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SOME OPTIMAL CONDITIONS FOR THE SOLVABILITY OF TWO-POINT SINGULAR BOUNDARY VALUE PROBLEMS *

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Abstract. For the differential equation

$$u'' = f(t,u),$$

where the function $f :]a,b[\times R \to R$ has non-integrable singularities at $t = a$ and $t = b$, we have found optimal sufficient conditions for the solvability and unique solvability of the boundary value problems

$$u(a) = c_1, \quad u(b) = c_2$$

and

$$u(a) = c_1, \quad u'(b) = c_2.$$

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§ 1. Formulation of the Main Results.

1.1. Statement of problems and the main notation. Consider the differential equation

$$u'' = f(t,u) \tag{1.1}$$

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with the boundary conditions

$$u(a) = c_1, \quad u(b) = c_2 \tag{1.2}$$

or

$$u(a) = c_1, \quad u'(b) = c_2. \tag{1.3}$$

Here $-\infty < a < b < +\infty$, $c_i \in R$ $(i = 1, 2)$ and the function $f :]a, b[\times R \to R$ satisfies the local Carathéodory conditions, i.e., $f(t, \cdot) : R \to R$ is continuous for almost all $t \in]a, b[$, $f(\cdot, x) :]a, b[\to R$ is measurable for every $x \in R$ and the function

$$f^*(t, \rho) = \max\{|f(t, s)| : \ |s| \le \rho\} \tag{1.4}$$

is integrable in the first argument on $[a + \varepsilon, b - \varepsilon]$ for arbitrary $\rho \in [0, +\infty[$ and $\varepsilon \in]0, (b - a)/2[$. We do not exclude case where the function f (and hence the function f^*) is non-integrable in the first argument on $[a, b]$, having singularities at the ends of this segment. In this sense, the problems under consideration are singular ones. Analogous problems for second order equations as well as for higher order ones have been are intensively studied starting from the 60s of the last century up to the present time (see, e.g., [1], [2], [6]–[25] and the references cited therein). In the present paper, we have found new optimal sufficient conditions for the solvability and unique solvability of the problems (1.1), (1.2) and (1.1), (1.3).

Along with (1.4), the use will be made of the following notation.

$R = -]\infty, +\infty[$, $R_+ = [0, +\infty[$.

If $x \in R$, then

$$[x]_+ = \frac{|x| + x}{2}, \quad [x]_- = \frac{|x| - x}{2}.$$

$L_{loc}(]a, b[)$ and $L_{loc}(]a, b])$ are the spaces of the functions $p :]a, b[\to R$ which are Lebesgue integrable on the segments $[a + \varepsilon, b - \varepsilon]$ and $[a + \varepsilon, b]$, respectively, for an arbitrary $\varepsilon \in]0, (b - a)/2[$.

$L_{\alpha,\beta}(]a, b[)$ is the space of the functions $p :]a, b[\to R$, integrable on $[a, b]$ with the weight $(t - a)^\alpha (b - t)^\beta$.

$M_{\alpha,\beta}(]a, b[\times R_+)$ is the set of functions $h :]a, b[\times R \to R_+$ such that $h(\cdot, x) \in L_{\alpha,\beta}(]a, b[)$ for every $x \in R$, and

$$h(t, x) \le h(t, y) \ \text{ for } \ t \in]a, b[, \ \ 0 \le x \le y.$$

If $\gamma > -2$, then $k_\gamma : R \setminus \{0, 2, 4, \dots\} \to R_+$ is the function given by the equalities

$$k_\gamma(s) = \begin{cases} (\gamma+3)s^\gamma - s^{2\gamma+2} & \text{for } \ 0 < s \le 1 \\ k_\gamma(2-s) & \text{for } \ 1 < s < 2 \\ k_\gamma(s+2) = k_\gamma(s) & \text{for } \ s \in R \setminus \{0, 2, 4, \dots\} \end{cases} . \tag{1.5}$$

For arbitrary functions $p_i :]a, b[\to R$ $(i = 1, 2)$ the writing $p_1(t) \not\equiv p_2(t)$ will mean that they are different from each other on a set of positive measure.

A solution of the problem (1.1), (1.2) (of problem the (1.1), (1.3)) is sought in the space of continuous functions $u : [a, b] \to R$ which are absolutely continuous together with their first derivative on every compact interval contained in $]a, b[$ (contained in $]a, b]$).

1.2. The problem (1.1), (1.2). We consider this problem in the case where

$$f^*(\cdot, \rho) \in L_{loc}(]a, b[) \ \text{ for } \ \rho \in R_+ \tag{1.6$_0$}$$

or

$$f^* \in M_{1,1}(]a, b[\times R_+). \tag{1.6}$$

Theorem 1.1. *Let there exist nonnegative functions $h_0 \in L_{1,1}(]a, b[)$ and $h \in M_{1,1}(]a, b[\times R_+)$ such that along with* (1.6$_0$) (*along with* (1.6)) *the conditions*

$$f(t, x)\operatorname{sgn} x \ge -h_0(t)|x| - h(t, |x|) \ \textit{ for } \ t \in]a, b[\ \ x \in R \tag{1.7}$$

and

$$\lim_{\rho \to +\infty} \frac{1}{\rho} \int_a^b (t-a)(b-t)h(t, \rho)\, dt = 0 \tag{1.8}$$

hold. Moreover, let either the function h_0 satisfy the inequality

$$\int_a^b (t-a)(b-t)h_0(t)\, dt \le b - a \tag{1.9}$$

or for some $\gamma \in]-2, +\infty[$ on the interval $]a, b[$ the condition

$$h_0(t) \le \frac{4}{(b-a)^2}\, k_\gamma\Big(\frac{2(t-a)}{b-a}\Big), \quad h_0(t) \not\equiv \frac{4}{(b-a)^2}\, k_\gamma\Big(\frac{2(t-a)}{b-a}\Big) \tag{1.10}$$

be fulfilled. Then for $c_1 = c_2 = 0$ (*for any* $c_i \in R$ $(i = 1,2)$), *problem* (1.1), (1.2) *is solvable.*

THEOREM 1.1′. *Let* $f(\cdot,0) \in L_{1,1}(]a,b[)$ *and there exist a nonnegative function* $h_0 \in L_{1,1}(]a,b[)$ *such that along with* (1.6_0) (*along with* (1.6)) *the condition*

$$f(t,x) - f(t,y) \geq -h_0(t)(x-y) \ \text{ for } \ t \in]a,b[\,, \quad x \geq y \tag{1.7′}$$

hold. Moreover, let either the function h_0 *satisfy the inequality* (1.9), *or for some* $\gamma \in]-2,+\infty[$ *on the interval* $]a,b[$ *the condition* (1.10) *be fulfilled. Then for* $c_1 = c_2 = 0$ (*for any* $c_i \in R$ $(i = 1,2)$), *problem* (1.1), (1.2) *is uniquely solvable.*

