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ON THE UNIQUE SOLVABILITY OF TWO-POINT BOUNDARY VALUE PROBLEMS FOR THIRD ORDER LINEAR DIFFERENTIAL EQUATIONS WITH SINGULARITIES

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Dedicated to the blessed memory of my friend, professor Tengiz Shervashidze

Abstract. Optimal in a certain sense conditions guaranteeing the unique solvability of the boundary value problem

$$u''' = p(t)u + q(t),$$

$$u(a+) = c_0, \quad u'(a+) = c_1, \quad \sum_{i=0}^{k} \ell_i u^{(i)}(b-) = c_2$$

are established in the case when the functions p and $q :]a,b[\to \mathbb{R}$ have nonintegrable singularities at the points a and b.

On a finite open interval $]a,b[$, we consider the differential equation

$$u''' = p(t)u + q(t) \tag{1}$$

with the boundary conditions

$$u(a+) = c_0, \quad u'(a+) = c_1, \quad \sum_{i=0}^{k} \ell_i u^{(i)}(b-) = c_2. \tag{2}$$

Here, p and $q :]a,b[\to \mathbb{R}$ are the functions, Lebesgue integrable on every closed interval contained in $]a,b[$,

$$k \in \{0,1,2\}, \quad \ell_i \geq 0 \ \ (i = 0,\dots,k), \quad \ell_k > 0,$$

and c_i $(i = 0,1,2)$ are arbitrary real numbers.

Particular cases of the boundary conditions (2) are the Dirichlet and Nicoletti boundary conditions

$$u(a+) = c_0, \quad u'(a+) = c_1, \quad u(b-) = c_2, \tag{2_1}$$

$$u(a+) = c_0, \quad u'(a+) = c_1, \quad u'(b-) = c_2, \tag{2_2}$$

$$u(a+) = c_0, \quad u'(a+) = c_1, \quad u''(b-) = c_2. \tag{2_3}$$

We are especially interested in the case when the functions p and q have nonintegrable singularities at the boundary points of the interval $]a,b[$, i.e. the case, where

$$\int_a^b \big(|p(t)| + |q(t)|\big)\, dt = +\infty.$$

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If the conditions

$$\int_a^b (t-a)(b-t)^{2-k}|p(t)|\,dt < +\infty, \tag{3}$$

$$\int_a^b (t-a)(b-t)^{2-k}|q(t)|\,dt < +\infty \tag{4}$$

are fulfilled, then according to T. Kiguradze's theorem [9, Theorem 1], problem (1), (2) is Fredholm's one. In the present paper, it is proved that this problem is likewise Fredholmian in the case, where $c_0 = 0$ and instead of (3) a weaker condition

$$\int_a^b (t-a)^2(b-t)^{2-k}|p(t)|\,dt < +\infty \tag{5}$$

is fulfilled. On the basis of this fact, the unimprovable in a certain sense effective conditions guaranteeing the unique solvability of problems (1), (2) and (1), (2_i) $(i = 1, 2, 3)$ are established. The obtained results are new not only in a singular case, but also in a regular case when

$$\int_a^b \big(|p(t)| + |q(t)|\big)\,dt < +\infty$$

(see [1–9] and the references therein).

In the paper, we use the following notation.

$$\delta_k = \sum_{i=0}^{k} \frac{3(b-a)^{3-i}}{(3-i)!}\,\ell_i \Big/ \sum_{i=0}^{k} \frac{(b-a)^{2-i}}{(2-i)!}\,\ell_i; \tag{6}$$

$$\Delta_k(t) = \sum_{i=0}^{k} \frac{(b-t)^{2-i}}{(2-i)!}\,\ell_i \Big/ \sum_{i=0}^{k} \frac{(b-a)^{2-i}}{(2-i)!}\,\ell_i; \tag{7}$$

$$g_k(t,s) = \begin{cases} \dfrac{1}{2}\big(\Delta_k(s)(t-a)^2 - (t-s)^2\big) & \text{for } a \le s < t \le b, \\[2mm] \dfrac{1}{2}\,\Delta_k(s)(t-a)^2 & \text{for } a \le t \le s \le b; \end{cases} \tag{8}$$

$$p_-(t) = \big(|p(t)| - p(t)\big)/2;$$

$v(a+)$ and $v(b-)$ are, respectively, the right and left limits of the function $v : \,]a,b[\, \to \mathbb{R}$ at the points a and b.

A solution of problem (1), (2) is sought in the class of twice continuously differentiable on $]a,b[$ real functions whose second derivative is absolutely continuous on every closed interval contained in $]a,b[$.

Theorem 1. *Let either conditions* (3) *and* (4) *hold, or* $c_0 = 0$ *and conditions* (4) *and* (5) *be fulfilled. Then for the unique solvability of problem* (1), (2) *it is necessary and sufficient that the corresponding homogeneous problem*

$$u''' = p(t)u, \tag{1_0}$$

$$u(a+) = 0, \quad u'(a+) = 0, \quad \sum_{i=0}^{k} \ell_i u^{(i)}(b-) = 0 \tag{2_0}$$

has only a trivial solution.

Before we proceed to proving the above theorem, we have to prove some auxiliary propositions.

