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Nonlinear Analysis 60 (2005) 241 – 256

Nonlinear Analysis

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On periodic solutions of two-dimensional nonautonomous differential systems

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Received 11 February 2004; accepted 9 July 2004

Abstract

Nonimprovable conditions are established for the existence and uniqueness of an ω-periodic solution of the nonautonomous differential systems

$$u_i' = p_{i1}(t)u_1 + p_{i2}(t)u_2 + q_i(t) \quad (i = 1, 2)$$

and

$$u_i' = f_i(t, u_1, u_2) \quad (i = 1, 2),$$

where $p_{ik} : R \to R$, $q_i : R \to R$ $(i, k = 1, 2)$ are ω-periodic functions, Lebesgue integrable on $[0, \omega]$, and $f_i : R \times R^2 \to R$ $(i = 1, 2)$ are functions from the Carathéodory class such that

$$f_i(t + \omega, x_1, x_2) \equiv f_i(t, x_1, x_2) \quad (i = 1, 2).$$

MSC: 34C25.

Keywords: Two-dimensional nonautonomous differential system; ω-Periodic solution; Existence and uniqueness theorems

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doi:10.1016/j.na.2004.07.053

1. Formulation of the main results. Examples

Problems on the existence and uniqueness of a periodic solution of systems of nonautonomous ordinary differential equations have long been attracting the attention of mathematicians and used as the subject of many studies (see, for example, [1–20] and the references therein). And all the same these problems still remain topical for two-dimensional linear and nonlinear differential systems

$$u_i' = p_{i1}(t)u_1 + p_{i2}(t)u_2 + q_i(t) \quad (i = 1, 2) \tag{1.1}$$

and

$$u_i' = f_i(t, u_1, u_2) \quad (i = 1, 2). \tag{1.2}$$

In this paper, an attempt is made to fill to a certain extent the gap existing in this area. More exactly, new and, in a certain sense, optimal sufficient conditions are established for the existence and uniqueness of a periodic solution of systems (1.1) and (1.2) with a period $\omega > 0$.

Throughout the paper the following notation is used.

R^m is the m-dimensional real Eucledean space.

L_ω is the space of ω-periodic and Lebesgue integrable on $[0, \omega]$ functions $p : R \to R$ with the norm

$$\|p\| = \int_0^\omega |p(t)|\,\mathrm{d}t.$$

$K_\omega(R \times R^m)$ is the space of functions $f : R \times R^m \to R$, which are ω-periodic in the first argument and satisfy the local Carathéodory conditions. Consequently, the notation $f \in K_\omega(R \times R^m)$ means that $f(t, \cdot, \dots, \cdot) : R^m \to R$ is continuous for any $t \in R$, $f(\cdot, x_1, \dots, x_m) \in L_\omega$ for any $(x_1, \dots, x_m) \in R^m$ and

$$\max\left\{|f(\cdot, x_1, \dots, x_m)| : \sum_{i=1}^m |x_i| \leqslant \rho\right\} \in L_\omega \quad \text{for any } \rho \in]0, +\infty[.$$

For any function $p : R \to R$ the notation $p(t) \not\equiv 0$ means that p is different from zero on the set of positive measure.

Everywhere in the sequel when we consider systems (1.1) and (1.2), it will be assumed that

$$p_{ik} \in L_\omega, \quad q_i \in L_\omega \quad (i, k = 1, 2) \tag{1.3}$$

and

$$f_i \in K_\omega(R \times R^2) \quad (i = 1, 2), \tag{1.4}$$

respectively.

1.1. Existence and uniqueness theorem for system (1.1)

Set

$$p_i(t) = p_{i\,3-i}(t)\exp\left(\int_0^t (p_{3-i\,3-i}(s) - p_{ii}(s))\,\mathrm{d}s\right) \quad (i=1,2), \tag{1.5}$$

$$\ell = \int_0^\omega |p_1(s)|\,\mathrm{d}s \int_0^\omega |p_2(s)|\,\mathrm{d}s, \tag{1.6}$$

$$\lambda_i = \exp\left(-\int_0^\omega p_{ii}(s)\,\mathrm{d}s\right) \quad (i=1,2), \tag{1.7}$$

$$\mu_1 = \min\{1, \lambda_1\lambda_2\}, \quad \mu_2 = \max\{1, \lambda_1\lambda_2\}. \tag{1.8}$$

Theorem 1.1. *Let* $p_i(t) \not\equiv 0$ $(i=1,2)$,

$$(\lambda_1 - 1)(\lambda_2 - 1) \notin \,]\ell\mu_1, \ell\mu_2[\tag{1.9}$$

and there exist $\sigma \in \{-1, 1\}$ *such that*

$$\sigma p_i(t) \geqslant 0 \quad for\ t \in R\ (i=1,2). \tag{1.10}$$

Then system (1.1) *has a unique* ω*-periodic solution*.