THEOREM 1.2. *Let there exist a natural number* n, *a number* $\gamma \in [4n-3,+\infty[$ *and nonnegative functions* $h_i \in L_{1,1}(]a,b[)$ $(i = 1,2)$, $h \in M_{1,1}(]a,b[\times R_+)$ *such that*

$$-h_2(t)|x| - h(t,|x|) \leq f(t,x)\operatorname{sgn} x \leq -h_1(t)|x| + h(t,|x|) \quad \text{for } \ t \in]a,b[\,, \ \ x \in R, \tag{1.11}$$

$$h_1(t) \geq \frac{4n^2}{(b-a)^2}\, k_\gamma\Big(\frac{2n(t-a)}{b-a}\Big), \quad h_1(t) \not\equiv \frac{4n^2}{(b-a)^2}\, k_\gamma\Big(\frac{2n(t-a)}{b-a}\Big) \tag{1.12}$$

and

$$\begin{gathered}\int_a^{a_n} (t-a)h_2(t)\,dt \leq 1, \quad \int_{a_n}^{b_n} h_2(t)\,dt \leq \frac{4n^2}{b_n - a_n}\,, \\ \int_{b_n}^{b} (b-t)h_2(t)\,dt \leq 1,\end{gathered} \tag{1.13}$$

where $a_n = a+(b-a)/4n$, $b_n = b-(b-a)/4n$. *If, moreover, the condition* (1.8) *is fulfilled, then the problem* (1.1), (1.2) *is solvable for any* $c_i \in R$ $(i = 1,2)$.

THEOREM 1.2′. *Let there exist a natural number* n, *a number* $\gamma \in [4n-3,+\infty[$ *and nonnegative functions* $h_i \in L_{1,1}(]a,b[)$ $(i = 1,2)$ *such that*

$$-h_2(t)(x-y) \leq f(t,x) - f(t,y) \leq -h_1(t)(x-y) \ \text{ for } \ t \in]a,b[\,, \ \ x \geq y, \tag{1.11′}$$

and let the conditions (1.12) *and* (1.13) *be fulfilled. If, moreover,* $f(\cdot,0) \in L_{1,1}(]a,b[)$, *then the problem* (1.1), (1.2) *is uniquely solvable for any* $c_i \in R$ $(i = 1,2)$.

REMARK 1.1. From the conditions (1.8) and (1.11) (from the condition (1.11′)) it follows that

$$h_1(t) \leq h_2(t) \ \text{ for } \ t \in]a,b[\,. \tag{1.14}$$

In this connection there naturally arises the question on the compatibility of the conditions (1.12)–(1.14), i.e. on the existence of functions $h_i \in L_{1,1}(]a,b[)$ $(i = 1,2)$, satisfying (1.12)–(1.14). Let us show that such are the functions given by the equalities

$$h_i(t) = \frac{\varepsilon_i}{(t-a)^\alpha (b-t)^\alpha} + \frac{4n^2}{(b-a)^2}\, k_\gamma\Big(\frac{2n(t-a)}{b-a}\Big) \quad (i = 1,2)$$

if $\gamma \geq 4n-3$, $0 \leq \alpha < 2$,

$$0 < \varepsilon_1 \leq \varepsilon_2 \leq \frac{2-\alpha}{8(2\gamma+3)}\,\Big(\frac{b-a}{4n}\Big)^{2\alpha-2}.$$

Indeed,

$$\begin{aligned}
\int_{a_n}^{b_n} h_2(t)\,dt &= \varepsilon_2 \int_{a_n}^{b_n} \frac{dt}{(t-a)^\alpha (b-t)^\alpha} + \frac{2n}{b-a}\int_{1/2}^{2n-1/2} k_\gamma(s)\,ds < \\
&< \varepsilon_2\Big(\frac{b-a}{4n}\Big)^{-2\alpha}(b-a) + \frac{4n^2}{b-a}\int_0^1 k_\gamma(s)\,ds = \\
&= \varepsilon_2\Big(\frac{4n}{b-a}\Big)^{2\alpha}(b-a) - \frac{4n^2}{(b-a)(2\gamma+3)} + \frac{4n^2}{b-a}\,\frac{\gamma+3}{\gamma+1} < \\
&< \frac{4n^2}{b-a}\,\frac{\gamma+3}{\gamma+1} = \frac{4n^2(2n-1)}{2n(b_n-a_n)}\,\frac{\gamma+3}{\gamma+1} \leq \frac{4n^2}{b_n-a_n}\,, \\
\int_a^{a_n} (t-a)h_2(t)\,dt &= \varepsilon_2 \int_a^{a_n} \frac{(t-a)^{1-\alpha}}{(b-t)^\alpha}\,dt + \int_0^{1/2} s k_\gamma(s)\,ds < \\
< \varepsilon_2\, \frac{(a_n-a)^{2-\alpha}}{(2-\alpha)(b-a_n)^\alpha} &+ \frac{\gamma+3}{\gamma+2}\,2^{-\gamma-2} < \frac{\varepsilon_2}{2-\alpha}\,\Big(\frac{b-a}{4n}\Big)^{2-2\alpha} + \frac{1}{6} < 1
\end{aligned}$$

and

$$\int_{b_n}^{b} (b-t)h_2(t)\,dt = \varepsilon_2 \int_{b_n}^{b} \frac{(b-t)^{1-\alpha}}{(t-a)^\alpha}\,dt + \int_0^{1/2} s k_\gamma(s)\,ds < 1.$$

Particular cases of the equation (1.1) are

$$u'' = \sum_{k=1}^{m} p_k(t)|u|^{\lambda_k}\operatorname{sgn} u + p(t)u + q(t), \tag{1.15}$$

$$u'' = p(t)u + q(t), \tag{1.16}$$

where p, q and $p_k \in L_{loc}(]a,b[)$ $(k = 1,\dots,m)$, and λ_k $(k = 1,\dots,m)$ are positive constants.

From Theorem 1.1 follows

COROLLARY 1.1. *Let*

$$p_k(t) \geq 0 \ \textit{for} \ a < t < b \ (k = 1, \dots, m), \quad q \in L_{1,1}(]a,b[), \tag{1.17}$$

and either the function p *satisfy the inequality*

$$\int_a^b (t-a)(b-t)[p(t)]_- \, dt \leq b - a \tag{1.18}$$

or for some $\gamma \in]-2, +\infty[$ *on the interval* $]a,b[$ *the condition*

$$[p(t)]_- \leq \frac{4}{(b-a)^2} k_\gamma\Big(\frac{2(t-a)}{b-a}\Big), \quad [p(t)]_- \not\equiv \frac{4}{(b-a)^2} k_\gamma\Big(\frac{2(t-a)}{b-a}\Big) \tag{1.19}$$

be fulfilled. Then for $c_1 = c_2 = 0$, *the problem* (1.15), (1.2) *is uniquely solvable, while for its unique solvability for any* $c_i \in R$ $(i = 1, 2)$ *it is necessary and sufficient that*

$$p_k \in L_{1,1}(]a,b[) \ (k = 1, \dots, m), \quad [p]_+ \in L_{1,1}(]a,b[). \tag{1.20}$$

Corollary 1.1 for the equation (1.14) takes the following form.