Lemma 1. *The number δ_k and the functions Δ_k and g_k satisfy the conditions*

$$\delta_0 = b-a, \quad \delta_k > b-a \ \ (k=1,2); \tag{9}$$

$$\Big(\frac{b-t}{b-a}\Big)^2 = \Delta_0(t) < \Delta_k(t) \le \Big(\frac{b-t}{b-a}\Big)^{2-k} \ \ \text{for } a<t<b \ \ (k=1,2); \tag{10}$$

$$\frac{(s-a)(b-s)^2}{2(b-a)^4}\,(t-a)^2(b-t) < g_0(t,s) < \begin{cases} r(s-a)(b-s)(b-t) & \text{for } a<s<t<b,\\ r(t-a)(b-s)^2 & \text{for } a<t\le s<b; \end{cases} \tag{11}$$

$$g_0(t,s) + \frac{\Delta_k(s)-\Delta_0(s)}{2}\,(t-a)^2 = g_k(t,s) < \begin{cases} r(s-a)(b-s)^{2-k} & \text{for } a<s<t<b,\\ r(t-a)(b-s)^{2-k} & \text{for } a<t\le s<b \end{cases} \ (k=1,2); \tag{12}$$

$$\Big|\frac{\partial^i g_k(t,s)}{\partial t^i}\Big| < \begin{cases} r(s-a)(b-s)^{2-i} & \text{for } a<s<t<b,\\ r(t-a)^{2-i}(b-s)^{2-k} & \text{for } a<t\le s<b \end{cases} \ (i=1,2;\ k=0,1); \tag{13}$$

$$\Big|\frac{\partial^i g_2(t,s)}{\partial t^i}\Big| < \begin{cases} r(s-a) & \text{for } a<s<t<b,\\ r(t-a)^{2-i} & \text{for } a<t\le s<b \end{cases} \ (i=1,2), \tag{14}$$

where $r = 1+b-a+(b-a)^{-2}$.

Proof. Conditions (9) and (10) follow directly from (6) and (7).

Owing to (7), we have

$$\Delta_0(s)(t-a)^2-(t-s)^2 = (b-a)^{-2}(s-a)(b-t)\big((b-s)(t-a)+(b-a)(t-s)\big).$$

However,

$$(b-s)(t-a)+(b-a)(t-s) > (b-s)(t-a) > \frac{(b-s)^2(t-a)^2}{(b-a)^2} \ \ \text{for } a<s<t<b,$$
$$(b-s)(t-a)+(b-a)(t-s) < 2(b-s)(b-a) \ \ \text{for } a<t,s<b.$$

Therefore,

$$\frac{(s-a)(b-s)^2}{(b-a)^4}(t-a)^2(b-t) < \Delta_0(s)(t-a)^2-(t-s)^2 < \frac{2}{b-a}\,(s-a)(b-s)(b-t) \ \ \text{for } a<s<t<b. \tag{15}$$

On the other hand, it follows from (10) that

$$\begin{aligned}\Delta_0(s)(t-a)^2-(t-s)^2 &< \Delta_k(s)(t-a)^2-(t-s)^2\\ &< (b-a)^{k-2}\big((b-s)^{2-k}(t-a)^2-(t-s)^2(b-a)^{2-k}\big)\\ &< (b-a)^{k-2}\big((b-s)^{2-k}(b-a)^2-(b-s)^2(b-a)^{2-k}\big) = (b-s)^{2-k}\big((b-a)^k-(b-s)^k\big)\\ &\le 2^{k-1}(b-a)^{k-1}(s-a)(b-s)^{2-k} \ \ \text{for } a<t\le s<b \ \ (k=1,2). \end{aligned}\tag{16}$$

If along with (15) and (16) we take into account the inequalities

$$\Delta_k(s)(t-a)^2 < (b-a)^{k-1}(t-a)(b-s)^{2-k} \ \ \text{for } a<t\le s<b \ \ (k=0,1,2),$$

then from equality (8) we obtain estimates (11) and (12).

The validity of estimates (12), (14) and (16) is proved analogously. □

Lemma 2. *Let $f:]a,b[\to \mathbb{R}$ be a measurable function such that*

$$\int_a^b (t-a)(b-t)^{2-k}|f(t)|\,dt < +\infty. \tag{17}$$

Then the differential equation

$$u''' = f(t) \tag{18}$$

has a unique solution u satisfying the boundary conditions (2) *and the representation*

$$u(t) = u_0(t) - \int_a^b g_k(t,s) f(s)\, ds \quad \textit{for} \quad a < t < b \tag{19}$$

is valid, where u_0 is a solution of the homogeneous equation

$$u''' = 0, \tag{18_0}$$

satisfying the boundary conditions (2).

Proof. Let us prove first that the semi-homogeneous problem $(18_0), (2)$ has a unique solution.

An arbitrary solution of equation (18_0) has the form

$$u(t) = x_0 + x_1(t-a) + x(t-a)^2,$$

where x_0, x_1 and x are real constants. This function satisfies the bounadry conditions (2) if and only if

$$x_0 = c_0, \quad x_1 = c_1, \quad x = c,$$

where

$$c = \left(c_2 - \sum_{i=0}^{k_0} \Big(\sum_{j=i}^{1} (b-a)^{j-i} c_j\Big) \ell_j\right) \Big/ \sum_{i=0}^{k} \frac{2(b-a)^{2-i}}{(2-i)!}\, \ell_i, \quad k_0 = \big(|k+1| - |k-1|\big)/2.$$

Consequently, the function

$$u_0(t) = c_0 + c_1(t-a) + c(t-a)^2$$

is a unique solution of problem $(18_0), (2)$.

Consider now the function

$$u_1(t) = -\int_a^b g_k(t,s) f(s)\, ds \quad \text{for} \quad a < t < b.$$

According to conditions (11)–(14), this function is a solution of the differential equation (18) satisfying the boundary conditions

$$u_1(a+) = 0, \quad u_1'(a+) = 0, \quad u_1^{(i)}(b-) = \frac{1}{(2-i)!} \int_a^b \Big(\Delta_k(s)(b-a)^{2-i} - (b-s)^{2-i}\Big) f(s)\, ds \quad (i = 0, \dots, k).$$

Therefore,

$$\sum_{i=0}^{k} \ell_i u_1^{(i)}(b-) = \int_a^b \left(\Delta_k(s) \sum_{i=0}^{k} \frac{(b-a)^{2-i}}{(2-i)!}\, \ell_i - \sum_{i=0}^{k} \frac{(b-s)^{2-i}}{(2-i)!}\right) f(s)\, ds = 0.$$

Thus u_1 is a solution of problem $(18), (2_0)$.

