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ORDINARY DIFFERENTIAL EQUATIONS

On a Resonance Periodic Problem for Nonautonomous Higher-Order Differential Equations

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Abstract—For nonlinear nonautonomous higher-order ordinary differential equations, we prove in a sense optimal criteria for the solvability and unique solvability of a resonance periodic problem.

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1. STATEMENT OF THE MAIN RESULTS

In the present paper, for differential equations of the form

$$u^{(n)} = f(t,u) + f_0(t), \tag{1.1}$$

we consider the problem on the existence of a periodic solution with a given period $\omega > 0$. Here $n \geq 1$, f is a function satisfying the local Carathéodory conditions,

$$f(t,0) = 0, \qquad f(t+\omega, x) = f(t,x) \quad \text{for} \quad (t,x) \in R^2, \quad \text{and} \quad f_0 \in L_\omega. \tag{1.2}$$

We are mainly interested in the little-studied case

$$\liminf_{|x|\to+\infty} \left| \frac{f(t,x)}{x} \right| = 0.$$

In this case, our periodic problem is a resonance problem, because the linear homogeneous differential equation $u^{(n)} = 0$ corresponding to (1.1) has infinitely many ω-periodic solutions.

Throughout the following, L_ω (respectively, L_ω^∞) stands for the space of ω-periodic real functions Lebesgue integrable (respectively, essentially bounded and measurable) on $[0,\omega]$. For arbitrary $p_i \in L_\omega$ $(i = 1, 2)$, we write $p_1(t) \not\equiv p_2(t)$ to indicate that the functions p_1 and p_2 differ from each other on a set of positive measure.

Theorem 1.1. *Let*

$$n = 2m-1, \qquad \sigma \in \{-1, 1\}$$

[*or* $n = 2m$ *and* $\sigma = (-1)^{m-1}$]. *Suppose that there exists a positive constant* r *and a function* $g \in L_\omega$ *such that*

$$\sigma f(t,x)\operatorname{sgn} x \geq g(t) \quad \textit{for} \quad (t,x) \in R^2, \quad |x| > r, \tag{1.3}$$

$$\left| \int_0^\omega f_0(t)dt \right| \leq \int_0^\omega g(t)dt. \tag{1.4}$$

Then Eq. (1.1) *has at least one ω-periodic solution.*

Theorem 1.2. *Let* $n = 2m - 1$ *and* $\sigma \in \{-1, 1\}$ [*or* $n = 2m$ *and* $\sigma = (-1)^{m-1}$]. *Suppose that there exists a function* $\ell \in L_\omega$ *and a continuous function* $h : R^2 \to [0, +\infty[$ *such that* $\ell(t) > 0$ *for* $t \in R$, $h(x, y) > 0$ *for* $x \neq y$, *and*

$$\sigma(f(t, x) - f(t, y)) \operatorname{sgn}(x - y) \geq \ell(t)h(x, y) \quad \text{for} \quad (t, x, y) \in R^3. \tag{1.5}$$

In addition, suppose that inequality (1.4) *with*

$$g(t) = \min\{|f(t, r)|, |f(t, -r)|\} \tag{1.6}$$

holds for some $r > 0$. *Then Eq.* (1.1) *has a unique* ω*-periodic solution.*

In forthcoming considerations, we need the following notion.

Definition 1.1. We say that a nonnegative function $p \in L_\omega$ belongs to the set $\mathcal{K}_\omega^m$ if $p(t) \not\equiv 0$ and for an arbitrary function $p_0 \in L_\omega$ satisfying the conditions

$$0 \leq p_0(t) \leq p(t) \quad \text{for} \quad t \in R, \quad p_0(t) \not\equiv 0 \tag{1.7}$$

the differential equation

$$u^{(2m)} = (-1)^m p_0(t)u \tag{1.8}$$

does not have nontrivial ω-periodic solutions.

Theorem 1.3. *Suppose that* $n = 2m$ *and there exists a positive constant* r *and functions* $g, p, q \in L_\omega$ *such that, along with* (1.4), *the conditions*

$$p \in \mathcal{K}_\omega^m, \tag{1.9}$$

$$g(t) \leq (-1)^m f(t, x) \operatorname{sgn} x \leq p(t)|x| + q(t) \quad \text{for} \quad (t, x) \in R^2, \quad |x| > r, \tag{1.10}$$

are satisfied. Then Eq. (1.1) *has at least one* ω*-periodic solution.*

Corollary 1.1. *Suppose that* $n = 2m$ *and there exists a positive constant* r *and functions* $g, p, q \in L_\omega$ *such that, along with* (1.4), *inequality* (1.10) *holds. In addition, let* p *and* m *satisfy one of the following three conditions*:

$$p(t) \leq \left(\frac{2\pi}{\omega}\right)^{2m} \quad \text{for} \quad t \in R, \qquad p(t) \not\equiv \left(\frac{2\pi}{\omega}\right)^{2m}, \tag{1.11}$$