COROLLARY 1.2. *Let* $q \in L_{1,1}(]a,b[)$ *and either the function* p *satisfy the inequality* (1.18), *or for some* $\gamma \in]-2, +\infty[$ *on the interval* $]a,b[$ *the condition* (1.19) *be fulfilled. Then for* $c_1 = c_2 = 0$, *the problem* (1.16), (1.2) *is uniquely solvable, while for its unique solvability for any* $c_i \in R$ $(i = 1, 2)$ *it is necessary and sufficient that* $[p]_+ \in L_{1,1}(]a,b[)$.

REMARK 1.2. If $[p]_- \in L_{1,1}(]a,b[)$ and $[p]_+ \notin L_{1,1}(]a,b[)$, then the problem (1.16), (1.2) has the "semi-Fredholm" property because the absence of a nontrivial solution in the homogeneous problem

$$u'' = p(t)u, \tag{1.16$_0$}$$

$$u(a) = 0, \quad u(b) = 0 \tag{1.2$_0$}$$

guarantees the unique solvability only of the semi-homogeneous problem (1.16), (1.2$_0$) for any $q \in L_{1,1}(]a,b[)$. As regards the non-homogeneous problem (1.16), (1.2), where $c_i \neq 0$ $(i = 1, 2)$, in this case it has no solution. For example, if $q \in L_{1,1}(]a,b[)$ and

$$p(t) \geq \delta(t-a)^{-\alpha}(b-t)^{-\beta},$$

where $\delta > 0$, $\alpha \geq 2$, $\beta \geq 2$, then by Corollary 1.2, the problem (1.16), (1.2$_0$) is uniquely solvable, while the problem (1.16), (1.2), where $c_i \neq 0$ $(i = 1, 2)$, has no solution.

Theorem 1.2$'$ leads to

COROLLARY 1.3. *Let $q \in L_{1,1}(]a, b[)$ and there exist a natural number n and a number $\gamma \in [4n - 3, +\infty[$ such that on the interval $]a, b[$ the condition*

$$p(t) \leq -\frac{4n^2}{(b-a)^2} k_\gamma\Big(\frac{2n(t-a)}{b-a}\Big), \quad p(t) \not\equiv -\frac{4n^2}{(b-a)^2} k_\gamma\Big(\frac{2n(t-a)}{b-a}\Big) \tag{1.21}$$

holds and

$$\begin{gathered} \int_a^{a_n} (t-a)|p(t)|\,dt \leq 1, \quad \int_{a_n}^{b_n} |p(t)|\,dt \leq \frac{4n^2}{b_n - a_n}, \\ \int_{b_n}^{b} (b-t)|p(t)|\,dt \leq 1. \end{gathered} \tag{1.22}$$

Then for arbitrary $c_i \in R$ $(i = 1, 2)$, the problem (1.16), (1.2) *is uniquely solvable.*

According to Remark 1.1, Corollary 1.3 results in

COROLLARY 1.4. *Let $q \in L_{1,1}(]a, b[)$, and there exist a natural number n and numbers $\alpha \in [0, 2[$ and $\gamma \in [4n-3, +\infty[$ such that on the interval $]a, b[$ the condition*

$$-\frac{\varepsilon}{(t-a)^\alpha (b-t)^\alpha} - \frac{4n^2}{(b-a)^2} k_\gamma\Big(\frac{2n(t-a)}{b-a}\Big) \leq p(t) < -\frac{4n^2}{(b-a)^2} k_\gamma\Big(\frac{2n(t-a)}{b-a}\Big)$$

holds, where

$$\varepsilon = \frac{2-\alpha}{8(2\gamma+3)} \Big(\frac{b-a}{4n}\Big)^{2\alpha - 2}.$$

Then for arbitrary $c_i \in R$ $(i = 1, 2)$, the problem (1.16), (1.2) *is uniquely solvable.*

EXAMPLE 1.1. Let n be a natural number, $\gamma > -2$,

$$v_\gamma(s) = \begin{cases} s \exp\Big(-\dfrac{s^{\gamma+2}}{\gamma+2}\Big) & \text{for } \ 0 \leq s \leq 1 \\ v_\gamma(2-s) & \text{for } \ 1 < s \leq 2 \\ -v_\gamma(2+s) & \text{for } \ s \in R \end{cases}, \tag{1.23}$$

$$p(t) = -\frac{4n^2}{(b-a)^2} k_\gamma\Big(\frac{2n(t-a)}{b-a}\Big), \quad q(t) = v_\gamma\Big(\frac{2n(t-a)}{b-a}\Big). \tag{1.24}$$

It immediately follows from the equalities (1.5), (1.23) and (1.24) that the homogeneous problem $(1.16_0), (1.2_0)$ has a nontrivial solution $u_0(t) \equiv q(t)$, while the semi-homogeneous problem $(1.16), (1.2_0)$ has no solution because

$$\int_a^b u_0(t)q(t)\,dt = \int_a^b q^2(t)\,dt > 0.$$

On the other hand, by Remark 1.1, if $\gamma \geq 4n-3$, then the function p satisfies all conditions of Corollary 1.3, except the inequality

$$p(t) \not\equiv -\frac{4}{(b-a)^2}\,k_\gamma\Big(\frac{2n(t-a)}{b-a}\Big), \tag{1.25}$$

and the functions $f(t,x) \equiv p(t)x + q(t)$, $h_i(t) \equiv |p(t)|$ $(i = 1,2)$, $h(t,x) \equiv |q(t)|$ satisfy all conditions of Theorems 1.2 and 1.2′, except the inequality

$$h_1(t) \not\equiv \frac{4n^2}{(b-a)^2}\,k_\gamma\Big(\frac{2n(t-a)}{b-a}\Big). \tag{1.26}$$

Thus the example constructed above shows that the condition (1.26) (condition (1.25)) in Theorems 1.2 and 1.2′ (in Corollary 1.3)) is essential and it cannot be neglected.

Consider now Example 1.1 for the case $n = 1$. For the preassigned $\varepsilon \in]0,1[$ we put $\gamma = 2/\varepsilon - 1$ and, taking into account (1.24), we find that

$$\int_a^b (t-a)(b-a)[p(t)]_-\,dt < \frac{(b-a)^2}{4}\int_a^b [p(t)]_-\,dt = \int_a^b k_\gamma\Big(\frac{2(t-a)}{b-a}\Big)\,dt =$$
$$= (b-a)\int_0^1 k_\gamma(s)\,ds = (b-a)\Big(\frac{\gamma+3}{\gamma+1} - \frac{1}{2\gamma+3}\Big) < (1+\varepsilon)(b-a).$$

Thus the condition (1.9) (condition (1.18)) in Theorems 1.1 and 1.1′ (in Corollary 1.2)) is unimprovable in the sense that it cannot be replaced by the condition

$$\int_a^b (t-a)(b-t)h_0(t)\,dt < (1+\varepsilon)(b-a)$$
$$\Big(\int_a^b (t-a)(b-t)[p(t)]_-\,dt < (1+\varepsilon)(b-a)\Big)$$

no matter how small $\varepsilon > 0$ is.