If

$$\int_0^\omega p_{11}(s)\,\mathrm{d}s \int_0^\omega p_{22}(s)\,\mathrm{d}s \leqslant 0, \tag{1.11}$$

then by virtue of (1.7) we have $(\lambda_1-1)(\lambda_2-1) \leqslant 0$ and therefore condition (1.9) is fulfilled. Thus Theorem 1.1 gives rise to

Corollary 1.1. *Let* $p_i(t) \not\equiv 0$ *and for some* $\sigma \in \{-1, 1\}$ *conditions* (1.10) *and* (1.11) *be fulfilled. Then system* (1.1) *has a unique* ω*-periodic solution.*

Example 1.1. For arbitrarily given $\varepsilon \in \,]0, 1[$, choose $\delta > 0$ such that

$$(\exp(\delta\omega) - 1)^2 < (1+\varepsilon)\delta^2\omega^2.$$

Consider the homogeneous system

$$u_i' = p_{i1}(t)u_1 + p_{i2}(t)u_2 \quad (i=1,2), \tag{1.1$_0$}$$

where $p_{11}(t) \equiv p_{22}(t) \equiv -\delta$, $p_{12}(t) \equiv p_{21}(t) \equiv \delta$. Then, by (1.6)–(1.8), we have

$$\ell = \delta^2\omega^2, \quad \lambda_i = \exp(\delta\omega)\ \ (i=1,2), \quad \mu_1 = 1, \quad \mu_2 = \exp(2\delta\omega).$$

Therefore,

$$(\lambda_1 - 1)(\lambda_2 - 1) \notin]\ell(\mu_1 + \varepsilon), \ell\mu_2[, \tag{1.12}$$

$$\int_0^\omega p_{11}(s)\,\mathrm{d}s \int_0^\omega p_{22}(s)\,\mathrm{d}s < \varepsilon \tag{1.13}$$

and inequalities (1.10), where $\sigma = 1$, are fulfilled.

On the other hand, system (1.1_0) has a nontrivial ω-periodic solution (u_1, u_2) with the components $u_i(t) \equiv 1$ $(i = 1, 2)$. Therefore, the nonhomogeneous system (1.1) either has no ω-periodic solution or has an infinite set of such solutions. The constructed example shows that in Theorem 1.1 (in Corollary 1.1), condition (1.9) (condition (1.11)) cannot be replaced by condition (1.12) (by condition (1.13)) no matter how small $\varepsilon > 0$ is.

1.2. Existence and uniqueness theorems for system (1.2)

Denote by M_ω the set of functions $h : R \times R \to R$ such that

$$h(t, x) \geqslant h(t, y) \quad \text{for } t \in R, \ \ x > y > 0,$$
$$h(t, x) \leqslant h(t, y) \quad \text{for } t \in R, \ \ y < x < 0,$$
$$h(\cdot, x) \in L_\omega \quad \text{for any } x \in R \text{ and } \inf\{h(\cdot, x) : x \neq 0\} \in L_\omega.$$

Theorem 1.2. *Let there exist a number $\sigma \in \{-1, 1\}$ and functions $h_{ik} \in M_\omega$ $(i, k = 1, 2)$ such that*

$$\lim_{x \to +\infty} \int_0^\omega h_{1k}(t, x)\,\mathrm{d}t > 0, \qquad \lim_{x \to -\infty} \int_0^\omega h_{1k}(t, x)\,\mathrm{d}t > 0 \quad (k = 1, 2) \tag{1.14}$$

and on $R \times R^2$ the inequalities

$$h_{1i}(t, x_{3-i}) \leqslant \sigma f_i(t, x_1, x_2) \operatorname{sgn} x_{3-i} \leqslant h_{2i}(t, x) \quad (i = 1, 2) \tag{1.15}$$

hold. Then system (1.1) *has at least one ω-periodic solution.*

The particular cases of (1.2) are the systems

$$u_i' = f_i(t, u_{3-i}) \quad (i = 1, 2), \tag{1.16}$$

$$u_i' = \sum_{k=1}^{m_i} p_{ik}(t)|u_{3-i}|^{\lambda_{ik}} \operatorname{sgn} u_{3-i} + q_i(t) \quad (i = 1, 2), \tag{1.17}$$

where

$$f_i \in K_\omega(R \times R) \quad (i = 1, 2)$$

and

$$\lambda_{ik} > 0, \quad p_{ik} \in L_\omega, \quad q_i \in L_\omega \quad (k = 1, \dots, m_i; \ i = 1, 2).$$

Theorem 1.3. *Let there exist a number* $\sigma \in \{-1, 1\}$ *and sets of positive measure* $I_k \subset R$ $(k = 1, 2)$ *such that*

$$\sigma f_k(t, x) \geqslant \sigma f_k(t, x) \quad for\ t \in R,\ x \geqslant y\ (k = 1, 2), \tag{1.18}$$

$$f_k(t, x) \neq f_k(t, y) \quad for\ t \in I_k,\ x \neq y\ (k = 1, 2). \tag{1.19}$$

Then system (1.16) *has at most one* ω*-periodic solution and for such a solution to exist it is necessary and sufficient that*

$$\lim_{x\to+\infty} \int_0^\omega \sigma f_k(t, x)\, \mathrm{d}t > 0, \qquad \lim_{x\to-\infty} \int_0^\omega \sigma f_k(t, x)\, \mathrm{d}t < 0 \quad (k = 1, 2). \tag{1.20}$$

Corollary 1.2. *Let* $p_{ik}(t) \not\equiv 0$ $(k=1, \ldots, m_i; i=1, 2)$ *and there exist a number* $\sigma \in \{-1, 1\}$ *such that*

$$\sigma p_{ik}(t) \geqslant 0 \quad for\ t \in R\ (k = 1, \ldots, m_i;\ i = 1, 2). \tag{1.21}$$