$$\int_0^\omega p(t)dt \leq \frac{4}{\omega}\left(\frac{2\pi}{\omega}\right)^{2m-2}, \tag{1.12}$$

$$m = 1, \qquad \int_0^\omega p(t)dt \leq \frac{16}{\omega}. \tag{1.13}$$

Then Eq. (1.1) *has at least one* ω*-periodic solution.*

Theorem 1.4. *Suppose that* $n = 2m$ *and there exist functions* $\ell \in L_\omega$ *and* $p \in \mathcal{K}_\omega^m$ *and a continuous function* $h : R^2 \to [0, +\infty[$ *such that* $\ell(t) \geq 0$ *for* $t \in R$, $\ell(t) \not\equiv 0$, $h(x, y) > 0$ *for* $x \neq y$, *and*

$$\ell(t)h(x, y) \leq (-1)^m (f(t, x) - f(t, y)) \operatorname{sgn}(x - y) \leq p(t)|x - y| \quad \text{for} \quad (t, x, y) \in R^3. \tag{1.14}$$

In addition, suppose that inequality (1.4) *holds for some* $r > 0$, *where* g *is the function defined in* (1.6). *Then Eq.* (1.1) *has a unique* ω*-periodic solution.*

Corollary 1.2. *Suppose that* $n = 2m$ *and there exists a positive constant* r, *nonnegative functions* $p \in L_\omega$ *and* $\ell \in L_\omega$, *and a continuous function* $h : R^2 \to [0, +\infty[$ *such that* $\ell(t) \not\equiv 0$, $h(x, y) > 0$ *for* $x \neq y$, *and conditions* (1.4) *and* (1.14) *are satisfied, where* g *is the function defined in* (1.6). *If, in addition,* m *and* p *satisfy one of conditions* (1.11), (1.12), *and* (1.13), *then Eq.* (1.1) *has a unique* ω-*periodic solution.*

By way of example, consider the differential equations

$$u^{(n)} = \sigma f_1(t) \exp\left(f_2(t)|u|^\nu\right) |\sin u| u + f_0(t), \tag{1.15}$$

$$u^{(n)} = \sigma f_1(t) \exp\left(f_2(t)|u|^\nu\right) u + f_0(t), \tag{1.16}$$

$$u^{(n)} = \sigma f_1(t) \left(1 + |u|^\nu\right)^{-1} |u|^\nu \operatorname{sgn} u + f_0(t), \tag{1.17}$$

where $\sigma \in \{-1, 1\}$, $\nu > 0$, $f_i \in L_\omega$ $(i = 0, 1)$, $f_1(t) \geq 0$ for $t \in R$, and $f_2 \in L_\omega^\infty$. By Theorem 1.1 and Corollary 1.1, Eq. (1.15) has at least one ω-periodic solution provided that $\int_0^\omega f_0(t)dt = 0$ and either one of the conditions

$$n = 2m - 1, \quad \sigma \in \{-1, 1\}, \tag{1.18}$$

$$n = 2m, \qquad \sigma = (-1)^{m-1} \tag{1.19}$$

is satisfied or $n = 2m$, $\sigma = (-1)^m$, $f_2(t) \leq 0$ for $t \in R$, and

$$\lim_{x \to +\infty} \int_0^\omega f_1(s) \exp\left(x f_2(s)\right) ds < \frac{4}{\omega} \left(\frac{2\pi}{\omega}\right)^{2m-2}.$$

By Theorem 1.2 and Corollary 1.2, we have the following assertions.

1. If $f_1(t) > 0$, $f_2(t) \geq 0$ for $t \in R$, and one of conditions (1.18) and (1.19) is satisfied, then Eq. (1.16) has a unique ω-periodic solution.

2. If $f_1(t) > 0$ for $t \in R$, $\left|\int_0^\omega f_0(s)ds\right| < \int_0^\omega f_1(s)ds$, and either one of conditions (1.18) and (1.19) is satisfied or $n = 2m$, $\sigma = (-1)^m$, $\nu \geq 1$, and $\nu f_1(t) < (2\pi/\omega)^n$ for $t \in R$, then Eq. (1.17) has a unique ω-periodic solution.

These facts do not follow from earlier-known theorems on the existence and uniqueness of ω-periodic solutions of Eq. (1.1) (see [1–15] and the bibliography therein).

Remark 1.1. In Theorems 1.1–1.4 and Corollaries 1.1 and 1.2, inequality (1.4) is sharp in the sense that it cannot be replaced by the inequality

$$\left| \int_0^\omega f_0(t)dt \right| \leq (1 + \varepsilon) \int_a^b g(t)dt \tag{1.20}$$

with a given (however small) $\varepsilon > 0$. Indeed, if

$$f_0(t) = (1 + \varepsilon)g(t), \qquad f(t, x) = \sigma(1 + \varepsilon)g(t)(1 + |x|)^{-1}x,$$

where $\varepsilon > 0$, $\sigma \in \{-1, 1\}$, $g \in L_\omega$, and $g(t) > 0$ for $t \in R$, then Eq. (1.1) has no ω-periodic solutions. On the other hand, if $n \in \{2m - 1, 2m\}$ and $\sigma = (-1)^{m-1}$ [respectively, $n = 2m$, $\sigma = (-1)^m$, and $(1+\varepsilon)g(t) < (2\pi/\omega)^{2m}$ for $t \in R$], then all assumptions of Theorem 1.2 (respectively, Corollary 1.2) are satisfied except for inequality (1.4), which is replaced by (1.20).

