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A presentation of the initial lift-algebra[1]

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Abstract

The object of study of the present paper may be considered as a model, in an elementary topos with a natural numbers object, of a non-classical variation of the Peano arithmetic. The new feature consists in admitting, in addition to the constant (zero) $s_0 \in \mathsf{N}$ and the unary operation (the successor map) $s_1 : \mathsf{N} \to \mathsf{N}$, arbitrary operations $s_u : \mathsf{N}^u \to \mathsf{N}$ of arities u 'between 0 and 1'. That is, u is allowed to range over subsets of a singleton set. © 1997 Elsevier Science B.V.

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In view of the Peano axioms, the set of natural numbers can be considered as a solution of the 'equation' $X + 1 = X$. Lawvere with his notion of natural numbers object (NNO) gave precise meaning to this statement: here X varies over objects of some category $\mathbf{S}$, 1 is a terminal object of this category, and $X + 1$ (let us call it the *decidable lift* of X) denotes coproduct of X and 1 in $\mathbf{S}$; this obviously defines an endofunctor of $\mathbf{S}$. Now for any $E : \mathbf{S} \to \mathbf{S}$ whatsoever, one solves the 'equation' $E(X) = X$ by considering *E-algebras*: an E-algebra structure on an object X is a morphism $E(X) \to X$, and one can form the category of these by defining a morphism from $E(X) \to X$ to another algebra $E(Y) \to Y$ to be an $X \to Y$ making

$$\begin{tikzcd} E(X) \arrow[r] \arrow[d] & E(Y) \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

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commute. Consider the initial E-algebra, i.e. an initial object I of this category. It was first observed by Lambek that its structure morphism $E(\mathsf{I}) \to \mathsf{I}$ is an isomorphism, so I is a solution of the above equation.

Returning to $E(X) = X + 1$, one easily sees that the initial decidable lift-algebra N has precisely the universal property of a NNO. Indeed, in accordance with the Peano arithmetic, structure of a decidable lift-algebra on an object X amounts to specifying just one unary and one nullary operation on it. An ultraintuitionist could demand to say it in a different way: there is one operation of each arity between 0 and 1. Or,

$$E(X) = \coprod_{U \subseteq 1} X^U,$$

which in presence of the law of excluded middle is $X^1 \sqcup X^0 = X + 1$ again. Now suppose our category $\mathbf{S}$ is a non-Boolean elementary topos; then, the coproduct above still exists, and is different from $X + 1$ – it is in fact $\tilde{X}$, the *partial map classifier* of X (see e.g. [8]). $\tilde{X}$ is characterized by a universal property: morphisms from any Y to it are in one-to-one correspondence with partial maps $Y \rightharpoonup X$, i.e. morphisms from subobjects of Y to X. Note that $X + 1$ has a similar universal property, but with all subobjects of Y replaced by the complemented ones only. One might think that the initial $\tilde{\ }$-algebra $\mathcal{N}$ is some kind of 'non-decidable NNO', relating to N in the same way as the subobject classifier $\Omega = \tilde{1}$ relates to $2{=}1{+}1$. From the point of view of non-classical logics, we are looking at some kind of ultraintuitionistic arithmetic. $\mathcal{N}$ might also serve needs of universal algebra: on objects of toposes, e.g. sheaves, one may indeed encounter algebraic operations with arities more general than numbers. There is hope that using $\mathcal{N}$ in place of N might enable one to extend methods of [10] from finitary to these more general operations.

There is another motivation to study objects like $\mathcal{N}$: it turned out that the work of Joyal and Moerdijk [11] on the algebraic foundation of set theory can be also formulated in terms of initial algebras for certain endofunctors of pretoposes. Moreover, although the case of our $\widetilde{(\ )}$ and $1 + (\ )$ is outside the situations considered in [11], their powerful method turns out to be still applicable. We will return to this matter after the main theorem.

Another field where initial algebras are welcome is denotational semantics of programming languages, and specifically synthetic domain theory (SDT); see [7, 16, 17, 19, 21]. Unfortunately author's acquaintance with the subject is insufficient to tell much about this. Let us just mention that it is of interest to consider endofunctors E which are in a sense intermediate between $1 + (\ )$ and $\widetilde{(\ )}$; they correspond to objects Σ between $1 + 1$ and $\tilde{1}$, $2 \subseteq \Sigma \subseteq \Omega$. Any such Σ gives rise to a distinguished class of subobjects: call a subobject *Σ-decidable* if its classifying map factors through Σ. Thus, for any X one may consider the object $X_\perp$, *Σ-lift* of X, such that morphisms from Y to $X_\perp$ are in one-to-one correspondence with morphisms from Σ-decidable subobjects of Y to X. In a topos, Σ-lifts are most easily defined by the pullback square

$$\begin{tikzcd} X_\perp \arrow[r] \arrow[d] & \tilde{X} \arrow[d] \\ \Sigma \arrow[r, hook] & \Omega \end{tikzcd},$$

where the vertical map on the right is determined by assigning to a partial map its domain. Or, as in [7], $X_\perp = \coprod_\Sigma \prod_{\mathrm{true}:1\to\Sigma}(X)$; identifying Σ with a set of subsets of a singleton set, this can be also written as

$$X_\perp = \coprod_{p\in\Sigma} X^p,$$

presenting $X_\perp$ as a *partial product* in the sense of [9].