Then system (1.17) *has one and only one* ω*-periodic solution.*

Remark 1.1. From Theorem 1.3 it follows that condition (1.14) in Theorem 1.2 is nonimprovable since it cannot be replaced by the condition

$$\lim_{x\to+\infty} \int_0^\omega h_{1k}(t, x)\, \mathrm{d}t \geqslant 0, \qquad \lim_{x\to-\infty} \int_0^\omega h_{1k}(t, x)\, \mathrm{d}t \geqslant 0 \quad (k = 1, 2).$$

Example 1.2. Let

$$f_1(t, x) = \begin{cases} x - 1 & \text{for } x < 1 + u_0(t), \\ u_0'(t) & \text{for } 1 \leqslant x - u_0'(t) \leqslant 4, \\ x - 4 & \text{for } x > 4 - u_0'(t), \end{cases}$$

$$f_2(t, x) = x - u_0(t),$$

where $u_0 : R \to R$ is a continuously differentiable ω-periodic function such that $|u_0'(t)| \leqslant 1$ for $t \in R$. Then system (1.16) has an infinite set of ω-periodic solutions since for arbitrary $c \in R$ the vector function (u_1, u_2) with the components $u_1(t) = u_0(t)$, $u_2(t) = c$ is its solution. On the other hand, the functions f_1 and f_2 satisfy all the conditions of Theorem 1.3 except (1.19).

The constructed example shows that condition (1.19) in Theorem 1.3 is the essential one and cannot be ignored.

2. Auxiliary propositions

2.1. Lemmas on ω*-periodic solutions of system (1.1)*

Let $\mathscr{U}$ be the fundamental matrix of the differential system (1.1_0), satisfying the initial condition

$$\mathscr{U}(0) = E,$$

where E is the unit 2×2 matrix. Then, by virtue of (1.3),

$$\mathscr{U}(t+\omega) = \mathscr{U}(t)\mathscr{U}(\omega) \quad \text{for } t \in R \tag{2.1}$$

and system (1.1_0) does not have a nontrivial ω-periodic solution if and only if the matrix $\mathscr{U}(\omega) - E$ is nondegenerate. If this condition is fulfilled, we assume that

$$\begin{pmatrix} g_{11}(t,s) & g_{12}(t,s) \\ g_{21}(t,s) & g_{22}(t,s) \end{pmatrix} = \mathscr{U}(t)(\mathscr{U}^{-1}(\omega) - E)^{-1}\mathscr{U}^{-1}(s). \tag{2.2}$$

It is clear that for arbitrarily fixed $s \in R$ the vector functions $(g_{1i}(\cdot, s), g_{2i}(\cdot, s))$ $(i = 1, 2)$ are solutions of system (1.1). On the other hand, taking into account (2.1), from (2.2) we find

$$g_{ik}(t+\omega, s+\omega) \equiv g_{ik}(t,s), \quad g_{ii}(t, t+\omega) \equiv 1 + g_{ii}(t,t),$$
$$g_{i\,3-i}(t, t+\omega) \equiv g_{i\,3-i}(t,t) \quad (i, k = 1, 2).$$

These identities immediately imply that the vector function (u_1, u_2) with the components

$$u_i(t) = \int_t^{t+\omega} [g_{i1}(t,s)q_1(s) + g_{i2}(t,s)q_2(s)]\,\mathrm{d}s \quad \text{for } t \in R \ (i = 1, 2) \tag{2.3}$$

is an ω-periodic solution of system (1.1).

Therefore the following statement is true.

Lemma 2.1. *System* (1.1) *has a unique ω-periodic solution if and only if the corresponding homogeneous system* (1.1_0) *does not have a nontrivial ω-periodic solution. In that case, the components of the ω-periodic solution of system* (1.1) *admit representation* (2.3), *where* g_{ik} $(i, k = 1, 2)$ *are the functions defined from equality* (2.2).

Let (u_1, u_2) be an arbitrary ω-periodic solution of system (1.1_0) and

$$v_i(t) = \exp\left(-\int_0^t p_{ii}(s)\,\mathrm{d}s\right) u_i(t) \quad (i = 1, 2).$$

Then (v_1, v_2) is a solution of the system

$$v_i' = p_i(t)v_{3-i} \quad (i = 1, 2), \tag{2.4}$$

satisfying the conditions

$$v_i(t+\omega) = \lambda_i v_i(t) \quad \text{for } t \in R \ (i = 1, 2), \tag{2.5}$$

where p_i and λ_i $(i = 1, 2)$ are the functions and numbers given by equalities (1.5) and (1.7). It also clearly follows that if (v_1, v_2) is a solution of problem (2.4), (2.5) and

$$u_i(t) = \exp\left(\int_0^t p_{ii}(s)\,\mathrm{d}s\right) v_i(t) \quad (i = 1, 2),$$

then (u_1, u_2) is an ω-periodic solution of system (1.1_0). Hence Lemma 2.1 gives rise to

Lemma 2.2. *System* (1.1) *has a unique ω-periodic solution if and only if problem* (2.4), (2.5) *has only a trivial solution*.