1.3. Problem (1.1), (1.3). We investigate this problem in the cases where

$$f^*(\cdot,\rho) \in L_{loc}(]a,b])\ \text{ for }\ \rho \in R_+, \tag{1.27_0}$$

or

$$f^* \in M_{1,0}(]a,b[\times R_+). \tag{1.27}$$

THEOREM 1.3. *Let there exist nonnegative functions* $h_0 \in L_{1,0}(]a,b[)$ *and* $h \in M_{1,0}(]a,b[\times R_+])$ *such that along with* (1.27_0) (*along with* (1.27)) *condition* (1.7) *is fulfilled and*

$$\lim_{\rho\to+\infty}\frac{1}{\rho}\int_a^b (t-a)h(t,\rho)\,dt = 0. \tag{1.28}$$

Let, moreover, either the function h_0 *satisfy the inequality*

$$\int_a^b (t-a)h_0(t)\,dt \le 1, \tag{1.29}$$

or for some $\gamma \in]-2,+\infty[$ *on the interval* $]a,b[$ *the condition*

$$h_0(t) \le \frac{1}{(b-a)^2}\,k_\gamma\Big(\frac{t-a}{b-a}\Big), \quad h_0(t) \not\equiv \frac{1}{(b-a)^2}\,k_\gamma\Big(\frac{t-a}{b-a}\Big) \tag{1.30}$$

be fulfilled. Then for $c_1 = 0$ *and any* $c_2 \in R$ (*for any* $c_i \in R$ $(i = 1,2)$) *the problem* (1.1), (1.3) *is solvable.*

THEOREM 1.3′. *Let* $f(\cdot,0) \in L_{1,0}(]a,b[)$ *and there exist a nonnegative function* $h_0 \in L_{1,0}(]a,b[)$ *such that along with* (1.27_0) (*along with* (1.27)) *condition* (1.7′) *is fulfilled. Let, moreover, either the function* h_0 *satisfy the inequality* (1.29), *or for some* $\gamma \in]-2,+\infty[$ *on the interval* $]a,b[$ *condition* (1.30) *be fulfilled. Then for* $c_1 = 0$ *and any* $c_2 \in R$ (*for arbitrary* $c_i \in R$ $(i = 1,2)$) *problem* (1.1), (1.3) *is uniquely solvable.*

THEOREM 1.4. *Let there exist a natural number* n, *a number* $\gamma \in [3n-1,+\infty[$ *and nonnegative functions* $h_i \in L_{1,0}(]a,b[)$ $(i = 1,2)$, $h \in M_{1,0}(]a,b[\times R_+])$ *such that along with* (1.11) *and* (1.28) *the conditions*

$$\begin{aligned} h_1(t) &\ge \frac{(2n-1)^2}{(b-a)^2}\,k_\gamma\Big(\frac{(2n-1)(t-a)}{b-a}\Big),\\ h_1(t) &\not\equiv \frac{(2n-1)^2}{(b-a)^2}\,k_\gamma\Big(\frac{(2n-1)(t-a)}{b-a}\Big) \end{aligned} \tag{1.31}$$

and

$$\int_a^{a_n} (t-a)h_2(t)\,dt \le 1, \quad \int_{a_n}^b h_2(t)\,dt \le \frac{4n^2}{b-a_n}\,, \tag{1.32}$$

where $a_n = a + (b-a)/(4n-2)$, *are fulfilled. Then for any* $c_i \in R$ $(i = 1,2)$ *the problem* (1.1), (1.3) *is solvable.*

THEOREM 1.4′. *Let there exist a natural number n, a number $\gamma \in [3n-1,+\infty[$ and nonnegative functions $h_i \in L_{1,0}(]a,b[)$ $(i=1,2)$ such that the conditions* (1.11′), (1.31) *and* (1.32) *are fulfilled. Then for any $c_i \in R$ $(i=1,2)$ the problem* (1.1), (1.3) *is uniquely solvable.*

From Theorem 1.3′ for the equation (1.15) we have the following

COROLLARY 1.5. *Let $p_k(t) \geq 0$ for $a < t < b$ $(k=1,\dots,m)$, $p_k \in L_{loc}(]a,b])$ $(k=1,\dots,m)$, $p \in L_{loc}(]a,b])$, $q \in L_{1,0}(]a,b[)$ and either the function $[p]_-$ satisfy the inequality*

$$\int_a^b (t-a)[p(t)]_-\,dt \leq 1, \tag{1.33}$$

or for some $\gamma \in]-2,+\infty[$ on the interval $]a,b[$ the condition

$$[p(t)]_- \leq \frac{1}{(b-a)^2}\,k_\gamma\Big(\frac{t-a}{b-a}\Big), \quad [p(t)]_- \not\equiv \frac{1}{(b-a)^2}\,k_\gamma\Big(\frac{t-a}{b-a}\Big) \tag{1.34}$$

be fulfilled. Then for $c_1 = 0$ and any $c_2 \in R$ the problem (1.15), (1.3) *is uniquely solvable, while for its unique solvability for any $c_i \in R$ $(i=1,2)$ it is necessary and sufficient that*

$$p_k \in L_{1,0}(]a,b[) \ \ (k=1,\dots,m), \quad [p]_+ \in L_{1,0}(]a,b[).$$

For equation (1.16) Corollary 1.5 takes the following form

COROLLARY 1.6. *Let $p \in L_{loc}(]a,b])$, $q \in L_{1,0}(]a,b[)$ and either $[p]_-$ satisfy the inequality* (1.33), *or for some $\gamma \in]-2,+\infty[$ on the interval $]a,b[$ the condition* (1.34) *be fulfilled. Then for $c_1 = 0$ and any $c_2 \in R$ the problem* (1.16), (1.3) *is uniquely solvable, while for its unique solvability for any $c_i \in R$ $(i=1,2)$ it is necessary and sufficient that $[p]_+ \in L_{1,0}(]a,b[)$.*

REMARK 1.3. If $[p]_- \in L_{1,0}(]a,b[)$, $[p]_+ \notin L_{1,0}(]a,b[)$ and $q \in L_{1,0}(]a,b[)$, then the problem (1.16), (1.3) has the "semi-Fredholm" property, because the absence of a nontrivial solution in the homogeneous equation (1.16_0), satisfying the boundary condition

$$u(a) = 0, \quad u'(b) = 0, \tag{1.35}$$

guarantees the unique solvability of the problem (1.16), (1.3) for $c_1 = 0$ only. For example, if

$$p(t) \geq \delta(t-a)^{-\alpha} \ \text{ for } \ a < t < b,$$

where $\delta > 0$ and $\alpha \geq 2$, then by Corollary 1.6, problem $(1.16), (1.3)$ is uniquely solvable if and only if $c_1 = 0$.