To sketch very roughly the rôle played by the Σ-lifts in domain theory, let us again consider the 'equation' $E(X) = X$, for a general endofunctor E. Note that the situation is not symmetric: instead of the initial E-algebra $E(\mathsf{I}) \to \mathsf{I}$ one might as well consider a *terminal E-coalgebra* $\mathsf{T} \to E(\mathsf{T})$, i.e. initial algebra for E considered as an endofunctor of the opposite category $\mathbf{S}^{\mathrm{op}}$. There always is a canonical morphism $\mathsf{I} \to \mathsf{T}$, but in all the obvious examples that come to mind, this morphism is not an isomorphism. According to Freyd's Versality Principle, domains for the denotational semantics are to be found in categories where it is (see [4–6]; in fact, Freyd shows that such categories occur naturally also in other situations, very far from computer science). Now the SDT approach to construct such categories is as follows: one considers the Σ-lift endofunctor of a topos $\mathbf{S}$ and tries to choose Σ in such a way that the morphism $\Sigma^{\mathsf{T}} \to \Sigma^{\mathsf{I}}$, induced by the $\mathsf{I} \to \mathsf{T}$ above, is an isomorphism. Moreover, the reflective subcategory of $\mathbf{S}$ 'cogenerated by Σ' (see [14] or [17] for the precise definition) must inherit the lift endofunctor and have the desired property, i.e. the morphism from the initial algebra to the terminal coalgebra must be an isomorphism there.

In the present paper, we are going to give one particular description of the initial algebra, in hope that it may be of some use in concrete calculations. We will discuss how this description relates to well-foundedness, and in particular to [11].

So, let us fix an elementary topos $\mathbf{S}$ and an object Σ as above, $2 \subseteq \Sigma \subseteq \Omega$, which moreover is a *dominance* in the sense of [16], i.e. if Y' is a Σ-decidable subobject of Y and Y'' is a Σ-decidable subobject of Y', then Y'' is a Σ-decidable subobject of Y.

As is by now customary, we will freely switch back and forth between the internal language of the topos and the external categorical language. For example subobjects of the terminal object, subsets of a singleton, terms of type Ω, truth values, and closed formulae will be used interchangeably as essentially equivalent concepts. Or, we will prove things by induction in the internal language, meaning by this 'Lawvere induction' using the universal property of the NNO.

All functors E considered in this paper will be supposed *indexed*, i.e. be in fact components $E = E_1$ over the terminal object of families of functors $E_I : S/I \to S/I$, $I \in \mathbf{S}$, such that for all $f : I \to J$ in $\mathbf{S}$, the square

$$\begin{tikzcd} S/J \arrow[r, "E_J"] \arrow[d, "f^*"'] & S/J \arrow[d, "f^*"] \\ S/I \arrow[r, "E_I"] & S/I \end{tikzcd}$$

commutes up to coherent canonical isomorphisms. Among other things, this additional structure enables one to internalize the lattice of subalgebras of any algebra $E(A) \to A$. That is, one may construct a subobject $\mathrm{Sub}(E(A) \to A)$ of Ω^A which is its complete meet-subsemilattice and has the needed universal property. In particular, any $E(A) \to A$ will have a smallest subalgebra, represented by the bottom element of $\mathrm{Sub}(E(A) \to A)$. See [15, V.2.1] for details.

Let us begin with a theorem stating existence of an initial E-algebra under some conditions on the endofunctor E. This seems to be a typical folklore theorem: I've heard versions of it from Alex Simpson, Paul Taylor, Pino Rosolini; see also the first proposition in [6]. To the author's knowledge, the earliest (and, it seems, the most general) version is Theorem V(2.2.2) of [15]. We shall extract from it the particular case we need. First let us recall the notion of *unique existentiation (u.e.) pullback* from [3, Proposition 2.21]. Given any $f : X \to Y$, there is an object Q_f, determined by $\{x \in X | f^{-1}f(x) = \{x\}\} \approx \{y \in Y | \exists!_x f(x) = y\}$, with the following universal property: there is a pullback square ("the u.e. pullback of f")

$$\begin{tikzcd} Q_f \arrow[r, hook] \arrow[d, equal] & X \arrow[d] \\ Q_f \arrow[r, hook] & Y \end{tikzcd},$$

such that for any other pullback

$$\begin{tikzcd} Q \arrow[r] \arrow[d, equal] & X \arrow[d] \\ Q \arrow[r] & Y \end{tikzcd},$$

the morphism $Q \to Y$ factors through Q_f. We say that a functor *preserves u.e. pullbacks*, if it carries the u.e. pullback of any morphism to a (not necessarily u.e.) pullback square. One has