To prove Theorem 1.1, along with Lemma 2.2 we need also the following:

Lemma 2.3. *Let*

$$p_i(t) \not\equiv 0, \quad \sigma_i p_i(t) \geqslant 0 \quad for\ t \in R\ (i = 1, 2), \tag{2.6}$$

where $\sigma_i \in \{-1, 1\}$, and problem (2.4), (2.5) *have a nontrivial solution (v_1, v_2). Then either v_1 and v_2 are functions with alternating sign on each interval of length ω or*

$$\sigma_i(\lambda_i - 1)v_1(t)v_2(t) > 0 \quad for\ t \in R\ (i = 1, 2). \tag{2.7}$$

Proof. First we note that, in view of conditions (1.3) and notations (1.5), (1.7), we have

$$p_1(t + \omega) = \frac{\lambda_1}{\lambda_2}\, p_1(t), \quad p_2(t + \omega) = \frac{\lambda_2}{\lambda_1}\, p_2(t) \quad \text{for } t \in R.$$

If along with this we take into account condition (2.6), then the validity of the following inequalities:

$$\sigma_i \int_t^{t+\omega} p_i(s)\, \mathrm{d}s > 0 \quad \text{for } t \in R\ (i = 1, 2) \tag{2.8}$$

becomes evident.

First it will be shown that if for some $k \in \{1, 2\}$ the function v_k has at least one zero, then v_{3-k} is a function with alternating sign. Let us assume the contrary. Then it can be assumed without loss of generality that

$$v_{3-k}(t) \geqslant 0 \quad \text{for } t \in R. \tag{2.9}$$

On the other hand, by (2.5), we have

$$v_k(t_0 + \omega) = v_k(t_0) = 0, \quad v_{3-k}(t_0) > 0$$

for some t_0. Using the latter conditions, from (2.4) we find

$$\int_{t_0}^{t_0+\omega} p_k(s)v_{3-k}(s)\, \mathrm{d}s = 0.$$

Hence on account of (2.6) and (2.9) it follows that $p_k(t)v_{3-k}(t) = 0$ for almost all $t \in [t_0, t_0 + \omega]$. Therefore,

$$v_k(t) = \int_{t_0}^{t} p_k(s)v_{3-k}(s)\, \mathrm{d}s = 0,$$

$$v_{3-k}(t) = v_{3-k}(t_0) + \int_{t_0}^{t} p_{3-k}(s)v_k(s)\, \mathrm{d}s = v_{3-k}(t_0) \quad \text{for } t \in [t_0, t_0 + \omega]$$

and

$$v_{3-k}(t_0)\int_{t_0}^{t_0+\omega} p_k(s)\,\mathrm{d}s = 0,$$

which contradicts condition (2.8). The contradiction obtained proves the sign alternation property of v_{3-k}. Hence, since $k \in \{1,2\}$ is arbitrary, it follows that either v_1 and v_2 are functions with alternating sign or

$$v_i(t)v_i(s) > 0 \quad \text{for } s,\ t \in R\ (i=1,2). \tag{2.10}$$

In case inequalities (2.10) are fulfilled, by (2.5), (2.6) and (2.8) we find from (2.4) that

$$\begin{aligned}&\sigma_i(\lambda_i - 1)v_i(t)v_{3-i}(t)\\ &\quad = \sigma_i \int_t^{t+\omega} p_i(s)(v_{3-i}(t)v_{3-i}(s))\,\mathrm{d}s > 0 \quad \text{for } t \in R\ (i=1,2).\end{aligned}$$

Therefore inequalities (2.7) are fulfilled. □

2.2. Lemma on the existence of a periodic solution of system (1.2)

Lemma 2.4. *Let there exist functions $p_{ik} \in L_\omega$ $(i,k=1,2)$ and a positive number ρ_0 such that system* (1.1_0) *does not have a nontrivial ω-periodic solution and for each $\lambda \in]0,1[$ an arbitrary ω-periodic solution of the differential system*

$$u_i' = (1-\lambda)(p_{i1}(t)u_1 + p_{i2}(t)u_2) + \lambda f_i(t,u_1,u_2) \quad (i=1,2) \tag{2.11}$$

admits the estimate

$$\sum_{i=1}^{n} |u_i(t)| < \rho_0 \quad \textit{for } t \in R. \tag{2.12}$$

Then system (1.1) *has at least one ω-periodic solution.*

Proof. Let B_ω be the Banach space of ω-periodic continuous vector functions $(u_1,u_2) : R \to R^2$ with the norm

$$\|(u_1,u_2)\| = \max\{|u_1(t)| + |u_2(t)| : t \in R\},$$

and $\delta : [0,+\infty[\to [0,+\infty[$ be the function given by the equality

$$\delta(s) = \begin{cases} 1 & \text{for } 0 \leqslant s \leqslant \rho_0,\\ 2 - s/\rho_0 & \text{for } \rho_0 < s < 2\rho_0,\\ 0 & \text{for } s \geqslant \rho_0.\end{cases} \tag{2.13}$$

For an arbitrary $(u_1,u_2) \in B_\omega$ we assume that

$$\begin{aligned}q_i(u_1,u_2)(t) = \delta(\|(u_1,u_2)\|)[&f_i(t,u_1(t),u_2(t))\\ &- p_{i1}(t)u_1(t) - p_{i2}(t)u_2(t)] \quad (i=1,2).\end{aligned} \tag{2.14}$$

