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ON $\lambda f-$STATISTICAL CONVERGENCE

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Abstract. The aim of this study is to define $\lambda f-$density, $\lambda f-$-statistical convergence and $\lambda f-$statistical boundedness in metric spaces using modulus functions under various conditions. In addition, we obtain some relations between the sets of sequences that are $\lambda f-$statistically convergent and $\lambda f-$statistically bounded in metric spaces.

1. Introduction and Preliminaries

The notion of statistical convergence was discovered by Zygmund [23] and first published in his monograph in Warsaw. Moreover, the concept of statistical convergence was fundamentally proposed by Steinhaus [20] and Fast [6]. Later, Schoenberg [19] independently rediscovered it. Numerous mathematicians have used statistical convergence as a method for solving a number of open issues in the areas of sequence spaces, summability theory and some other applications. Statistical convergence has been studied in many fields and under several titles during the past few decades, including number theory, ergodic theory Banach spaces, measure theory, Fourier analysis, cone metric space, trigonometric series, topological space and time scale theory.

Statistical convergence depends on the natural density of subsets of $\mathbb{N}$. The number $\delta(A)$ of a subset A of $\mathbb{N}$ is called a natural density of A and is defined by

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} \left|\{a \le n : a \in A\}\right|,$$

in the case the limit exists, where $|\{a \le n : a \in A\}|$ is the number of elements of A that are less than or equal to n (see [17]).

A sequence (x_k) of numbers is said to be statistically convergent (or $S-$convergent) to the number x_0 if

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \le n : |x_k - x_0| \ge \varepsilon\}\right| = 0$$

for each $\varepsilon > 0$ (see [17]). In this case, we write $S - \lim x_k = x_0$ or $x_k \to x_0\,(S)$ and S denotes the set of all $S-$convergent sequences.

To generalize this idea, Mursaleen [11] proposed the notion of $\lambda-$statistical convergence using the sequence $\lambda = (\lambda_n)$, where $\lambda = (\lambda_n)$ denotes a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_1 = 1$ and $\lambda_{n+1} \le \lambda_n + 1$. We write I_n to denote the closed and bounded interval $[n - \lambda_n + 1, n]$. Also, we write Λ to denote the set of all such sequences $\lambda = (\lambda_n)$.

A sequence $x = (x_k)$ of numbers is called $\lambda-$statistically convergent (or $S_\lambda-$convergent) to the number x_0 if for every $\varepsilon > 0$, the following condition:

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left|\{k \in I_n : |x_k - x_0| \ge \varepsilon\}\right| = 0$$

holds. In this case, we write $S_\lambda - \lim x_k = x_0$ or $x_k \to x_0\,(S_\lambda)$, and the set of all sequences that are $\lambda-$statistically convergent will be denoted by S_λ (see [11]). In the case $\lambda_n = n$ for each $n \in \mathbb{N}$, $S_\lambda-$convergence reduces to $S-$convergence.

A sequence (x_k) of numbers is said to be statistically bounded (or $S-$bounded) if there is $M > 0$ such that

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \le n : |x_k| \ge M\}\right| = 0.$$

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We write $S(b)$ to symbolize the class of all $S-$bounded sequences (see [21]).

In 1953, Nakano [13] developed the concept of a modulus function. A function $f:[0,\infty)\rightarrow[0,\infty)$ is called a modulus function (or modulus) if

(1) $f(x)=0\Leftrightarrow x=0$,
(2) $f(x+y)\leq f(x)+f(y)$ for every $x,y\in[0,\ \infty)$,
(3) f is increasing,
(4) f is continuous from the right at 0.

From these properties, it is clear that a modulus function must be continuous everywhere on $[0,\infty)$. A modulus function may be bounded or unbounded. For instance, $f(x)=x^p$, where $p\in(0\ ,1]$ is an unbounded modulus, but $f(x)=\frac{x}{x+1}$ is a bounded modulus.

Using modulus functions, some authors have introduced and established several sequence spaces such as Ruckle [16], Gosh and Srivastava [7], Altin and Et [2], Savas and Patterson [18], Candan [4], Prakash et al. [15], and others.