Proposition 1. *Let $E : \mathbf{S} \to \mathbf{S}$ be an indexed endofunctor of an elementary topos $\mathbf{S}$ preserving u.e. pullbacks. Suppose there is an E-algebra whose structure morphism $E(B) \to B$ is a monomorphism. Then, its smallest E-subalgebra $E(\mathsf{I}) \to \mathsf{I}$ is an initial E-algebra.*

Proof. (We follow the proof from [15] closely; in fact that proof is in turn adapted from [18].) Let $E(A) \to A$ be any E-algebra. Since the forgetful functor from E-algebras to $\mathbf{S}$ creates all the available limits, there is a unique algebra structure on $A \times B$ turning the projections $A \times B \to A$, $A \times B \to B$ into algebra morphisms. Let $E(C) \to C$ be the smallest E-subalgebra of $E(A \times B) \to A \times B$ (existing by the indexing requirement on E – see above); then also $a : C \hookrightarrow A \times B \to A$ and $b : C \hookrightarrow A \times B \to B$ are algebra morphisms. If we show that b is a monomorphism, we will be done. Indeed, this will mean that there is a morphism to A from a subalgebra of B, hence also from its smallest subalgebra I. Such a morphism is then unique, since an equalizer of any two algebra morphisms from I to A will be a subalgebra of I, hence the whole of I.

First note that the structure morphism $E(C) \to C$ is epi: if it factors through $X \subseteq C$, then $E(X) \to E(C) \to X$ is a subalgebra of C, hence $X = C$. Now consider the u.e. pullback of b,

$$\begin{tikzcd} Q_b \arrow[r, hook] \arrow[d, equal] & C \arrow[d] \\ Q_b \arrow[r, hook] & B \end{tikzcd} .$$

By assumption on E, the square

$$\begin{tikzcd} E(Q_b) \arrow[r, hook] \arrow[d, equal] & E(C) \arrow[d] \\ E(Q_b) \arrow[r, hook] & E(B) \end{tikzcd}$$

is also pullback; also 1

$$\begin{tikzcd} E(B) \arrow[r, equal] \arrow[d, equal] & E(B) \arrow[d] \\ E(B) \arrow[r] & B \end{tikzcd}$$

is pullback because $E(B) \to B$ is mono. Hence, the compound square

$$\begin{tikzcd} E(Q_b) \arrow[r, hook] \arrow[d, equal] & E(C) \arrow[r, equal] \arrow[d] & E(C) \arrow[d] \\ E(Q_b) \arrow[r, hook] \arrow[d, equal] & E(B) \arrow[r, equal] \arrow[d, equal] & E(B) \arrow[d] \\ E(Q_b) \arrow[r, hook] & E(B) \arrow[r] & B \end{tikzcd}$$

is pullback too. But since b is an algebra morphism, $E(C) \to E(B) \to B$ equals $E(C) \to C \to B$; so pulling back in stages one gets

$$\begin{tikzcd} E(Q_b) \arrow[r, hook] \arrow[d] & E(C) \arrow[d] \\ \bullet \arrow[r, hook] \arrow[d] & C \arrow[d] \\ E(Q_b) \arrow[r, hook] & B \end{tikzcd},$$

with vertical composition on the left being identity. So $E(Q_b) \to \bullet$ is split mono; on the other hand, it is pullback of the epi $E(C) \to C$, hence epi itself, hence iso. It follows that there is a pullback square

$$\begin{tikzcd} E(Q_b) \arrow[r, hook] \arrow[d, equal] & C \arrow[d] \\ E(Q_b) \arrow[r, hook] & B \end{tikzcd},$$

hence by the universal property of u.e. pullbacks $E(Q_b) \subseteq Q_b$ is a subalgebra of C, hence $Q_b = C$ and b is mono as required. $\square$

Remark. Without the indexing requirement on E, the above proposition is no longer true; here is an example. Let $\mathbf{S}$ be, say, the category of sets, and consider the subcategory $\mathbf{E}$ of $\mathbf{S}^{\mathsf{N}}$ consisting of those morphisms $(f_n)_{n \in \mathsf{N}}$ of $\mathbf{S}^{\mathsf{N}}$ satisfying $f_m = f_n$ for all sufficiently large m and n. $\mathbf{E}$ is evidently closed under all the topos structure, so is a logical subtopos of $\mathbf{S}^{\mathsf{N}}$. Now consider the endofunctor $E : \mathbf{E} \to \mathbf{E}$ given by $E(X_0, X_1, \ldots) = (1, X_1, X_0, X_3, X_2, X_5, X_4, \ldots)$. Note that E has a left adjoint, hence preserves limits. Consider the sequence U_N, $N = 0, 1, \ldots$ of objects of $\mathbf{E}$, given by $U_N = (X_0, X_1, \ldots)$, where for any $n \geq 0$, $X_{2n} = 1$ and

$$X_{2n+1} = \begin{cases} \emptyset, & n < N, \\ 1, & n \geq N. \end{cases}$$

Since each of the U_N has a (unique) E-algebra structure, it follows that an initial E-algebra must admit a morphism to all of the U_N; consequently, if some object $(X_0, X_1, \ldots)$ has an initial algebra structure, then $X_{2n+1} = \emptyset$ for all n. On the other hand, it is easy to see that if an object $(X_0, X_1, \ldots)$ has a structure of an E-algebra, then all the X_{2n} must be inhabited; if this is an object of $\mathbf{E}$, this will imply that all but a finite number of the X_{2n+1} are inhabited too. Hence, there is no initial E-algebra in this case.