From all the above-proven it follows that the function u given by equality (19) is a solution of problem $(18), (2)$. This problem has no other solution since the corresponding homogeneous problem $(18_0), (2_0)$ has only a trivial solution. □

Lemma 3. *Let $t_0 \in\,]a,b[$ and the functions p and q be such that*

$$(t_0 - a) \int_a^{t_0} (t-a)|p(t)|\, dt < 1, \tag{20}$$

$$\int_a^{t_0} (t-a)|q(t)|\, dt < +\infty. \tag{21}$$

Then for any $c_i \in \mathbb{R}$ $(i = 0,1,2)$, equation (1) *has a unique solution satisfying the conditions*

$$u(a+) = c_0, \quad u'(a+) = c_1, \quad u''(t_0-) = c_2. \tag{22}$$

However, if instead of (20), *along with* (21) *the condition*

$$\int\limits_a^{t_0} (t-a)^2|p(t)|\,dt < 1 \tag{23}$$

is fulfilled, then for any $c_i \in \mathbb{R}$ $(i=1,2)$, *equation* (1) *has a unique solution satisfying the conditions*

$$u(a+) = 0, \quad u'(a+) = c_1, \quad u''(t_0-) = c_2. \tag{24}$$

Proof. Let $C(]a,t_0[)$ be the Banach space of continuous and bounded functions $u :]a,t_0[\to \mathbb{R}$ with the norm

$$\|u\|_C = \sup\big\{|u(t)| : \ a < t < t_0\big\},$$

and let $C_1(]a,t_0[)$ be the Banach space of continuous and bounded with the weight $1/(t-a)$ functions with the norm

$$\|u\|_{C_1} = \sup\Big\{\frac{|u(t)|}{t-a} \ : \ a < t < t_0\Big\} < +\infty.$$

Introduce the function

$$g_{20}(t,s) = \begin{cases} \dfrac{1}{2}\big((t-a)^2 - (t-s)^2\big) & \text{for } \ a \le s < t \le t_0, \\[2mm] \dfrac{1}{2}\,(t-a)^2 & \text{for } \ a \le t \le s \le t_0, \end{cases}$$

and in the space $C(]a,t_0[)$ (in the space $C_1(]a,t_0[)$) let us consider the operator G given by the equality

$$G(u)(t) = c_0 + c_1(t-a) + \frac{1}{2}\,c_2(t-a)^2 - \int\limits_a^{t_0} g_{20}(t,s)\big(p(s)u(s) + q(s)\big)\,ds \ \text{ for } \ a < t < t_0$$

$$\left(G(u)(t) = c_1(t-a) + \frac{1}{2}\,c_2(t-a)^2 - \int\limits_a^{t_0} g_{20}(t,s)\big(p(s)u(s) + q(s)\big)\,ds \ \text{ for } \ a < t < t_0 \right).$$

If conditions (20) and (21) (conditions (21) and (23)) are fulfilled, then for any $u \in C(]a,t_0[)$ (for any $u \in C_1(]a,t_0[)$) the function $f(t) \equiv p(t)u(t) + q(t)$ satisfies the condition

$$\int\limits_a^{t_0} (t-a)|f(t)|\,dt < +\infty.$$

By virtue of Lemma 2, this implies that the function $u :]a,b[\to \mathbb{R}$ is a solution of problem $(1),(22)$ (of problem $(1),(24)$) if and only if its restriction to $]a,t_0[$ is a solution of the integral equation

$$u(t) = G(u)(t) \ \text{ for } \ a < t < t_0, \tag{25}$$

while the restriction to $[t_0,b[$ is a solution of the differential equation (1) satisfying the initial conditions

$$u^{(i)}(t_0) = \lim_{t<t_0,\, t\to t_0} \frac{\partial^i G(u)(t)}{\partial t^i} \ \ (i=0,1,2). \tag{26}$$

If we take into account the estimate

$$|g_{20}(t,s)| \le (t-a)(s-a) \ \text{ for } \ a \le t,s \le t_0,$$

then it becomes clear that the operator G maps into itself the space $C(]a,t_0[)$ (the space $C_1(]a,t_0[)$) and for any two functions u_1 and u_2 of that space it satisfies the inequality

$$\|G(u_1) - G(u_2)\| \le \alpha\|u_1 - u_2\|_C \quad \Big(\|G(u_1) - G(u_2)\|_{C_1} \le \alpha\|u_1 - u_2\|_{C_1} \Big),$$

where

$$\alpha = (t_0 - a)\int\limits_a^{t_0} (s-a)|p(s)|\,ds < 1 \quad \left(\alpha = \int\limits_a^{b} (s-a)^2|p(s)|\,ds < 1 \right).$$

By virtue of the Banach theorem on the contracted mapping, the integral equation (25) in the space $C(]a,t_0[)$ (in the space $C_1(]a,t_0[)$) has a unique solution u. However, by the above-said, u is likewise a unique solution of the boundary value problem (1), (22) (problem (1), (24)) in the interval $]a,t_0[$. Extension of that function to the whole interval $]a,b[$, as a solution of the initial problem (1), (26), is a solution of problem (1), (22) (problem (1), (24)) in the interval $]a,b[$. □