By virtue of condition (1.4)

$$q_i : B_\omega \to L_\omega \quad (i = 1, 2)$$

are continuous operators satisfying, for arbitrary $t \in R$ and $(u_1, u_2) \in R^2$, the inequalities

$$|q_i(u_1, u_2)(t)| \leqslant \widetilde{q}_i(t) \quad (i = 1, 2), \tag{2.15}$$

where

$$\widetilde{q}_i(t) = 2\rho_0 \sum_{k=1}^{2} |p_{ik}(t)| + \max\{|f_i(t, x_1, x_2)| : |x_1| + |x_2| \leqslant 2\rho_0\}$$

and $\widetilde{q}_i \in L_\omega$ $(i = 1, 2)$.

Let us first of all show that an arbitrary ω-periodic solution (u_1, u_2) of the system of functional differential equations

$$u_i' = p_{i1}(t)u_1 + p_{i2}(t)u_2 + q_i(u_1, u_2)(t) \quad (i = 1, 2) \tag{2.16}$$

is a solution of system (1.1). Assume the contrary. Then either the inequality

$$\|(u_1, u_2)\| > 2\rho_0 \tag{2.17}$$

or

$$\rho_0 \leqslant \|(u_1, u_2)\| \leqslant 2\rho_0 \tag{2.18}$$

is fulfilled by virtue of equalities (2.13) and (2.14).

Inequality (2.17) cannot take place since in that case (u_1, u_2) would be a solution of system (1.1_0) by virtue of equalities (2.13) and (2.14). But this system does not have a nontrivial ω-periodic solution. It remains to consider the case where condition (2.18) is fulfilled. Then (u_1, u_2) is a solution of system (2.11), where

$$\lambda = \delta(\|(u_1, u_2)\|) \in]0, 1[\,.$$

Hence by the condition of the lemma it follows that $\|(u_1, u_2)\| < \rho_0$, which contradicts inequality (2.18). The obtained contradiction proves that (u_1, u_2) admits estimate (2.12) and thus is a solution of system (1.1).

By the facts proved above, to complete the proof of the lemma, it suffices to establish that system (2.16) has at least one ω-periodic solution.

Let $\mathscr{U}$ be the fundamental matrix of system (1.1_0), satisfying the initial condition $\mathscr{U}(0) = E$, and $g_{ik} : R^2 \to R$ $(i, k = 1, 2)$ be the functions given by the matrix equality (2.2). Following Lemma 2.1, system (2.16) has at least one ω-periodic solution if the operator equation

$$(u_1(t), u_2(t)) = (g_1(u_1, u_2)(t), g_2(u_1, u_2)(t)), \tag{2.19}$$

where

$$g_i(u_1, u_2)(t) = \int_t^{t+\omega} [g_{i1}(t, s)q_1(u_1, u_2)(s) + g_{i2}(t, s)q_2(u_1, u_2)(s)]\, \mathrm{d}s \quad (i = 1, 2) \tag{2.20}$$

has at least one solution in the space B_ω.

Assume that

$$u_{i0}(t)=\int_t^{t+\omega}[|g_{i1}(t,s)|\widetilde{q}_1(s)+|g_{i2}(t,s)|\widetilde{q}_2(s)]\,\mathrm{d}s \quad (i=1,2),$$
$$B^0_\omega=\{(u_1,u_2)\in B_\omega : |u_1(t)|\leqslant u_{10}(t),\ \ |u_2(t)|\leqslant u_{20}(t)\ \text{for}\ t\in R\}.$$

Obviously, $u_{i0}\in L_\omega$ $(i=1,2)$ and B^0_ω is the closed convex set of B_ω. On the other hand, if, along with the continuity of the operators $q_i : B_\omega\to L_\omega$ $(i=1,2)$, we take into account inequalities (2.15) and apply the Arzella–Ascoli lemma, then from representation (2.20) we conclude that $(g_1,g_2) : B_\omega\to B_\omega$ is a continuous operator that transforms B^0_ω into its compact subset. Hence, by Schauder's theorem it follows that the operator equation (2.19) has a solution $(u_1,u_2)\in B^0_\omega$. $\square$

2.3. *Lemma on an a priori estimate*

Let $\sigma\in\{-1,1\}$, ρ_1 be a positive number, $h_{0i}\in L_\omega$ $(i=1,2)$ be nonnegative functions and $h : R\times[0,+\infty[\,\to[0,+\infty[$ be a nondecreasing with respect to the second argument function such that $h(\cdot,\rho)\in L_\omega$ for any $\rho\in[0,+\infty[$. We will consider the system of differential inequalities

$$\sigma u_i'(t)u_{3-i}(t)\geqslant -h_{0i}(t)|u_{3-i}(t)| \quad (i=1,2), \tag{2.21$_1$}$$

$$|u_i'(t)|\leqslant h(t,|u_{3-i}(t)|) \quad (i=1,2) \tag{2.21$_2$}$$

with the additional conditions

$$\min\{|u_i(t)| : t\in R\}<\rho_1 \quad (i=1,2). \tag{2.22}$$

We call an ω-periodic vector function $(u_1,u_2) : R\to R^2$ an ω-periodic solution of system (2.21$_1$), (2.21$_2$) if it is absolutely continuous and satisfies, almost everywhere on R, inequalities (2.21$_1$) and (2.21$_2$).