Remark 1.2. In Corollaries 1.1 and 1.2, condition (1.11) cannot be replaced by the condition $p(t) \equiv (2\pi/\omega)^{2m}$, because if $n = 2m$ and

$$f(t, x) \equiv (-1)^m \left(\frac{2\pi}{\omega}\right)^{2m} x \quad \text{and} \quad \int_0^\omega f_0(t) \sin \frac{2\pi t}{\omega} dt \neq 0,$$

then Eq. (1.1) has no ω-periodic solutions.

2. AUXILIARY ASSERTIONS

By C_ω (respectively, C_ω^{n-1}) we denote the space of continuous (respectively, $n-1$ times continuously differentiable) ω-periodic functions $u: R \to R$ with the norm

$$\|u\|_{C_\omega} = \max\{\|u(t)\| : \ 0 \le t \le \omega\} \qquad \left(\|u\|_{C_\omega^{n-1}} = \sum_{k=1}^{n} \left\|u^{(k-1)}\right\|_{C_\omega}\right),$$

and by $\tilde{C}_\omega^{n-1}$ we denote the space of functions $u \in C_\omega^{n-1}$ such that $u^{(n-1)}$ is an absolutely continuous function.

For $v \in L_\omega$, we introduce the norm

$$\|v\|_{L_\omega} = \int_0^\omega |v(t)| dt.$$

For arbitrary $u \in C_\omega$, we set $\mu(u) = \min\{|u(t)| : \ 0 \le t \le \omega\}$.

Let us introduce the numbers

$$\alpha_{nk} = \frac{\omega}{4}\left(\frac{\omega}{2\pi}\right)^{n-1-k} \quad (k = 1, \dots, n-1), \qquad \alpha_{nn} = \frac{1}{2}, \qquad \alpha_n = \sum_{k=1}^{n} \alpha_{nk}. \tag{2.1}$$

Lemma 2.1. *If $u \in \tilde{C}_\omega^{n-1}$, then*

$$\|u\|_{C_\omega} \le \mu(u) + \alpha_{n1}\left\|u^{(n)}\right\|_{L_\omega}, \qquad \|u\|_{C_\omega^{n-1}} \le \mu(u) + \alpha_n\left\|u^{(n)}\right\|_{L_\omega}. \tag{2.2}$$

If, in addition, u is not of constant sign, then

$$\|u\|_{C_\omega} < \alpha_{n1}\left\|u^{(n)}\right\|_{L_\omega}, \qquad \|u\|_{C_\omega^{n-1}} < \alpha_n\left\|u^{(n)}\right\|_{L_\omega}. \tag{2.3}$$

Proof. Take $t_0 \in [0, \omega]$ and $t^* \in]t_0, t_0 + \omega[$ so as to ensure that $\mu(u) = |u(t_0)|$ and $\|u\|_{C_\omega} = \|u(t^*)\|$. Then

$$\|u\|_{C_\omega} = \left|u(t_0) + \int_{t_0}^{t^*} u'(t)dt\right| \le \mu(u) + \int_{t_0}^{t^*} |u'(t)|\, dt,$$

$$\|u\|_{C_\omega} = \left|u(t_0) - \int_{t^*}^{t_0+\omega} u'(t)dt\right| \le \mu(u) + \int_{t^*}^{t_0+\omega} |u'(t)|\, dt.$$

Therefore,

$$2\|u\|_{C_\omega} \le 2\mu(u) + \int_{t_0}^{t_0+\omega} |u'(t)|\, dtz,$$

and consequently,

$$\|u\|_{C_\omega} \le \mu(u) + \frac{1}{2}\|u'\|_{L_\omega}. \tag{2.4}$$

This inequality for the function $u^{(n-1)}$ acquires the form

$$\left\|u^{(n-1)}\right\|_{C_\omega} \le \frac{1}{2}\left\|u^{(n)}\right\|_{C_\omega}, \tag{2.5}$$

because $\mu\left(u^{(n-1)}\right) = 0$.