From Theorem 1.4′ for the equation (1.16) we have

Corollary 1.7. *Let $q \in L_{1,0}(]a, b[)$ and there exist a natural number n and a number $\gamma \in [3n-1, +\infty[$ such that on the interval $]a, b[$ the condition*

$$p(t) \leq -\frac{(2n-1)^2}{(b-a)^2}\, k_\gamma\Big(\frac{(2n-1)(t-a)}{b-a}\Big),$$
$$p(t) \not\equiv -\frac{(2n-1)^2}{(b-a)^2}\, k_\gamma\Big(\frac{(2n-1)(t-a)}{b-a}\Big)$$

is satisfied, and

$$\int_a^{a_n} (t-a)|p(t)|\,dt \leq 1, \quad \int_{a_n}^b |p(t)|\,dt \leq \frac{4n^2}{b-a_n},$$

where $a_n = a + (b-a)/(4n-2)$. Then for arbitrary $c_i \in R$ $(i = 1, 2)$ the problem $(1.16), (1.3)$ *is uniquely solvable.*

From this corollary we arrive immediately at

Corollary 1.8. *Let $q \in L_{1,0}(]a, b[)$ and there exist a natural number n and numbers $\gamma \in [3n-1, +\infty[$ and $\alpha \in [0, 2[$ such that on the interval $]a, b[$ the condition*

$$-\frac{\varepsilon}{(t-a)^\alpha} - \frac{(2n-1)^2}{(b-a)^2}\, k_\gamma\Big(\frac{(2n-1)(t-a)}{b-a}\Big) \leq p(t) <$$
$$< -\frac{(2n-1)^2}{(b-a)^2}\, k_\gamma\Big(\frac{(2n-1)(t-a)}{b-a}\Big)$$

be fulfilled, where

$$\varepsilon = \frac{2-\alpha}{8(2\gamma+3)}\Big(\frac{b-a}{4n-2}\Big)^{\alpha-2}.$$

Then for arbitrary $c_i \in R$ $(i = 1, 2)$ the problem $(1.16), (1.3)$ *is uniquely solvable.*

Example 1.2. Let n be a natural number, $\gamma > -2$, and let v_γ be the function given by (1.23). Suppose that

$$p(t) = -\frac{(2n-1)^2}{(b-a)^2}\, k_\gamma\Big(\frac{(2n-1)(t-a)}{b-a}\Big), \quad q(t) = v_\gamma\Big(\frac{(2n-1)(t-a)}{b-a}\Big).$$

Then the homogeneous problem $(1.16_0), (1.3_0)$ has a nontrivial solution $u_0(t) \equiv q(t)$ and the problem $(1.16), (1.3_0)$ has no solution because

$$\int_a^b q(t)u_0(t)\,dt = \int_a^b q^2(t)\,dt > 0.$$

This example shows that in Theorems 1.4 and 1.4′ (in Corollary 1.7) the inequality

$$h_1(t) \not\equiv \frac{(2n-1)^2}{(b-a)^2}\, k_\gamma\Big(\frac{(2n-1)(t-a)}{b-a}\Big)$$
$$\Big(p(t) \not\equiv -\frac{(2n-1)^2}{(b-a)^2}\, k_\gamma\Big(\frac{(2n-1)(t-a)}{b-a}\Big)\Big)$$

is essential and we cannot neglect it.

On the other hand, if $n=1$, $\varepsilon \in]0,1/2[$ and $\gamma = 1/\varepsilon - 2$, we have

$$\int_a^b (t-a)[p(t)]_-\,dt = \int_0^1 s k_\gamma(s)\,ds = \frac{\gamma+3}{\gamma+2} - \frac{1}{2\gamma+4} < 1+\varepsilon.$$

Consequently, Example 1.2 also shows that in Theorems 1.4 and 1.4′ (in Corollary 1.6), condition (1.29) (condition (1.33)) cannot be replaced by the condition

$$\int_a^b (t-a)h_0(t)\,dt < 1+\varepsilon \quad \Big(\int_a^b (t-a)[p(t)]_-\,dt < 1+\varepsilon\Big)$$

no matter how small $\varepsilon > 0$ is.

§ 2. Auxiliary Propositions.

2.1. Radon's Lemma. Below we will use the following

Lemma 2.1. *If $\lambda > 1$, $\alpha_k > 0$, $\beta_k > 0$ $(k=1,\dots,m)$, then*

$$\sum_{k=1}^m \frac{\alpha_k^\lambda}{\beta_k^{\lambda-1}} \geq \frac{(\sum\limits_{k=1}^m \alpha_k)^\lambda}{(\sum\limits_{k=1}^m \beta_k)^{\lambda-1}}\,.$$

This lemma, due to J. Radon (see [3], Theorem 65), is a simple modification of the well-known O. Hölder's lemma. Indeed, if we put

$$\mu = \frac{\lambda}{\lambda-1}\,,$$

then by means of Hölder's inequality we can find that

$$\sum_{k=1}^m \alpha_k = \sum_{k=1}^m \frac{\alpha_k}{\beta_k^{1/\mu}}\,\beta_k^{1/\mu} \leq \Big(\sum_{k=1}^m \frac{\alpha_k^\lambda}{\beta_k^{\lambda-1}}\Big)^{1/\lambda}\Big(\sum_{k=1}^m \beta_k\Big)^{\frac{\lambda-1}{\lambda}}.$$

2.2. Lemmas on the properties of solutions of second order linear singular differential equations. Consider a linear differential equation

$$u'' = p(t)u \tag{2.1}$$

with the coefficient $p \in L_{loc}(]a, b[)$.

Lemma 1.1 from [15] leads to

LEMMA 2.2. *If $p \in L_{1,1}(]a, b[)$, then:*

(i) *an arbitrary solution u of the equation* (2.1) *at the points a and b has, respectively, finite right and left limits $u(a+)$ and $u(b-)$; moreover if $u(a+) = 0$ $((u(b-) = 0)$, then there exists the finite limit $u(a+)$ $((u(b-))$;*

(ii) *the equation* (2.1) *has a unique solution satisfying the initial conditions*

$$u(a+) = 0, \quad u'(a+) = 1; \tag{2.2}$$

(iii) *the equation* (2.1) *has a unique solution satisfying the initial conditions*

$$u(b-) = 0, \quad u'(b-) = -1. \tag{2.3}$$

On the basis of the above lemma, we can easily prove that Sturm's lemmas remain true for singular differential equations as well. More precisely, the following two lemmas are valid.

LEMMA 2.3. *Let p and $p_0 \in L_{1,1}(]a, b[)$,*

$$p(t) \geq p_0(t) \ \ \text{for} \ \ a < t < b, \quad p(t) \not\equiv p_0(t), \tag{2.4}$$

and the equation (2.1) *have a non-trivial solution u satisfying the boundary conditions*

$$u(a+) = 0, \quad u(b-) = 0. \tag{2.5}$$

Then an arbitrary solution of the equation (2.1)*, linearly independent of u, and an arbitrary solution of the equation*

$$v'' = p_0(t)v \tag{2.6}$$

have at least one zero in the interval $]a, b[$.

Lemma 2.4. *Let p and $p_0 \in L_{1,0}(]a,b[)$, the condition* (2.4) *be fulfilled and the equation* (2.1) *have a nontrivial solution u, satisfying the boundary conditions*

$$u(a+) = 0, \quad u(b)u'(b) \le 0.$$

Then an arbitrary solution of the equation (2.6) *either has at least one zero in the interval $]a,b[$, or satisfies the inequality*

$$v(b)v'(b) < 0.$$

Applying Lemma 2.1, we can prove just in the same way as in the classical case (see [5], or [4], Ch. XI, § 5) that the following lemmas of Liapounoff–Hartman–Wintner type are valid.

