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ORDINARY DIFFERENTIAL EQUATIONS

Nonlinear Nonlocal Problems for Second-Order Differential Equations Singular with Respect to the Phase Variable

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Abstract—For second-order differential equations singular with respect to the phase variable, we obtain in a sense optimal criteria for the existence and uniqueness of positive solutions of nonlinear nonlocal boundary value problems.

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1. STATEMENT OF THE MAIN RESULTS

Let $-\infty < a < b < +\infty$, let

$$I \subset \,]a, b[, \qquad \operatorname{mes} I = b - a,$$

and let $f : I \times]0, +\infty[\to \mathbb{R}_-$ be a function measurable in the first argument and continuous in the second argument. Consider the differential equation

$$u'' = f(t, u) \tag{1.1}$$

with boundary conditions of one of the following two forms:

$$u(a) = \varphi_1(u), \qquad u(b) = \varphi_2(u), \tag{1.2}$$

$$u(a) = \varphi_1(u), \qquad u'(b) = \varphi_2(u), \tag{1.3}$$

where the $\varphi_i : C([a, b]; \mathbb{R}_+) \to \mathbb{R}_+$ $(i = 1, 2)$ are continuous functionals bounded on each bounded subset of $C([a, b]; \mathbb{R}_+)$.

Such problems are of interest from both the purely theoretical and the practical viewpoint. This is especially true for the case in which Eq. (1.1) is singular with respect to the phase variable, that is, the case in which

$$\lim_{x \to 0} f(t, x) = -\infty \quad \text{for} \quad t \in I_0, \tag{1.4}$$

where $I_0 \subset I$ is a subset of positive measure. For example, the problem

$$u'' = -\frac{t^2}{32u^2}, \qquad u(0) = u(1) = 0$$

arises in the membrane and plate theory (see [1–3]), and the problem

$$u'' = -\frac{1-t}{u}, \qquad u(0) = u(1) = 0$$

arises in the boundary layer theory for a viscous incompressible fluid (see [4–6]).

Mainly, two-point boundary value problems were considered for Eq. (1.1) in the above-mentioned singular case (e.g., see [7–20] and the bibliography therein). Nonlinear nonlocal problems remain so far unstudied. The present paper is intended to fill the gap.

The theorems on the solvability and unique solvability of problem (1.1), (1.2) [respectively, problem (1.1), (1.3)] proved in the present paper cover the case in which the function f satisfies condition (1.4) and has nonintegrable singularities (respectively, a nonintegrable singularity) with respect to the first argument at the points a and b (respectively, at the point a).

We use the following notation:

$$f^*(t,x,y) = \max\{|f(t,s)| :\ x \le s \le y\} \quad \text{for} \quad t \in I, \quad 0 < x \le y < +\infty,$$

$\mathbb{R}_+ = [0,+\infty[$, $\mathbb{R}_- =]-\infty,0]$, $C([a,b];\mathbb{R})$ is the Banach space of continuous functions $u:\ [a,b] \to \mathbb{R}$ with norm $\|u\| = \max\{|u(t)| :\ a \le t \le b\}$, $C([a,b];\mathbb{R}_+)$ is the set of all nonnegative functions in $C([a,b];\mathbb{R})$, and $\widetilde{C}^1_{\mathrm{loc}}(]a,b[;\mathbb{R})$ is the space of continuously differentiable functions $u:]a,b[\to \mathbb{R}$ whose first derivative is absolutely continuous on each closed interval contained in $]a,b[$; if $u \in C([a,b];\mathbb{R})$, then

$$\|u\|_{[a,t]} = \max\{|u(s)| :\ a \le s \le t\} \quad \text{if} \quad a < t \le b,$$
$$\mu(u;t_1,t_2) = \min\{u(t) :\ t_1 \le t \le t_2\} \quad \text{if} \quad a \le t_1 < t_2 \le b.$$

We say that a function $g:\ I \times]0,+\infty[\to \mathbb{R}_+$ belongs to the set M_+ if the function $g(\cdot,x): I \to \mathbb{R}_+$ is measurable for each $x \in]0,+\infty[$ and the function $g(t,\cdot):]0,+\infty[\to \mathbb{R}_+$ is continuous and nonincreasing for each $t \in I$.

We say that a function $g:\ I \times]0,+\infty[\to \mathbb{R}_-$ belongs to the set M_- if $-g \in M_+$.

A function $u \in C([a,b];\mathbb{R}) \cap \widetilde{C}_{\mathrm{loc}}(]a,b[;\mathbb{R})$ is called a *positive solution* of the differential equation (1.1) if it is positive on the open interval $]a,b[$ and satisfies the equation almost everywhere on $]a,b[$.

A positive solution u of the differential equation (1.1) is called a positive solution of problem (1.1), (1.2) [respectively, a positive solution of problem (1.1), (1.3)] if relations (1.2) are satisfied [respectively, there exists a finite limit $u'(b) = \lim_{t\to b} u'(t)$ and relation (1.3) holds].

We assume that the function f satisfies the inequality

$$f(t,x) \le -p_0(t,x) \tag{1.5}$$

on $I \times]0,+\infty[$, where $p_0 \in M_+$. In addition, we study problem (1.1), (1.2) in the case where

$$0 < \int_a^b (s-a)(b-s)p_0(s,x)\,ds \le \int_a^b (s-a)(b-s)f^*(s,x,y)\,ds < +\infty \quad \text{for} \quad 0 < x \le y < +\infty \tag{1.6}$$

and problem (1.1), (1.3) in the case where

$$0 < \int_a^b (s-a)p_0(s,x)\,ds \le \int_0^b (s-a)f^*(s,x,y)\,ds < +\infty \quad \text{for} \quad 0 < x \le y < +\infty. \tag{1.7}$$

Along with problems (1.1), (1.2) and (1.1), (1.3), we need to consider the differential equation

$$u'' = (\lambda-1)p_0(t,u) + \lambda f(t,u) \tag{1.8}$$

depending on the parameter $\lambda \in [0,1]$ with boundary conditions of one of the following two forms:

$$u(a) = \lambda\varphi_1(u) + \delta, \qquad u(b) = \lambda\varphi_2(u) + \delta, \tag{1.9}$$
$$u(a) = \lambda\varphi_1(u) + \delta, \qquad u'(b) = \lambda\varphi_2(u). \tag{1.10}$$

Proposition 1.1 (the a priori boundedness principle). Let a function f satisfy inequality (1.5) on the set $I \times]0,+\infty[$, where $p_0 \in M_+$. In addition, suppose that condition (1.6) [respectively, condition (1.7)] is satisfied and there exist numbers $\delta_0 > 0$ and $\varrho > \delta_0$ such that, for arbitrary $\lambda \in [0,1]$ and $\delta \in]0,\delta_0]$, each solution of problem (1.8), (1.9) [respectively, problem (1.8), (1.10)] satisfies the estimate

$$\|u\| \le \varrho. \tag{1.11}$$

Then problem (1.1), (1.2) [respectively, problem (1.1), (1.3)] has at least one solution.