Lemma 4. *If*

$$\int\limits_a^{t_0} \big((t-a)^2|p(t)|+|q(t)|\big)\,dt<+\infty \ \ \textit{for} \ \ t_0\in\,]a,b[\,, \tag{27}$$

then for any $c\in\mathbb{R}$, equation (1) *has a unique solution satisfying the initial conditions*

$$u(0+)=0, \quad u'(a+)=0, \quad u''(a+)=c. \tag{28}$$

Proof. Let

$$r(t)=\exp\bigg(\int\limits_a^t \big((s-a)^2|p(s)|+|q(s)|\big)\,ds\bigg) \ \ \text{for} \ \ a\le t<b,$$

and let $C_2(]a,b[)$ be the Banach space of continuous functions $u:\,]a,b[\,\to\mathbb{R}$ satisfying the conditions

$$\limsup_{t\to a}\frac{|u(t)|}{(t-a)^2}<+\infty, \quad \limsup_{t\to+\infty}\frac{|u(t)|}{r(t)}<+\infty,$$

with the norm

$$\|u\|_{C_2}=\sup\Big\{\frac{|u(t)|}{(t-a)^2r(t)} \ : \ a<t<b\Big\}.$$

In the space $C_2(]a,b[)$, we introduce the operator

$$G(u)(t)=\frac{c}{2}\,(t-a)^2+\frac{1}{2}\int\limits_a^t (t-s)^2\big[p(s)u(s)+q(s)\big]\,ds \ \ \text{for} \ \ a<t<b.$$

According to condition (27), the function $u:\,]a,b[\,\to\mathbb{R}$ is a solution of problem (1), (28) if and only if it belongs to the space $C_2(]a,b[)$ and satisfies the integral equation

$$u(t)=G(u)(t) \ \ \text{for} \ \ a<t<b. \tag{29}$$

On the other hand, for any functions u and $v\in C_2(]a,b[)$, we have

$$\begin{aligned}|G(u)(t)|&\le\frac{|c|}{2}\,(t-a)^2+\frac{1}{2}\,(t-a)^2\int\limits_a^t \Big[(s-a)^2r(s)|p(s)|\,\|u\|_{C_2}+|q(s)|\Big]\,ds\\ &\le\frac{|c|}{2}\,(t-a)^2+\frac{1}{2}\,(t-a)^2\big(1+\|u\|_{C_2}\big)\int\limits_a^t r'(s)\,ds\\ &<\frac{1}{2}\,\big(1+|c|+\|u\|_{C_2}\big)(t-a)^2r(t) \ \ \text{for} \ \ a<t<b,\end{aligned}$$

$$\begin{aligned}\big|G(u)(t)-G(v)(t)\big|&\le\frac{1}{2}(t-a)^2\int\limits_a^t |p(s)|\,|u(s)-v(s)|\,ds\\ &\le\frac{1}{2}\,\|u-v\|_{C_2}(t-a)^2\int\limits_a^t (s-a)^2|p(s)|r(s)\,ds\le\frac{1}{2}\,\|u-v\|_{C_2}(t-a)^2r(t) \ \ \text{for} \ \ a<t<b.\end{aligned}$$

Therefore,

$$\|G(u)\|_{C_2}<+\infty, \quad \|G(u)-G(v)\|_{C_2}\le\frac{1}{2}\,\|u-v\|_{C_2}.$$

Consequently, the operator G maps tersely the space $C_2(]a,b[)$ into itself.

By the Banach theorem, the integral equation (29) has a unique solution which according to the above-said is a unique solution of problem (1), (28). □

Lemma 5. *If $m \in \{0,1,2\}$ and for $t_0 \in]0,b[$ the condition*

$$\int_a^{t_0} (t-a)^{2-m}\big(|p(t)|+|q(t)|\big)\,dt < +\infty \quad \Big(\int_{t_0}^b (b-t)^{2-m}\big(|p(t)|+|q(t)|\big)\,dt < +\infty\Big) \tag{30}$$

is fulfilled, then every solution of equation (1) *satisfies the condition*

$$\int_a^{t_0} |u^{(m+1)}(t)|\,dt < +\infty \quad \Big(\int_{t_0}^b |u^{(m+1)}(t)|\,dt < +\infty\Big).$$

If $m \in \{0,1\}$ and for $t_0 \in]a,b[$ the condition

$$\int_a^{t_0} \Big[(t-a)^2|p(t)|+(t-a)^{1-m}|q(t)|\Big]\,dt < +\infty \quad \Big(\int_{t_0}^b \Big[(b-t)^2|p(t)|+(b-t)^{1-m}|q(t)|\Big]\,dt < +\infty\Big)$$

holds, then an arbitrary solution of equation (1)*, satisfying the conditions*

$$u^{(i)}(a+)=0 \ \ (i=0,\dots,m) \quad \big(u^{(i)}(b-)=0 \ \ (i=0,\dots,m)\big),$$

satisfies also the condition

$$\int_a^{t_0} |u^{(m+2)}(t)|\,dt < +\infty \quad \Big(\int_{t_0}^b |u^{(m+2)}(t)|\,dt < +\infty\Big).$$

Proof. Let $m \in \{0,1,2\}$ and for $t_0 \in]a,b[$ the condition (30) be fulfilled. For the sake of definiteness, we assume that

$$\int_a^{t_0} (t-a)^{2-m}\big(|p(t)|+|q(t)|\big)\,dt < +\infty \ \text{ for } \ a<t_0<b, \tag{31}$$

since the case, where

$$\int_{t_0}^b (b-t)^{2-m}\big(|p(t)|+|q(t)|\big)\,dt < +\infty \ \text{ for } \ a<t_0<b,$$

is considered analogously.

An arbitrary solution u of equation (1) satisfies the inequalities

$$|u(t)| \le \sum_{i=0}^{2} |u^{(i)}(t_0)|(t_0-a)^i + \frac{1}{2}\int_t^{t_0} (s-a)^2\Big[|p(s)|\,|u(s)|+|q(s)|\Big]\,ds \ \text{ for } \ a<t<t_0, \tag{32}$$

$$\int_t^{t_0} |u^{(m+1)}(s)|\,ds \le r_0(t_0) + \int_t^{t_0} (s-a)^{2-m}\Big[|p(s)|\,|u(s)|+|q(s)|\Big]\,ds \ \text{ for } \ a<t<t_0, \tag{33}$$

where

$$r_0(t_0) = \sum_{i=1}^{2-m} |u^{(m+i)}(t_0)|(t_0-a)^i \ \text{ for } \ m\in\{0,1\}, \quad r_0(t_0)=0 \ \text{ for } \ m=2.$$