We now concentrate on the Σ-lift endofunctors (they have evident indexings). Note that to have an initial Σ-lift algebra I, it is necessary to have an NNO: indeed, $\mathsf{I} + 1 \subseteq \mathsf{I}_\perp \cong \mathsf{I}$, so the smallest decidable lift-subalgebra of I is an NNO. So from now on we assume that our topos $\mathbf{S}$ has a natural numbers object, N.

As already mentioned, the Σ-lift functors are partial product functors, hence according to [9] they preserve pullbacks, in particular the u. e. pullbacks. So to find an initial Σ-lift algebra, one can try to find an object containing its own Σ-lift (and then describe its smallest subalgebra). In particular, a terminal Σ-lift coalgebra will do, as by the dual of the Lambek's observation above, its structure morphism will be an isomorphism. We now turn to its construction.

Specifying a Σ-lift coalgebra structure on an object X, i.e. a map $X \to X_\perp$, is equivalent to specifying a Σ-decidable partial map from X to itself, i.e. a diagram $X \hookleftarrow X_0 \to X$, where $X_0 \subseteq X$ is a Σ-decidable subobject. Consider the object

$$\mathsf{T} = \{(p_n)_{n \in \mathsf{N}} \in \Sigma^{\mathsf{N}} \mid \forall_n \; p_{n+1} \Rightarrow p_n\};$$

in other words, this is the object $\downarrow \mathsf{N}$ of those Σ-decidable subobjects $D \subset \mathsf{N}$ which are *downdeals*, i.e. satisfy $\forall_{m \geq n} \; m \in D \Rightarrow n \in D$. It can be also identified with the object of order-reversing maps from N (with its natural ordering) to Σ (with ordering via $\Rightarrow$). Consider $\mathsf{T}_0 = \{p_* \in \mathsf{T} \mid p_0\}$. This is a Σ-decidable subobject of T: it is classified by the map $(p_* \mapsto p_0) : \mathsf{T} \to \Sigma$. Hence, the map $(p_* \mapsto p_{*+1}) : \mathsf{T}_0 \to \mathsf{T}$, where $p_{*+1} = (p_{n+1})_{n \in \mathsf{N}}$, determines a Σ-lift coalgebra structure $i : \mathsf{T} \to \mathsf{T}_\perp$ on T.

Proposition 2. *The $i : \mathsf{T} \to \mathsf{T}_\perp$ above is a terminal Σ-lift coalgebra.*

Proof. Given a coalgebra determined by $f : X_0 \to X$, for a Σ-decidable subobject $X_0 \subseteq X$ as above, define $F : \mathsf{N} \to \Sigma^X$ inductively by $F(0) = X_0$, $F(n+1) = f^{-1}(F(n))$. This is legitimate, as for any Σ-decidable subobject X' of X with the classifying map $p : X \to \Sigma$, $f^{-1}(X') \subseteq X_0$ is obviously Σ-decidable, with the classifying map pf; and since $f^{-1}(X') \subseteq X_0$ and $X_0 \subseteq X$ are both Σ-decidable, $f^{-1}(X') \subseteq X$ also is (recall that Σ is a dominance).

Now let us show that the exponential transpose $\bar{F} : X \to \Sigma^{\mathsf{N}}$ of F lands in $\mathsf{T} \subseteq \Sigma^{\mathsf{N}}$. Since $\bar{F}$ is given by $\bar{F}(x) = \{n \mid x \in F(n)\}$, this means that $F(n+1) \subseteq F(n)$ for all $n \in \mathsf{N}$. This is true for $n = 0$, as $F(1) = f^{-1}(X_0) \subseteq X_0 = F(0)$. Then by induction, having proved for n, one has $F(n+2) = f^{-1}(F(n+1)) \subseteq f^{-1}(F(n)) = F(n+1)$.

Now uniqueness; given any $\bar{F}' : X \to \mathsf{T}$ making the square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\bar{F}'"'] & X_\perp \arrow[d, "\bar{F}'_\perp"] \\ \mathsf{T} \arrow[r] & \mathsf{T}_\perp \end{tikzcd}$$

commute, we have to show $\bar{F}' = \bar{F}$, or, equivalently, $F' = F$, where F' is the transpose of $\bar{F}' : X \to \mathsf{T} \hookrightarrow \Sigma^{\mathsf{N}}$. One checks that the two ways from X to $\mathsf{T}_\perp$ in the diagram above are given by $x \mapsto (\text{“}x \in F'(n)\text{”})_{n \in \mathsf{N}} \mapsto (\text{“}x \in F'(n+1)\text{”})_{n \in \mathsf{N}}$ and $x \mapsto \{f(x) \mid x \in X_0\} \mapsto \{(\text{“}f(x) \in F'(n)\text{”})_{n \in \mathsf{N}} \mid x \in X_0\}$. Comparing these gives $F'(0) = X_0$ and $F'(n+1) = \{x \mid f(x) \in F'(n)\}$, so indeed $F' = F$. $\square$

Remark. T has appeared in several places, e.g. [7] or [21]; but the author could not find an explicit mention of the fact that it is the terminal coalgebra. In [17], for certain endofunctors E (including our Σ-lifts) it is shown that $E(1)^{\mathsf{N}}$ has a structure of a *weakly terminal* E-coalgebra, i.e. such that every coalgebra has a (possibly non-unique) morphism to it.