We use this proposition and a priori estimates for solutions of singular differential inequalities of second order in nonlocal boundary conditions [21] to obtain in a sense optimal criteria for the solvability and unique solvability of problems (1.1), (1.2) and (1.1), (1.3).

First, let us present the results for problem (1.1), (1.2).

Theorem 1.1. *Let the inequalities*

$$-(1+x)p_1(t,x) \le f(t,x) \le -p_0(t,x), \tag{1.12}$$

$$\varphi_i(u) \le \ell\|u\| + r \qquad (i=1,2) \tag{1.13}$$

hold on the sets $I \times]0,+\infty[$ *and* $C([a,b];\mathbb{R}_+)$, *respectively, where* $p_i \in M_+$ $(i=0,1)$, $\ell \in [0,1[$, *and* $r \ge 0$. *If, in addition,*

$$0 < \int_a^b (s-a)(b-s)p_i(s,x)\,ds < +\infty \quad \textit{for} \quad x>0 \quad (i=0,1), \tag{1.14}$$

$$\lim_{x\to+\infty} \int_a^b (s-a)(b-s)p_1(s,x)\,ds < (1-\ell)(b-a), \tag{1.15}$$

then problem (1.1), (1.2) *has at least one positive solution.*

Theorem 1.2. *Let the conditions*

$$-p_0(t,x) - p(t)(1+x) \le f(t,x) \le -p_0(t,x), \tag{1.16}$$

$$(f(t,x)-f(t,y))\operatorname{sgn}(x-y) \ge -p(t)|x-y| \tag{1.17}$$

be satisfied on the set $I \times]0,+\infty[$, *and let the conditions*

$$|\varphi_i(u)-\varphi_i(v)| \le \ell\|u-v\| \qquad (i=1,2) \tag{1.18}$$

be satisfied on the set $C([a,b];\mathbb{R}_+)$, *where* $p_0 \in M_+$, $p: I \to \mathbb{R}_+$ *is a measurable function, and* $\ell \in [0,1[$. *If, in addition,*

$$\begin{gathered} 0 < \int_a^b (s-a)(b-s)p_0(s,x)\,ds < +\infty \quad \textit{for} \quad x>0, \\ \int_a^b (s-a)(b-s)p(s)\,ds < (1-\ell)(b-a), \end{gathered} \tag{1.19}$$

then problem (1.1), (1.2) *has exactly one positive solution.*

Corollary 1.1. *Let*

$$f \in M_-, \quad 0 < \int_a^b (s-a)(b-s)|f(s,x)|\,ds < +\infty \quad \textit{for} \quad x>0, \tag{1.20}$$

and let conditions (1.13) [*respectively*, (1.18)] *be satisfied on the set* $C([a,b];\mathbb{R}_+)$, *where* $\ell \in [0,1[$ *and* $r \ge 0$. *Then problem* (1.1), (1.2) *has at least one* (*respectively*, *exactly one*) *positive solution.*

In this corollary, the inequality
$$\ell < 1 \tag{1.21}$$
is sharp. Moreover, the following assertion holds.

Corollary 1.2. *Let the function f satisfy condition* (1.20), *and let the functionals φ_i $(i = 1, 2)$ satisfy the inequalities*
$$\ell\mu(u; a_i, b_i) \le \varphi_i(u) \le \ell\|u\| + r \qquad (i = 1, 2) \tag{1.22}$$
on the set $C([a, b]; \mathbb{R}_+)$, where $a < a_i < b_i < b$ $(i = 1, 2)$ and ℓ and r are nonnegative constants. Then problem (1.1), (1.2) *has at least one positive solution if and only if inequality* (1.21) *is valid.*

Remark 1.1. For example, the functionals
$$\varphi_i(u) = \int\limits_{a_i}^{b_i} \psi_i(u(s))\, d\sigma_i(s) \qquad (i = 1, 2)$$
satisfy inequalities (1.22), where the $\psi_i :\ \mathbb{R}_+ \to \mathbb{R}_+$ $(i = 1, 2)$ are continuous functions and the $\sigma_i :\ [a_i, b_i] \to \mathbb{R}$ $(i = 1, 2)$ are nondecreasing functions such that
$$\ell x \le \psi_i(x) \le \ell x + r \quad \text{for} \quad x \in \mathbb{R}_+, \quad \sigma(b_i) - \sigma(a_i) = 1 \quad (i = 1, 2).$$

The following two theorems and their corollaries deal with problems (1.1), (1.2) as well but for the case in which conditions (1.13) and (1.18) are replaced by the conditions
$$\varphi_1(u) \le \ell\|u\| + r, \quad \varphi_2(u) \le \|u\|_{[a,b_0]}, \tag{1.23}$$
$$|\varphi_1(u) - \varphi_1(v)| \le \ell\|u - v\|, \quad |\varphi_2(u) - \varphi_2(v)| \le \|u - v\|_{[a,b_0]}, \quad \varphi_2(0) = 0, \tag{1.24}$$
respectively, where $b_0 \in\,]a, b[$.

Theorem 1.3. *Let inequalities* (1.12) *and* (1.13) *be satisfied on the sets $I \times\,]0, +\infty[$ and $C([a, b];\, \mathbb{R}_+)$, respectively, where the $p_i \in M_+$ $(i = 0, 1)$ are functions satisfying condition* (1.14), *$b_0 \in\,]a, b[$, $\ell \in [0, 1[$, and $r \ge 0$. If, in addition,*
$$\lim_{x\to+\infty} \int\limits_a^b (s - a)(b - s)p_1(s, x)\, ds < (1 - \ell)(b - b_0), \tag{1.25}$$
then problem (1.1), (1.2) *has at least one positive solution.*

Theorem 1.4. *Suppose that the function f satisfies condition* (1.20). *If, in addition, conditions* (1.23) [*respectively,* (1.24)], *where $\ell \in [0, 1[$ and $r \ge 0$, are satisfied on the set $C([a, b]; \mathbb{R}_+)$, then problem* (1.1), (1.2) *has at least one* (*respectively, exactly one*) *positive solution.*

Corollary 1.3. *Let the function f satisfy condition* (1.20), *and let the functionals φ_i $(i = 1, 2)$ satisfy the inequalities*
$$\ell\mu(u; a_1, b_1) \le \varphi_1(u) \le \ell\|u\| + r, \qquad \mu(u; a_2, b_2) \le \varphi_2(u) \le \|u\|_{[a,b_0]} \tag{1.26}$$
on the set $C([a, b]; \mathbb{R}_+)$, where $a < a_1 < b_1 < b$, $a < a_2 < b_2 \le b_0$, and ℓ and r are nonnegative constants. Then inequality (1.21) *is a necessary and sufficient condition for the existence of at least one positive solution of problem* (1.1), (1.2).