In 2014, using unbounded modulus functions, Aizpuru et al. [1] presented the definition of $f-$statistical convergence as follows. Let f be an unbounded modulus function. Then, the sequence (x_k) of numbers is said to be $f-$statistically convergent (or S^f-convergent) to the number x_0 if for every $\varepsilon>0$,

$$\lim_{n\rightarrow\infty}\frac{1}{f(n)}f(|\{k\leq n:|x_k-x_0|\geq\varepsilon\}|)=0$$

holds. Some other applications and generalizations on the notion of statistical convergence are available in [3,5,8–10,12,14,22].

2. Main Results

Definition 2.1. Let f be an unbounded modulus, $\lambda=(\lambda_n)\in\Lambda$ and $A\subset\mathbb{N}$. Then, the number $\delta_\lambda^f(A)$ of the set A is called the $\lambda f-$density of A and is defined by

$$\delta_\lambda^f(A)=\lim_{n\rightarrow\infty}\frac{1}{f(\lambda_n)}f(|\{a\in I_n:a\in A\}|),$$

in the case this limit exists.

Remark 2.1. It is not necessary for the equality $\delta_\lambda^f(A)+\delta_\lambda^f(\mathbb{N}\backslash A)=1$ to remain true, in general. Indeed, takeing $f(x)=\log(x+1)$, $(\lambda_n)=(n)$ and $A=\{1,3,5,\ldots\}$, we get $\delta_\lambda^f(A)=\delta_\lambda^f(\mathbb{N}\backslash A)=1$. That is, $\delta_\lambda^f(A)+\delta_\lambda^f(\mathbb{N}\backslash A)\neq 1$.

Definition 2.2. Let (x_k) be a sequence in a metric space (X,d), f be an unbounded modulus and $\lambda=(\lambda_n)\in\Lambda$. Then (x_k) is called $\lambda f-$statistically convergent in a metric space (X,d) (or simply $S_\lambda^f(X,d)-$convergent) if there exists some $x_0\in X$ such that

$$\lim_{n\rightarrow\infty}\frac{1}{f(\lambda_n)}f(|\{k\in I_n:d(x_k,x_0)\geq\varepsilon\}|)=0$$

for each $\varepsilon>0$, where $f(\lambda_n)$ denotes the n-th term of the sequence $(f(\lambda_n))$. In this case, we write $x_k\rightarrow x_0\left(S_\lambda^f(X,d)\right)$ or $x_k\xrightarrow{S_\lambda^f}x_0(X,d)$. Throughout this study, the class of all $\lambda f-$statistically convergent sequences in a metric space (X,d) will be denoted by $S_\lambda^f(X,d)$, that is,

$$S_\lambda^f(X,d)=\left\{(x_k):\lim_{n\rightarrow\infty}\frac{1}{f(\lambda_n)}f(|\{k\in I_n:d(x_k,x_0)\geq\varepsilon\}|)=0\text{ for some }x_0\in X\right\}.$$

$S_\lambda^f(X,d)-$convergence reduces to $S_\lambda(X,d)-$convergence in the case $f(x)=x$, and $S_\lambda^f(X,d)-$convergence reduces to $S^f(X,d)-$convergence in the case $(\lambda_n)=(n)$. Also, in the special case $f(x)=x$ and $(\lambda_n)=(n)$, $S_\lambda^f(X,d)-$convergence reduces to $S(X,d)-$convergence.

We write $S_{\lambda,0}^f(X,d)$ to denote the class of all $S_{\lambda,0}^f(X,d)-$null sequences. It is clear that $S_{\lambda,0}^f(X,d)\subset S_\lambda^f(X,d)$.

Theorem 2.1. *Let (x_k) and (y_k) be sequences in a metric space (X,d), f be an unbounded modulus and $\lambda \in \Lambda$.*

(1) *If $x_k \xrightarrow{S_\lambda^f} x_0\,(X,d)$, then $cx_k \xrightarrow{S_\lambda^f} cx_0\,(X,d)$ for each $c \in \mathbb{C}$.*

(2) *If $x_k \xrightarrow{S_\lambda^f} x_0\,(X,d)$ and $y_k \xrightarrow{S_\lambda^f} y_0\,(X,d)$, then $(x_k+y_k) \xrightarrow{S_\lambda^f} (x_0+y_0)\,(X,d)$.*

