U}, \boldsymbol{U})_{\mathscr{S}} \\ &\geq C_1 \big\|\boldsymbol{U}\big|\mathbb{H}^1(\mathscr{S})\big\|^2 - C_2 \big\|\boldsymbol{U}\big|\mathbb{H}^{-r}(\mathscr{S})\big\|^2,\end{aligned}$$

which proves (3.12). □

Remark 4.1. Gårding's inequality (3.12), but in a weaker form $r = 0$, is a direct consequence of the inequality (4.5) and Korn's inequality (2.8) for $p = 2$.

Theorem 4.2. *Let $\mathscr{S}$ be a ℓ-smooth ~~closed hypersurface,~~ $\ell \geq 2$ and $\mathscr{B} \in C^\ell(\mathbb{R}^n)$ be a real-valued and non-negative $\mathscr{B} \geq 0$ function with non-trivial support* $\operatorname{mes} \operatorname{supp} \mathscr{B} \neq 0$.

The perturbed operator

$$\mathscr{L}_{\mathscr{S}}(\mathscr{X}, \mathscr{D}) + \mathscr{B}I \; : \; \mathbb{H}^{\theta+1}_p(\mathscr{S}) \to \mathbb{H}^{\theta-1}_p(\mathscr{S}) \tag{4.8}$$

is invertible for all $|\theta| \leq \ell - 1$ and all $1 < p < \infty$.

Proof. The principal symbol of the operator $\mathscr{L}_{\mathscr{S}}(\mathscr{X}, \mathscr{D}) + \mathscr{B}I$ in (4.8) ignores lower-order terms and coincides with $\mathscr{L}_{\mathscr{S}}(\mathscr{X}, \xi)$ and is elliptic (cf. Theorem 3.5). Therefore on the ~~closed hypersurface~~ $\mathscr{S}$ the operator $\mathscr{L}_{\mathscr{S}}(\mathscr{X}, \mathscr{D}) + \mathscr{B}I$ in (4.8) is Fredholm for all $\theta = 0, 1, \ldots$ (cf. Theorem 3.5). On the other hand, if $(\mathscr{L}_{\mathscr{S}}(\mathscr{X}, \mathscr{D}) + \mathscr{B}I)\boldsymbol{U} = 0$, then

$$0 = ((\mathscr{L}_{\mathscr{S}}(\mathscr{X}, \mathscr{D}) + \mathscr{B})\boldsymbol{U}, \boldsymbol{U})_{\mathscr{S}} = (\mathscr{L}_{\mathscr{S}}(\mathscr{X}, \mathscr{D})\boldsymbol{U}, \boldsymbol{U})_{\mathscr{S}} + (\mathscr{B}\boldsymbol{U}, \boldsymbol{U})_{\mathscr{S}}$$

and (3.10) implies that $(\mathscr{B}\boldsymbol{U},\boldsymbol{U})_{\mathscr{S}}=0$. Since $\mathscr{B}\geq 0$, the obtained equality implies $\boldsymbol{U}=0$ for all $\mathscr{X}\in\operatorname{supp}\mathscr{B}$ and, due to the strong unique continuation property $\boldsymbol{U}=0$ (cf. Lemma 3.8).

Thus, the operator

$$\mathscr{L}_{\mathscr{S}}(\mathscr{X},\mathscr{D})+\mathscr{B}I\,:\,\mathbb{H}^1(\mathscr{S})\to\mathbb{H}^{-1}(\mathscr{S}) \tag{4.9}$$

has the trivial kernel $\operatorname{Ker}(\mathscr{L}_{\mathscr{S}}(\mathscr{X},\mathscr{D})+\mathscr{B}I)=\{0\}$. Since $\mathscr{L}_{\mathscr{S}}(\mathscr{X},\mathscr{D})+\mathscr{B}I$ is formally self-adjoint (cf. Theorem 3.5), the same is true for the dual operator and $\operatorname{Coker}(\mathscr{L}_{\mathscr{S}}(\mathscr{X},\mathscr{D})+\mathscr{B}I)=\{0\}$. The invertibility of the Fredholm operator $\mathscr{L}_{\mathscr{S}}(\mathscr{X},\mathscr{D})+\mathscr{B}I$ in (4.9) for $p=2$ and $\theta=0$ follows.

The invertibility of $\mathscr{L}_{\mathscr{S}}(\mathscr{X},\mathscr{D})+\mathscr{B}I$ in (4.9) for arbitrary p and θ is a consequence of the ellipticity of $\mathscr{L}_{\mathscr{S}}(\mathscr{X},\mathscr{D})+\mathscr{B}I$ (cf. a similar arguments in the proof of Theorem 3.5). □

Corollary 4.3. *Let $\mathscr{S}$ be a C^∞-smooth hypersurface in $\mathbb{R}^n$ and $\mathscr{C}\subset\mathscr{S}$ be a proper subsurface $\mathscr{S}\setminus\overline{\mathscr{C}}\neq\emptyset$. Then $\mathscr{L}_{\mathscr{S}}(\mathscr{X},\mathscr{D})$ has a fundamental solution on $\mathscr{S}$, which we call a local fundamental solution on $\mathscr{C}$, viewed as the Schwartz kernel of the inverse operator to $\mathscr{L}_{\mathscr{S}}(\mathscr{X},\mathscr{D})+\mathscr{B}I$, where* $\operatorname{supp}\mathscr{B}\subset\mathscr{S}\setminus\overline{\mathscr{C}}$.

Proof. The Schwartz kernel $\mathscr{K}_{\mathscr{S}}(\mathscr{X},\tau)$ of the inverse operator to $\mathscr{L}_{\mathscr{S}}(\mathscr{X},\mathscr{D})+\mathscr{B}I$, satisfies the equality

$$\mathscr{L}_{\mathscr{S}}(\mathscr{X},\mathscr{D})\mathscr{K}_{\mathscr{S}}=\delta(\mathscr{X})I,\qquad \mathscr{X}\in\mathscr{C}$$

since $\mathscr{B}(\mathscr{X})=0$ for $\mathscr{X}\in\mathscr{C}$, and can be viewed as a local fundamental solution of $\mathscr{L}_{\mathscr{S}}(\mathscr{X},\mathscr{D})$ on $\mathscr{C}$. □

Remark 4.4. The operator $\mathscr{L}_{\mathscr{S}}(\mathscr{X},\mathscr{D})$ itself has a fundamental solution on the entire hypersurface $\mathscr{S}$ if and only if the space of Killing's vector fields on $\mathscr{S}$ is trivial $\mathscr{R}(\mathscr{S})=\big\{0\big\}$. The situation is essentially different from the case of the Euclidean space $\mathbb{R}^n$, where the condition at infinity

$$\boldsymbol{U}(\mathscr{X})=\mathscr{O}(1)\qquad\text{as}\quad|\mathscr{X}|\to\infty$$

eliminates the kernel of any linear partial differential operator with constant coefficients and the fundamental solution (the inverse operator) exists.