Remark 1.2. Inequalities (1.26) hold, for example, for the functionals

$$\varphi_1(u) = \int_{a_1}^{b_1} \psi(u(s))\, d\sigma_1(s), \qquad \varphi_2(u) = \int_{a_2}^{b_2} u(s)\, d\sigma_2(s),$$

where $\psi:\ \mathbb{R}_+ \to \mathbb{R}_+$ $(i = 1, 2)$ is a continuous function and the $\sigma_i:\ [a_i, b_i] \to \mathbb{R}_+$ $(i = 1, 2)$ are nondecreasing functions such that

$$\ell x \le \psi(x) \le \ell x + r \quad \text{for} \quad x \in \mathbb{R}_+, \quad \sigma_i(b_i) - \sigma_i(a_i) = 1 \quad (i = 1, 2).$$

Now let us proceed to the theorems on the solvability and unique solvability for problem (1.1), (1.3).

Theorem 1.5. *Let inequality* (1.12) *be satisfied on the set* $I \times]0, +\infty[$, *and let the inequality*

$$\varphi_1(u) + (b - a)\varphi_2(u) \le \ell\|u\| + r, \tag{1.27}$$

where $p_i \in M_+$ $(i = 0, 1)$, $\ell = [0, 1[$, *and* $r \ge 0$, *hold on the set* $C([a, b]; \mathbb{R}_+)$. *If, in addition,*

$$0 < \int_a^b (s - a)p_i(s, x)\, ds < +\infty \quad \textit{for} \quad x > 0 \quad (i = 0, 1), \tag{1.28}$$

$$\lim_{x\to+\infty} \int_a^b (s - a)p_1(s, x)\, ds < 1 - \ell, \tag{1.29}$$

then problem (1.1), (1.3) *has at least one positive solution.*

Theorem 1.6. *Let conditions* (1.16) *and* (1.17) *be satisfied on the set* $I \times]0, +\infty[$, *and let the condition*

$$|\varphi_1(u) - \varphi_1(v)| + (b - a)|\varphi_2(u) - \varphi_2(v)| \le \ell\|u - v\| \qquad (i = 1, 2), \tag{1.30}$$

where $p_0 \in M_+$, $p:\ I \to \mathbb{R}_+$ *is a measurable function, and* $\ell = [0, 1[$, *hold on the set* $C([a, b]; \mathbb{R}_+)$. *If, in addition,*

$$0 < \int_a^b (s - a)p_0(s, x)\, ds < +\infty \quad \textit{for} \quad x > 0, \quad \int_a^b (s - a)p(s)\, ds < 1 - \ell,$$

then problem (1.1), (1.3) *has exactly one positive solution.*

Corollary 1.4. *Let*

$$f \in M_-, \quad 0 < \int_a^b (s - a)|f(s, x)|\, ds < +\infty, \tag{1.31}$$

and let condition (1.27) [*respectively, condition* (1.30)], *where* $\ell \in [0, 1[$ *and* $r \ge 0$, *be satisfied on the set* $C([a, b]; \mathbb{R}_+)$. *Then problem* (1.1), (1.3) *has at least one* (*respectively, exactly one*) *positive solution.*

Corollary 1.5. *Let the function* f *satisfy condition* (1.31), *and let the functionals* φ_i $(i = 1, 2)$ *satisfy the inequalities*

$$\ell u(b) \le \varphi_1(u) + (b - a)\varphi_2(u) \le \ell u(b) + r$$

on the set $C([a, b]; \mathbb{R}_+)$, *where* ℓ *and* r *are nonnegative constants. Then inequality* (1.21) *is a necessary and sufficient condition for the existence of at least one positive solution of problem* (1.1), (1.3).

In conclusion of this section, by way of example, consider the differential equation

$$u'' = -\frac{p(t)}{h(u)} \tag{1.32}$$

with boundary conditions of one of the following three kinds:

$$u(a) = \sum_{i=1}^{m} \ell_{1k} u(t_k), \qquad u(b) = \sum_{i=1}^{m} \ell_{2k} u(t_k), \tag{1.33}$$

$$u(a) = \sum_{k=1}^{m} \ell_{1k} u(t_k), \qquad u'(b) = \sum_{k=1}^{m} \ell_{2k} u(t_k), \tag{1.34}$$

$$u(a) = \ell_1 u(b), \qquad u'(b) = \ell_2 u(b), \tag{1.35}$$

where $p: \ I \to \mathbb{R}_+$ is a measurable function that is nonzero on a set of positive measure, $h:\ ]0,+\infty[\ \to\ ]0,+\infty[$ is a continuous function, $a < t_1 < \cdots < t_m < b$, and ℓ_{ik} and ℓ_i ($i = 1, 2$; $k = 1, \dots, m$) are nonnegative numbers.

Note that singular boundary value problems arising in applications and considered in [1–6, 20] are special cases of problem (1.32), (1.33).

Theorems 1.3–1.6 imply the following assertions.

Corollary 1.6. *Let the function h satisfy the conditions*

$$\limsup_{x\to 0} h(x) < +\infty, \qquad \liminf_{x\to+\infty} h(x) > 0 \tag{1.36}$$

(*respectively, let h be a nondecreasing function*), *and let*

$$\sum_{k=1}^{m} \ell_{ik} < 1 \qquad (i = 1, 2). \tag{1.37}$$

Then the condition

$$\int_a^b (s-a)(b-s)p(s)\, ds < +\infty \tag{1.38}$$

is necessary and sufficient for the existence of at least one (*respectively, a unique*) *positive solution of problem* (1.32), (1.33).

Corollary 1.7. *Let the function h satisfy condition* (1.36) (*respectively, be nondecreasing*), *and suppose that*

$$\text{either} \quad \sum_{k=1}^{m} \ell_{1k} = \sum_{k=1}^{m} \ell_{2k} \quad \text{or} \quad \sum_{k=1}^{m} \ell_{2k} = 1.$$

Then the conditions

$$\sum_{k=1}^{m} \ell_{1k} < 1, \qquad \int_a^b (s-a)(b-s)p(s)\, ds < +\infty$$

are necessary and sufficient for the existence of at least one (*respectively, a unique*) *positive solution of problem* (1.32), (1.33).