We now turn to the initial algebra. We already noted that $i : \mathsf{T} \to \mathsf{T}_\perp$ is an isomorphism; its inverse j determines an algebra structure on T, and we have to determine its smallest subalgebra. It would be useful to find out which subsets of T are subalgebras with respect to this structure. One has

Lemma 3. *A subobject $A \subseteq \mathsf{T}$ is a subalgebra of $j : \mathsf{T}_\perp \to \mathsf{T}$ iff*

$$\forall_{p_* \in \mathsf{T}}(p_0 \Rightarrow (p_{*+1} \in A)) \Rightarrow (p_* \in A).$$

Proof. Let us describe j more explicitly. Since $\mathsf{T}_\perp = \coprod_{p \in \Sigma} \mathsf{T}^p$, the map j is determined by a family $(j_p : \mathsf{T}^p \to \mathsf{T})_{p \in \Sigma}$. Proving the previous proposition we have seen that for any X, maps $\bar{F} : X \to \mathsf{T}$ are in one-to-one correspondence with maps $F : \mathsf{N} \to \Sigma^X$ satisfying $F(n+1) \subseteq F(n)$. Using this, let us identify T^p with $\{p_* \in \mathsf{T} \mid p_0 \Rightarrow p\}$. Moreover, this identification carries $A^p \subseteq \mathsf{T}^p$ to $\{p_* \in \mathsf{T}^p \mid p \Rightarrow (p_* \in A)\}$.

In terms of the above identification also, for any $p_* \in \mathsf{T}$, one has that $i(p_*)_n = p_{n+1}$, $i(p_*) \in \mathsf{T}^{p_0}$. The fact that j is inverse to i thus forces $j_p(p_*)_0 = p$, and $j_p(p_*)_{n+1} = p_n$ (which uniquely determines j_p by induction). Hence, j_p carries A^p to $\{p_* \in \mathsf{T} \mid p_0 \Rightarrow (p_{*+1} \in A)\}$. The lemma follows. $\square$

Our next task is the identification of the smallest subalgebra in T. It is instructive to see what happens in the "classical" case, when $\mathbf{S}$ is the category of sets, and $2 = \Sigma = \Omega$. Recall that T can be identified with the set $\downarrow \mathsf{N}$ of all downdeals of N. These are either $\{0, 1, \ldots, n-1\}$ for $n \in \mathsf{N}$, or the whole N. Then $\mathsf{I} = \mathsf{N} \subset \mathsf{T}$ consists of all downdeals except that last one. In other words, $\mathsf{I} = \{D \in \mathsf{T} \mid \exists_n\ n \notin D\}$. This suggests to try in general case $\omega_\vee = \{p_* \in \mathsf{T} \mid \exists_n \neg p_n\}$ (see [7]). One can show that $\omega_\vee$ is contained in any subalgebra of $j : \mathsf{T}_\perp \to \mathsf{T}$. Unfortunately, it is not always a subalgebra of T. For example, let $\mathbf{S}$ be the topos of sheaves on some space X, with the frame of opens $\mathcal{O}(X)$; recall that Ω is given by $\Omega(U) = \{U' \in \mathcal{O}(X) \mid U' \subseteq U\}$. Fix some subsheaf Σ of Ω, and write $U' \leq U$ for $U' \in \Sigma(U)$. Then, T is easily seen to be the sheaf given by

$$\mathsf{T}(U) = \{(U_n)_{n \in \mathsf{N}} \in \mathcal{O}(X) \mid \cdots \leq U_{n+1} \leq U_n \leq \cdots \leq U_0 \leq U\}.$$

The condition on subalgebras from the lemma translates here as follows: a subsheaf $A \subseteq \mathsf{T}$ is a Σ-lift subalgebra of T if for any $\cdots \leq U_{n+1} \leq U_n \leq \cdots \leq U_0 \leq U$, one has

$$\begin{aligned}&(\cdots \leq U_{n+2} \leq U_{n+1} \leq \cdots \leq U_1 \leq U_0 \in A(U_0))\\ &\quad \Rightarrow (\cdots \leq U_{n+1} \leq U_n \leq \cdots \leq U_0 \leq U \in A(U)).\end{aligned}$$

Choose X to be the subspace $\{0, \frac{1}{2}, \frac{2}{3}, \ldots, n/(n+1), \ldots, 1\}$ of the real line, and consider the sequence of its open sets $p_n = \{n/(n+1), (n+1)/(n+2), \ldots\}$, $1 \notin p_n$, so that $\neg p_n = \{0, \frac{1}{2}, \ldots, (n-1)/n\}$. Then, $\exists_n \neg p_{n+1} = \exists_n \neg p_n = p_0 = X - \{1\}$. Hence, $((p_0 \Rightarrow p_{*+1} \in \omega_\vee) \Rightarrow p_* \in \omega_\vee) = p_0 \neq X$ and $\omega_\vee$ is not a Σ-lift subalgebra of T, for any Σ containing all the p_n. In fact, this example is not quite relevant for SDT; however, it has been pointed out by Alex Simpson [20] that $\omega_\vee$ also fails to be an initial Σ-lift algebra in the Effective topos with its standard dominance Σ.