Lemma 2.5. *There exists a positive number ρ_0 such that an arbitrary ω-periodic solution (u_1,u_2) of system* (2.21$_1$), (2.21$_2$), *satisfying conditions* (2.22), *admits estimate* (2.12).

Proof. By inequalities (2.21$_1$) we have

$$\begin{aligned}|u_i'(t)u_{3-i}(t)| &= |\sigma u_i'(t)u_{3-i}(t)+h_{0i}(t)|u_{3-i}(t)|-h_{0i}(t)|u_{3-i}(t)||\\ &\leqslant|\sigma u_i'(t)u_{3-i}(t)+h_{0i}(t)|u_{3-i}(t)||+h_{0i}(t)|u_{3-i}(t)|\\ &=\sigma u_i'(t)u_{3-i}(t)+2h_{0i}(t)|u_{3-i}(t)| \quad (i=1,2).\end{aligned}$$

Therefore,

$$|u_1'(t)u_2(t)|+|u_2'(t)u_1(t)|\leqslant\sigma(u_1(t)u_2(t))'+2(h_{01}(t)|u_2(t)|+h_{02}(t)|u_1(t)|).$$

By integrating both parts of the latter inequality from 0 to ω and taking into account the ω-periodicity of the functions u_1 and u_2 we obtain

$$\int_0^\omega (|u_1'(t)u_2(t)| + |u_2'(t)u_1(t)|)\,\mathrm{d}t \leqslant 2\int_0^\omega (h_{01}(t)|u_2(t)| + h_{02}(t)|u_1(t)|)\,\mathrm{d}t. \tag{2.23}$$

Set

$$\rho_2 = 1 + 2(1+\rho_1)\int_0^\omega (h_{01}(t) + h_{02}(t))\,\mathrm{d}t \tag{2.24}$$

and

$$I_k = \{t \in [0,\omega] : |u_{3-k}(t)| > \rho_2\} \quad (k=1,2).$$

Then by inequalities (1.21_2) we have

$$\rho_2\int_0^\omega |u_k'(t)|\,\mathrm{d}t = \rho_2\int_{[0,\omega]\setminus I_k} |u_k'(t)|\,\mathrm{d}t + \rho_2\int_{I_k} |u_k'(t)|\,\mathrm{d}t \leqslant \rho_2\int_{[0,\omega]\setminus I_k} h(t,\rho_2)\,\mathrm{d}t + \int_{I_k} |u_k'(t)u_{3-k}(t)|\,\mathrm{d}t \quad (k=1,2).$$

If along with these inequalities we take into account (2.23), then we find

$$\rho_2\int_0^\omega (|u_1'(t)| + |u_2'(t)|)\,\mathrm{d}t \leqslant \rho_2\int_0^\omega h(t,\rho_2)\,\mathrm{d}t + 2\int_0^\omega (h_{01}(t)|u_2(t)| + h_{02}(t)|u_1(t)|)\,\mathrm{d}t. \tag{2.25}$$

By (2.22) we have

$$|u_i(t)| \leqslant \rho_1 + \int_0^\omega |u_i'(t)|\,\mathrm{d}t \quad \text{for } t \in R \quad (i=1,2). \tag{2.26}$$

Using these estimates and notation (2.24), from (2.25) we obtain

$$\rho_2\int_0^\omega (|u_1'(t)| + |u_2'(t)|)\,\mathrm{d}t < \rho_2\int_0^\omega h(t,\rho_2)\,\mathrm{d}t + \rho_2 + (\rho_2 - 1)\int_0^\omega (|u_1'(t)| + |u_2'(t)|)\,\mathrm{d}t$$

and, therefore,

$$\int_0^\omega (|u_1'(t)| + |u_2'(t)|)\,\mathrm{d}t < \rho_2\int_0^\omega h(t,\rho_2)\,\mathrm{d}t + \rho_2.$$

By virtue of this inequality, from (2.26) follows estimate (2.12), where

$$\rho_0 = 2\rho_1 + \rho_2\int_0^\omega h(t,\rho_2)\,\mathrm{d}t + \rho_2$$

is the positive constant not depending on (u_1, u_2). □

3. Proofs of the main results

Proof of Theorem 1.1. Assume the contrary that the theorem is not true. Then by Lemma 2.2 problem (2.4), (2.5) has a nontrivial solution (v_1, v_2). In that case, by condition (1.10) and Lemma 2.3 either

$$\sigma(\lambda_i - 1)v_1(t)v_2(t) > 0 \quad \text{for } t \in R \ (i = 1, 2) \tag{3.1}$$

or v_1 and v_2 are functions with alternating sign on each interval of length ω.

Let inequality (3.1) be fulfilled. Then

$$(\lambda_1 - 1)(\lambda_2 - 1) > 0.$$

Assume for simplicity that

$$\lambda_1 > 1, \quad \lambda_2 > 1, \tag{3.2}$$

since the case with $\lambda_1 < 1$ and $\lambda_2 < 1$ is considered similarly.