Proof. (1) If $c=0$, then the proof is clear. If $c \neq 0$, then for every $\varepsilon>0$, the proof follows from the fact

$$f\left(\left|\left\{k \in I_n : d\left(cx_k, cx_0\right) \geq \varepsilon\right\}\right|\right) = f\left(\left|\left\{k \in I_n : d\left(x_k, x_0\right) \geq \frac{\varepsilon}{|c|}\right\}\right|\right).$$

(2) The proof follows from the fact that

$$f\left(\left|\left\{k \in I_n : d\left(x_k+y_k, x_0+y_0\right) \geq \varepsilon\right\}\right|\right) \leq f\left(\left|\left\{k \in I_n : d\left(x_k, x_0\right) \geq \frac{\varepsilon}{2}\right\}\right|\right) + f\left(\left|\left\{k \in I_n : d\left(y_k, y_0\right) \geq \frac{\varepsilon}{2}\right\}\right|\right).$$

□

Theorem 2.2. *Let (x_k) be a sequence in a metric space (X,d), f be an unbounded modulus, $\lambda \in \Lambda$ and $x_0 \in X$. Then $x_k \xrightarrow{S_\lambda^f} x_0\,(X,d)$ implies $x_k \xrightarrow{S_\lambda} x_0\,(X,d)$.*

Proof. Suppose $x_k \xrightarrow{S_\lambda^f} x_0\,(X,d)$. Then for every $\varepsilon>0$,

$$\lim_{n\to\infty} \frac{1}{f\left(\lambda_n\right)} f\left(\left|\left\{k \in I_n : d\left(x_k, x_0\right) \geq \varepsilon\right\}\right|\right) = 0.$$

So, for any $m \in \mathbb{N}$, there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$f\left(\left|\left\{k \in I_n : d\left(x_k, x_0\right) \geq \varepsilon\right\}\right|\right) \leq \frac{1}{m} f\left(\lambda_n\right) \leq \frac{1}{m} m f\left(\frac{1}{m}\lambda_n\right) = f\left(\frac{1}{m}\lambda_n\right).$$

Since f is a modulus function, we have

$$\left|\left\{k \in I_n : d\left(x_k, x_0\right) \geq \varepsilon\right\}\right| \leq \frac{1}{m}\lambda_n.$$

Since the above inequality holds for each $m \in \mathbb{N}$, we get

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left|\left\{k \in I_n : d\left(x_k, x_0\right) \geq \varepsilon\right\}\right| = 0.$$

Therefore, $x_k \xrightarrow{S_\lambda} x_0\,(X,d)$. □

Remark 2.2. The converse of Theorem 2.2 is, in grneral, not true. That is, $x_k \xrightarrow{S_\lambda} x_0\,(X,d)$ does not imply $x_k \xrightarrow{S_\lambda^f} x_0\,(X,d)$ for every unbounded modulus f and for each $\lambda \in \Lambda$. The following example provides this situation.

Example. Consider the sequence (x_k) in a usual metric space $(\mathbb{R},d)$ of the form

$$x_k = \begin{cases} k & \text{if } k = m^2 \\ & \qquad\qquad m \in \mathbb{N}. \\ \frac{k}{4k+2} & \text{if } k \neq m^2 \end{cases}$$

If we take $f(x) = \log(x+1)$ and $(\lambda_n) = (n)$, then for every $\varepsilon>0$, we have

$$\left|\left\{k \in I_n : d\left(x_k, \frac{1}{4}\right) \geq \varepsilon\right\}\right| \leq \sqrt{\lambda_n}.$$

That is,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left|\left\{k \in I_n : d\left(x_k, \frac{1}{4}\right) \geq \varepsilon\right\}\right| \leq \lim_{n\to\infty} \frac{1}{\lambda_n}\sqrt{\lambda_n} = 0.$$

This means that $x_k \xrightarrow{S_\lambda} \frac{1}{4}\left(\mathbb{R},d\right)$. However, for every $\varepsilon>0$, we have

$$\lim_{n\to\infty}\frac{1}{f\left(\lambda_n\right)}f\left(\left|\left\{k\in I_n: d\left(x_k,\frac{1}{4}\right)\geq\varepsilon\right\}\right|\right)\geq\lim_{n\to\infty}\frac{1}{f\left(\lambda_n\right)}f\left(\sqrt{\lambda_n}-1\right)=\frac{1}{2}.$$

Thus, $x_k \overset{S_\lambda^f}{\nrightarrow} \frac{1}{4}\left(\mathbb{R},d\right)$.