Corollary 1.8. *Let the function h satisfy condition* (1.36) (*respectively, be nondecreasing*), *and let*

$$\sum_{k=1}^{m} (\ell_{1k} + (b-a)\ell_{2k}) < 1.$$

Then there exists at least one (*respectively, a unique*) *positive solution of problem* (1.32), (1.34) *if and only if*

$$\int_a^b (s-a)p(s)\,ds < +\infty.$$

Corollary 1.9. *If the function* h *satisfies conditions* (1.36) (*respectively, is nondecreasing*), *then the conditions*

$$\ell_1 + (b-a)\ell_2 < 1, \qquad \int_a^b (s-a)p(s)\,ds < +\infty$$

are necessary and sufficient for the existence of at least one (*respectively, a unique*) *solution of problem* (1.32), (1.35).

2. AUXILIARY ASSERTIONS

2.1. Lemmas on A Priori Estimates

Consider the differential inequalities

$$u''(t) \le -p_0(t, u(t)), \tag{2.1}$$
$$-(1+u(t))p_1(t, u(t)) \le u''(t) \le -p_0(t, u(t)), \tag{2.2}$$

where $p_i \in M_+$ $(i = 0, 1)$.

A function $u \in C([a,b];\mathbb{R}) \cap \widetilde{C}^1_{\mathrm{loc}}(]a,b[;\mathbb{R})$ is called a *positive solution* of the differential inequality (2.1) [respectively, the differential inequality (2.2)] if it is positive on the open interval $]a,b[$ and satisfies this differential inequality almost everywhere on $]a,b[$.

Lemma 2.1. *Suppose that*

$$p_0 \in M_+, \quad \operatorname{mes}\{t \in I :\ p_0(t,x) > 0\} > 0 \quad \textit{for} \quad x > 0, \tag{2.3}$$

and the differential inequality (2.1) *has a positive solution* u. *Then*

$$0 < \int_a^b (s-a)(b-s)p_0(s,x)\,ds < +\infty \quad \textit{for} \quad x \ge \|u\| \tag{2.4}$$

and

$$u(t) \ge u_0 + (t-a)(b-t)(b-a)^{-3} \int_a^b (s-a)(b-s)p_0(s,\|u\|)\,ds > u_0 \quad \textit{for} \quad a < t < b, \tag{2.5}$$

where $u_0 = \min\{u(a), u(b)\}$.

Proof. Let $a_k \in\,]a,b[$ and $b_k \in\,]a_k,b[$ $(k = 1, 2, \ldots)$ be some sequences satisfying the conditions

$$\lim_{k\to+\infty} a_k = a, \qquad \lim_{k\to+\infty} b_k = b.$$

By Lemma 2.2 in [21] and inequality (2.1), the function u can be estimated as

$$u(t) \geq u_{0k} + (t-a_k)(b_k-t)(b_k-a_k)^{-3}\int\limits_{a_k}^{b_k}(s-a_k)(b_k-s)|u''(s)|\,ds$$
$$\geq u_{0k} + (t-a_k)(b_k-t)(b_k-a_k)^{-3}\int\limits_{a_k}^{b_k}(s-a_k)(b_k-s)p_0(s,u(s))\,ds$$

on the interval $[a_k, b_k]$ for each k, where $u_{0k} = \min\{u(a_k), u(b_k)\}$. If we pass to the limit as $k \to +\infty$ in this inequality and use condition (2.3), then inequalities (2.4) and (2.5) become obvious. The proof of the lemma is complete.

The following assertion can be proved in a similar way.

Lemma 2.2. *Let condition* (2.3) *be satisfied, and let the differential inequality* (2.1) *have a positive solution u. If, in addition, there exists a finite limit*

$$u'(b) = \lim_{t\to b} u'(t),$$

then

$$0 < \int\limits_a^b (b-s)p_0(s,x)\,ds < +\infty \quad \text{for} \quad x \geq \|u\|$$

and

$$u(t) \geq u(a) + (t-a)u'(b) + \frac{t-a}{b-a}\int\limits_a^b (s-a)p_0(s,\|u\|)\,ds > u(a) + (t-a)u'(b) \quad \text{for} \quad a < t \leq b.$$

Lemmas 2.3–2.5 below deal with a priori estimates for positive solutions of the differential inequality (2.2) with boundary conditions of one of the following three forms:

$$u(a) \leq \ell\|u\| + r_0, \qquad u(b) \leq \ell\|u\| + r_0, \tag{2.6}$$
$$u(a) \leq \ell\|u\| + r_0, \qquad u(b) \leq \ell\|u\|_{[a,b_0]}, \tag{2.7}$$
$$u(a) + (b-a)u'(b) \leq \ell\|u\|, \qquad u'(b) \geq 0, \tag{2.8}$$

where $\ell \geq 0$, $r_0 \geq 0$, and $b_0 \in\,]a,b[$. The proof of these lemmas can be found in [21].

Lemma 2.3. *Let $p_i \in M_+$ $(i = 0,1)$, let $\ell < 1$, and let conditions* (1.14) *and* (1.15) *be satisfied. Then there exists a positive constant ϱ and continuous functions $\varepsilon_i : [a,b] \to [0,\varrho]$ $(i = 0,1,2)$ such that*

$$\varepsilon_1(a) = 0, \quad \varepsilon_2(b) = 0, \quad \varepsilon_0(t) > 0 \quad \text{for} \quad a < t < b, \tag{2.9}$$

and an arbitrary positive solution u of problem (2.2), (2.6) *satisfies the estimates*

$$\varepsilon_0(t) \leq u(t) \leq \varrho \quad \text{for} \quad a \leq t \leq b, \tag{2.10}$$
$$|u(t) - u(a)| \leq \varepsilon_1(t), \quad |u(t) - u(b)| \leq \varepsilon_2(t) \quad \text{for} \quad a \leq t \leq b. \tag{2.11}$$

Lemma 2.4. *Let $p_i \in M_+$ $(i = 0,1)$, let $\ell < 1$, and let conditions* (1.14) *and* (1.25) *be satisfied. Then there exists a positive constant ϱ and continuous functions $\varepsilon_i : [a,b] \to [0,\varrho]$ $(i = 0,1,2)$ satisfying conditions* (2.9) *such that an arbitrary positive solution u of problem* (2.2), (2.7) *satisfies the estimates* (2.10) *and* (2.11).