By virtue of (1.10), (3.1) and (3.2), $|u_1|$ and $|u_2|$ are nondecreasing functions,

$$|v_i(0)| < |v_i(\omega)| \quad (i = 1, 2)$$

and

$$(\lambda_1 - 1)|v_1(0)| = \int_0^\omega |p_1(s)|\,|v_2(s)|\,ds, \quad (\lambda_2 - 1)|v_2(0)| = \int_0^\omega |p_2(s)|\,|v_1(s)|\,ds.$$

Hence by the conditions $p_i(t) \not\equiv 0$ $(i = 1, 2)$ and notation (1.6) we find

$$(\lambda_1 - 1)(\lambda_2 - 1)|v_1(0)v_2(0)| < \ell|v_1(\omega)v_2(\omega)| = \lambda_1\lambda_2\ell|v_1(0)v_2(0)|,$$
$$(\lambda_1 - 1)(\lambda_2 - 1)|v_1(0)v_2(0)| > \ell|v_1(0)v_2(0)|.$$

Therefore $(\lambda_1 - 1)(\lambda_2 - 1) \in]\ell, \ell\lambda_1\lambda_2[$, which contradicts condition (1.9). We have thereby proved that v_1 and v_2 are functions with alternating sign.

Let $t_0 \in [0, \omega]$ be such that $v_1(t_0)v_2(t_0) = 0$. Then

$$v_1(t_0 + \omega)v_2(t_0 + \omega) = 0.$$

On the other hand,

$$\sigma(v_1(t)v_2(t))' = |p_1(t)|v_2^2(t) + |p_2(t)|v_1^2(t) \geqslant 0 \quad \text{for } t_0 < t < t_0 + \omega.$$

Therefore,

$$v_1(t)v_2(t) = 0 \quad \text{for } t \in [t_0, t_0 + \omega].$$

Since v_1 and v_2 are functions with alternating sign, this implies that there exists $s_0 \in [t_0, t_0 + \omega]$ such that

$$v_i(s_0) = 0 \quad (i = 1, 2). \tag{3.3}$$

But this is impossible since problem (2.4), (3.3) has only a trivial solution. The contradiction obtained proves the theorem. $\square$

Proof of Theorem 1.2. First we note that by virtue of the conditions $h_{ik} \in M_\omega$ $(i,k=1,2)$ from inequalities (1.15) follow the inequalities

$$\sigma f_i(t,x_1,x_2)x_{3-i} \geqslant -h_{0i}(t)|x_{3-i}| \quad (i=1,2), \tag{3.4}$$

$$|f_i(t,x_1,x_2)| \leqslant h(t,|x_{3-i}|) - |x_{3-i}|, \tag{3.5}$$

where $h_{0i}(t) = -\inf\{h_{1i}(t,x) : x \neq 0\}$, $h_{0i} \in L_\omega$ $(i=1,2)$ and

$$h(t,\rho) = \sum_{i=1}^{2} [h_{2i}(t,\rho) + h_{2i}(t,-\rho) + 2h_{0i}(t)] + \rho \geqslant 0 \quad \text{for } \rho \geqslant 0.$$

Moreover, h does not decrease with respect to the second argument and $h(\cdot,\rho) \in L_\omega$ for any $\rho \in]0,+\infty[$. On the other hand, in view of (1.14) there exists a positive number ρ_1 such that

$$\int_0^\omega h_{1k}(t,x)\,\mathrm{d}t > 0 \quad \text{for } |x| = \rho_1 \ (k=1,2). \tag{3.6}$$

By Lemma 2.5, there exists a positive number ρ_0 such that an arbitrary ω-periodic solution of system $(2.21_1), (2.21_2)$, satisfying conditions (2.22), admits estimate (2.12).

According to Corollary 1.1, the differential system

$$u_i' = \sigma u_{3-i} \quad (i=1,2)$$

does not have a nontrivial ω-periodic solution. By virtue of this fact and Lemma 2.4, to prove Theorem 1.2 it suffices to establish that for each $\lambda \in]0,1[$ an arbitrary ω-periodic solution (u_1,u_2) of the differential system

$$u_i' = (1-\lambda)\sigma u_{3-i} + \lambda f_i(t,u_1,u_2) \quad (i=1,2) \tag{3.7}$$

admits estimate (2.12).

First we show that u_1 and u_2 satisfy inequalities (2.22). Assume the contrary that for some $i \in \{1,2\}$ the inequality

$$\sigma_0 u_i(t) \geqslant \rho_1 \quad \text{for } t \in R, \quad \text{where } \sigma_0 = \operatorname{sgn} u_i(0)$$

holds. If along with this we take into account condition (1.15), then from (3.7) we find

$$\sigma_0\sigma u_{3-i}'(t) > \lambda h_{1i}(t,u_i(t)) \geqslant \lambda h_{1i}(t,\sigma_0\rho_1).$$

Therefore,

$$0 = \sigma_0\sigma \int_0^\omega u_{3-i}'(t)\,\mathrm{d}t > \lambda \int_0^\omega h_{1i}(t,\sigma_0\rho_1)\,\mathrm{d}t,$$

which is impossible in view of condition (3.6). The obtained contradiction proves the validity of inequality (2.22). On the other hand, due to conditions (3.4) and (3.5), from (3.7) it follows that (u_1,u_2) is a solution of the system of differential inequalities $(2.21_1), (2.21_2)$. Hence, taking into account how the number ρ_0 is chosen, we have estimate (2.12). □