Lemma 2.5. *Let $p \in L_{1,1}(]a,b[)$ and the inequality* (1.18) *be fulfilled. Then any solution of the problem* (2.1), (2.2) *satisfies the condition*

$$u(t) > 0 \ \ for \ \ a < t < b, \quad u(b-) > 0,$$

while any solution of problem (2.1), (2.3) *satisfies the condition*

$$u(t) > 0 \ \ for \ \ a < t < b, \quad u(a+) > 0.$$

Lemma 2.6. *Let $p \in L_{1,0}(]a,b[)$ and the inequality* (1.33) *be fulfilled* ($p \in L_{1,0}([a,b])$ *and* $\int_a^b (b-t)[p(t)]_- \, dt \le 1$). *Then any solution of the problem* (2.1), (2.2) (*of the problem* (2.1), (2.3)) *satisfies the condition*

$$u'(t) > 0 \ \ for \ \ a < t \le b \quad (u'(t) > 0 \ \ for \ \ a \le t < b).$$

Lemma 2.7. *Let the function $p : [a_0, b_0] \to R$ be integrable and there exist a nontrivial solution u of the equation* (2.1) *having at least n zeros in the interval $]a_0, b_0[$, such that*

$$u'(a_0) = 0, \quad u'(b_0) = 0. \tag{2.7}$$

Then

$$\int_{a_0}^{b_0} [p(t)]_- \, dt > \frac{4n^2}{b_0 - a_0}. \tag{2.8}$$

Proof. Let $t_i \in]a_0, b_0[$ $(i = 1, \dots, n)$ be the zeros of the function u; moreover if $n > 1$, then

$$t_i < t_{i+1} \ \ (i = 1, \dots, n-1).$$

From Lemma 2.6 it follows that

$$\int_{a_0}^{t_1} (t_1 - t)[p(t)]_- \, dt > 1, \quad \int_{t_n}^{b_0} (t - t_n)[p(t)]_- \, dt > 1$$

and hence

$$\int_{a_0}^{t_1} [p(t)]_- \, dt > \frac{1}{t_1 - a_0}, \quad \int_{t_n}^{b_0} [p(t)]_- \, dt > \frac{1}{b_0 - t_n}.$$

On the other hand, if $n > 1$, then by Lemma 2.5 we have

$$\int_{t_i}^{t_{i+1}} (t_{i+1} - t)(t - t_i)[p(t)]_- \, dt > t_{i+1} - t_i \ \ (i = 1, \dots, n-1).$$

However,

$$(t_{i+1} - t)(t - t_i) \le (t_{i+1} - t_i)^2/4 \ \text{ for } \ t_i \le t \le t_{i+1}.$$

Therefore

$$\int_{t_i}^{t_{i+1}} [p(t)]_- \, dt > \frac{4}{t_{i+1} - t_i} \ \ (i = 1, \dots, n-1).$$

Thus if $n = 1$, we have

$$\int_{a_0}^{b_0} [p(t)]_- \, dt > \frac{1}{t_1 - a_0} + \frac{1}{b_0 - t_1},$$

while if $n > 1$, we get

$$\int_{a_0}^{b_0} [p(t)]_- \, dt > \frac{1}{t_1 - a} + \sum_{i=1}^{n-1} \frac{4}{t_{i+1} - t_i} + \frac{1}{b_0 - t_n}.$$

By virtue of Lemma 2.1, the above inequalities result in (2.8). □

Let us now introduce

Definition 2.1. *We say that the function $p :]a, b[\to R$ belongs to the set $\mathcal{U}_0(]a, b[)$ (to the set $\mathcal{U}_0'(]a, b[)$) if $p \in L_{1,1}(]a, b[)$ ($p \in L_{1,0}(]a, b[)$), and the solution of the problem* (2.1), (2.2) *satisfies the condition*

$$u(t) > 0 \ \text{ for } \ a < t < b, \quad u(b-) > 0 \quad (\, u'(t) > 0 \ \text{ for } \ a < t \le b \,).$$

Definition 2.2. *Let k be a natural number. We say that the function $p :]a,b[\to R$ belongs to the set $\mathcal{U}_k(]a,b[)$ (to the set $\mathcal{U}'_k(]a,b[)$) if $p \in L_{1,1}(]a,b[)$ ($p \in L_{1,0}(]a,b[)$), the solution u of problem* (2.1), (2.2) *has exactly k zeros* (*not less than $k-1$ and no more than k zeros*) *in the interval $]a,b[$ and $u(b-) > 0$* (*$(-1)^k u'(b) > 0$*).

Lemma 2.8. *Let $p \in L_{1,1}(]a,b[)$ and there exist a natural number n and numbers $a_* \in]a,b[$ and $b_* \in]a_0,b[$ such that*

$$\int_a^{a_*} (t-a)[p(t)]_- \, dt \le 1, \tag{2.9}$$

$$\int_{a_*}^{b_*} [p(t)]_- \, dt \le \frac{4n^2}{b_* - a_*}\,, \tag{2.10}$$

$$\int_{b_*}^{b} (b-t)[p(t)]_- \, dt \le 1. \tag{2.11}$$

Let, moreover, the solution u of the problem (2.1), (2.2) *has at least n zeros in the interval $]a,b[$. Then $p \in \mathcal{U}_n(]a,b[)$.*

Proof. Suppose that the above lemma is invalid. Then there would exist numbers $t^* \in]a,b]$ and $t_i \in]a,t^*[$ $(i = 1,\dots,n)$ such that

$$u(t_i) = 0 \;\; (i = 1,\dots,n) \quad \text{and} \quad u(t^*-) = 0.$$

In case $n > 1$, we will assume that $t_i < t_{i+1}$ $(i = 1,\dots,n-1)$.

By Roll's theorem and Lemma 2.7, there exist $a_0 \in]a,t_1[$ and $b_0 \in]t_n,t^*[$ such that conditions (2.7) and (2.8) are fulfilled. On the other hand, by Lemma 2.6 and the inequality (2.9), we have

$$a_0 > a_*. \tag{2.12}$$

Taking this fact into account, the inequalities (2.8) and (2.10) imply that

$$b_* < b_0 < t^*. \tag{2.13}$$

By Lemma 2.6,

$$\int_{b_0}^{t^*} (t^* - t)[p(t)]_- \, dt > 1.$$

On the other hand, by the inequalities (2.11) and (2.13) we have

$$\int_{b_0}^{t^*} (t^* - t)[p(t)]_- \, dt \le \int_{b_*}^{b} (b-t)[p(t)]_- \, dt \le 1.$$

The obtained contradiction proves the lemma. □

LEMMA 2.9. *Let* $p \in L_{1,0}(]a,b[)$ *and there exist a natural number* n *and a number* $a_* \in]a,b[$ *such that along with* (2.9) *the inequality*

$$\int_{a_*}^{b} [p(t)]_- \, dt \le \frac{4n^2}{b - a_*} \tag{2.14}$$

is fulfilled and the solution u *of the problem* (2.1), (2.2) *has at least* n *zeros in the interval* $]a,b[$. *Then* $p \in \mathcal{U}'_n(]a,b[)$.