Lemma 2.5. *Let $p_i \in M_+$ $(i = 0, 1)$, let $\ell < 1$, and let conditions* (1.28) *and* (1.29) *be satisfied. Then there exists a positive constant ϱ and continuous functions $\varepsilon_i : [a, b] \to [0, \varrho]$ $(i = 0, 1)$ and $\varrho_1 : \,]a, b] \to \,]0, +\infty[$ such that*

$$\varepsilon_1(a) = 0, \quad \varepsilon_0(t) > 0 \quad \text{for} \quad a < t \le b, \tag{2.12}$$

and an arbitrary positive solution u of problem (2.2), (2.8) *satisfies the estimates*

$$\varepsilon_0(t) \le u(t) \le \varrho, \quad 0 \le u'(t) \le \varrho_1(t) \quad \text{for} \quad a < t \le b, \tag{2.13}$$
$$0 \le u(t) - u(a) \le \varepsilon_1(t) \quad \text{for} \quad a \le t \le b. \tag{2.14}$$

2.2. Lemmas on the Solvability of Nonlocal Problems for Equation (1.1)

Consider the differential equation (1.1) with boundary conditions of one of the following two forms:

$$u(a) = \varphi_1(u) + \delta, \quad u(b) = \varphi_2(u) + \delta, \tag{2.15}$$
$$u(a) = \varphi_1(u) + \delta, \quad u(b) = \varphi_2(u), \tag{2.16}$$

where δ is some positive constant.

Just as above, we assume that $f : I \times]0, +\infty[\to \mathbb{R}$ is a function measurable in the first argument and continuous in the second argument and the $\varphi_i : C([a, b]; \mathbb{R}_+) \to \mathbb{R}_+$ $(i = 1, 2)$ are continuous functionals bounded on each bounded subset of $C([a, b]; \mathbb{R}_+)$.

Lemma 2.6. *Let the function f satisfy inequality* (1.5), *where $p_0 \in M_+$, on the set $I \times]0, +\infty[$. In addition, suppose that condition* (1.6) *is satisfied and there exist numbers $\delta_0 > 0$ and $\varrho > \delta_0$ such that, for arbitrary $\lambda \in [0, 1]$ and $\delta \in \,]0, \delta_0]$, each solution of problem* (1.8), (1.9) *satisfies the estimate* (1.11). *Then there exist continuous functions $\varepsilon_i : [a, b] \to \mathbb{R}_+$ $(i = 0, 1, 2)$ satisfying conditions* (2.9) *such that, for each $\delta \in \,]0, \delta_0]$, problem* (1.1), (2.15) *has at least one positive solution satisfying the estimates* (2.10) *and* (2.11).

Proof. First, let us show that, for an arbitrarily fixed $\delta \in \,]0, \delta_0]$, problem (1.1), (2.15) has at least one positive solution.

Let

$$\chi(x) = \begin{cases} 1 & \text{for} \quad 0 \le x \le \varrho, \\ 1 - x/(2\varrho) & \text{for} \quad \varrho < x < 2\varrho, \\ 0 & \text{for} \quad x \ge 2\varrho. \end{cases} \tag{2.17}$$

For an arbitrary continuous function $u : [a, b] \to \,]0, +\infty[$, set

$$\widetilde{f}(u)(t) = (\chi(\|u\|) - 1)p_0(t, u(t)) + \chi(\|u\|)f(t, u(t)) \quad \text{for} \quad t \in I, \tag{2.18}$$
$$\widetilde{\varphi}_i(u) = \chi(\|u\|)\varphi_i(u) \quad (i = 1, 2) \tag{2.19}$$

and consider the auxiliary problem

$$u''(t) = \widetilde{f}(u)(t), \tag{2.20}$$
$$u(a) = \widetilde{\varphi}_1(u) + \delta, \qquad u(b) = \widetilde{\varphi}_2(u) + \delta. \tag{2.21}$$

By notation (2.17) and (2.18) and condition (1.5), each positive solution of the functional-differential equation (2.20) is simultaneously a solution of the differential inequality (2.1). This, together with Lemma 2.1 and condition (2.3), implies that if u is a positive solution of problem (2.20), (2.21), then

$$u(t) > \delta \quad \text{for} \quad a < t < b.$$

On the other hand, by conditions (1.5) and (1.6), for an arbitrary continuous function $u: [a,b] \to [\delta, +\infty[$ we have

$$0 \le -\widetilde{f}(u)(t) \le f^*(t;\delta,2\varrho) \quad \text{for} \quad t \in I, \tag{2.22}$$

$$\int_a^b (s-a)(b-s) f^*(s;\delta,2\varrho)\,ds < +\infty. \tag{2.23}$$

It follows from the preceding that the set of positive solutions of problem (2.20), (2.21) coincides with the set of positive solutions of the operator equation

$$u(t) = F(u)(t), \tag{2.24}$$

where

$$F(u)(t) = \delta + \frac{b-t}{b-a}\,\widetilde{\varphi}_1(u) + \frac{t-a}{b-a}\,\widetilde{\varphi}_2(u) + \int_a^b g_0(t,s)\widetilde{f}(u)(s)\,ds, \tag{2.25}$$

$$g_0(t,s) = \frac{1}{a-b}\begin{cases} (t-a)(b-s) & \text{for} \quad t \le s, \\ (s-a)(b-t) & \text{for} \quad t > s. \end{cases} \tag{2.26}$$

Let

$$r_0 = \delta_0 + \sup\{\varphi_1(u) + \varphi_2(u) :\ u \in C([a,b];\mathbb{R}_+),\ \|u\| \le 2\varrho\}, \tag{2.27}$$

$$\varrho_0 = r_0 + (b-a)^{-1}\int_a^b (s-a)(b-s) f^*(s;\delta,2\varrho)\,ds,$$

$$\varrho_1(t) = \frac{r_0}{b-a} + \int_a^t \frac{s-a}{b-a}\, f^*(s;\delta,2\varrho)\,ds + \int_t^b \frac{b-s}{b-a}\, f^*(s;\delta,2\varrho)\,ds \quad \text{for} \quad a < t < b.$$

By condition (2.23), it is obvious that $\varrho_1 :\ ]a,b[\to \mathbb{R}_+$ is a continuous function such that

$$\int_a^b \varrho_1(s)\,ds < +\infty. \tag{2.28}$$

By D we denote the set of functions $u \in C([a,b];\mathbb{R})$ satisfying the inequality

$$\delta \le u(t) \le \varrho_0 \quad \text{for} \quad a \le t \le b.$$

Let $u \in D$ and $v(t) = F(u)(t)$. Then v is a continuously differentiable function on the open interval $]a,b[$. On the other hand, by virtue of inequality (2.22) and notation (2.19) and (2.26), from relation (2.25), we obtain

$$\delta \le v(t) \le \delta_0 + \widetilde{\varphi}_1(u) + \widetilde{\varphi}_2(u) + (b-a)^{-1}\int_a^b (s-a)(b-s) f^*(s;\delta,2\varrho)\,ds \le \varrho_0 \quad \text{for} \quad a \le t \le b,$$

$$|v'(t)| \le \varrho_1(t) \quad \text{for} \quad a < t < b.$$

These estimates, together with condition (2.28) and the Arzelá–Ascoli lemma, imply that the operator F maps the set D into a compact subset of itself. By the Schauder theorem, the operator equation (2.24) and hence problem (2.20), (2.21) have a solution $u \in D$.