Another possibility could be to try $\omega^\wedge = \{p_* \in \mathsf{T} \mid \neg\forall_n\ p_n\}$. But this is now too big: once again consider the example of sheaves on a space X. Then, $\omega^\wedge$ is the subsheaf of T given by

$$\omega^\wedge(U) = \{(\cdots \leq U_{n+1} \leq U_n \leq \cdots \leq U_0 \leq U) \in \mathsf{T}(U) \mid \operatorname{int}\left(\bigcap_n U_n\right) = \emptyset\}.$$

This is clearly a subalgebra; however, it may contain proper subalgebras, e.g. $\{(\cdots \leq U_{n+1} \leq U_n \leq \cdots \leq U_0 \leq U) \in \mathsf{T} \mid (\bigcap_n U_n) = \emptyset\}$. Once again, Alex Simpson [20] has a counterexample more interesting to domain theorists.

The idea that works is easiest to grasp on this topological example. To each open $U \subseteq X$ corresponds the sublocale (i.e. frame quotient) ${}_{\mathcal{O}(X)}U = \{V \in \mathcal{O}(X) \mid (U \Rightarrow V) = V\}$ of $\mathcal{O}(X)$. Then, a good candidate for the initial algebra is the subsheaf I of T given by

$$\mathsf{I}(U) = \left\{(\cdots \leq U_{n+1} \leq U_n \leq \cdots \leq U_0 \leq U) \in \mathsf{T}(U) \mid \bigcap_n {}_{\mathcal{O}(X)}U_n = {}_{\mathcal{O}(X)}\emptyset\right\}.$$

Here clearly ${}_{\mathcal{O}(X)}\emptyset$ is the trivial sublocale $\{X\}$. To keep closer to the general case, let us rewrite this as follows:

$$\mathsf{I}(U) = \{U_* \in \mathsf{T}(U) \mid \forall_{V \in \mathcal{O}(X)}(\forall_n\ V \in {}_{\mathcal{O}(X)}U_n) \Rightarrow V = X\}.$$

Or,

$$\mathsf{I}(U) = \{U_* \in \mathsf{T}(U) \mid \forall_{V \in \mathcal{O}(X)}(\forall_n\ ((U_n \Rightarrow V) = V)) \Rightarrow V = X\}.$$

We can translate this to the internal language of any topos:

$$\mathsf{I} = \{p_* \in \mathsf{T} \mid \forall_{\phi \in \Omega}(\forall_n((p_n \Rightarrow \phi) \Rightarrow \phi)) \Rightarrow \phi\}.$$

It is clear that before proceeding further it is absolutely necessary to improve readability of this expression. One solution can be to introduce an auxiliary notation as follows: denote $\psi \Rightarrow \phi$ by $\underset{\phi}{\neg}\psi$; then, the above nightmare rewrites as follows:

$$\mathsf{I} = \{p_* \in \mathsf{T} \mid \forall_{\phi \in \Omega}\ \underset{\phi}{\neg}\forall_n\ \underset{\phi}{\neg}\underset{\phi}{\neg}p_n\}.$$

To understand this expression better, let us return to the fact that T may be identified with the set $\downarrow \mathsf{N}$ of all downdeals of the NNO. Call an embedding $i : X' \hookrightarrow X$ *strict*, if there is a unique geometric morphism f (namely, from the degenerate topos to $\mathbf{S}$)

for which $f^*(i)$ is an isomorphism. Under these circumstances, the corresponding subobject $X' \subset X$ will be also called strict. Since surjective geometric morphisms reflect isomorphisms, i is strict iff the topology that forces it to be iso is the largest one – or, the sublocale ${}_{\Omega}i$ of Ω corresponding to this topology is degenerate. There is an explicit formula for such forcing topology (see e.g. [8]), which gives an explicit description of that sublocale:

$$\phi \in {}_{\Omega}i \ \text{ iff } \ \forall_{x \in X} \neg_\phi \neg_\phi (x \in X');$$

hence $X' \subset X$ is strict iff

$$\forall_{\phi \in \Omega} \neg_\phi \forall_{x \in X} \neg_\phi \neg_\phi (x \in X').$$

Comparing this with our expression for I reveals the following:

Proposition 4. *I is isomorphic to the set of strict downdeals of N, i.e. those downdeals which are strict as subobjects of N.*

Note also that the strictness condition above is equivalent to

$$\forall_{\phi \in \Omega} \neg_\phi \neg_\phi \exists_{x \in X} \neg_\phi (x \in X').$$