Proof. By (2.14), there exists $b_* \in]a,b[$ such that the inequalities (2.10) and (2.11) are fulfilled. This, by Lemma 2.8, implies that $p \in \mathcal{U}_n(]a,b[)$. Hence the function u in the interval $]a,b[$ has exactly n zeros and $u(b) \neq 0$.

Let $t_i \in]a,b[$ $(i = 1,\dots,n)$ be the zeros of the function u numbered in increasing order.Then

$$(-1)^n u'(t_n) > 0 \tag{2.15}$$

and there exists $a_0 \in]a,t_1[$ such that $u'(a_0) = 0$. Moreover, by condition (2.9) and Lemma 2.6, the inequality (2.12) is satisfied.

To complete the proof of our lemma, it remains to show that

$$(-1)^n u'(b) > 0.$$

Suppose to the contrary that $(-1)^n u'(b) \le 0$. Then by (2.15) there exists $b_0 \in]t_n, b]$ such that $u'(b_0) = 0$. Consequently, the conditions of Lemma 2.7 are fulfilled which guarantee the fulfilment of the inequality (2.8). But that impossible in view of the inequalities (2.12) and (2.14). The obtained contradiction proves the lemma. □

2.3. Lemmas on solvability and unique solvability of the problem (1.1), (1.2).

LEMMA 2.10. *Let there exist functions* $h_0 \in L_{1,1}(]a,b[)$ *and* $h \in M_{1,1}(]a,b[\times R_+)$ *such that along with* (1.6_0) (*along with* (1.6)) *the conditions* (1.7), (1.8) *and*

$$-h_0 \in \mathcal{U}_0(]a,b[) \tag{2.16}$$

are fulfilled. Then for $c_1 = c_2 = 0$ (*for any* $c_i \in R$ $(i = 1,2)$) *the problem* (1.1), (1.2) *is solvable.*

Proof. For $c_1 = c_2 = 0$, the validity of the lemma follows from Theorem 11.4 of the monograph [9]. It remains for us to prove the case where the condition (1.6) is fulfilled and c_i $(i=1,2)$ are arbitrary. In this case the problem (1.1), (1.2) is equivalent to the problem

$$v''=\widetilde{f}(t,v);\quad v(a)=0,\ \ v(b)=0, \tag{2.17}$$

where

$$\widetilde{f}(t,x)=f(t,x+u_0(t)),\quad u_0(t)=\frac{b-t}{b-a}\,c_1+\frac{t-a}{b-a}\,c_2.$$

Set

$$h_*(t)=\max\{|f(t,x)|:\ 0\le x\le 2|c_1|+2|c_2|\},$$
$$\widetilde{h}(t,\rho)=h_*(t)+(|c_1|+|c_2|)|h_0(t)|+h(t,\rho+|c_1|+|c_2|).$$

Then on the basis of the inequality (1.7), we find that

$$\widetilde{f}(t,x)\operatorname{sgn}x\ge -h_0(t)|x|-|h_0(t)|\,|x|-|f(t,x+u_0(t))|\ge$$
$$\ge -h_0(t)|x|-(|c_1|+|c_2|)|h_0(t)|-h_*(t)\ \text{ for }\ a<t<b,\ \ |x|\le|c_1|+|c_2|,$$
$$\widetilde{f}(t,x)\operatorname{sgn}x=f(t,x+u_0(t))\operatorname{sgn}(x+u_0(t))\ge$$
$$\ge -h_0(t)|x+u_0(t)|-h(t,|x+u_0(t)|)\ge$$
$$\ge -h_0(t)|x|-(|c_1|+|c_2|)|h_0(t)|-h(t,|x|+|c_1|+|c_2|)$$
$$\text{for }\ a<t<b,\ \ |x|>|c_1|+|c_2|,$$

and hence

$$\widetilde{f}(t,x)\operatorname{sgn}x\ge -h_0(t)|x|-\widetilde{h}(t,|x|)\ \text{ for }\ a<t<b,\ \ x\in R. \tag{2.18}$$

On the other hand, the conditions (1.6) and (1.8) imply that

$$\widetilde{h}\in M_{1,1}(]a,b[\times R_+),\quad \lim_{\rho\to+\infty}\frac{1}{\rho}\int_a^b(t-a)(t-b)\widetilde{h}(t,\rho)\,dt=0.$$

However, these conditions, as is said above, along with the conditions (2.16) and (2.18) guarantee the solvability of the problem (2.17). □

LEMMA 2.10′. *Let $f(\cdot,0)\in L_{1,1}(]a,b[)$ and there exist $h_0\in L_{1,1}(]a,b[)$ such that along with* (1.6_0) (*along with* (1.6)) *the conditions* (1.7′) *and* (2.16) *are fulfilled. Then for $c_1=c_2=0$ (for any $c_i\in R$ $(i=1,2)$) the problem* (1.1), (1.2) *is uniquely solvable.*

Proof. The condition (1.7) with $h(t,\rho)\equiv|f(t,0)|$, follows from the condition (1.7′). Moreover, $h\in M_{1,1}(]a,b[\times R_+)$ and (1.8) is fulfilled because $f(\cdot,0)\in L_{1,1}(]a,b[)$. If we now apply Lemma 2.10, the solvability of the problem (1.1), (1.2) becomes evident. As for the uniqueness of the solution, it follows from Theorem 11.6 of the monograph [9]. □

Theorems 11.5 and 11.17 of the monograph [9] result in

LEMMA 2.11. *Let there exist functions* $h_i \in L_{1,1}(]a,b[)$ $(i=1,2)$ *and* $h \in M_{1,1}(]a,b[\times R_+)$ (*let* $f(\cdot,0) \in L_{1,1}(]a,b[)$ *and there exist functions* $h_i \in L_{1,1}(]a,b[)$ $(i=1,2)$) *and a natural number* n *such that along with* (1.8) *and* (1.11) (*along with* (1.11′)) *the conditions*

$$-h_i \in \mathcal{U}_n(]a,b[) \quad (i=1,2) \tag{2.19}$$

are fulfilled. Then for any $c_i \in R$ $(i=1,2)$ *the problem* (1.1), (1.2) *is solvable* (*uniquely solvable*).

2.4. Lemmas on solvability and unique solvability of the problem (1.1), (1.3).

LEMMA 2.12. *Let there exist functions* $h_0 \in L_{1,0}(]a,b[)$ *and* $h \in M_{1,0}(]a,b[\times R_+)$ *such that along with* (1.27_0) (*along with* (1.27)) *the conditions* (1.7), (1.28) *and*

$$-h_0 \in \mathcal{U}_0'(]a,b[) \tag{2.20}$$

are fulfilled. Then for $c_1 = 0$ *and any* $c_2 \in R$ (*for any* $c_i \in R$ $(i=1,2)$) *the problem* (1.1), (1.3) *is solvable.*

LEMMA 2.12′. *Let* $f(\cdot,0) \in L_{1,0}(]a,b[)$ *and there exist a function* $h_0 \in L_{1,0}(]a,b[)$ *such that along with* (1.27_0) (*along with* (1.27)) *the conditions* (1.7′) *and* (2.20) *are fulfilled. Then for* $c_1 = 0$ *and any* $c_2 \in R$ (*for arbitrary* $c_i \in R$ $(i=1,2)$) *the problem* (1.1), (1.3) *is uniquely solvable.*

LEMMA 2.13. *Let there exist functions* $h_i \in L_{1,0}(]a,b[)$ $(i=1,2)$ *and* $h \in M_{1,0}(]a,b[\times R_+)$ (*let* $f(\cdot,0) \in L_{1,0}(]a,b[)$ *and there exist functions* $h_i \in L_{1,0}(]a,b[)$ $(i=1,2)$) *and a natural number* n *such that along with* (1.8) *and* (1.11) (*along with* (1.11′)) *the conditions*

$$-h_i \in \mathcal{U}_n'(]a,b[) \quad (i=1,2) \tag{2.21}$$

are fulfilled. Then for arbitrary $c_i \in R$ $(i=1,2)$ *the problem* (1.1), (1.3) *is solvable* (*uniquely solvable*).