By virtue of relations (2.18) and (2.19), the function u is a positive solution of problem (1.8), (1.9), where $\lambda = \chi(\|u\|)$. By one of the assumptions of the lemma, the function u admits the estimate (1.11). This estimate, together with relation (2.17), implies that $\lambda = 1$. Consequently, u is a solution of problem (1.1), (2.15).

Let S_δ be the set of all solutions of problem (1.1), (2.15), and let $S = \bigcup_{0\le\delta\le\delta_0} S_\delta$. Obviously, each function $u \in S$ satisfies the estimate (1.11). If, in addition, we use condition (1.5), then we find that u is a positive solution of problem (2.2), (2.6), where

$$p_1(t,x) = \frac{1}{1+x}\begin{cases} f^*(t;x,\varrho) & \text{for} \quad 0 < x < \varrho, \\ f^*(t;\varrho,\varrho) & \text{for} \quad x \ge \varrho, \end{cases}$$

$p_i \in M_+$ $(i = 0,1)$, $\ell = 0$, and r_0 is the number given by relation (2.27). In addition, relations (1.14) and (1.15) follow from conditions (1.16), because

$$\lim_{x\to+\infty} \int_a^b (s-a)(b-s)p_1(s,x) = 0.$$

By Lemma 2.3, there exists a positive constant $\overline{\varrho}$ and continuous functions $\overline{\varepsilon}_i: \ [a,b] \to [0,\overline{\varrho}]$ $(i = 0,1,2)$ such that

$$\overline{\varepsilon}_1(a) = 0, \quad \overline{\varepsilon}_2(b) = 0, \quad \overline{\varepsilon}_0(t) > 0 \quad \text{for} \quad a < t < b$$

and an arbitrary positive solution of problem (2.2), (2.6) satisfies the estimates

$$\overline{\varepsilon}_0(t) \le u(t) \le \overline{\varrho} \quad \text{for} \quad a \le t \le b,$$
$$|u(t) - u(a)| \le \overline{\varepsilon}_1(t), \quad |u(t) - u(b)| \le \overline{\varepsilon}_2(t) \quad \text{for} \quad a \le t \le b.$$

Now if we set

$$\varepsilon_i(t) = \min\{\varrho, \overline{\varepsilon}_i(t)\} \quad \text{for} \quad a \le t \le b,$$

then we find that each function $u \in S$ satisfies the estimates (2.10) and (2.11), where the $\varepsilon_i: \ [a,b] \to \varrho$ $(i = 1,2)$ are continuous functions satisfying conditions (2.9). The proof of the lemma is complete.

Lemma 2.7. *Let the function f satisfy inequality* (1.5), *where $p_0 \in M_+$, on the set $I \times]0,+\infty[$. In addition, suppose that condition* (1.7) *is satisfied and there exist numbers $\delta_0 > 0$ and $\varrho > \delta_0$ such that, for arbitrary $\lambda \in [0,1]$ and $\delta \in]0,\delta_0]$, each solution u of problem* (1.8), (1.10) *satisfies the estimate* (2.11). *Then there exist continuous functions $\varepsilon_i: \ [a,b] \to \mathbb{R}_+$ $(i = 0,1)$ satisfying conditions* (2.12) *and a continuous function $\varrho_1: \ ]a,b] \to]0,+\infty[$ such that, for each $\delta \in]0,\delta_0]$, problem* (1.1), (2.16) *has at least one positive solution that admits the estimates* (2.13) *and* (2.14).

This lemma can be proved by analogy with Lemma 2.6 with the only difference that Lemma 2.5 rather than Lemma 2.3 is used in the proof.

2.3. Lemmas on the Unique Solvability of Nonlocal Problems for Differential Inequalities

Consider the differential inequality

$$w''(t)\,\mathrm{sgn}(w(t)) \ge -p(t)|w(t)| \tag{2.29}$$

with nonlinear nonlocal boundary conditions of one of the following two forms:

$$|w(a)| \le \ell\|w\|, \qquad |w(b)| \le \ell\|w\|, \tag{2.30}$$
$$|w(a)| + (b-a)|w'(b)| \le \ell\|w\|, \tag{2.31}$$

where $p: \ ]a,b[\to \mathbb{R}_+$ is a measurable function and $\ell \in [0,1[$.

We seek a solution of problem (2.29), (2.30) in the set $C([a,b];\mathbb{R}) \cap \widetilde{C}^1_{\mathrm{loc}}(]a,b[;\mathbb{R})$ and a solution of problem (2.29), (2.31) in the set of functions $w \in C([a,b];\mathbb{R}) \cap \widetilde{C}^1_{\mathrm{loc}}(]a,b[;\mathbb{R})$ whose derivative has a finite limit $w'(b) = \lim_{t\to b} w'(t)$.

Lemma 2.8. *If*

$$\int_a^b (s-a)(b-s)p(s)\,ds < (1-\ell)(b-a), \tag{2.32}$$

then problem (2.29), (2.30) *has only the trivial solution.*

Proof. By inequality (2.32), there exists a number $\ell_1 \in \,]\ell, 1[$ such that

$$\int_a^b (s-a)(b-s)p(s)\,ds < (1-\ell_1)(b-a). \tag{2.33}$$

Now assume that problem (2.29), (2.30) has a nontrivial solution w. Then, by virtue of inequalities (2.30), we can assume without loss of generality that the conditions

$$w(t_0) = \|w\|, \quad w(t) > 0 \quad \text{if} \quad a_1 < t < b_1, \quad 0 \le w(a_1) \le \ell_1\|w\|, \quad 0 \le w(b_1) \le \ell_1\|w\|$$

are satisfied for some $a_1 \in \,]a, b[$, $b_1 \in \,]a_1, b[$, and $t_0 \in \,]a_1, b_1[$. On the other hand, by using these conditions, from the differential inequality (2.29), we obtain

$$\begin{aligned} \|w\| = w(t_0) &= \frac{b_1 - t_0}{b_1 - a_1}\, w(a_1) + \frac{t_0 - a_1}{b_1 - a_1}\, w(b_1) + \int_{a_1}^{b_1} g_0(t_0, s) w''(s)\,ds \\ &\le \left(\ell_1 + \int_{a_1}^{b_1} |g_0(t_0, s)| p(s)\,ds\right)\|w\|, \end{aligned} \tag{2.34}$$

where

$$g_0(t, s) = \frac{1}{a_1 - b_1}\begin{cases} (t-a_1)(b_1-s) & \text{for} \quad a_1 \le t \le s \le b_1, \\ (s-a_1)(b_1-t) & \text{for} \quad a_1 \le s < t \le b_1 \end{cases}$$

and

$$|g_0(t, s)| \le (b-a)^{-1}(s-a)(b-s) \quad \text{for} \quad a_1 \le s, t \le b_1.$$

By virtue of this estimate and condition (2.32), it follows from inequality (2.34) that

$$\|w\| \le \left(\ell_1 + (b-a)^{-1}\int_{a_1}^{b_1} (s-a)(b-s)p(s)\,ds\right)\|w\| < \|w\|.$$

The contradiction thus obtained implies that problem (2.29), (2.30) has only the trivial solution. The proof of the lemma is complete.