Proof of Theorem 1.3. Let (u_1, u_2) and $(\overline{u}_1, \overline{u}_2)$ be arbitrary ω-periodic solutions of system (1.16). Set

$$v_i(t) = \overline{u}_i(t) - u_i(t) \quad (i = 1, 2),$$
$$I_{0k} = \{t \in [0, \omega] : v_{3-k}(t) \neq 0\} \quad (k = 1, 2).$$

Then

$$v_k'(t) = 0 \quad \text{for almost all } t \in [0, \omega] \setminus I_{0k} \ (k = 1, 2), \tag{3.8}$$

$$v_k'(t) \neq 0 \quad \text{for almost all } t \in I_{0k} \cap I_k \ (k = 1, 2). \tag{3.9}$$

On the other hand, in view of condition (1.18), for almost all $t \in [0, \omega]$ we have

$$\sigma v_k'(t) v_{3-k}(t)$$
$$= [f_k(t, \overline{u}_{3-k}(t)) - f_k(t, u_{3-k}(t))](\overline{u}_{3-k}(t) - u_{3-i}(t)) \geqslant 0 \quad (k = 1, 2).$$

Therefore,

$$\sigma(v_1(t)v_2(t))' = |v_1'(t)v_2(t)| + |v_2'(t)v_2(t)|.$$

Integrating this identity from 0 to ω, we find

$$\int_0^\omega (|v_1'(t)v_2(t)| + |v_2'(t)v_2(t)|)\, \mathrm{d}t = 0.$$

Thus,

$$\int_{I_{0k}} |v_k'(t)|\, \mathrm{d}t = 0 \quad (k = 1, 2)$$

and, consequently,

$$v_k'(t) = 0 \quad \text{for almost all } t \in I_{0k} \ (k = 1, 2).$$

Hence, due to (3.8) and (3.9), follows

$$v_k'(t) = 0 \quad \text{for almost all } t \in R \ (k = 1, 2) \tag{3.10}$$

and

$$\operatorname{mes}(I_{0k} \cap I_k) = 0 \quad (k = 1, 2).$$

However, $\operatorname{mes} I_k > 0$ $(k = 1, 2)$. Thus there exist $t_k \in I_{3-k}$ $(k = 1, 2)$ such that

$$v_k(t_k) = 0 \quad (k = 1, 2).$$

By virtue of these conditions, (3.10) results in $v_k(t) = 0$ $(k = 1, 2)$, and consequently, $\overline{u}_k(t) \equiv u_k(t)$ $(k = 1, 2)$. Thus we have proved that system (1.16) has at most one solution.

Now we prove that for the existence of an ω-periodic solution of system (1.16) it is necessary inequalities (1.20) to be fulfilled. Indeed, if this system has an ω-periodic solution (u_1, u_2), then

$$\int_0^\omega f_k(t, u_{3-k}(t))\, \mathrm{d}t = 0 \quad (k = 1, 2).$$

Hence, by virtue of conditions (1.18) and (1.19), follows

$$\int_0^\omega \sigma f_k(t,\rho)\,\mathrm{d}t > 0, \qquad \int_0^\omega \sigma f_k(t,-\rho)\,\mathrm{d}t < 0 \quad (k=1,2),$$

where $\rho = 1 + \max\{|u_1(t)| + |u_2(t)| : t \in R\}$. If now we apply again condition (1.18), then the validity of inequalities (1.20) becomes evident.

To complete the proof it remains to show that if along with (1.18) and (1.19) condition (1.20) holds, then system (1.16) has an ω-periodic solution. Indeed, system (1.16) is obtained from system (1.2) in the case, where

$$f_i(t,x_1,x_2) \equiv f_i(t,x_{3-i}) \quad (i=1,2).$$

In that case on $R \times R^2$ inequalities (1.15) are satisfied, where

$$h_{1i}(t,x) \equiv h_{2i}(t,x) \equiv \sigma f_i(t,x)\,\mathrm{sgn}\,x \quad (i=1,2).$$

On the other hand, in view of (1.18) and (1.20) it is obvious that $h_{ik} \in M_\omega$ $(i,k=1,2)$ and the functions h_{1k} $(k=1,2)$ satisfy inequalities (1.20). Therefore all the conditions of Theorem 1.2 are fulfilled, which guarantees the existence of an ω-periodic solution of system (1.16). $\square$

To convince ourselves of the validity of Corollary 1.2, it suffices to note that according to inequalities (1.21) and $p_{ik}(t) \not\equiv 0$ $(k=1,\dots,m_i;\ i=1,2)$, the functions f_1 and f_2, given by the equalities

$$f_i(t,x) = \sum_{k=1}^{m_i} p_{ik}(t)|x|^{\lambda_{ik}}\,\mathrm{sgn}\,x + q_i(t) \quad (i=1,2),$$

satisfy conditions (1.18)–(1.20).

Acknowledgements

The work of I. Kiguradze was supported by the INTAS Grant No. 00136. The work of S. Mukhigulashvili was supported by a fellowship from the INTAS YS 2001–2/80 as visiting professor at University of Udine, Italy.

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