We will briefly mention relation of strictness to another important notion, well-foundedness; but before that, let us prove

Theorem 5. *The object I above is the smallest Σ-lift subalgebra of T, hence the initial Σ-lift algebra.*

Proof. First let us show that I is indeed a subalgebra. According to the lemma, this means that for any $p_* \in \mathsf{T}$ satisfying $p_0 \Rightarrow \forall_{\phi \in \Omega} \neg_\phi \forall_n \neg_\phi \neg_\phi p_{n+1}$, one has $\forall_{\phi \in \Omega} \neg_\phi \forall_n \neg_\phi \neg_\phi p_n$. Now $p_0 \Rightarrow \forall_{\phi \in \Omega} \neg_\phi \forall_n \neg_\phi \neg_\phi p_{n+1}$ is equivalent to $\forall_{\phi \in \Omega} (\forall_n \neg_\phi \neg_\phi p_{n+1}) \Rightarrow \neg_\phi p_0$, since one trivially has $(\psi \Rightarrow \neg_\phi \chi) = (\chi \Rightarrow \neg_\phi \psi)$. Now $\neg_\phi p_0 = \neg_\phi \neg_\phi \neg_\phi p_0$, hence we have $\forall_{\phi \in \Omega} (\forall_n \neg_\phi \neg_\phi p_{n+1}) \Rightarrow \neg_\phi \neg_\phi \neg_\phi p_0$. Using $(\psi \Rightarrow \neg_\phi \chi) = \neg_\phi (\psi \& \chi)$, this may be rewritten as $\forall_{\phi \in \Omega} \neg_\phi ((\forall_n \neg_\phi \neg_\phi p_{n+1}) \& \neg_\phi \neg_\phi p_0)$. The expression in outermost parentheses has the form $\forall_n \psi_{n+1} \& \psi_0$; such expressions may be rewritten as follows: $\forall_m ((\exists_n m = n+1) \Rightarrow \psi_m) \& ((m = 0) \Rightarrow \psi_m)$, which is equivalent to $\forall_m ((\exists_n m = n+1) \vee (m = 0)) \Rightarrow \psi_m$. But by decidability properties of N, $(\exists_n m = n+1) \vee (m = 0)$ is true for all m, hence this gives $\forall_m \ \psi_m$. In our case this gives $\forall_{\phi \in \Omega} \neg_\phi \forall_n \neg_\phi \neg_\phi p_n$, so I is indeed a subalgebra.

It remains to prove that any subalgebra $A \subseteq \mathsf{T}$ contains I. That is, given $p_* \in \mathsf{T}$ with $\forall_\phi \neg_\phi \forall_n \neg_\phi \neg_\phi p_n$, we must prove $p_* \in A$. We can as well prove the statement $\phi =$ "$\forall_m \ (p_m \Rightarrow p_{*+m+1} \in A)$": specializing to $m = 0$ will give $p_0 \Rightarrow p_{*+1} \in A$, which, since A is a subalgebra, implies $p_* \in A$. Now since p_* is in I, one can deduce ϕ from $\forall_n \neg_\phi \neg_\phi p_n$. To verify this latter statement, we must, for any n and k, deduce from $\neg_\phi p_n$ and from p_k, that $p_{*+k+1} \in A$. Now by decidability properties of N, we may deal separately with cases $k \geq n$ and $\exists_x\ n = k + x$. In the first case,

$$\begin{aligned}(p_k \& \neg_{\phi} p_n) &= (p_k \& (p_n \Rightarrow \forall_m\ p_m \Rightarrow (p_{*+m+1} \in A))) \\ &\Rightarrow p_k \& (p_n \Rightarrow (p_k \Rightarrow (p_{*+k+1} \in A))) \\ &\Rightarrow p_k \& (p_k \Rightarrow (p_{*+k+1} \in A)) \\ &\Rightarrow p_{*+k+1} \in A.\end{aligned}$$

In the second,

$$\begin{aligned}&p_k \& (p_n \Rightarrow \forall_m\ p_m \Rightarrow (p_{*+m+1} \in A)) \& \exists_x\ n = k + x \\ &\quad\Rightarrow p_k \& \exists_x\ (p_{k+x} \Rightarrow \forall_m\ p_m \Rightarrow (p_{*+m+1} \in A)) \\ &\quad\Rightarrow p_k \& \exists_x\ (p_{k+x} \Rightarrow (p_{*+k+x+1} \in A)) \\ &\quad\Rightarrow \exists_x\ p_k \& (p_{k+x} \Rightarrow (p_{*+k+x+1} \in A)).\end{aligned}$$

We have to deduce from this that $p_{*+k+1} \in A$, i.e. we have to prove the statement $\forall_x [(p_{k+x} \Rightarrow (p_{*+k+x+1} \in A)) \Rightarrow (p_k \Rightarrow (p_{*+k+1} \in A))]$. We will use induction on x. For $x = 0$, this is trivial. Then, having proved for x, we have, since A is a subalgebra, $(p_{k+x+1} \Rightarrow (p_{*+k+x+2} \in A)) \Rightarrow (p_{*+k+x+1} \in A) \Rightarrow (p_{k+x} \Rightarrow (p_{*+k+x+1} \in A))$, and we are done. □

Remark. There is a close connection here with the work of Joyal and Moerdijk [11]. There, initial objects are constructed in categories of algebras over a *monad*, equipped with additional structure in form of an arbitrary endomap. According to [11, Appendix], the objects of their study are equivalent to initial algebras for endofunctor parts of the corresponding monads. It turns out that, although Σ-lifts do not satisfy some of the conditions required by Joyal and Moerdijk, their construction of initial algebras using certain well-founded trees still applies.