We get the following result by taking $(\lambda_n)=(n)$ from Theorem 2.2.

Corollary. *Let (x_k) be a sequence in a metric space (X,d), f be any unbounded modulus, and $x_0\in X$. Then $x_k \xrightarrow{S^f} x_0\left(X,d\right)$ implies $x_k \xrightarrow{S} x_0\left(X,d\right)$.*

Theorem 2.3. *Let (x_k) be a sequence in a metric space (X,d), f be an unbounded modulus, $\lambda\in\Lambda$ and $x_0\in X$. If $\liminf_{n\to\infty}\frac{f(\lambda_n)}{\lambda_n}>0$, then $x_k \xrightarrow{S_\lambda} x_0\left(X,d\right)$ implies $x_k \xrightarrow{S_\lambda^f} x_0\left(X,d\right)$.*

Proof. Suppose $x_k \xrightarrow{S_\lambda} x_0\left(X,d\right)$. Since f is a modulus, we have

$$\left|\left\{k\in I_n: d\left(x_k,x_0\right)\geq\varepsilon\right\}\right|\geq\frac{1}{f\left(1\right)}f\left(\left|\left\{k\in I_n: d\left(x_k,x_0\right)\geq\varepsilon\right\}\right|\right)$$

for every $\varepsilon>0$. That is,

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\left|\left\{k\in I_n: d\left(x_k,x_0\right)\geq\varepsilon\right\}\right|\geq\lim_{n\to\infty}\frac{f\left(\lambda_n\right)}{\lambda_n}\frac{1}{f\left(1\right)}\frac{f\left(\left|\left\{k\in I_n: d\left(x_k,x_0\right)\geq\varepsilon\right\}\right|\right)}{f\left(\lambda_n\right)}.$$

Since $x_k \xrightarrow{S_\lambda} x_0\left(X,d\right)$ and $\liminf_{n\to\infty}\frac{f(\lambda_n)}{\lambda_n}>0$, we get

$$\lim_{n\to\infty}\frac{1}{f\left(\lambda_n\right)}f\left(\left|\left\{k\in I_n: d\left(x_k,x_0\right)\geq\varepsilon\right\}\right|\right)=0.$$

Therefore, $x_k \xrightarrow{S_\lambda} x_0\left(X,d\right)$ implies $x_k \xrightarrow{S_\lambda^f} x_0\left(X,d\right)$ if $\liminf_{n\to\infty}\frac{f(\lambda_n)}{\lambda_n}>0$. □

We get the following result taking $(\lambda_n)=(n)$ in Theorem 2.3.

Corollary. *Let (x_k) be a sequence in a metric space (X,d), f be an unbounded modulus and $x_0\in X$. If $\liminf_{n\to\infty}\frac{f(n)}{n}>0$, then $x_k \xrightarrow{S} x_0\left(X,d\right)$ implies $x_k \xrightarrow{S^f} x_0\left(X,d\right)$.*

From Theorem 2.2 and Theorem 2.3, we get the following result.

Corollary. *Let (x_k) be a sequence in a metric space (X,d), $\lambda\in\Lambda$ and $x_0\in X$. Then for any unbounded modulus f, providing $\liminf_{n\to\infty}\frac{f(\lambda_n)}{\lambda_n}>0$,we have $x_k \xrightarrow{S_\lambda} x_0\left(X,d\right)$ if and only if $x_k \xrightarrow{S_\lambda^f} x_0\left(X,d\right)$.*

Theorem 2.4. *Let (x_k) be a sequence in a metric space (X,d), f be an unbounded modulus, and $\lambda\in\Lambda$ and $x_0\in X$. If $\liminf_{n\to\infty}\frac{f(\lambda_n)}{n}>0$, then $x_k \xrightarrow{S} x_0\left(X,d\right)$ implies $x_k \xrightarrow{S_\lambda^f} x_0\left(X,d\right)$.*

Proof. Suppose $x_k \xrightarrow{S} x_0\left(X,d\right)$. Obviously, for every $\varepsilon>0$, we have

$$\left|\left\{k\leq n: d\left(x_k,x_0\right)\geq\varepsilon\right\}\right|\geq\left|\left\{k\in I_n: d\left(x_k,x_0\right)\geq\varepsilon\right\}\right|.$$