A compact hypersurface with certain symmetry might possess non-trivial Killing's vector fields. ~~For example, vector fields $a+b\times\mathscr{X}$ with arbitrary vectors $a,b\in\mathbb{R}^3$ and the variable $\mathscr{X}$ are tangential Killing's vector fields on the unit sphere $\mathbb{S}^2\subseteq\mathbb{R}^3$.~~

5. BVPs for the Lamé equation and Green's formulae

Throughout the present section, if not stated otherwise, $\mathscr{S}$ is a C^2-smooth surface, $\mathscr{C}\subset\mathscr{S}$ denotes a C^2-smooth subsurface with the Lipschitz boundary $\partial\mathscr{C}=\Gamma\neq\emptyset$ and $r_{\mathscr{C}}$ is the restriction to the surface $\mathscr{C}$. Under the operation $r_{\mathscr{C}}\mathscr{L}_{\mathscr{C}}(\mathscr{X},\mathscr{D})\boldsymbol{U}$ on a function (distribution) $\boldsymbol{U}\in\mathbb{H}^s_p(\mathscr{C})$ is meant that the operator $r_{\mathscr{C}}\mathscr{L}_{\mathscr{C}}(\mathscr{X},\mathscr{D})$ acts on a vector function $\boldsymbol{U}$ extended to a function $\widetilde{\boldsymbol{U}}\in\mathbb{H}^s_p(\mathscr{S})$ on the entire surface, $r_{\mathscr{C}}\widetilde{\boldsymbol{U}} = \boldsymbol{U}$. Since $\mathscr{L}_{\mathscr{C}}(\mathscr{X},\mathscr{D})$ is a local (differential) operator, the result is, after restriction, independent of the extension. Moreover, $\mathscr{L}_{\mathscr{C}}(\mathscr{X},\mathscr{D})$ does not extends supports of vector functions: if $\operatorname{supp}\boldsymbol{U}\subset\mathscr{C}$ then $\operatorname{supp}\mathscr{L}_{\mathscr{C}}(\mathscr{X},\mathscr{D})\boldsymbol{U}\subset\mathscr{C}$. Therefore we will drop the restriction operator $r_{\mathscr{C}}$ and write $(\mathscr{L}_{\mathscr{C}}(\mathscr{X},\mathscr{D})\boldsymbol{U})(\mathscr{X})$ for all $\mathscr{X}\in\mathscr{C}$.

We can not relax the constraint on a surface $\mathscr{C}$ (we remind that the underlying surface is C^2-smooth), because in the definition of equation

$$\mathscr{L}_{\mathscr{C}}(\mathscr{X},\mathscr{D})\boldsymbol{U} = \boldsymbol{F}, \qquad \boldsymbol{U}\in\mathbb{H}^1(\mathscr{C}), \quad \boldsymbol{F}\in\widetilde{\mathbb{H}}^{-1}(\mathscr{C}), \tag{5.1}$$

is participating the gradient $\nabla_{\mathscr{S}}\boldsymbol{\nu} = [\mathscr{D}_j\nu_k]_{n\times n}$ of the unit normal vector field $\boldsymbol{\nu}$ (see (0.20)–(0.24)). $\boldsymbol{\nu}(\mathscr{X})$ is defined almost everywhere on $\mathscr{C}$ is just C^1-smooth. We can actually require that $\mathscr{S}$ is $\mathbb{H}^2_\infty$ (i.e., corresponding parameterizations of the surface have, instead of continuous, bounded second derivatives).

Equation (5.1) is actually understood in a weak sense:

$$(\mathscr{L}_{\mathscr{C}}(\mathscr{X},\mathscr{D})\boldsymbol{U},\boldsymbol{V})_{\mathscr{C}} := (\mathbb{T}\operatorname{Def}_{\mathscr{C}}\boldsymbol{U},\operatorname{Def}_{\mathscr{C}}\boldsymbol{V})_{\mathscr{C}} = (\boldsymbol{F},\boldsymbol{V})_{\mathscr{C}}\,, \tag{5.2}$$

$$\forall\,\boldsymbol{U}\in\mathbb{H}^1(\mathscr{C}), \boldsymbol{V}\in\widetilde{\mathbb{H}}^1(\mathscr{C})\,.$$

In particular, for the Lamé operator in isotropic media we have

$$(\mathscr{L}_{\mathscr{C}}(\mathscr{X},\mathscr{D})\boldsymbol{U},\boldsymbol{V})_{\mathscr{C}} := \lambda(\nabla_{\mathscr{C}}\boldsymbol{U},\nabla_{\mathscr{C}}\boldsymbol{V})_{\mathscr{C}} + \mu(\operatorname{Def}_{\mathscr{C}}\boldsymbol{U},\operatorname{Def}_{\mathscr{C}}\boldsymbol{V})_{\mathscr{C}} = (\boldsymbol{F},\boldsymbol{V})_{\mathscr{C}}\,, \tag{5.3}$$

$$\forall\,\boldsymbol{V}\in\widetilde{\mathbb{H}}^1_2(\mathscr{S})$$

(cf. (0.23)).

Let $\boldsymbol{\nu}_\Gamma = \left(\nu^1_\Gamma,\dots,\nu^n_\Gamma\right)^\top$ be the tangential to $\mathscr{C}$ and outer unit normal vector field to Γ.

If a tangential vector field $\boldsymbol{U}\in\mathbb{H}^1_p(\mathscr{C})\cap\mathscr{V}(\mathscr{C})$ denotes the displacement, the natural boundary value problems for $\mathscr{L}_{\mathscr{C}}$ are the following:

I. The Dirichlet problem when the displacement is prescribed on the boundary

$$\begin{cases} (\mathscr{L}_{\mathscr{C}}(\mathscr{X},\mathscr{D})\boldsymbol{U})(\mathscr{X}) = \boldsymbol{F}(\mathscr{X}), & \mathscr{X}\in\mathscr{C},\\ \boldsymbol{U}^+(\tau) = \boldsymbol{G}(\tau), & \tau\in\Gamma, \end{cases} \tag{5.4}$$

$$\boldsymbol{F}\in\widetilde{\mathbb{H}}^{-1}(\mathscr{C})\,,\quad \boldsymbol{G}\in\mathbb{H}^{1/2}(\Gamma)\,,\quad \boldsymbol{U}\in\mathbb{H}^1(\mathscr{C});$$

the first (basic) equation in the domain is understood in a weak sense (see (5.2), (5.3)) and

$$\gamma^+_D\boldsymbol{U} := \boldsymbol{U}^+ \tag{5.5}$$

is the Dirichlet trace operator on the boundary.