By virtue of Gronwall's lemma and condition (31), it follows from condition (32) that

$$|u(t)| \le r_1(t_0) \ \text{ for } \ a<t<b,$$

where

$$r_1(t_0) = \left[\sum_{i=0}^{2} |u^{(i)}(t_0)|(t_0-a)^i + \frac{1}{2}\int_a^{t_0} (s-a)^2|q(s)|\,ds\right] \exp\left(\frac{1}{2}\int_a^{t_0} (s-a)^2|p(s)|\,ds\right) < +\infty.$$

If along with the above estimate we take into account condition (31), then from inequality (33) we find

$$\int_a^{t_0} |u^{(m+1)}(s)|\,ds \le r_0(t_0) + \int_a^{t_0} (s-a)^{2-m}\Big[r_1(t_0)|p(s)| + |q(s)|\Big]\,ds < +\infty.$$

Thus we have proved the validity of the first part of the lemma.

We now proceed to proving the second part of the lemma. For the sake of definiteness, we assume that u is a solution of equation (1) satisfying the conditions

$$u^{(i)}(a+) = 0 \quad (i = 0, \dots, m), \tag{34}$$

and the functions p and q are such that

$$\int_a^{t_0} \Big[(t-a)^2|p(t)| + (t-a)^{1-m}|q(t)|\Big]\,dt < +\infty \quad \text{for} \quad a < t_0 < b. \tag{35}$$

Our aim is to prove that

$$\int_a^{t_0} |u^{(m+2)}(t)|\,dt < +\infty. \tag{36}$$

The function u is a solution of the differential equation

$$u''' = q_0(t), \tag{37}$$

where $q_0(t) \equiv p(t)u(t) + q(t)$. On the other hand, by conditions (34) and (35), we have

$$r(t_0) = \sup\left\{\frac{|u(t)|}{(t-a)^m} \,:\, a < t < t_0\right\} < +\infty,$$

$$\int_a^{t_0} (t-a)^{2-m}|q_0(t)|\,dt \le \int_a^{t_0} \Big[r(t_0)(t-a)^2|p(t)| + (t-a)^{2-m}|q(t)|\Big]\,dt < +\infty \quad \text{for} \quad a < t_0 < b,$$

whence by virtue of the above-proven first part of the lemma it follows that

$$\int_a^{t_0} |u^{(m+1)}(t)|\,dt < +\infty \quad \text{for} \quad a < t_0 < b.$$

Suppose

$$v(t) \equiv \int_a^{t} |u^{(m+1)}(s)|\,ds$$

and choose $t_1 \in \,]a, b[$ in such a way that the inequality

$$\int_a^{t_1} (s-a)^2|p(s)|\,ds < \frac{1}{2}$$

is fulfilled. Then from the equalities

$$u(t) = \int_a^t (t-s)^m u^{(m+1)}(s)\,ds \quad \text{for } a < t < b,$$

$$u^{(m+1)}(t) = \sum_{i=1}^{2-m} u^{(m+i)}(t_1)(t-t_1)^{i-1} + \int_{t_1}^t (t-s)^{1-m}\big[p(s)u(s) + q(s)\big]\,ds \quad \text{for } a < t < b$$

it follows that

$$|u(t)| \le (t-a)^m v(t) \quad \text{for } a < t < b, \tag{38}$$

$$\begin{aligned} v(t) &\le r_1(t-a) + (t-a)\int_t^{t_1} (s-a)|p(s)|v(s)\,ds + \int_a^t (s-a)^2|p(s)|v(s)\,ds \\ &\le r_1(t-a) + (t-a)\int_t^{t_1} (s-a)|p(s)|v(s)\,ds + \frac{v(t)}{2} \quad \text{for } a < t \le t_1, \end{aligned}$$

where

$$r_1 = \sum_{i=1}^{2-m} |u^{(m+i)}(t_1)|(t_1-a)^{i-1} + \int_a^{t_1} (s-a)^{1-m}|q(s)|\,ds.$$

Consequently,

$$\frac{v(t)}{t-a} \le 2r_1 + 2\int_t^{t_1} (s-a)^2|p(s)|\,\frac{v(s)}{s-a}\,ds \quad \text{for } a < t \le t_1.$$

From the last inequality, by Gronwell's lemma, we obtain the estimate

$$\frac{v(t)}{t-a} \le 2r_1 \exp\left(2\int_t^{t_1} (s-a)^2|p(s)|\,ds\right) < 2r_1\exp(1) \quad \text{for } a < t \le t_1.$$

Thus we have proved that

$$w(t_0) = \sup\left\{\frac{v(t)}{t-a} : a < t \le t_0\right\} < +\infty \quad \text{for } a < t \le t_0.$$

Therefore inequality (38) results in the estimate

$$|u(t)| \le w(t_0)(t-a)^{m+1} \quad \text{for } a < t \le t_0 < b.$$

If along with the obtained estimate we take into account condition (35), we find that

$$\int_a^{t_0} (t-a)^{1-m}|q_0(t)|\,dt < \int_a^{t_0} \Big[w(t_0)(t-a)^2|p(t)| + (t-a)^{1-m}|q(t)|\Big]\,dt < +\infty \quad \text{for } a < t_0 < b.$$

By virtue of this inequality and the first part of the lemma, the function u, as a solution of the differential equation (37), satisfies condition (36). □

Proof of Theorem 1. If the nonhomogeneous problem $(1),(2)$ is uniquely solvable, then obviously the homogeneous problem $(1_0),(2_0)$ has only a trivial solution. It is also clear that if problem $(1_0),(2_0)$ has only a trivial solution, then problem $(1),(2)$ has no more than one solution. Thus to prove the theorem, it remains to prove that if conditions (3) and (4) are fulfilled ($c_0 = 0$ and conditions (4) and (5) are fulfilled), then the unique solvability of problem $(1_0),(2_0)$ guarantees the solvability of problem $(1),(2)$ as well.