Thus,

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\left|\left\{k\leq n: d\left(x_k,x_0\right)\geq\varepsilon\right\}\right| &\geq\lim_{n\to\infty}\frac{1}{n}\left|\left\{k\in I_n: d\left(x_k,x_0\right)\geq\varepsilon\right\}\right|\\ &\geq\lim_{n\to\infty}\frac{1}{n}\frac{1}{f(1)}f\left(\left|\left\{k\in I_n: d\left(x_k,x_0\right)\geq\varepsilon\right\}\right|\right)\\ &=\lim_{n\to\infty}\frac{f\left(\lambda_n\right)}{n}\frac{1}{f(1)}\frac{f\left(\left|\left\{k\in I_n: d\left(x_k,x_0\right)\geq\varepsilon\right\}\right|\right)}{f\left(\lambda_n\right)}.\end{aligned}$$

Since $x_k \xrightarrow{S} x_0 \,(X,d)$ and $\liminf\limits_{n\to\infty} \frac{f(\lambda_n)}{n} > 0$, we get

$$\lim_{n\to\infty} \frac{1}{f(\lambda_n)} f\left(\left|\{k \in I_n : d(x_k, x_0) \geq \varepsilon\}\right|\right) = 0.$$

Therefore, $x_k \xrightarrow{S} x_0 \,(X,d)$ implies $x_k \xrightarrow{S_\lambda^f} x_0 \,(X,d)$ if $\liminf\limits_{n\to\infty} \frac{f(\lambda_n)}{n} > 0$. □

From Theorem 2.4, we get the following result taking $f(x) = x$.

Corollary. *Let (x_k) be a sequence in a metric space (X,d), $\lambda \in \Lambda$ and $x_0 \in X$. If $\liminf\limits_{n\to\infty} \frac{\lambda_n}{n} > 0$, then $x_k \xrightarrow{S} x_0 \,(X,d)$ implies $x_k \xrightarrow{S_\lambda} x_0 \,(X,d)$.*

From Theorem 2.4, we get the following result taking $(\lambda_n) = (n)$.

Corollary. *Let (x_k) be a sequence in a metric space (X,d), f be an unbounded modulus and $x_0 \in X$. If $\liminf\limits_{n\to\infty} \frac{f(n)}{n} > 0$, then $x_k \xrightarrow{S} x_0 \,(X,d)$ implies $x_k \xrightarrow{S^f} x_0 \,(X,d)$.*

Definition 2.3. Let (x_k) be a sequence in a metric space (X,d), f be an unbounded modulus and $\lambda = (\lambda_n) \in \Lambda$. Then (x_k) is called $\lambda f-$statistically bounded in a metric space (X,d) (or simply $S_\lambda^f(X,d)-$bounded) if there exists $q \in X$ such that

$$\lim_{n\to\infty} \frac{1}{f(\lambda_n)} f\left(\left|\{k \in I_n : d(x_k, q) \geq M\}\right|\right) = 0$$

for some $M \in \mathbb{R}^+$. Throughout the study, the set of all $S_\lambda^f(X,d)-$bounded sequences will be denoted by $BS_\lambda^f(X,d)$.

In the case $f(x) = x$, we write $BS_\lambda(X,d)$ instead of $BS_\lambda^f(X,d)$. In the case $(\lambda_n) = (n)$, we write $BS^f(X,d)$ instead of $BS_\lambda^f(X,d)$. Also, in the special case $f(x) = x$ and $(\lambda_n) = (n)$, we write $BS(X,d)$ instead of $BS_\lambda^f(X,d)$.

Theorem 2.5. *Let (x_k) be a sequence in a metric space (X,d), f be an unbounded modulus and $\lambda \in \Lambda$. If (x_k) is $S_\lambda^f(X,d)-$convergent, then it is $S_\lambda^f(X,d)-$bounded.*

Proof. Suppose (x_k) is $S_\lambda^f(X,d)-$convergent and $x_k \xrightarrow{S_\lambda^f} x_0 \,(X,d)$. Obviously, for every $\varepsilon > 0$, we have

$$\left|\{k \in I_n : d(x_k, x_0) \geq \varepsilon\}\right| \geq \left|\{k \in I_n : d(x_k, x_0) > M\}\right|$$

for some $M \in \mathbb{R}^+$ such that $M > \varepsilon$. Since f is a modulus, the above inequality implies

$$\frac{1}{f(\lambda_n)} f\left(\left|\{k \in I_n : d(x_k, x_0) \geq \varepsilon\}\right|\right) \geq \frac{1}{f(\lambda_n)} f\left(\left|\{k \in I_n : d(x_k, x_0) > M\}\right|\right).$$