If $n > 1$, then, by the Wirtinger theorem [16, Th. 258] and inequality (2.5), we have

$$\|u'\|_{L_\omega} \le \omega^{1/2}\left(\int_0^\omega |u'(t)|^2\,dt\right)^{1/2} \le \omega^{1/2}\left(\frac{\omega}{2\pi}\right)^{n-2}\left(\int_0^\omega \left|u^{(n-1)}(t)\right|^2 dt\right)^{1/2}$$
$$\le \omega\left(\frac{\omega}{2\pi}\right)^{n-2}\left\|u^{(n-1)}\right\|_{C_\omega} \le 2\alpha_{n1}\left\|u^{(n)}\right\|_{L_\omega}, \qquad (2.6)$$

which, together with inequality (2.4), implies that

$$\|u\|_{C_\omega} \le \mu(u) + \alpha_{n1}\left\|u^{(n)}\right\|_{L_\omega}.$$

If we apply this inequality to the functions $u^{(k-1)}$ $(k = 2, \ldots, n)$, then we obtain

$$\left\|u^{(k-1)}\right\|_{C_\omega} \le \alpha_{nk}\left\|u^{(n)}\right\|_{L_\omega} \qquad (k = 2, \ldots, n). \qquad (2.7)$$

We have thereby proved inequalities (2.2), where α_{nk} $(k = 1, \ldots, n)$ and α_n are the numbers defined in (2.1).

Let us proceed to the analysis of the case in which u is not of constant sign. Then there exist $t_0 \in [0, \omega]$, $t^* \in\,]t_0, t_0 + \omega[$, and $\sigma \in \{-1, 1\}$ such that

$$u(t_0) = 0, \qquad \|u\|_{C_\omega} = \sigma\int_{t_0}^{t^*} u'(s)ds, \qquad \|u\|_{C_\omega} = -\sigma\int_{t^*}^{t_0+\omega} u'(s)ds.$$

It follows from these relations that either

$$\|u\|_{C_\omega} < \frac{1}{2}\|u'\|_{L_\omega} \qquad (2.8)$$

or

$$\sigma u'(t)(t^* - t) \ge 0 \quad \text{for almost all} \quad t \in [t_0, t_0 + \omega].$$

However, the latter inequality cannot be true, because $u(t_0) = u(t_0 + \omega) = 0$ and u is not of constant sign. Consequently, inequality (2.8) holds. On the other hand, as was shown above, if $n > 1$, then the function u satisfies inequalities (2.6) and (2.7). Inequalities (2.6)–(2.8) imply (2.3). The proof of the lemma is complete.

Lemma 2.2. *Suppose that $p \in L_\omega$, $p(t) \ge 0$ for $t \in R$, $p(t) \not\equiv 0$, and one of conditions* (1.11), (1.12), *and* (1.13) *is satisfied. Then condition* (1.9) *is also satisfied.*

Proof. Let $p_0 \in L_\omega$ be an arbitrary function satisfying inequalities (1.7). If, in addition, condition (1.11) [respectively, condition (1.13)] is satisfied, then, by Theorem 1.1 in [11] (respectively, by the lemma proved in [1, Sec. 3]), Eq. (1.8) does not have nontrivial ω-periodic solutions. It remains to show that this equation does not have nontrivial ω-periodic solutions in the case of condition (1.12) as well. Suppose the contrary: let Eq. (1.8) have a nontrivial ω-periodic solution u. Then $\int_0^\omega p_0(t)u(t)dt = 0$. This, together with condition (1.7), implies that u is not of constant sign. By Lemma 2.1, this provides the validity of inequalities (2.3). If, along with (2.3), we use inequalities (1.7) and (1.12), then from (1.8), we obtain

$$\left\|u^{(2m)}\right\|_{L_\omega} = \|p_0 u\|_{L_\omega} \le \|p_0\|_{L_\omega}\|u\|_{C_\omega} < \alpha_{2m\,1}\|p\|_{L_\omega}\left\|u^{(2m)}\right\|_{L_\omega} \le \left\|u^{(2m)}\right\|_{L_\omega}.$$

The resulting contradiction proves the lemma.

Now consider the system of differential inequalities

$$\sigma u^{(n)}(t)\operatorname{sgn} u(t) \ge \left|u^{(n)}(t)\operatorname{sgn} u(t)\right| - q(t), \qquad \left|u^{(n)}(t)\right| \le q_0(t, |u(t)|) \qquad (2.9)$$

and the two-sided differential inequality

$$-q(t) \le (-1)^m u^{(2m)}(t) \operatorname{sgn} u(t) \le p(t)|u(t)| + q(t), \tag{2.10}$$

where $\sigma \in \{-1, 1\}$, p and $q \in L_\omega$ are nonnegative functions, and $q_0 : R \times [0, +\infty[\to [0, +\infty[$ is a function such that $q_0(\cdot, x) \in L_\omega$ for arbitrary $x \in [0, +\infty[$ and $q_0(t, \cdot) : [0, +\infty[\to [0, +\infty[$ is a continuous nondecreasing function for almost all $t \in R$.

A function $u \in \tilde{C}_\omega^{n-1}$ is called an *ω-periodic solution* of system (2.9) of differential inequalities [respectively, of the differential inequality (2.10)] if it satisfies this system (respectively, this inequality) almost everywhere on R.