In the case under consideration, there is $t_0 \in]a,b[$ such that along with (20) and (21) (along with (21) and (23)) the condition

$$\int_{t_0}^{b} (b-t)^{2-k}\big(|p(t)|+|q(t)|\big)\,dt < +\infty$$

is fulfilled.

By Lemmas 3 and 5, the differential equation (1) has a unique solution u_1 satisfying the conditions

$$u_1(a+) = c_0, \quad u_1'(a+) = c_1, \quad u_1''(t_0) = 1,$$
$$\int_{t_0}^{b} |u_1^{(k+1)}(t)|\,dt < +\infty. \tag{39}$$

On the other hand, by Lemmas 4 and 5, the homogeneous equation (1_0) has a solution u_0 such that

$$u_0(a+) = 0, \quad u_0'(a+) = 0, \quad u_0''(a_0+) = 1,$$
$$\int_{t_0}^{b} |u_0^{(k+1)}(t)|\,dt < +\infty. \tag{40}$$

According to conditions (39) and (40), the functions $u_0^{(i)}$ and $u_1^{(i)}$ $(i = 0,\dots,k)$ have at the point b the finite left limits $u_0^{(i)}(b-)$ and $u_1^{(i)}(b-)$ $(i = 0,\dots,k)$. Moreover,

$$\gamma_0 = \sum_{i=0}^{k} \ell_i u_0^{(i)}(b-) \neq 0,$$

because the homogeneous problem $(1_0), (2_0)$ has only a trivial solution.

Let

$$\gamma = -\Big(\sum_{i=0}^{n} \ell_i u_1^{(i)}(b-)\Big)\Big/\gamma_0.$$

Then the function

$$u(t) = \gamma u_0(t) + u_1(t) \ \text{ for } \ a < t < b$$

is a solution of problem (1), (2). □

Remark 1. The restrictions in Theorem 1 imposed on the functions p and q are unimprovable. Indeed, it can be easily verified that if condition (3) holds, or $c_0 = 0$ and condition (5) is fulfilled, while the function q in the neighborhood of the points a and b is of constant sign, then the fulfilment of condition (4) is necessary for problem (1), (2) to be solvable. On the other hand, if $k = 0$, $c_0 \neq 0$, $c_2 \neq 0$ ($k = 0$, $c_0 = 0$, $c_1 \neq 0$, $c_2 \neq 0$), the function p in the neighborhood of the points a and b is of constant sign, while the function q satisfies condition (4), then the fulfilment of condition (3) (of condition (5)) is necessary for problem (1), (2) to be solvable.

Corollary 1. *Let either conditions* (3) *and* (4) *be fulfilled, or* $c_0 = 0$ *and conditions* (4) *and* (5) *hold. Moreover, let* $f :]a,b[\to [0,+\infty[$ *be some measurable function satisfying the condition*

$$0 < \int_{a}^{b} (t-a)(b-t)^{2-k} f(t)\,dt < +\infty. \tag{41}$$

Then for the unique solvability of problem (1), (2) *it is necessary that the inequality*

$$\operatorname{mes}\Big\{t \in]a,b[:\ G_k(f)(t)p(t) \neq -f(t)\Big\} > 0 \tag{42}$$

is fulfilled, where

$$G_k(f)(t)=\int\limits_a^b g_k(t,s)f(s)\,ds \ \ \text{for} \ \ a<t<b. \tag{43}$$

Proof. By Lemma 1, the functions Δ_k and g_k $(k=0,1,2)$ satisfy inequalities (10)–(12). If along with this we take into account conditions (41) and (43), then by Lemma 2 it becomes clear that the function

$$u(t)=G_k(f)(t) \ \ \text{for} \ \ a<t<b$$

is a solution of problem $(18),(2_0)$ such that

$$u(t)>\eta(t-a)^2(b-t) \ \ \text{for} \ \ a<t<b,$$

where

$$\eta=\int\limits_a^b \frac{(s-a)(b-s)^2}{2(b-a)^4}\,f(s)\,ds>0.$$

On the other hand, if inequality (42) is violated, then

$$p(t)=-\frac{f(t)}{G_k(f)(t)} \ \ \text{for almost all} \ \ t\in\,]a,b[\,.$$

Consequently, the function u is a solution of the homogeneous problem $(1_0),(2_0)$, and by Theorem 1, the nonhomogeneous problem $(1),(2)$ is not uniquely solvable. □

Theorem 2. *Let either conditions* (3) *and* (4) *hold, or* $c_0=0$ *and conditions* (4) *and* (5) *be fulfilled. Moreover, let there exist a continuous function* $w:\,]a,b[\,\to\,]0,+\infty[$ *such that*

$$\liminf_{t\to a}\frac{w(t)}{(t-a)^2}>0, \quad \liminf_{t\to b}\frac{w(t)}{(b-t)^{m_k}}>0, \tag{44}$$

where $m_k=(1-k+|1-k|)/2$*, and*

$$\sup\left\{\int\limits_a^b \frac{g_k(t,s)}{w(t)}\,w(s)p_-(s)\,ds:\ a<t<b\right\}<1. \tag{45}$$

Then problem $(1),(2)$ *has a unique solution.*

Proof. By Theorem 1, to prove Theorem 2 it suffices to state that the homogeneous problem $(1_0),(2_0)$ has only a trivial solution.

Suppose that problem $(1_0),(2_0)$ has a nontrivial solution u. Then by Lemma 5 and condition (5), we have

$$\int\limits_a^{t_0}|u'''(t)|\,dt<+\infty, \quad \int\limits_{t_0}^b |u^{(k+m_k+1)}(t)|\,dt<+\infty.$$

Therefore there exist finite one-sided limits

$$u''(a+), \quad u^{(i)}(b-) \ \ (i=0,\dots,k+m_k).$$

Moreover, $u''(a+)\not\equiv 0$, since by Lemma 4 equation (1_0) under the initial conditions

$$u^{(i)}(a+)=0 \ \ (i=0,1,2)$$

has only a trivial solution.