II. The Neumann problem when the traction is prescribed on the boundary:

$$\begin{cases} (\mathscr{L}_{\mathscr{C}}(\mathscr{X},\mathscr{D})\boldsymbol{U})(\mathscr{X}) = \boldsymbol{F}(\mathscr{X}), & \mathscr{X}\in\mathscr{C},\\ (\mathfrak{T}_{\mathscr{C}}(\boldsymbol{\nu}_\Gamma,\mathscr{D})\;\boldsymbol{U})^+(\tau) = \boldsymbol{H}(\tau), & \tau\in\Gamma\,, \end{cases} \tag{5.6}$$

$$\boldsymbol{F}\in\widetilde{\mathbb{H}}^{-1}(\mathscr{C})\,,\quad \boldsymbol{H}\in\mathbb{H}^{-1/2}(\Gamma),\quad \boldsymbol{U}\in\mathbb{H}^1(\mathscr{C});$$

here

$$\gamma_N^+\boldsymbol{U} := (\mathfrak{T}_{\mathscr{C}}(\boldsymbol{\nu}_\Gamma, \mathscr{D})\ \boldsymbol{U})^+, \tag{5.7}$$

$$\mathfrak{T}_{\mathscr{C}}(\boldsymbol{\nu}_\Gamma, \mathscr{D})\boldsymbol{U} := -\lambda(\mathrm{div}_{\mathscr{C}}\boldsymbol{U})\boldsymbol{\nu}_\Gamma - 2\mu\sum_{j=1}^{n}\left\{(\nu_\Gamma^j + \mathscr{H}_{\mathscr{C}}^0\nu_j)\mathfrak{D}_{jk}(\boldsymbol{U})\right\}_{k=1}^n \tag{5.8}$$

$$= -\mu\mathscr{D}_{\boldsymbol{\nu}_\Gamma}\boldsymbol{U} - (\lambda+\mu)(\mathrm{div}_{\mathscr{C}}\boldsymbol{U})\boldsymbol{\nu}_\Gamma \tag{5.9}$$

is the Neumann trace operator on the boundary (the traction) with

$$\mathscr{D}_{\boldsymbol{\nu}_\Gamma}\varphi := \sum_{j=1}^{n}\nu_\Gamma^j\mathscr{D}_j\varphi. \qquad \varphi \in \mathbb{H}^1(\mathscr{C}). \tag{5.10}$$

In Lemma 7.3 below it will be shown that the trace $\gamma_N^+\boldsymbol{U}$ exists provided that $\boldsymbol{U}$ is a solution to the basic (first) equation in (5.6).

A crucial role in the investigation of BVPs (5.4)–(5.10) belongs to the Green formula.

Theorem 5.1. *Let $\mathscr{B} \in C^1(\mathscr{S})$ and $\mathscr{C}^+ := \mathscr{C}$, $\mathscr{C}^- := \mathscr{C}^c = \mathscr{S}\setminus\overline{\mathscr{C}}$ denote the complemented ~~open surfaces~~, $\mathscr{S} = \mathscr{C}^+ \cup \overline{\mathscr{C}^-}$*

For a solution to the equation

$$\mathscr{L}_{\mathscr{C}}\boldsymbol{U} + \mathscr{B}\boldsymbol{U} = \boldsymbol{F}, \qquad \boldsymbol{F} \in \widetilde{\mathbb{H}}^{-1}(\mathscr{C}^{\pm}), \quad \boldsymbol{U} \in \mathbb{H}^1(\mathscr{C}^{\pm}) \tag{5.11}$$

(for $\mathscr{B} = 0$ cf. (5.1) and the basic equations in (5.4)–(5.10)) the following Green formula are valid

$$\begin{aligned}\big((\mathscr{L}_{\mathscr{C}} + \mathscr{B}I)\boldsymbol{U}, \boldsymbol{V}\big)_{\mathscr{C}^{\pm}} &= \int_{\mathscr{C}^{\pm}}\langle(\mathscr{L}_{\mathscr{C}} + \mathscr{B}I)\boldsymbol{U}(\mathscr{x}), \boldsymbol{V}(\mathscr{x})\rangle\, dS \\ &= \pm\oint_\Gamma\langle\gamma_N^{\pm}\boldsymbol{U}(\tau), \gamma_D^{\pm}\boldsymbol{V}(\tau)\rangle\, d\mathfrak{s} \ + \ \mathscr{E}_{\pm}(\boldsymbol{U}, \boldsymbol{V}),\end{aligned} \tag{5.12}$$

$$\begin{aligned}&\mathscr{E}_{\pm}(\boldsymbol{U}, \boldsymbol{V}) \\ &:= \int_{\mathscr{C}^{\pm}}\Big[\lambda\langle\mathrm{div}_{\mathscr{C}}\,\boldsymbol{U}, \mathrm{div}_{\mathscr{C}}\,\boldsymbol{V}\rangle + 2\mu\,\langle\mathrm{Def}_{\mathscr{C}}\,\boldsymbol{U}, \mathrm{Def}_{\mathscr{C}}\,\boldsymbol{V}\rangle + \mathscr{B}\langle\boldsymbol{U}, \boldsymbol{V}\rangle\Big]\, dS, \\ &\mathscr{D}_{\nu_\Gamma} := \sum_{m=1}^{n}\nu_\Gamma^j\mathscr{D}_j, \quad \boldsymbol{U} = \sum_{j=1}^{n}U_j^0\boldsymbol{d}^j, \quad \boldsymbol{V} = \sum_{j=1}^{n}V_j^0\boldsymbol{d}^j \in \mathbb{H}^1(\mathscr{C}) \cap \mathscr{V}(\mathscr{C}),\end{aligned} \tag{5.13}$$

Here the index $^{\pm}$ denotes the traces on Γ from the surfaces $\mathscr{C}^{\pm}$ and the scalar product of matrices is defined as follows:

$$\langle M, N\rangle := \mathrm{Tr}\big[MN^\top\big], \qquad M = [M_{jk}]_{n\times n}, \quad N = [N_{jk}]_{n\times n}. \tag{5.14}$$

Proof. We apply the integration by parts formula

$$\int_{\mathscr{C}^{\pm}}\langle(\mathscr{D}_j\boldsymbol{U}), \boldsymbol{V}\rangle\, dS = \pm\oint_\Gamma\nu_\Gamma^j\langle\boldsymbol{U}^{\pm}, \boldsymbol{V}^{\pm}\rangle\, d\mathfrak{s} + \int_{\mathscr{C}^{\pm}}\langle\boldsymbol{U}, \big(\mathscr{D}_j^*\boldsymbol{V}\big)\rangle\, dS, \tag{5.15}$$

$$\boldsymbol{U},\ \boldsymbol{V} \in \mathbb{H}^1(\mathscr{C}^{\pm}) \qquad j = 1, \ldots, n,$$

proved in [DMM1] (cf. (1.5) for the formal adjoint $\mathscr{D}_j^*$), and proceed as follows

$$((\mathscr{L}_{\mathscr{C}} + \mathscr{B}I)\boldsymbol{U},\boldsymbol{V})_{\mathscr{C}^\pm} = \int_{\mathscr{C}^\pm} \langle \lambda\nabla_{\mathscr{C}}\operatorname{div}_{\mathscr{C}}\boldsymbol{U} + 2\mu\operatorname{Def}^*_{\mathscr{C}}\operatorname{Def}_{\mathscr{C}}\boldsymbol{U} + \mathscr{B}\boldsymbol{U},\boldsymbol{V}\rangle\, dS$$

$$= \int_{\mathscr{C}^\pm} \langle \lambda\nabla_{\mathscr{C}}\operatorname{div}_{\mathscr{C}}\boldsymbol{U} + 2\mu\sum_{j=1}^n \left\{\mathscr{D}_j^*\mathfrak{D}_{jk}(\boldsymbol{U})\right\}_{k=1}^n + \mathscr{B}\boldsymbol{U},\boldsymbol{V}\rangle\, dS$$