Lemma 2.3. *If one of conditions* (1.18) *and* (1.19) *holds, then an arbitrary ω-periodic solution u of system* (2.9) *admits the estimate*

$$\|u\|_{C_\omega^{n-1}} \le \varrho_1 \mu(u) + \varrho_2, \tag{2.11}$$

where

$$\varrho_1 = 1 + \alpha_n \|q\|_{L_\omega}, \qquad \varrho_2 = \alpha_n \varrho_1 \int\limits_0^\omega q_0\,(t, \varrho_1)\, dt. \tag{2.12}$$

Proof. First, note that the function u admits the estimates (2.2) by Lemma 2.1.

If we multiply the first inequality in system (2.9) by $|u(t)|$ and integrate the resulting relation from 0 to ω, then we obtain the inequality

$$-(n+1-2m) \int\limits_0^\omega \left|u^{(m)}(t)\right|^2 dt \ge \int\limits_0^\omega \left|u^{(n)}(t) u(t)\right| dt - \int\limits_0^\omega q(t)|u(t)| dt.$$

This, together with (2.2), implies that

$$\int\limits_0^\omega \left|u^{(n)}(t) u(t)\right| dt \le \left(\mu(u) + \alpha_n \left\|u^{(n)}\right\|_{L_\omega}\right) \|q\|_{L_\omega}. \tag{2.13}$$

Let $I_1 = \{t \in [0, \omega] : \ |u(t)| > \varrho_1\}$ and $I_2 = [0, \omega] \backslash I_1$. Then, by the second inequality in system (2.9) and inequality (2.13), we obtain

$$\begin{aligned}
\varrho_1 \left\|u^{(n)}\right\|_{L_\omega} &\le \int\limits_{I_1} \left|u^{(n)}(t) u(t)\right| dt + \varrho_1 \int\limits_{I_2} \left|u^{(n)}(t)\right| dt \\
&\le \left(\mu(u) + \alpha_n \left\|u^{(n)}\right\|_{L_\omega}\right) \|q\|_{L_\omega} + \varrho_1 \int\limits_0^\omega q_0\,(t, \varrho_1)\, dt \\
&= (\varrho_1 - 1) \left(\left\|u^{(n)}\right\|_{L_\omega} + \mu(u)/\alpha_n\right) + \varrho_2/\alpha_n,
\end{aligned}$$

and consequently,

$$\left\|u^{(n)}\right\|_{L_\omega} \le \left(\left(\varrho_1 - 1\right) \mu(u) + \varrho_2\right)/\alpha_n,$$

where ϱ_1 and ϱ_2 are the numbers defined in (2.12). If, along with this, we use inequalities (2.2), then the estimate (2.11) becomes obvious. The proof of the lemma is complete.

Lemma 2.4. *If condition* (1.9) *is satisfied, then there exists a positive constant ϱ_0 such that, for an arbitrary nonnegative function $q \in L_\omega$, each solution u of the differential inequality* (2.10) *admits the estimate*

$$\|u\|_{C_\omega^{2m-1}} \le \varrho_0 \left(\mu(u) + \|q\|_{L_\omega}\right). \tag{2.14}$$

Proof. Suppose that the lemma is not true. Then, for an arbitrary positive integer k, there exists a nonnegative function $q_k \in L_\omega$ and an ω-periodic solution u_k of the differential inequality

$$0 \le (-1)^m u_k^{(2m)}(t) \operatorname{sgn} u_k(t) + q_k(t) \le p(t)\,|u_k(t)| + 2q_k(t) \tag{2.15}$$

such that

$$\|u_k\|_{C_\omega^{2m-1}} > k\left(\mu(u_k) + \|q_k\|_{L_\omega}\right). \tag{2.16}$$

Let

$$u_{0k}(t) = u_k(t)/\|u_k\|_{C_\omega^{2m-1}}, \quad q_{0k}(t) = q_k(t)/\|u_k\|_{C_\omega^{2m-1}}, \quad \delta_k(t) = p(t)\,|u_{0k}(t)| + 2q_{0k}(t),$$

$$\eta_k(t) = \begin{cases} 0 & \text{for} \quad \delta_k(t) = 0 \\ \Big((-1)^m u_{0k}^{(2m)}(t) \operatorname{sgn} u_{0k}(t) + q_{0k}(t)\Big)/\delta_k(t) & \text{for} \quad \delta_k(t) > 0, \end{cases}$$

$$p_k(t) = \eta_k(t)p(t), \qquad \mathcal{P}_k(t) = \int_0^t p_k(s)ds, \qquad q_{1k}(t) = (2\eta_k(t) - 1)\, q_{0k}(t) \operatorname{sgn} u_{0k}(t).$$