Without loss of generality we can assume that

$$u''(a+)=1,$$

and consequently,

$$\lim_{t\to a}\frac{u(t)}{(t-a)^2}=1. \tag{46}$$

On the other hand, it is clear that

$$\lim_{t\to b}\frac{|u(t)|}{(b-t)^{m_k}}=|u^{(m_k)}(b-)|<+\infty.$$

Therefore,

$$\sup\left\{\frac{|u(t)|}{(t-a)^2(b-t)^{m_k}}\ :\ a<t<b\right\}<+\infty. \tag{47}$$

By condition (46), either

$$u(t)>0 \ \text{ for } \ a<t<b, \tag{48}$$

or there exists $b_0\in\,]a,b[$ such that

$$u(t)>0 \ \text{ for } \ a<t<b_0, \quad u(b_0)=0. \tag{49}$$

Let inequality (48) be fulfilled. Then by conditions (5), (44) and (47) we have

$$0<r=\sup\left\{\frac{u(t)}{w(t)}\ :\ a<t<b\right\}<+\infty, \tag{50}$$

$$\int_a^b (b-t)^{2-k}|f(t)|\,dt<+\infty,$$

where $f(t)\equiv p(t)u(t)$. This, by virtue of Lemma 2, implies that the identity

$$u(t)=-\int_a^b g_k(t,s)p(s)u(s)\,ds \ \text{ for } \ a<t<b$$

is valid.

Taking into account inequalities (45), (48), (50), and the positiveness of the function g_k, this identity results in

$$\frac{u(t)}{w(t)}\le\int_a^b\frac{g_k(t,s)}{w(t)}\,p_-(s)u(s)\,ds\le r\int_a^b\frac{g_k(t,s)}{w(t)}\,w(s)p_-(s)\,ds\le\alpha r \ \text{ for } \ a<t<b,$$

where

$$\alpha=\sup\left\{\int_a^b\frac{g_k(t,s)}{w(t)}\,w(s)p_-(s)\,ds:\ a<t<b\right\}<1. \tag{51}$$

Therefore,

$$0<r\le\alpha r<r.$$

The obtained contradiction proves that inequality (48) does not hold.

It remains to consider the case when condition (49) is fulfilled. In this case the function u satisfies the condition

$$0<r_0=\sup\left\{\frac{u(t)}{w(t)}\ :\ a<t<b_0\right\}<+\infty, \tag{52}$$

and the function $f(t)=p(t)u(t)$ satisfies the condition

$$\int_a^{b_0}|f(t)|\,dt<+\infty.$$

On the other hand, by Lemma 2, the last condition guarantees the validity of the following identity

$$u(t)=-\int_a^{b_0}\widetilde{g}_0(t,s)p(s)u(s)\,ds \ \text{ for } \ a<t<b_0, \tag{53}$$

where

$$\widetilde{g}_0(t,s) = \begin{cases} \dfrac{1}{2}\left(\left(\dfrac{b_0-s}{b_0-a}\right)^2(t-a)^2-(t-s)^2\right) & \text{for } a \le s < t \le b_0, \\ \dfrac{1}{2}\left(\dfrac{b_0-s}{b_0-a}\right)^2(t-a)^2 & \text{for } a \le t \le s \le b_0. \end{cases}$$

It can be easily seen that

$$0 < \widetilde{g}_0(t,s) \le g_0(t,s) \text{ for } a < t, s \le b_0.$$

On the other hand, by Lemma 1, we have

$$g_0(t,s) < g_k(t,s) \text{ for } a < t, s < b.$$

If along with the above inequalities we take into account conditions (49) and (52), then from the identity (53) we find

$$0 < u(t) \le r_0 \int_a^{b_0} g_k(t,s)w(s)[p(s)]_+\,ds \text{ for } a < t < b_0.$$

This, in view of inequality (51), implies

$$0 < r_0 \le \alpha r_0 < r_0.$$

The obtained contradiction proves that problem $(1_0),(2_0)$ has no nontrivial solution. □

Corollary 2. *Let either conditions* (3), (4) *hold, or* $c_0 = 0$ *and conditions* (4) *and* (5) *be fulfilled. Let, moreover, there exist a measurable function* $f :]a,b[\to [0,+\infty[$ *such that along with* (41) *and* (42) *the inequality*

$$p(t) \ge -\frac{f(t)}{G_k(f)(t)} \text{ for almost all } t \in]a,b[\tag{54}$$

is satisfied. Then problem (1), (2) *has a unique solution.*

Proof. First, we note that by virtue of Lemma 1, the function g_k admits the estimate

$$g_k(t,s) \ge \gamma_k(s)(t-a)^2(b-t)^{m_k} \text{ for } a < t, s < b, \tag{55}$$

where $m_k = (1-k+|1-k|)/2$,

$$\gamma_0(s) = \frac{(s-a)(b-s)^2}{2(b-a)^4} > 0, \quad \gamma_k(s) = \frac{\Delta_k(s)-\Delta_0(s)}{2} > 0 \text{ for } a < s < b \ (k=1,2).$$

Put

$$w(t) = G_k(f)(t) \text{ for } a < t < b.$$

Then by conditions (41) and (55), we have

$$w(t) \ge \rho(t-a)^2(b-t)^{m_k} \text{ for } a < t < b, \tag{56}$$

where

$$\rho = \int_a^b \gamma_k(s)f(s)\,ds > 0.$$

Consequently, the function w satisfies inequalities (44). By Theorem 2, to prove Corollary 2 it suffices to establish that the function p satisfies inequality (45).