Lemma 2.9. *If*

$$\int_a^b (s-a)p(s)\,ds < 1-\ell,$$

then problem (2.29), (2.31) *has only the trivial solution.*

Proof. Take $\ell_1 \in \,]0, \ell[$ such that

$$\int_a^b (s-a)p(s)\,ds < 1-\ell_1. \tag{2.35}$$

Now assume that problem (2.29), (2.31) has a nontrivial solution w. Then, by virtue of inequality (2.31), we can assume without loss of generality that the conditions

$$w(b_1) = \|w\|, \quad w(t) > 0 \quad \text{for} \quad a_1 < t \le b_1, \quad w(a_1) + (b_1 - a_1)w'(b_1) \le \ell_1\|w\|$$

are true for some $a_1 \in]a, b[$ and $b_1 \in]a_1, b]$. If, in addition, we use inequality (2.35), then, from the differential inequality (2.29), we obtain

$$\|w\| = w(b_1) = w(a_1) + (b_1 - a_1)w'(b_1) - \int\limits_{a_1}^{b_1}(s - a_1)w''(s)\,ds \le \left(\ell_1 + \int\limits_{a_1}^{b_1}(s - a)p(s)\,ds\right)\|w\| < \|w\|.$$

The contradiction thus obtained implies that problem (2.29), (2.31) has only the trivial solution. The proof of the lemma is complete.

In conclusion, consider the differential inequality

$$w''(t)\,\mathrm{sgn}(w(t)) \ge 0 \tag{2.36}$$

with the boundary conditions

$$|w(a)| \le \ell\|w\|, \qquad |w(b)| \le \|w\|_{[a,b_0]}, \tag{2.37}$$

where $\ell \in [0, 1[$ and $b_0 \in]a, b[$.

Lemma 2.10. *Problem* (2.36), (2.37) *has only the trivial solution.*

Proof. First, note that, by Lemmas 2.8 and 2.9, for an arbitrary $t_0 \in]a, b]$, the differential inequality (2.36) on the interval $[a, t_0]$ does not have a nontrivial solution such that either

$$|w(a)| \le \ell\|w\|, \qquad w(t_0) = 0$$

or

$$|w(a)| \le \ell\|w\|, \qquad w'(t_0) = 0.$$

Now assume that problem (2.36), (2.37) has a nontrivial solution w. Then, by the preceding, either

$$w(t)w'(t) \ne 0 \quad \text{for} \quad a < t \le b, \tag{2.38}$$

or there exists an $a_1 \in]a, b[$ such that

$$w(t) = 0 \quad \text{for} \quad a \le t \le a_1, \quad w(t)w'(t) > 0 \quad \text{for} \quad a_1 < t \le b. \tag{2.39}$$

However, both condition (2.38) and condition (2.39) contradict inequality (2.37). The contradiction thus obtained completes the proof of the lemma.

3. PROOF OF THE MAIN RESULTS

Proof of Proposition 1.1. By Lemma 2.6, for each positive integer k, the differential equation (1.1) has a positive solution u_k satisfying the conditions

$$u_k(a) = \varphi_1(u_k) + \frac{\delta_0}{k}, \quad u_k(b) = \varphi_2(u_k) + \frac{\delta_0}{k}, \tag{3.1}$$

$$\varepsilon_0(t) \le u_k(t) \le \varrho \quad \text{for} \quad a \le t \le b, \tag{3.2}$$

$$|u_k(t) - u_k(a)| \le \varepsilon_0(t), \quad |u_k(t) - u_k(b)| \le \varepsilon_1(t) \quad \text{for} \quad a \le t \le b, \tag{3.3}$$

where the $\varepsilon_i : \; [a, b] \to [0, \varrho]$ $(i = 0, 1, 2)$ are continuous functions independent of k and satisfying conditions (2.9).

Let

$$\varrho_0(t) = f^*(t;\varepsilon_0(t),\varrho) \quad \text{for} \quad t \in I.$$

By conditions (1.6) and (2.9), the function ϱ_0 is Lebesgue integrable on each closed interval contained in $]a,b[$. Take arbitrary numbers $a_0 \in \,]a,b[$ and $b_0 \in \,]a_0,b[$ and introduce the function

$$\varrho_1(t) = \frac{\varrho}{b_0 - a_0} + \int\limits_{a_0}^{b_0} \varrho_0(s)\,ds + \left|\int\limits_{a_0}^{t} \varrho_0(s)\,ds\right| \quad \text{for} \quad a < t < b.$$

By virtue of the estimate (3.2), we have

$$|u_k''(t)| = |f(t,u_k(t))| \le \varrho_0(t) \quad \text{for almost all} \quad t \in \,]a,b[, \tag{3.4}$$

$$|u_k'(t)| \le \varrho_1(t) \quad \text{for} \quad a < t < b. \tag{3.5}$$

The estimates (3.2), (3.3), and (3.5) ensure the uniform boundedness and equicontinuity of the sequence $(u_k)_{k=1}^{+\infty}$ on $[a,b]$, and the estimates (3.4) and (3.5) guarantee the uniform boundedness and equicontinuity of the sequence $(u_k')_{k=1}^{+\infty}$ on each closed interval contained in $]a,b[$.

By the Arzelá–Ascoli lemma, the sequence $(u_k)_{k=1}^{+\infty}$ contains a subsequence $(u_{k_m})_{m=1}^{+\infty}$ uniformly converging on $[a,b]$ and such that $(u_{k_m}')_{m=1}^{+\infty}$ uniformly converges on each closed interval contained in $]a,b[$. Set

$$u(t) = \lim_{m\to+\infty} u_{k_m}(t) \quad \text{for} \quad a \le t \le b.$$

Then

$$u'(t) = \lim_{m\to+\infty} u_{k_m}'(t) \quad \text{for} \quad a < t < b.$$

If we pass in the relation

$$u_{k_m}'(t) = u_{k_m}'(a_0) + \int\limits_{a_0}^{t} f(s,u_{k_m}(s))\,ds \quad \text{for} \quad a < t < b$$

to the limit as $m \to +\infty$, then, by using the dominated convergence theorem and condition (3.4), we obtain

$$u'(t) = u'(a_0) + \int\limits_{a_0}^{t} f(s,u(s))\,ds \quad \text{for} \quad a < t < b.$$

On the other hand, it follows from relation (3.1) and inequality (3.2) that the function u satisfies the boundary conditions (1.2) and the conditions

$$\varepsilon_0(t) \le u(t) \le \varrho \quad \text{for} \quad a \le t \le b.$$

Consequently, u is a positive solution of problem (1.1), (1.2).