$$= \int_{\mathscr{C}^\pm} \langle \lambda\nabla_{\mathscr{C}}\operatorname{div}_{\mathscr{C}}\boldsymbol{U} - 2\mu\sum_{j=1}^n \left\{(\mathscr{D}_j + \nu_\Gamma^j\mathscr{H}^0_{\mathscr{C}})\mathfrak{D}_{jk}(\boldsymbol{U})\right\}_{k=1}^n + \mathscr{B}\boldsymbol{U},\boldsymbol{V}\rangle\, dS$$

$$= \pm\oint_\Gamma \langle \overset{\pm}{\gamma}_N\boldsymbol{U}(\tau), \overset{\pm}{\gamma}_D\boldsymbol{V}(\tau)\rangle\, d\mathfrak{s} \;+\; \mathscr{E}_\pm(\boldsymbol{U},\boldsymbol{V}),$$

where $\mathfrak{T}_{\mathscr{C}}(\boldsymbol{\nu}_\Gamma,\mathscr{D})$ is gives by formula (5.8). We have applied formulae (1.5), (0.21), (0.15) and the equalities

$$\langle \pi_{\mathscr{S}}\boldsymbol{U},\boldsymbol{V}\rangle = \langle \boldsymbol{U},\boldsymbol{V}\rangle, \qquad \langle(\mathscr{D}_j^{\mathscr{C}})^*\boldsymbol{U},\boldsymbol{V}\rangle = \langle \pi_{\mathscr{S}}\mathscr{D}_j^*\boldsymbol{U},\boldsymbol{V}\rangle = \langle \mathscr{D}_j^*\boldsymbol{U},\boldsymbol{V}\rangle \quad \forall\, \boldsymbol{V}\in\mathscr{V}.$$

To obtain another representation (5.9) of $\mathfrak{T}_{\mathscr{C}}(\boldsymbol{\nu}_\Gamma,\mathscr{D})$ we start by second representation of $\mathscr{L}_{\mathscr{C}}$ in (0.23) and proceed similarly. □

6. The Dirichlet BVP for the Lamé equation

Throughout this section $\mathscr{C}$ is a C^2-smooth hypersurface with the Lipschitz boundary $\Gamma = \partial\mathscr{C}$.

Theorem 6.1. *The Dirichlet problem* (5.4) *has a unique solution* $\boldsymbol{U}\in\mathbb{H}^1(\mathscr{C})$ *for arbitrary data* $\boldsymbol{F}\in\widetilde{\mathbb{H}}^{-1}(\mathscr{C})$ *and* $\boldsymbol{G}\in\mathbb{H}^{1/2}(\Gamma)$.

The proof will be exposed at the end of the section after we prove some auxiliary results.

Lemma 6.2. (Gårding's inequality "with boundary condition"). *The Lamé operator*

$$\mathscr{L}_{\mathscr{C}}(\mathscr{X},\mathscr{D}) \;:\; \widetilde{\mathbb{H}}^1(\mathscr{C}) \to \mathbb{H}^{-1}(\mathscr{C}) \tag{6.1}$$

is positive definite: there exists some constant $C>0$ *such that*

$$(\mathscr{L}_{\mathscr{C}}(\mathscr{X},\mathscr{D})\boldsymbol{U},\boldsymbol{U})_{\mathscr{C}} \geq C\left\|\boldsymbol{U}\middle|\mathbb{H}^1(\mathscr{C})\right\|^2 \qquad \forall\,\boldsymbol{U}\in\widetilde{\mathbb{H}}^1(\mathscr{C}). \tag{6.2}$$

Proof. Due to (3.11) inequality (6.1) holds for all $\boldsymbol{U}\in\mathbb{H}^1_{\mathscr{R}}(\mathscr{S})$, i.e., for $\boldsymbol{U}\in\mathbb{H}^1(\mathscr{S})$ and $\boldsymbol{U}\notin\mathscr{R}(\mathscr{S})$. Since $\boldsymbol{U}\in\widetilde{\mathbb{H}}^1(\mathscr{C})$ due to the strong unique continuation from the boundary (cf. Lemma 3.12), all Killing's vector fields $\boldsymbol{K}\in\widetilde{\mathbb{H}}^1(\mathscr{C})$ are identically 0. Therefore, (3.11) holds for all $\boldsymbol{U}\in\widetilde{\mathbb{H}}^1(\mathscr{C})$. □

Corollary 6.3. *The Lamé operator* $\mathscr{L}_{\mathscr{C}}(\mathscr{X},\mathscr{D})$ *in* (6.1) *is invertible.*

Proof. From the inequality (6.2) follows that $\mathscr{L}_{\mathscr{C}}(\mathscr{X},\mathscr{D})$ is normally solvable (has the closed range) and the trivial kernel $\operatorname{Ker}\mathscr{L}_{\mathscr{C}}(\mathscr{X},\mathscr{D})=\{0\}$. Since $\mathscr{L}_{\mathscr{C}}(\mathscr{X},\mathscr{D})$ is self-adjoint, the co-kernel (the kernel of the adjoint operator) is trivial as well $\operatorname{Ker}\mathscr{L}^*_{\mathscr{C}}(\mathscr{X},\mathscr{D})=\operatorname{Ker}\mathscr{L}_{\mathscr{C}}(\mathscr{X},\mathscr{D})=\{0\}$. Therefore $\mathscr{L}_{\mathscr{C}}(\mathscr{X},\mathscr{D})$ is invertible. □

Definition 6.4. (see [LM1, Ch.2, § 1.4]). A partial differential operator

$$\mathbf{A}(x,\mathscr{D}):=\sum_{|\alpha|\le m} a_\alpha(x)\nabla^\alpha_{\mathscr{C}},\quad \nabla^\alpha_{\mathscr{C}}u=\mathscr{D}_1^{\alpha_1}\cdots\mathscr{D}_n^{\alpha_n},\quad a_\alpha\in C(\mathscr{C},C^{N\times N})\tag{6.3}$$

is called normal on Γ if

$$\inf|\det\mathscr{A}_0(\mathscr{X},\boldsymbol{\nu}(\mathscr{X}))|\neq 0,\quad \mathscr{X}\in\Gamma\,,\quad |\xi|=1\,,\tag{6.4}$$

where $\mathscr{A}_0(x,\xi)$ is the homogeneous principal symbol of $\mathbf{A}$

$$\mathscr{A}_0(x,\xi):=\sum_{|\alpha|=m} a_\alpha(x)(-i\xi)^\alpha,\quad x\in\overline{\mathscr{C}}\,,\quad \xi\in\mathbb{R}^n.\tag{6.5}$$

Definition 6.5. A system $\{\mathbf{B}_j(\mathscr{X},D)\}_{j=0}^{k-1}$ of differential operators with matrix $N\times N$ coefficients is called a Dirichlet system of order k if all participating operators are normal on Γ (see Definition 6.4) and $\operatorname{ord}\mathbf{B}_j=j$, $j=0,1,\dots,k-1$.