We omit the proofs of Lemmas 2.12 and 2.12′ (of Lemma 2.13) because they are similar to those of Lemmas 2.10 and 2.10′ (of the Theorems 11.5 and 11.17 of monograph [9]).

3. Proof of the Main Results.

Proof of Theorems 1.1 *and* 1.1′. According to Lemmas 2.10 and 2.10′, to prove Theorems 1.1 and 1.1′ it suffices to show that if the function $h_0 \in$

$L_{1,1}(]a,b[)$ satisfies one of conditions (1.9) or (1.10), then it satisfies the condition (2.16) as well.

By Lemma 2.5, the inequality (1.9) guarantees the validity of the inclusion (2.16). Thus it remains for us to consider the case where the condition (1.10) is fulfilled.

Let v_γ be the function given by (1.23) and let $v(t) = v_\gamma(\frac{2(t-a)}{b-a})$. Then v is a solution of the equation

$$v'' = -\frac{4}{(b-a)^2}\,k_\gamma\Big(\frac{2(t-a)}{b-a}\Big)v$$

satisfying the conditions

$$v(a+) = 0,;v(b-) = 0, \quad v(t) > 0 \ \text{ for } \ a < t < b.$$

This, by Lemma 2.3 and (1.10), implies that the solution u of the initial value problem

$$u'' = -h_0(t)u; \quad u(a+) = 0, \ \ u(a+) = 1$$

satisfies the condition

$$u(t) > 0 \ \text{ for } \ a < t, b, \quad u(b-) > 0.$$

Consequently, the inclusion (2.16) is valid. □

Proof of Theorems 1.2 *and* 1.2′. First of all, it should be noted that the inequality (1.14) follows from the (1.8) and (1.11) (from (1.11′)). For every $i \in \{1,2\}$ we denote by u_i the solution of the initial value problem

$$u'' = -h_i(t)u; \quad u(a+) = 0, \ \ u'(a+) = 1.$$

Let v_γ be the function given by (1.23), and $v(t) = v_\gamma(\frac{2n(t-a)}{b-a})$. Then v is a non-trivial solution of the problem

$$v'' = -\frac{4n^2}{(b-a)^2}\,k_\gamma\Big(\frac{2n(t-a)}{b-a}\Big)v, \quad v(a+) = 0, \ \ v(b-) = 0$$

which has exactly $n-1$ zeros in the interval $]a,b[$. This, by Lemma 2.3 and the conditions (1.12) and (1.14), implies that the function u_1 and hence the function u_2 have at least n zeros in the interval $]a,b[$. However, according to Lemma 2.8, the inequalities (1.13) and (1.14) and the fact that every function u_1 and u_2 has at least n zeros in the interval $]a,b[$ guarantee the validity of the inclusions (2.19). If we now apply Lemma 2.11, the validity of Theorems 1.2 and 1.2′ becomes evident. □

Proof of Corollary 1.1. Equation (1.15) follows from (1.1) in the case where

$$f(t,x)=\sum_{k=1}^{m}p_k(t)|x|^{\lambda_k}\operatorname{sgn}x+p(t)x+q(t).$$

In this case by (1.17), the condition (1.7′) is fulfilled, where $h_0(t)=[p(t)]_-$, and $f(\cdot,0)\in L_{1,1}(]a,b[)$. According to Theorem 1.2′, this implies that if the condition (1.18) (condition (1.19)) is fulfilled, then for $c_1=c_2=0$ the problem (1.15), (1.2) is uniquely solvable. If, moreover, the condition (1.20) is also fulfilled, then this problem is uniquely solvable for arbitrary $c_i\in R$ $(i=1,2)$.

To complete the proof of our corollary, it remains to show that if the conditions (1.17), (1.18) (conditions (1.17), (1.19)) are fulfilled, then the condition (1.20) is necessary for the solvability of the problem (1.15), (1.2) for any $c_i\in R$ $(i=1,2)$. Indeed, let u be a solution of (1.15), (1.2) under the boundary conditions

$$u(a)=2,\quad u(b)=2.$$

Set

$$q_0(t)=q(t)-[q(t)]_-u(t),\quad \widetilde{q}(t)=\sum_{k=1}^{m}p_k(t)|u(t)|^{\lambda_k}\operatorname{sgn}u(t)+[p(t)]_+u(t)$$

and choose the numbers $t_i\in\,]a,b[$ $(i=1,2)$ and $r>0$ in such a way that

$$\widetilde{q}(t)\geq\sum_{k=1}^{m}p_k(t)+[p(t)]_+\ \text{ for }\ t\in\,]a,t_1[\,\cup\,]t_2,b[\,,$$
$$|u(t)|+|u(t_i)|+|u'(t_i)|(b-a)+$$
$$\int_a^{t_1}(s-a)|q_0(s)|\,ds+\int_{t_2}^{b}(b-s)|q_0(s)|\,ds<r\ \text{ for }\ a<t<b\ \ (i=1,2).$$

Then from the equalities

$$u(t)=u(t_i)+(t-t_i)u'(t_i)+\int_{t_i}^{t}(t-s)(q_0(s)+\widetilde{q}(s))\,ds\ \ (i=1,2)$$

we find that

$$\int_a^{t_1}(s-a)\Big(\sum_{k=1}^{m}p_k(s)+[p(s)]_+\Big)\,ds<r,$$
$$\int_{t_1}^{b}(b-s)\Big(\sum_{k=1}^{m}p_k(s)+[p(s)]_+\Big)\,ds<+\infty.$$

Consequently, the condition (1.20) is fulfilled. □

Proof of Corollary 1.3. Let

$$f(t,x) = p(t)x + q(t), \quad h_i(t) = -p(t) \ \ (i = 1,2).$$

Then from (1.21) and (1.22) we obtain the conditions (1.7′), (1.12) and (1.13). If now we apply Theorem 1.2′, the validity of Corollary 1.3 becomes evident. □

Theorems 1.3 and 1.3′ (Theorems 1.4 and 1.4′) can be proved just in the same way as Theorems 1.1 and 1.1′ (as Theorems 1.2 and 1.2′). The only difference is that instead of Lemmas 2.5, 2.10 and 2.10′ (instead of Lemmas 2.8 and 2.11) we apply Lemmas 2.6, 2.12 and 2.12′ (Lemmas 2.9 and 2.13), and along with Lemma 2.3 we apply Lemma 2.4.

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