On the other hand, significantly enough the Σ-lift endofunctors have canonical monad structure for any dominance Σ (explanation of this fact is to be found in [2]). So our I carries also an algebra structure over this monad. For $\Sigma = \Omega$, such algebras have been studied in [13].

Just to sketch the connection, let us explain how our notion of strict subobject relates to aspects of well-foundedness, as studied in [1, 11], or [22].

Using the proposition below one can show that applying the aforementioned construction from [11] in our situation gives precisely I, i.e. the method works, although some of the axioms from ibid. are not satisfied.

First recall that a subobject $X' \subseteq X$ is called *inductive* with respect to the binary relation $R \subseteq X \times X$ if the following holds:

$$\forall_{x \in X} (\forall_{y \in X}\ yRx \Rightarrow y \in X') \Rightarrow x \in X';$$

the relation R is called *well-founded* if it admits a unique inductive subobject, namely, the whole X. Let us consider the relation R on N given by yRx iff $y = x + 1$. The universal property of NNO shows that the opposite relation R° is well-founded; R however is not, e.g. the empty subobject is inductive w.r.t. R. Then, one has

Proposition 6. *A downdeal $D \in \mathsf{T}$ is strict, i.e. belongs to I, iff restriction of R to it is well-founded.*

Proof. Let us write the inductivity condition on a $D' \subseteq D$ as follows:

$$\forall_{n \in D}((n+1 \in D) \Rightarrow (n+1 \in D')) \Rightarrow (n \in D')$$

Equivalently, this may be written as

$$\forall_n[(n+1 \in D) \Rightarrow (n+1 \in D')] \Rightarrow [(n \in D) \Rightarrow (n \in D')].$$

It is well known that for any subset D of any set N whatsoever, assigning to $D' \subseteq D$ the set $(D \rightarrow D') = \{n \in N \mid n \in D \Rightarrow n \in D'\}$, establishes the one-to-one correspondence between subsets of D and D-*perfect* sets, that is, those subsets D'' of N satisfying $(D \rightarrow D'') = D''$, i.e. $\forall_n\ (n \in D \Rightarrow n \in D'') \Rightarrow n \in D''$; the assignment in the reverse direction just carries a D-perfect D'' to $D \cap D''$. Hence, inductive subsets of our D above are in one-to-one correspondence with those $D'' \subseteq \mathsf{N}$ satisfying

$$\forall_{n \in \mathsf{N}}\ (n \in D \Rightarrow n \in D'') \Rightarrow n \in D''$$

and

$$\forall_{n \in \mathsf{N}}\ (n+1 \in D'') \Rightarrow n \in D'',$$

i.e. D'' must be a D-perfect downdeal. It follows that R is a well-founded relation on D iff the only D-perfect downdeal is the whole N.

Now call a subset $D'' \subseteq \mathsf{N}$ *constant* if it has the form $D_\phi = \{n|\phi\}$ for some fixed $\phi \in \Omega$. Since all the constant subsets are trivially downdeals, the condition "the only constant D-perfect downdeal is the whole N" is equivalent to "the only constant D-perfect set is the whole N". Translating this into the formal language gives

$$\forall_\phi\ (\forall_n((n \in D \Rightarrow \phi) \Rightarrow \phi)) \Rightarrow \phi,$$

i.e. it exactly means that D is a strict subobject of N. So we just have to show that when ensuring absence of inductive subsets to check well-foundedness, it is enough to look only at constant ones. That is, we have to show that if there is only one constant inductive subset, there cannot be any non-constant ones. So suppose we have some inductive subset D''. Consider the constant set $C = \{n \mid D'' = \mathsf{N}\}$. If we show that C is D-perfect, we are done, since it will follow that $C = \mathsf{N}$, i.e. $D'' = \mathsf{N}$. So we have to show $\forall_n\ (n \in D \Rightarrow n \in C) \Rightarrow n \in C$, that is, given an n with $(n \in D) \Rightarrow (D'' = \mathsf{N})$ and any k, we have to show $k \in D''$. Now condition on n gives $(n \in D) \Rightarrow (D'' = \mathsf{N}) \Rightarrow (n \in D'')$, and since D'' is D-perfect, $n \in D''$; since D'' is a downdeal, this captures those k with $k \leq n$. Whereas if $k \geq n$, then $(k \in D) \Rightarrow (n \in D) \Rightarrow (D'' = \mathsf{N}) \Rightarrow (k \in D'')$, and since D'' is D-perfect, this implies $k \in D''$. $\square$

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