Let us assume $\mathscr{C}$ is k-smooth and $m\le k$ $(m,k=1,2,\dots)$ and define the trace operator (cf. (5.10)):

$$\mathscr{R}_m u:=\{\gamma_\Gamma\mathbf{B}_1u,\dots,\gamma_\Gamma\mathbf{B}_mu\}^\top\,,\qquad u\in\mathbb{C}^k_0(\overline{\mathscr{C}})\,.\tag{6.6}$$

Proposition 6.6. *Let $\mathscr{C}$ be k-smooth, $1\le p\le\infty$, $m=1,2,\dots$, $m\le k$ and $m<s-1/p\notin\mathbb{N}_0$. The trace operator*

$$\mathscr{R}_m\ :\ \mathbb{H}^s_p(\overline{\mathscr{C}})\to\bigotimes_{j=0}^{m}\mathbb{W}_p^{s-1/p-j}(\Gamma)\,,\tag{6.7}$$

where $\mathbb{W}^r_p(\overline{\mathscr{C}})=\mathbb{B}^r_{p,p}(\overline{\mathscr{C}})$ is the Sobolev-Slobodecki-Besov space (*cf.* [Tr1] *for details*) *is a retraction, i.e., is continuous and has a continuous right inverse, called a coretraction*

$$\begin{aligned}(\mathscr{R}_m)^{-1}\ :\ &\bigotimes_{j=0}^{m}\mathbb{W}_p^{s-1/p-j}(\mathscr{S})\to\mathbb{H}^s_p(\overline{\Omega})\\ \mathscr{R}_m(\mathscr{R}_m)^{-1}\Phi=\Phi\,,\qquad &\forall\Phi\in\bigotimes_{j=0}^{m}\mathbb{W}_p^{s-1/p-j}(\mathscr{S})\,.\end{aligned}\tag{6.8}$$

Proof. The result was proved in [Tr1, Theorem 2.7.2, Theorem 3.3.3] for a domain $\Omega\subset\mathbb{R}^{n-1}$ and the classical Dirichlet trace operator $\mathscr{R}_mu:=\{\gamma_\Gamma\partial_{\boldsymbol{\nu}}u,\dots,\gamma_\Gamma\partial^m_{\boldsymbol{\nu}}u\}^\top$. In [Du3] the theorem was proved for a domain $\Omega\subset\mathbb{R}^{n-1}$ and for arbitrary trace operator $\mathscr{R}_mu$.

A surface $\mathscr{C}=\cup_{j=1}^N\mathscr{C}_j$ is covered by a finite number of local coordinate charts $\varkappa_j\ :\ \Omega_j\to\mathscr{C}_j$, $\Omega_j\subset\mathbb{R}^{n-1}$. After transformation, the Dirichlet trace operator $\mathscr{R}_mu$ on a portion $\mathscr{C}_j$ of the surface transform into another Dirichlet trace operator on the coordinate domains Ω_j. Therefore, we prove the assertion locally on each coordinate chart $\mathscr{C}_j \subset \mathscr{C}$ and, by applying a partition of unity, extend it to the entire surface $\mathscr{C}$. □

Proof of Theorem 6.1. Let $\widetilde{\boldsymbol{G}} = (\mathscr{R}_0)^{-1}\boldsymbol{G} \in \mathbb{H}^1(\mathscr{C})$ be the continuation of the Dirichlet boundary data $\boldsymbol{G} \in \mathbb{H}^{1/2}(\Gamma)$ from BVP (5.4) into the surface $\mathscr{C}$ from the boundary Γ, found with the help of a coretraction from Proposition 6.6. Then the Dirichlet BVP

$$\begin{cases} (\mathscr{L}_{\mathscr{C}}(\mathscr{X},\mathscr{D})\widetilde{\boldsymbol{U}})(\mathscr{X}) = \boldsymbol{F}_0(\mathscr{X}), & \mathscr{X} \in \mathscr{C}, \\ \widetilde{\boldsymbol{U}}^+(\tau) = 0, & \tau \in \Gamma, \end{cases} \tag{6.9}$$

$$\boldsymbol{F}_0 := \boldsymbol{F} - \mathscr{L}_{\mathscr{C}}(\mathscr{X},\mathscr{D})\widetilde{\boldsymbol{G}} \in \widetilde{\mathbb{H}}^{-1}(\mathscr{C}),$$

is an equivalent reformulation of BVP (5.4) and the solutions are related by the equality $\widetilde{\boldsymbol{U}} := \boldsymbol{U} - \widetilde{\boldsymbol{G}}$. On the other hand, since

$$\widetilde{\mathbb{H}}^{-1}(\mathscr{C}) := \big\{\boldsymbol{U} \in \mathbb{H}^{-1}(\mathscr{C}) \ : \ \boldsymbol{U}^+ = 0\big\},$$

the solvability of BVP (6.9) is equivalent to the invertibility of the operator $\mathscr{L}_{\mathscr{C}}(\mathscr{X},\mathscr{D})$ in (6.1). Now the unique solvability of BVP (6.9) (and of the equivalent BVP (5.4)) follows from Corollary 6.3. □

7. The Neumann BVP for the Lamé equation

Throughout this section $\mathscr{C}$ is a C^2-smooth hypersurface with the Lipschitz boundary $\Gamma = \partial\mathscr{C}$.

Theorem 7.1. *The Neumann problem* (5.6) *has a solution* $\boldsymbol{U} \in \mathbb{H}^1(\mathscr{C})$ *only for those right-hand sides* $\boldsymbol{F} \in \widetilde{\mathbb{H}}^{-1}(\Gamma)$ *and* $\boldsymbol{H} \in \mathbb{H}^{-1/2}(\Gamma)$ *which satisfy the equality*

$$\int_{\mathscr{C}} \boldsymbol{F}(\mathscr{X})\boldsymbol{K}(\mathscr{X})dS = \oint_{\Gamma} \boldsymbol{H}(\tau)\gamma_D^+\boldsymbol{K}(\tau)d\mathfrak{s} \qquad \forall \boldsymbol{K} \in \mathscr{R}(\mathscr{C}). \tag{7.1}$$

If the condition (7.1) *holds, the Neumann problem has a general solution* $\boldsymbol{U} = \boldsymbol{U}^0 + \boldsymbol{K} \in \mathbb{H}^1(\mathscr{C})$, *where* $\boldsymbol{U}^0 \in \mathbb{H}^1(\mathscr{C})$ *is a particular solution and* $\boldsymbol{K} \in \mathscr{R}(\mathscr{C})$ *is a Killing's vector field.*

The proof will be exposed at the end of the section after we prove some auxiliary results. The proof is based on the celebrated Lax-Milgram lemma.

Lemma 7.2. (Lax-Milgram). *Let* $\mathfrak{B}$ *be a Banach space and* $A(\varphi,\psi)$ *be a bilinear* $A(\cdot,\cdot) : \mathfrak{B} \times \mathfrak{B} \to \mathbb{R}$, *positive definite form: the inequality*

$$A(\varphi,\varphi) \geq C\|\varphi\big|\mathfrak{B}\|^2 \tag{7.2}$$

holds for some constant $C > 0$ *and all* $\varphi \in \mathfrak{B}$. *Further let* $L(\cdot) : \mathfrak{B} \to \mathbb{R}$ *be a continuous linear form* (*a functional*).