Then, by virtue of (2.15) and (2.16), we have $0 \le \eta_k(t) \le 1$ for almost all $t \in R$ and

$$\|u_{0k}\|_{C_\omega^{2m-1}} = 1, \quad \mu(u_{0k}) < 1/k, \tag{2.17}$$

$$\|q_{1k}\|_{L_\omega} < 1/k. \tag{2.18}$$

In addition, it is clear that, for each positive integer k, the function u_{0k} is an ω-periodic solution of the differential equation

$$u_{0k}^{(2m)}(t) = (-1)^m p_{0k}(t) u_{0k}(t) + q_{1k}(t) \tag{2.19}$$

and the function $\mathcal{P}_k$ satisfies the conditions

$$\begin{gathered} \mathcal{P}_k(0) = 0, \qquad \mathcal{P}_k(t+\omega) = \mathcal{P}_k(\omega) + \mathcal{P}_k(t), \\ 0 \le \mathcal{P}_k(t) - \mathcal{P}_k(\tau) \le \int_\tau^t p(s)ds \quad \text{for} \quad t \in R, \quad \tau \le t. \end{gathered} \tag{2.20}$$

By (2.17) and (2.18), from (2.19), we obtain

$$\left\|u_{0k}^{(2m)}\right\|_{L_\omega} \le \|p\|_{L_\omega} + 1. \tag{2.21}$$

On the other hand, by the Arzelá–Ascoli lemma and conditions (2.17), (2.20), and (2.21), one can assume without loss of generality that the sequences $\left(u_{0k}^{(i-1)}\right)_{k=1}^{\infty}$ $(i = 1, \dots, n)$ and $(\mathcal{P}_k)_{k=1}^{\infty}$ are uniformly convergent on R; i.e.,

$$\lim_{k\to\infty} \|u_{0k} - u\|_{C_\omega^{n-1}} = 0, \qquad \lim_{k\to\infty} \mathcal{P}_k(t) = \mathcal{P}_0(t) \quad \text{uniformly on} \quad R, \tag{2.22}$$

where $u \in C_\omega^{n-1}$ and $\mathcal{P}_0 : R \to R$ is a continuous function. Therefore, it follows from (2.17) and (2.20) that

$$\|u\|_{C_\omega^{n-1}} = 1, \qquad \mu(u) = 0, \tag{2.23}$$

$$\begin{gathered} \mathcal{P}_0(0) = 0, \qquad \mathcal{P}_0(t+\omega) = \mathcal{P}_0(\omega) + \mathcal{P}_0(t), \\ 0 \le \mathcal{P}_0(t) - \mathcal{P}_0(\tau) \le \int_\tau^t p(s)ds \quad \text{for} \quad t \in R, \quad \tau \in\,]-\infty, t]. \end{gathered} \tag{2.24}$$

By conditions (2.24), $\mathcal{P}_0$ is an absolutely continuous function, and

$$\mathcal{P}_0(t) = \int_0^t p_0(s)ds \quad \text{for} \quad t \in R, \tag{2.25}$$

where the function $p_0 \in L_\omega$ either is identically zero or satisfies inequalities (1.7).

By Lemma 1.1 in [15] and conditions (2.22) and (2.25), we have

$$\lim_{k\to\infty} \int_0^t p_k(s)u_{0k}(s)ds = \int_0^t p_0(s)u(s)ds \quad \text{for} \quad t \in R.$$

If, along with these relations, we use condition (2.18), then from the relation

$$u_{0k}^{(2m-1)}(t) = u_{0k}^{(2m)}(0) + \int_0^t \left[(-1)^m p_k(s)u_{0k}(s) + q_{1k}(s)\right] ds \quad \text{for} \quad t \in R,$$

we obtain

$$u^{(2m-1)}(t) = u^{(2m-1)}(0) + (-1)^m \int_0^t p_0(s)u(s)ds \quad \text{for} \quad t \in R.$$

Consequently, u is an ω-periodic solution of Eq. (1.8). If p_0 satisfies inequalities (1.7), then, by condition (1.9), Eq. (1.8) does not have nontrivial ω-periodic solutions. Therefore, it remains to consider the case in which $p_0(t) \equiv 0$. In this case, $u(t) \equiv \text{const}$, which contradicts condition (2.23). The resulting contradiction proves the lemma.

Along with (1.1), consider the differential equation

$$u^{(n)} = (1-\lambda)a(t)u + \lambda\left[f(t,u) + f_0(t)\right], \tag{2.26}$$

which depends on the parameter $\lambda \in \,]0,1[$ and the function $a \in L_\omega$. The following assertion holds.

Lemma 2.5. *Suppose that there exists an $a \in L_\omega$ and a positive constant ϱ such that the linear homogeneous equation*

$$u^{(n)} = a(t)u \tag{2.27}$$

does not have nontrivial ω-periodic solutions and for each $\lambda \in \,]0,1[$ an arbitrary ω-periodic solution u of the differential equation (2.26) *admits the estimate*

$$\|u\|_{C_\omega^{n-1}} \le \varrho. \tag{2.28}$$

Then Eq. (1.1) *has at least one ω-periodic solution.*

Proof. If a function $u : [0,\omega] \to R$ is a solution of Eq. (1.1) [respectively, Eq. (2.26)] with the boundary conditions

$$u^{(i-1)}(\omega) = u^{(i-1)}(0) \qquad (i = 1, \dots, n), \tag{2.29}$$

then, by virtue of conditions (1.2), its ω-periodic extension to R is an ω-periodic solution of the respective equation. Therefore, to prove the lemma, it suffices to show that problem (1.1), (2.29) has at least one solution.