Since $x_k \xrightarrow{S_\lambda^f} x_0 \,(X,d)$, we get

$$\lim_{n\to\infty} \frac{1}{f(\lambda_n)} f\left(\left|\{k \in I_n : d(x_k, x_0) > M\}\right|\right) = 0.$$

Therefore, (x_k) is $S_\lambda^f(X,d)-$bounded. □

Remark 2.3. The converse of the above theorem is, in general, not true. That is, an $S_\lambda^f(X,d)-$bounded sequence does not imply $S_\lambda^f(X,d)-$convergent for every unbounded modulus f and for each $\lambda \in \Lambda$. The following example provides this situation.

Example. Consider the sequence (x_k) in a usual metric space $(\mathbb{R}, d)$ of the form

$$x_k = \begin{cases} 1 & \text{if } k \text{ is odd}, \\ 0 & \text{if } k \text{ is even}. \end{cases}$$

Then (x_k) is $S_\lambda^f(\mathbb{R},d)-$bounded, but (x_k) is not $S_\lambda^f(\mathbb{R},d)-$convergent if we take $(\lambda_n)=(n)$ and $f(x)=\sqrt{x}$.

From Theorem 2.5, we get the following result taking $f(x)=x$.

Corollary. *Let (x_k) be a sequence in a metric space (X,d) and $\lambda\in\Lambda$. If (x_k) is $S_\lambda(X,d)-$convergent, then it is $S_\lambda(X,d)-$bounded, but the converse is not true, in general.*

From Theorem 2.5, we get the following result taking $(\lambda_n)=(n)$.

Corollary. *Let (x_k) be a sequence in a metric space (X,d) and f be an unbounded modulus. If (x_k) is $S^f(X,d)-$convergent, then it is $S^f(X,d)-$bounded.*

Theorem 2.6. *Let (x_k) be a sequence in a metric space (X,d), f be an unbounded modulus and $\lambda\in\Lambda$. If (x_k) is $S_\lambda^f(X,d)-$bounded, then it is $S_\lambda(X,d)-$bounded.*

Proof. Suppose (x_k) is $S_\lambda^f(X,d)-$bounded. Then there is $q\in X$ such that

$$\lim_{n\to\infty}\frac{1}{f(\lambda_n)}f(|\{k\in I_n : d(x_k,q)\geq M\}|)=0$$

for some $M\in\mathbb{R}^+$. So, for any $m\in\mathbb{N}$, there is $n_0\in\mathbb{N}$ such that for all $n\geq n_0$,

$$f(|\{k\in I_n : d(x_k,q)\geq M\}|)\leq\frac{1}{m}f(\lambda_n)\leq\frac{1}{m}mf\left(\frac{1}{m}\lambda_n\right)=f\left(\frac{1}{m}\lambda_n\right).$$

Since f is a modulus function, we have

$$|\{k\in I_n : d(x_k,q)\geq M\}|\leq\frac{1}{m}\lambda_n.$$

This means that

$$\lim_{n\to\infty}\frac{1}{\lambda_n}|\{k\in I_n : d(x_k,q)\geq M\}|=0.$$

Therefore, (x_k) is $S_\lambda(X,d)-$bounded. □

Remark 2.4. The converse of the above theorem is, in general, not true. That is, an $S_\lambda(X,d)-$bounded sequence does not imply $S_\lambda^f(X,d)-$bounded for every unbounded modulus f and each $\lambda\in\Lambda$. The following example provides this situation.

Example. Consider the sequence $(x_k)=(1,0,0,4,0,0,0,0,9,\ldots)$ in a usual metric space $(\mathbb{R},d)$. Then for any $M>0$ and each $q\in\mathbb{R}$, we have $\{k\in\mathbb{N} : d(x_k,q)>M\}=A\backslash H$, where $A=\{1,4,9,\ldots\}$, and H is a finite subset of $\mathbb{N}$. Now, if we take $f(x)=\log(x+1)$ and $(\lambda_n)=(n)$, we get

$$\delta_\lambda^f(A)=\lim_{n\to\infty}\frac{1}{f(\lambda_n)}f(|\{k\in I_n : k\in\{1,4,9,\ldots\}\}|)=\frac{1}{2}\neq 0.$$

This means that (x_k) is not $S_\lambda^f(X,d)-$bounded. However,

$$\delta(A)=\lim_{n\to\infty}\frac{1}{\lambda_n}|\{k\in I_n : k\in\{1,4,9,\ldots\}\}|=0.$$

Consequently, (x_k) is $S_\lambda(X,d)-$bounded.