A linear equation

$$A(\varphi,\psi) = L(\psi) \tag{7.3}$$

has a unique solution $\varphi \in \mathfrak{B}$ *for arbitrary* $\psi \in \mathfrak{B}$.

Proof. The proof can be retrieved from many sources (cf., e.g., [Ci3, § 6.3]), mostly for symmetric forms (we have dropped this requirement). For a non-symmetric form the proof can be found in the original paper [LaM1]. □

Lemma 7.3. *If* $\boldsymbol{U} \in \mathbb{H}^1_p(\mathscr{C})$, $1 < p < \infty$, *is a solution to the first equations in* (5.6), *then the Neumann trace on the boundary exists and* $\gamma_N^+ \boldsymbol{U} = (\mathfrak{T}_{\mathscr{C}}(\boldsymbol{\nu}_\Gamma, \mathscr{D})\ \boldsymbol{U})^+ \in \mathbb{H}^{-1/p}_p(\Gamma)$.

Proof. For $\mathscr{B} = 0$, $\mathscr{C}^+ = \mathscr{C}$ the Green's formulae (5.12), (5.13) become:

$$(\mathscr{L}_{\mathscr{C}} \boldsymbol{U}, \boldsymbol{V})_{\mathscr{C}} = (\gamma_N^+ \boldsymbol{U}, \gamma_D^+ \boldsymbol{V})_\Gamma + \mathscr{E}(\boldsymbol{U}, \boldsymbol{V}), \tag{7.4}$$

$$\mathscr{E}(\boldsymbol{U}, \boldsymbol{V}) = \lambda(\operatorname{div}_{\mathscr{C}} \boldsymbol{U}, \operatorname{div}_{\mathscr{C}} \boldsymbol{V})_{\mathscr{C}} + 2\mu\,(\operatorname{Def}_{\mathscr{C}} \boldsymbol{U}, \operatorname{Def}_{\mathscr{C}} \boldsymbol{V})_{\mathscr{C}} \tag{7.5}$$

Introducing the value $\mathscr{L}_{\mathscr{C}}(\mathscr{X}, \mathscr{D})\ \boldsymbol{U} = \boldsymbol{F}$ into the Green formula (7.4) we rewrite it

$$(\gamma_N^+ \boldsymbol{U}, \gamma_D^+ \boldsymbol{V})_\Gamma = (\boldsymbol{F}, \boldsymbol{V})_{\mathscr{C}} - \mathscr{E}(\boldsymbol{U}, \boldsymbol{V}),$$

where $\boldsymbol{V} \in \mathbb{H}^1_p(\mathscr{C})$ is arbitrary. The bilinear forms $(\boldsymbol{F}, \boldsymbol{V})_{\mathscr{C}}$ and $\mathscr{E}(\boldsymbol{U}, \boldsymbol{V})$ are continuous for $\boldsymbol{F} \in \widetilde{\mathbb{H}}^{-1}_p(\mathscr{C})$ and $\boldsymbol{U} \in \mathbb{H}^1_p(\mathscr{C})$, $\boldsymbol{V} \in \mathbb{H}^1_{p'}(\mathscr{C})$, $p' := p/(p-1)$; the bilinear form $\big((\mathfrak{T}_{\mathscr{C}}(\boldsymbol{\nu}_\Gamma, \mathscr{D})\boldsymbol{U})^+, \boldsymbol{V}^+\big)_\Gamma$ is well defined and, by a duality argument, $\big(\mathfrak{T}_{\mathscr{C}}(\boldsymbol{\nu}_\Gamma, \mathscr{D})\boldsymbol{U}\big)^+ \in \mathbb{H}^{-1/p}_p(\Gamma)$ since $\boldsymbol{V}^+ \in \mathbb{H}^{1-1/p'}_{p'}(\mathscr{C}) = \mathbb{H}^{1/p}_{p'}(\mathscr{C})$ is arbitrary. □

Lemma 7.4. *The condition* (7.1) *is necessary for the Neumann problem* (5.6) *to have a solution* $\boldsymbol{U} \in \mathbb{H}^1(\mathscr{C})$.

Proof. First note that for a Killing's vector field $\boldsymbol{K} \in \mathscr{R}(\mathscr{C})$,

$$\mathscr{L}_{\mathscr{C}}(\mathscr{X}, \mathscr{D})\boldsymbol{K} = 0 \quad \text{and} \qquad \gamma_N^+ \boldsymbol{K} = \big(\mathfrak{T}_{\mathscr{C}}(\boldsymbol{\nu}_\Gamma, \mathscr{D})\boldsymbol{K}\big)^+ = 0. \tag{7.6}$$

Indeed, if $\boldsymbol{K} \in \mathscr{R}(\mathscr{C})$ is naturally extended to $\widetilde{\boldsymbol{K}} \in \mathscr{R}(\mathscr{S})$, then $\mathscr{L}_{\mathscr{C}}(\mathscr{X}, \mathscr{D})\boldsymbol{K}(\mathscr{X}) = \mathscr{L}_{\mathscr{C}}(\mathscr{X}, \mathscr{D})\widetilde{\boldsymbol{K}}(\mathscr{X}) = 0$ for $\mathscr{X} \in \mathscr{C}$ (cf. (3.9)) and the first equality follows.

The second equality in (7.6) follows from (5.8) if we recall that $\operatorname{Def}_{\mathscr{C}}(\boldsymbol{U}) = 0$ (cf. Definition 3.1) and this also implies $\operatorname{div}_{\mathscr{C}} \boldsymbol{U} = 0$ (cf. (4.6)).

~~From (4.6) and the second equality in (7.6) follows~~

$$\begin{aligned} \mathscr{E}(\boldsymbol{K}, \boldsymbol{U}) &= \mathscr{E}(\boldsymbol{U}, \boldsymbol{K}) \\ &= \int_{\mathscr{C}} \Big[\lambda \langle \operatorname{div}_{\mathscr{C}} \boldsymbol{U}, \operatorname{div}_{\mathscr{C}} \boldsymbol{K}\rangle + 2\mu\,\langle \operatorname{Def}_{\mathscr{C}} \boldsymbol{U}, \operatorname{Def}_{\mathscr{C}} \boldsymbol{K}\rangle\Big]\, dS = 0 \\ &\qquad \text{for all} \quad \boldsymbol{U} \in \mathbb{H}^1(\mathscr{C}) \quad \text{and all} \quad \boldsymbol{K} \in \mathscr{R}(\mathscr{C}). \end{aligned} \tag{7.7}$$

Introducing into the Green formula (5.12) $\mathscr{B} = 0$, $\boldsymbol{F} = \mathscr{L}_{\mathscr{C}}(\mathscr{X}, \mathscr{D})\boldsymbol{U}$, $\boldsymbol{V} = \boldsymbol{K} \in \mathscr{R}(\mathscr{C})$ and the obtained equality, we get the claimed orthogonality condition (7.1). □