Let $\lambda \in \,]0,1[$, and let u be an arbitrary solution of problem (2.26), (2.29). Then, as was mentioned above, the ω-periodic extension of u to R is also a solution of Eq. (2.26), and, by one of the assumptions of the lemma, the estimate (2.28) holds. However, by virtue of Corollary 2 in [17], this estimate, together with the existence of only the trivial solution of the linear homogeneous problem (2.27), (2.29), provides the solvability of problem (1.1), (2.29). The proof of the lemma is complete.

Lemma 2.6. *Suppose that there exist numbers* $\sigma \in \{-1, 1\}$ *and* $r \geq 0$ *and a function* $g \in L_\omega$ *such that, along with* (1.3) *and* (1.4), *the inequalities*

$$\sigma a(t) \geq 0 \quad \text{for} \quad t \in R, \quad a(t) \not\equiv 0, \tag{2.30}$$

hold. Then for each $\lambda \in]0, 1[$, *an arbitrary* ω*-periodic solution of Eq.* (2.26) *admits the estimate*

$$\mu(u) \leq r. \tag{2.31}$$

Proof. Suppose the contrary: $\mu(u) > r$. Then, by (1.3) and (2.30), from (2.26), we obtain

$$\begin{aligned} \sigma_0 u^{(n)}(t) &= (1-\lambda)\sigma a(t)|u(t)| + \lambda \left[\sigma f(t, u(t)) \operatorname{sgn} u(t) + \sigma_0 f_0(t)\right] \\ &\geq r(1-\lambda)|a(t)| + \lambda \left(g(t) + \sigma_0 f_0(t)\right), \end{aligned}$$

where $\sigma_0 = \sigma \operatorname{sgn} u(0)$. If we integrate this inequality from 0 to ω, then, by virtue of condition (1.4), we obtain

$$0 \geq r(1-\lambda) \int_0^\omega |a(t)| dt + \lambda \left[\int_0^\omega g(t) dt + \sigma_0 \int_0^\omega f_0(t) dt \right] > 0.$$

The resulting contradiction implies the estimate (2.31). The proof of the lemma is complete.

3. PROOF OF THE MAIN RESULTS

Proof of Theorem 1.1. Let

$$\begin{aligned} &a(t) = \sigma, \qquad q_0(t, y) = y + |f_0(t)| + \max\{|f(t, x)| : \ |x| \leq y\}, \\ &q(t) = 2q_0(t, r) + 2|g(t)| \quad \text{for} \quad t \in R, \quad y \geq 0. \end{aligned}$$

Then

$$a(t)|x| + |f(t, x) + f_0(t)| \leq q_0(t, |x|) \quad \text{for} \quad (t, x) \in R^2. \tag{3.1}$$

On the other hand, by virtue of inequality (1.3), we have

$$\begin{aligned} &\sigma \left[f(t, x) + f_0(t)\right] \operatorname{sgn} x \\ &\quad = |\sigma \left[f(t, x) + f_0(t)\right] \operatorname{sgn} x + q_0(t, r) + |g(t)|\,| - q_0(t, r) - |g(t)| \\ &\quad \geq |\sigma \left[f(t, x) + f_0(t)\right] \operatorname{sgn} x| - q(t) \quad \text{for} \quad (t, x) \in R^2. \end{aligned} \tag{3.2}$$

By Proposition 1.1 in [11], Eq. (2.27) does not have nontrivial ω-periodic solutions, because $\sigma a(t) \equiv 1$ and one of conditions (1.18) and (1.19) is satisfied. This, together with Lemma 2.5, implies that to prove the theorem, it suffices to show that for each $\lambda \in]0, 1[$ an arbitrary ω-periodic solution u of the differential equation (2.26) admits the estimate (2.28), where ϱ is a positive constant independent of λ and u.

By conditions (3.1) and (3.2), the inequalities

$$\begin{aligned} \left|u^{(n)}(t)\right| &= |(1-\lambda)a(t)u(t) + \lambda \left[f(t, u(t)) + f_0(t)\right]| \leq q_0(t, |u(t)|), \\ \sigma u^{(n)}(t) \operatorname{sgn} u(t) &= (1-\lambda)|u(t)| + \lambda\sigma \left[f(t, u(t)) + f_0(t)\right] \operatorname{sgn} u(t) \\ &\geq (1-\lambda)|u(t)| + \lambda \left|\sigma \left[f(t, u(t)) + f_0(t)\right] \operatorname{sgn} u(t)\right| - \lambda q(t) \\ &\geq \left|u^{(n)}(t) \operatorname{sgn} u(t)\right| - q(t) \end{aligned}$$

hold almost everywhere on R. Consequently, u is an ω-periodic solution of system (2.9), which, together with Lemma 2.3, implies the estimate (2.11), where ϱ_1 and ϱ_2 are the numbers defined in (2.12). On the other hand, by Lemma 2.6, the function u admits the estimate (2.31). It follows from the estimates (2.11) and (2.31) that the estimate (2.28) holds, where $\varrho = \varrho_1 r + \varrho_2$ is a positive constant independent of λ and u. The proof of the theorem is complete.