If we use Lemma 2.7 instead of Lemma 2.6, then, in a similar way, we can show that problem (1.1), (1.3) has at least one solution. The proof of the proposition is complete.

Proof of Theorem 1.1. First, note that, by conditions (1.12) and (1.14), the function f^* admits the estimate

$$p_0(t,x) \le f^*(t,x,y) \le p_1(t,x)(1+y) \quad \text{for} \quad t \in I, \quad 0 < x \le y < +\infty$$

and satisfies condition (1.6).

Let $r_0 = r+1$, let ϱ be the positive constant introduced in Lemma 2.3, and let u be a positive solution of problem (1.8), (1.9) for some $\lambda \in [0,1]$ and $\delta \in \,]0,1]$. By inequalities (1.12) and (1.13), the function u also is a positive solution of problem (2.2), (2.6) and satisfies the estimate (1.11) by Lemma 2.3.

Now if we use Proposition 1.1, then Theorem 1.1 becomes obvious. The proof of the theorem is complete.

Proof of Theorem 1.2. Conditions (1.16), (1.18), and (1.19) imply conditions (1.12)–(1.15), where $r = \max\{\varphi_1(0), \varphi_2(0)\}$ and $p_1(t,x) = p(t) + p_0(t,x)(1+x)^{-1}$. On the other hand, by Theorem 1.1, these conditions ensure the existence of a positive solution u of problem (1.1), (1.2).

It remains to show that problem (1.1), (1.2) has no solution other than u.

Let v be an arbitrary positive solution of problem (1.1), (1.2), and let

$$w(t) = u(t) - v(t) \quad \text{for} \quad a \le t \le b. \tag{3.6}$$

Then, by conditions (1.17) and (1.18), the function w is a solution of problem (2.29), (2.30). However, by Lemma 2.8 and condition (2.32), this problem has only the trivial solution. Consequently, $v(t) \equiv u(t)$. The proof of the theorem is complete.

Proof of Corollary 1.1. Condition (1.20) implies conditions (1.12), (1.14), (1.16), and (1.17), where

$$p(t) = 0, \quad p_0(t,x) = |f(t,x)|, \quad p_1(t,x) = (1+x)^{-1}|f(t,x)| \qquad \text{for} \quad t \in I, \quad x > 0.$$

On the other hand,

$$p_i \in M_+ \quad (i = 0,1), \quad \lim_{x \to +\infty} \int_a^b (s-a)(b-s)p_1(s,x)\,ds = 0.$$

Now if condition (1.13) [respectively, (1.18)] is satisfied on the set $C([a,b];\mathbb{R}_+)$, then, by Theorem 1.1 (respectively, Theorem 1.2), problem (1.1), (1.2) has at least one (respectively, exactly one) positive solution. The proof of the corollary is complete.

Proof of Corollary 1.2. If inequality (1.21) holds, then, by Corollary 1.1, problem (1.1), (1.2) has at least one positive solution.

It remains to show that inequality (1.21) is also necessary for the existence of at least one positive solution of problem (1.1), (1.2). Indeed, if there exists such a solution, then, by Lemma 2.1,

$$\mu(u; a_i, b_i) > \min\{\varphi_1(u), \varphi_2(u)\} \qquad (i = 1,2).$$

This, together with inequalities (1.22), implies inequality (1.21). The proof of the corollary is complete.

The proof of Theorem 1.3 (respectively, Theorem 1.5) can be carried out by analogy with that of Theorem 1.1. The only difference is that Lemma 2.4 (respectively, Lemma 2.5) is used instead of Lemma 2.3.

Theorem 1.4 follows from Theorem 1.3 and Lemma 2.10, and Theorem 1.6 follows from Theorem 1.5 and Lemma 2.9.

Corollary 1.3 follows from Theorem 1.4 and Lemma 2.1, Corollary 1.4 follows from Theorems 1.5 and 1.6, and Corollary 1.5 follows from Corollary 1.4 and Lemma 2.2.

Proof of Corollary 1.6. Set

$$f(t,x) = -\frac{p(t)}{h(x)} \quad \text{for} \quad t \in I, \quad x > 0, \quad \varphi_i(u) = \sum_{k=1}^{m} \ell_{ik} u(t_k) \quad \text{for} \quad u \in C([a,b];\mathbb{R}_+) \quad (i = 1,2).$$

Then problem (1.32), (1.33) acquires the form (1.1), (1.2).

First, consider the case in which the function h satisfies condition (1.36). In this case, there exist nondecreasing functions $h_i : \;]0,+\infty[\; \to \;]0,+\infty[$ $(i = 0,1)$ such that

$$h_1(x) \le h(x) \le h_0(x) \quad \text{for} \quad x > 0.$$

By the preceding, the function f and the functionals φ_i $(i = 1, 2)$ satisfy inequalities (1.12) and (1.13) on the sets $I \times]0, +\infty[$ and $C([a, b]; \mathbb{R}_+)$, respectively, where

$$p_0(t, x) = \frac{p(t)}{h_0(x)}, \qquad p_1(t, x) = (1 + x)^{-1} \frac{p(t)}{h_1(x)}, \qquad \ell = \max\left\{\sum_{k=1}^{m} \ell_{ik} :\ i = 1, 2\right\}, \quad r = 0.$$

Moreover, $p_i \in M_+$, and condition (2.3) holds.

Each positive solution of problem (1.32), (1.33) is simultaneously a solution of the differential inequality (2.1). This, together with Lemma 2.1, implies that condition (1.38) is necessary for the existence of at least one solution of this form. On the other hand, if, in addition to (1.36) and (1.37), condition (1.38) is satisfied, then the functions p_i $(i = 0, 1)$ satisfy conditions (1.14) and (1.15), and problem (1.1), (1.2) has at least one positive solution by Theorem 1.1.

Let us proceed to the case in which h is a nondecreasing function. Then $f \in M_-$, and, by Corollary 1.1, conditions (1.37) and (1.38) ensure the existence of a unique positive solution of problem (1.32), (1.33). The proof of the corollary is complete.

We omit the proof of Corollaries 1.7–1.9, because it is similar to that of Corollary 1.6. Note only that Corollary 1.7 is proved on the basis of Theorems 1.3 and 1.4 and Lemma 2.1, and Corollaries 1.8 and 1.9 are proved on the basis of Theorem 1.5, Corollary 1.4, and Lemma 2.2.

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