Lemma 7.5. *The bilinear form* (*cf.* (7.4) *and* (7.5))

$$\mathbb{A}_N(\boldsymbol{U}, \boldsymbol{V}) := (\mathscr{L}_{\mathscr{C}}(\mathscr{X}, \mathscr{D})\boldsymbol{U}, \boldsymbol{V})_{\mathscr{C}} - (\gamma_N^+ \boldsymbol{U}, \gamma_D^+ \boldsymbol{V})_\Gamma = \mathscr{E}(\boldsymbol{U}, \boldsymbol{V}) \tag{7.8}$$

is well defined, symmetric $\mathbb{A}_N(\boldsymbol{U},\boldsymbol{V}) = \mathbb{A}_N(\boldsymbol{V},\boldsymbol{U})$ *for all* $\boldsymbol{U},\ \boldsymbol{V} \in \mathbb{H}^1(\mathscr{C})$ *and non-negative* $\mathbb{A}_N(\boldsymbol{U},\boldsymbol{U}) \geq 0$ *for* $\boldsymbol{U} \in \mathbb{H}^1(\mathscr{S})$ *(cf.* (5.13)*). Moreover, the form is positive definite*

$$\mathbb{A}_N(\boldsymbol{U},\boldsymbol{U}) \geq M_3\big\|\boldsymbol{U}\big|\mathbb{H}^1(\mathscr{S})\big\|^2 \qquad \forall\, \boldsymbol{U} \in \mathbb{H}^1_{\mathscr{R}}(\mathscr{S}) \tag{7.9}$$

on the orthogonal complement $\mathbb{H}^1_{\mathscr{R}}(\mathscr{S})$ *to the finite-dimensional subspace of Killing's vector fields* $\mathscr{R}(\mathscr{C})$ *in the Hilbert-Sobolev space* $\mathbb{H}^1(\mathscr{C})$*.*

Proof. The estimate

$$\big|\mathbb{A}_N(\boldsymbol{U},\boldsymbol{V})\big| = \big|\mathscr{E}(\boldsymbol{U},\boldsymbol{V})\big| \leq \big\|\boldsymbol{U}\big|\mathbb{H}^1(\mathscr{S})\big\|\,\big\|\boldsymbol{V}\big|\mathbb{H}^1(\mathscr{S})\big\|$$

follows from the definition of the form $\mathscr{E}(\boldsymbol{U},\boldsymbol{V})$ in (5.13) and proves that $\mathbb{A}_N(\boldsymbol{U},\boldsymbol{V})$ is well defined. Moreover, the equality proves that the form is symmetric and non-negative

$$\mathbb{A}_N(\boldsymbol{U},\boldsymbol{V}) = \mathscr{E}(\boldsymbol{U},\boldsymbol{V}) = \mathscr{E}(\boldsymbol{V},\boldsymbol{U}) = \mathbb{A}_N(\boldsymbol{V},\boldsymbol{U}),$$

$$\mathbb{A}_N(\boldsymbol{U},\boldsymbol{U}) = \mathscr{E}(\boldsymbol{U},\boldsymbol{U}) \geq 0.$$

From (5.12) and (5.13) follows

$$\begin{aligned}\mathbb{A}_N(\boldsymbol{U},\boldsymbol{U}) = \mathscr{E}(\boldsymbol{U},\boldsymbol{U}) &= \lambda\big\|\mathrm{div}_{\mathscr{C}}\boldsymbol{U}\big|\mathbb{L}_2(\mathscr{S})\big\|^2 + 2\mu\big\|\mathrm{Def}_{\mathscr{C}}\,\boldsymbol{U}\big|\mathbb{L}_2(\mathscr{S})\big\|^2\\ &\geq 2\mu\big\|\mathrm{Def}_{\mathscr{C}}\,\boldsymbol{U}\big|\mathbb{L}_2(\mathscr{S})\big\|^2 \geq 2\mu\, c^2\big\|\boldsymbol{U}\big|\mathbb{H}^1(\mathscr{S})\big\|^2 \qquad \forall\, \boldsymbol{U} \in \mathbb{H}^1_{\mathscr{R}}(\mathscr{S})\end{aligned} \tag{7.10}$$

and accomplishes the proof. □

Proof of Theorem 7.1. The space of Killing's vector fields $\mathscr{R}(\mathscr{S})$ is finite-dimensional and consists of continuous vector-fields with bounded second derivatives (these fields are actually as smooth as the surface $\mathscr{C}$, i.e., are infinitely smooth if $\mathscr{S}$ is infinitely smooth; see Theorem 3.5). Let $\boldsymbol{K}_1,\dots,\boldsymbol{K}_m$ be the finite-dimensional orthonormal basis in $\mathscr{R}(\mathscr{C})$, $(\boldsymbol{K}_j,\boldsymbol{K}_r)_{\mathscr{C}} = \delta_{jr}$, $j,r = 1,\dots,m$. Consider the finite rank smoothing operator $\boldsymbol{T}\boldsymbol{U}$ introduced in (4.7). As we already know the operator $\boldsymbol{T}$ is symmetric and non-negative:

$$(\boldsymbol{T}\boldsymbol{U},\boldsymbol{V})_{\mathscr{C}} = (\boldsymbol{T}\boldsymbol{V},\boldsymbol{U})_{\mathscr{C}}. \quad (\boldsymbol{T}\boldsymbol{U},\boldsymbol{U})_{\mathscr{C}} = \sum_{j=1}^{m}(\boldsymbol{U},\boldsymbol{K}_j)^2_{\mathscr{C}} \geq 0 \qquad \forall\, \boldsymbol{U},\ \boldsymbol{V} \in \mathbb{H}^1(\mathscr{C}). \tag{7.11}$$

Consider the modified bilinear form

$$\begin{aligned}\mathbb{A}^{\#}_N(\boldsymbol{U},\boldsymbol{V}) &:= ((\mathscr{L}_{\mathscr{C}}(t,\mathscr{D}) + \boldsymbol{T})\boldsymbol{U},\boldsymbol{V})_{\mathscr{C}} - (\gamma^+_N\boldsymbol{U},\gamma^+_D\boldsymbol{V})_\Gamma\\ &= \mathscr{E}(\boldsymbol{U},\boldsymbol{V}) + (\boldsymbol{T}\boldsymbol{U},\boldsymbol{V})_{\mathscr{C}} \qquad \boldsymbol{U},\ \boldsymbol{V} \in \mathbb{H}^1(\mathscr{C})\end{aligned} \tag{7.12}$$

(cf. (7.4)). The form is symmetric because both summands are

$$\mathbb{A}^{\#}_N(\boldsymbol{U},\boldsymbol{V}) = \mathscr{E}(\boldsymbol{U},\boldsymbol{V}) + (\boldsymbol{T}\boldsymbol{U},\boldsymbol{V})_{\mathscr{C}} = \mathscr{E}(\boldsymbol{V},\boldsymbol{U}) + (\boldsymbol{T}\boldsymbol{V},\boldsymbol{U})_{\mathscr{C}} = \mathbb{A}^{\#}_N(\boldsymbol{V},\boldsymbol{U})$$

(cf. Lemma 7.5 and the first equality in (7.11)).