Proof of Theorem 1.2. Conditions (1.2) and (1.5) imply condition (1.3), where g is the function given by (1.6). Consequently, all assumptions of Theorem 1.1 are valid, which provides the existence of at least one ω-periodic solution of Eq. (1.1). It remains to show that if u_1 and u_2 are arbitrary ω-periodic solutions of Eq. (1.1) and $u(t) = u_1(t) - u_2(t)$, then $u(t) \equiv 0$. Suppose the contrary: $u(t) \not\equiv 0$. Then, by condition (1.5), the inequality

$$\sigma u^{(n)}(t)u(t) \geq \ell_0(t)$$

holds almost everywhere on R, where $\ell_0(t) = \ell(t)h\left(u_1(t), u_2(t)\right)|u(t)| \geq 0$ for $t \in R$ and $\ell_0(t) \not\equiv 0$. If we integrate both sides of this inequality from 0 to ω and use one of conditions (1.18) and (1.19), then we obtain

$$-(n+1-2m)\int_0^\omega \left|u^{(m)}(t)\right|^2 dt \geq \int_0^\omega \ell_0(t)dt > 0.$$

But this is impossible, since $n \geq 2m-1$. The resulting contradiction proves the theorem.

Proof of Theorem 1.3. By condition (1.10), one can assume without loss of generality that

$$|f_0(t)| - q(t) \leq (-1)^m f(t,x)\operatorname{sgn} x \leq p(t)|x| + q(t) - |f_0(t)| \quad \text{for} \quad (t,x) \in R^2. \tag{3.3}$$

Let ϱ_0 be the number occurring in Lemma 2.4, let $\varrho = \varrho_0\left(r + \|q\|_{L_\omega}\right)$, let

$$a(t) = (-1)^m p(t), \tag{3.4}$$

and let u be an ω-periodic solution of Eq. (2.26) for some $\lambda \in]0,1[$. By virtue of condition (1.9) and Lemma 2.5, to prove the theorem, it suffices to show that u admits the estimate (2.28).

By using conditions (3.3) and (3.4), from (2.26), we find that u is an ω-periodic solution of the differential inequality (2.10). Therefore, it admits the estimate (2.14). On the other hand, the estimate (2.31) also holds by Lemma 2.6. However, the estimate (2.28) follows from the estimates (2.14) and (2.31). The proof of the theorem is complete.

Proof of Corollary 1.1. It follows from inequality (1.10) that $p(t) \geq 0$ for $t \in R$. Without loss of generality, one can assume that $p(t) \not\equiv 0$. By Lemma 2.2, in this case, condition (1.9) is also satisfied, because p and m satisfy one of conditions (1.11), (1.12), and (1.13). If we now use Theorem 1.3, then Corollary 1.1 becomes obvious.

Proof of Theorem 1.4. From conditions (1.2) and (1.14), we obtain condition (1.10), where g is the function given by (1.6). Consequently, all assumptions of Theorem 1.3 hold, which provides the existence of an ω-periodic solution of Eq. (1.1).

It remains to prove the uniqueness. Suppose the contrary. Then there exist ω-periodic solutions u_1 and u_2 of Eq. (1.1) such that $u(t) = u_1(t) - u_2(t) \not\equiv 0$. By condition (1.14), the function u is a solution of the differential inequality

$$0 \leq (-1)^m u(t)\operatorname{sgn} u(t) \leq p(t)|u(t)|.$$

This, together with Lemma 2.4 and condition (1.9), implies that

$$0 < \|u\|_{C_\omega^{n-1}} \leq \varrho_0 \mu(u),$$

where $\varrho_0 = \text{const} > 0$. Consequently, $\mu(u) > 0$. If, in addition, we use condition (1.14), then it becomes clear that the inequality

$$\sigma_0 u^{(n)}(t) \geq \ell_0(t)$$

holds almost everywhere on R, where $\sigma_0 = (-1)^m \operatorname{sgn} u(0)$, $\ell_0(t) = \ell(t)\eta\left(u_1(t), u_2(t)\right) \geq 0$ for $t \in R$, and $\ell_0(t) \not\equiv 0$. The integration of this inequality from 0 to ω results in the relations

$$0 = \sigma_0\left(u^{(n-1)}(\omega) - u^{(n-1)}(0)\right) \geq \int_0^\omega \ell_0(t)dt > 0.$$

The resulting contradiction proves the theorem.

By virtue of Lemma 2.2, from Theorem 1.4, we obtain Corollary 1.2.

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