According to inequalities (42) and (54), there exists an integrable function $f_0 :]a,b[\to]0,+\infty[$ such that

$$\int_a^b f_0(t)\,dt > 0,$$

$$p_-(t) \le \frac{f(t)-f_0(t)}{w(t)} \text{ for almost all } t \in]a,b[\,.$$

If along with this we take into account inequalities (55) and (56), then we obtain

$$\int_a^b g_k(t,s)p_-(s)w(s)\,ds \le \int_a^b g_k(t,s)f(s)\,ds - \rho_0 w(t) \le w(t)(1-\rho_0) \ \text{ for } \ a<t<b_0,$$

where

$$\rho_0 = \frac{1}{\rho}\int_a^b \gamma_k(s)f_0(s)\,ds > 0.$$

Consequently, inequality (45) is fulfilled. □

Corollary 3. *Let the function q satisfy condition (4) and the function p admit the representation*

$$p(t) = \frac{\alpha f(t)}{G_k(f)(t)} \ \text{ for almost all } \ t \in]a,b[\,, \tag{57}$$

where α is a constant, $f :]a,b[\to [0,+\infty[$ is a measurable function, other than zero on the set of positive measure and $G_k(f)$ is the function given by equality (43). If, moreover, $\alpha > -1$ and the condition

$$\int_a^b \frac{(b-t)^{2-k-m_k}}{t-a}\,f(t)\,dt < +\infty \tag{58}$$

holds, or $\alpha > -1$, $c_0 = 0$ and the condition

$$\int_a^b (b-t)^{2-k-m_k} f(t)\,dt < +\infty \tag{59}$$

is fulfilled, where $m_k = (1-k+|1-k|)/2$, then problem (1), (2) has a unique solution. However, if $\alpha = -1$ and condition (59) is fulfilled, then problem (1), (2) either has no solution, or has an infinite set of solutions.

Proof. As it has been mentioned above, the function g_k admits estimate (55). Therefore,

$$G_k(f)(t) \ge \rho(t-a)^2(b-t)^{m_k} \ \text{ for } \ a<t<b,$$

where ρ is a positive constant. According to this estimate, it follows from (57) that

$$|p(t)| \le \frac{|\alpha| f(t)}{\rho(t-a)^2(b-t)^{m_k}} \ \text{ for } \ a<t<b.$$

Hence it is clear that if condition (58) (condition (59)) is fulfilled, then the function p satisfies condition (3) (condition (5)). On the other hand, in the case where $\alpha > -1$ ($\alpha = -1$), inequality (42) is fulfilled (is violated). If now we take into account Corollaries 1 and 2, then the validity of Corollary 3 becomes evident. □

Remark 2. Let $\alpha = -1$ ($\alpha = -1$, $c_0 = 0$) and along with (57) and (58) (along with (57) and (59)) condition (4) be fulfilled. If, moreover,

$$w(t) = G_k(f)(t) \ \text{ for } \ a<t<b,$$

then all the conditions of Theorem 2 are fulfilled with the exception of inequality (45) instead of which we have

$$\int_a^b \frac{g_k(t,s)}{w(t)}\,w(s)p_-(s)\,ds \le 1 \ \text{ for } \ a<t<b. \tag{60}$$

Consequently, condition (45) in Theorem 2 is unimprovable and it cannot be replaced by condition (60).

In the case where $f(t) \equiv 1$, from Corollary 2 it follows

Corollary 4. *Let either the condition*

$$\int_a^b (t-a)(b-t)^{2-k}\big(|p(t)|+|q(t)|\big)\,dt < +\infty \tag{61}$$

hold, or

$$c_0 = 0, \quad \int_a^b (b-t)^{2-k}\big((t-a)^2|p(t)|+(t-a)|q(t)|\big)\,dt < +\infty \tag{62}$$

be fulfilled. If, moreover,

$$p(t) \geq -\frac{6}{(t-a)^2(\delta_k - t + a)} \ \text{ for almost all } \ t \in]a,b[\,,$$
$$\operatorname{mes}\Big\{t \in]a,b[:\ (t-a)^2(\delta_k - t + a)p(t) \neq -6\Big\} > 0,$$

then problem (1), (2) *has a unique solution.*

For an arbitrarily fixed $i \in \{1,2,3\}$, from Corollary 4 we have the following proposition concerning the unique solvability of problem $(1), (2_i)$.

Corollary 4$_i$. *Let either condition* (61) *or condition* (62) *be fulfilled, where* $k = i-1$. *If, moreover,*

$$p(t) \geq -\frac{24-6i}{(t-a)^2((4-i)(b-t)+(i-1)(b-a))} \ \text{ for almost all } \ t \in]a,b[\,,$$
$$\operatorname{mes}\Big\{t \in]a,b[:\ (t-a)^2((4-i)(b-t)+(i-1)(b-a))p(t) > 24-6i\Big\} > 0,$$

then problem $(1), (2_i)$ *has a unique solution.*

Remark 3. According to Remark 1 and Corollary 1, conditions of Corollaries 4 and 4_i are in a certain sense unimprovable.

Corollary 5. *Let either condition* (61) *or condition* (62) *be fulfilled. If, moreover,*

$$\int_a^b (t-a)^2\Delta_k(t)p_-(t)\,dt < 2, \tag{63}$$

then problem (1), (2) *has a unique solution.*

Proof. By equality (8) and condition (10), we have

$$0 < g_k(t,s) \leq \frac{\Delta_k(s)}{2}\,(t-a)^2 \ \text{ for } \ a < t, s < b. \tag{64}$$

Let

$$w(t) \equiv (t-a)^2.$$

Then it is clear that inequalities (44) are fulfilled. On the other hand, inequalities (63) and (64) result in inequality (45). Consequently, all the conditions of Theorem 2 are fulfilled which guarantees the unique solvability of problem (1), (2). □

The above proven corollary for arbitrary $i \in \{1,2,3\}$ leads to the following

Corollary 5$_i$. *Let either condition* (61) *or condition* (62) *be fulfilled, where* $k = i-1$. *If, moreover,*

$$\int_a^b (t-a)(b-t)^{3-i}p_-(t)\,dt < 2(b-a)^{3-i},$$

then problem $(1), (2_i)$ *has a unique solution.*

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