Moreover, the corresponding quadratic form is strongly positive

$$\mathbb{A}_N^{\#}(\boldsymbol{U},\boldsymbol{U}) = \mathscr{E}(\boldsymbol{U},\boldsymbol{U}) + (\boldsymbol{T}\boldsymbol{U},\boldsymbol{U})_{\mathscr{C}} \geq C\big\|\boldsymbol{U}\big|\mathbb{H}^1(\mathscr{C})\big\| \tag{7.13}$$

for some $C > 0$. Indeed, $\mathbb{A}_N^{\#}(\boldsymbol{U},\boldsymbol{U}) = 0$ due to the positivity of the summands implies: $\mathscr{E}(\boldsymbol{U},\boldsymbol{U}) = 0$, and further $\boldsymbol{U} \in \mathscr{R}(\mathscr{C})$ (cf. Lemma 7.5), $(\boldsymbol{T}\boldsymbol{U},\boldsymbol{U})_{\mathscr{C}} = 0$ and further $(\boldsymbol{U},\boldsymbol{K}_j) = 0$ for all $j = 1,\dots,m$. Then $\boldsymbol{U} = \sum_{j=1}^m (\boldsymbol{U},\boldsymbol{K}_j)\boldsymbol{K}_j = 0$. A non-negative symmetric form with the property $\mathbb{A}_N^{\#}(\boldsymbol{U},\boldsymbol{U}) = 0$ if and only if $\boldsymbol{U} = 0$ is positive definite.

According to Lax-Milgram's Lemma 7.2 the equation

$$\mathbb{A}_N^{\#}(\boldsymbol{U},\boldsymbol{V}) = (\boldsymbol{F},\boldsymbol{V})_{\mathscr{C}} - (\boldsymbol{H},\boldsymbol{V}^+)_{\Gamma} \tag{7.14}$$

has a unique solution $\boldsymbol{U} \in \mathbb{H}^1(\mathscr{C})$ for all $\boldsymbol{V} \in \mathbb{H}^1(\mathscr{C})$. This solves the problem

$$\begin{cases} (\mathscr{L}_{\mathscr{C}}(t,\mathscr{D})\boldsymbol{U})(t) + \boldsymbol{T}\boldsymbol{U}(t) = \boldsymbol{F}(t), & t \in \mathscr{C}, \\ (\mathfrak{T}_{\mathscr{C}}(\boldsymbol{\nu}_\Gamma,\mathscr{D})\ \boldsymbol{U})^+(\tau) = \boldsymbol{H}(\tau), & \tau \in \Gamma, \end{cases} \tag{7.15}$$

which is a modified Neumann's problem (5.6).

Now assume that the vector functions $\boldsymbol{F} \in \widetilde{\mathbb{H}}^{-1}(\mathscr{C})$ and $\boldsymbol{H} \in \mathbb{H}^{-1/2}(\Gamma)$ satisfy the orthogonality condition (7.1) from Theorem 7.1 and $\boldsymbol{U}^0 \in \mathbb{H}^1(\mathscr{C})$ be a solution of (7.15). Since

$$(\boldsymbol{T}\boldsymbol{U}^0,\boldsymbol{K}_k)_{\mathscr{C}} = (\boldsymbol{U}^0,\boldsymbol{K}_k)_{\mathscr{C}}, \quad \mathbb{A}_N(\boldsymbol{U}^0,\boldsymbol{K}_k) = \mathscr{E}(\boldsymbol{U}^0,\boldsymbol{K}_k) = 0 \qquad k = 1,2,\dots,m$$

(cf. (7.5)) from (7.14) we get

$$\begin{aligned} 0 = (\boldsymbol{F},\boldsymbol{K}_k)_{\mathscr{C}} - (\boldsymbol{H},\boldsymbol{K}_k)_{\Gamma} = \mathbb{A}_N^{\#}(\boldsymbol{U}^0,\boldsymbol{K}_k) &= \mathbb{A}_N(\boldsymbol{U}^0,\boldsymbol{K}_k) + (\boldsymbol{T}\boldsymbol{U}^0,\boldsymbol{K}_k)_{\mathscr{C}} \\ &= (\boldsymbol{U}^0,\boldsymbol{K}_k)_{\mathscr{C}} \qquad k = 1,2,\dots,m. \end{aligned}$$

Therefore, $\boldsymbol{T}\boldsymbol{U}^0 = \sum_{k=1}^m (\boldsymbol{U}^0,\boldsymbol{K}_k)_{\mathscr{C}}\boldsymbol{K}_k = 0$ and BVP (7.15), which is uniquely solvable, coincides with BVP (5.6) provided that the right-hand sides satisfy the orthogonality condition (7.1). Since the kernel of BVP (5.6) coincides with the space of Killing's vector fields $\mathscr{R}(\mathscr{C})$, a general solution of BVP (5.6) has the form $\boldsymbol{U} = \boldsymbol{U}^0 + \boldsymbol{K}$ with arbitrary $\boldsymbol{K} \in \mathscr{R}(\mathscr{C})$. □

Remark 7.6. If the surface is smooth, by invoking a local fundamental solution to the Lamé equation (cf. Corollary 4.3) and the potential method, it is possible to prove that BVPs (5.4), (5.6) and (7.17) have the same solvability properties if the constraints (5.4) and (5.6) are replaced by the following non-classical constraints

$$\boldsymbol{F} \in \widetilde{\mathbb{H}}_p^{s-2}(\mathscr{C}), \quad \boldsymbol{G} \in \mathbb{H}_p^{s-1/p}(\Gamma), \quad \boldsymbol{H} \in \mathbb{H}_p^{s-1/p-1}(\Gamma), \tag{7.16}$$
$$1 < p < \infty, \quad s \geq 1$$

and $\boldsymbol{U} \in \mathbb{H}_p^s(\mathscr{C})$ is unknown.

Moreover, by the potential method we can investigate *the mixed problem*: find the tangential displacement vector field $\boldsymbol{U} \in \mathbb{H}_p^s(\mathscr{C})$, prescribed on the part Γ_D of the boundary, while on the remainder part $\Gamma_N := \Gamma\setminus\Gamma_D$ is prescribed the traction:

$$\begin{cases} (\mathscr{L}_{\mathscr{C}}(\mathscr{X},\mathscr{D})\,\boldsymbol{U})(\mathscr{X}) = \boldsymbol{F}(\mathscr{X}), & \mathscr{X}\in\mathscr{C},\\ \boldsymbol{U}^+(\tau) = \boldsymbol{G}_0(\tau), & \tau\in\Gamma_D,\\ (\mathfrak{T}_{\mathscr{C}}(\boldsymbol{\nu}_\Gamma,\mathscr{D})\,\boldsymbol{U})^+(\tau) = \boldsymbol{H}_0(\tau), & \tau\in\Gamma_N\,. \end{cases} \tag{7.17}$$

The unique solvability of the mixed problem follows under the following conditions

$$\boldsymbol{F}\in\widetilde{\mathbb{H}}^{s-2}_p(\mathscr{C})\,,\quad \boldsymbol{G}_0\in\mathbb{H}^{s-1/p}_p(\Gamma_D)\,,\quad \boldsymbol{H}_0\in\mathbb{H}^{s-1/p-1}_p(\Gamma_N), \tag{7.18}$$

$$1<p<\infty,\quad s\geq 1,\quad \frac{1}{p}-\frac{1}{2}<s<\frac{1}{p}+\frac{1}{2}.$$

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