From Theorem 2.6, taking $(\lambda_n)=(n)$, we get the following result.

Corollary. *Let (x_k) be a sequence in a metric space (X,d) and f be an unbounded modulus. If (x_k) is $S^f(X,d)-$bounded, then it is $S(X,d)-$bounded,but the converse is not true, in general.*

Theorem 2.7. *Let (x_k) be a sequence in a metric space (X,d), f be an unbounded modulus and $\lambda\in\Lambda$. If $\liminf\limits_{n\to\infty}\frac{f(\lambda_n)}{\lambda_n}>0$ and the sequence (x_k) is $S_\lambda(X,d)-$bounded, then it is $S_\lambda^f(X,d)-$bounded.*

Proof. Suppose (x_k) is $S_\lambda(X,d)-$bounded. Then there is $q \in X$ such that

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left|\{k \in I_n : d(x_k, q) \geq M\}\right| = 0$$

for some $M \in \mathbb{R}^+$. Since f is a modulus, we have

$$\begin{aligned} \lim_{n\to\infty} \tfrac{1}{\lambda_n} \left|\{k \in I_n : d(x_k, q) \geq M\}\right| &\geq \lim_{n\to\infty} \tfrac{1}{\lambda_n} \tfrac{1}{f(1)} f\left(\left|\{k \in I_n : d(x_k, q) \geq M\}\right|\right) \\ &\geq \lim_{n\to\infty} \tfrac{f(\lambda_n)}{\lambda_n} \tfrac{1}{f(1)} \tfrac{f(|\{k\in I_n : d(x_k,q)\geq M\}|)}{f(\lambda_n)}. \end{aligned}$$

Since (x_k) is $S_\lambda(X,d)-$bounded and $\liminf_{n\to\infty} \frac{f(\lambda_n)}{\lambda_n} > 0$, we get

$$\lim_{n\to\infty} \frac{1}{f(\lambda_n)} f\left(\left|\{k \in I_n : d(x_k, q) \geq M\}\right|\right) = 0.$$

Thus, (x_k) is $S_\lambda^f(X,d)-$bounded. □

From Theorem 2.6 and Theorem 2.7, we get the following result.

Corollary. *Let (x_k) be a sequence in a metric space (X,d), f be an unbounded modulus such that $\liminf_{n\to\infty} \frac{f(\lambda_n)}{\lambda_n} > 0$, and $\lambda \in \Lambda$. Then (x_k) is $S_\lambda(X,d)-$bounded if and only if it is $S_\lambda^f(X,d)-$bounded.*

Theorem 2.8. *Let (x_k) be a sequence in a metric space (X,d), f be an unbounded modulus, and $\lambda \in \Lambda$. If $\liminf_{n\to\infty} \frac{f(\lambda_n)}{f(n)} > 0$ and the sequence (x_k) is $S^f(X,d)-$bounded, then it is $S_\lambda^f(X,d)-$bounded.*

Proof. Suppose (x_k) is $S^f(X,d)-$bounded. Then there is $q \in X$ such that

$$\lim_{n\to\infty} \frac{1}{f(n)} f\left(\left|\{k \leq n : d(x_k, q) \geq M\}\right|\right) = 0$$

for some $M \in \mathbb{R}^+$. Obviously, we have

$$\left|\{k \leq n : d(x_k, q) \geq M\}\right| \geq \left|\{k \in I_n : d(x_k, q) \geq M\}\right|.$$

Since f is a modulus, we can write

$$\begin{aligned} \frac{1}{f(n)} f\left(\left|\{k \leq n : d(x_k, q) \geq M\}\right|\right) &\geq \frac{1}{f(n)} f\left(\left|\{k \in I_n : d(x_k, q) \geq M\}\right|\right) \\ &= \frac{f(\lambda_n)}{f(n)} \frac{1}{f(\lambda_n)} f\left(\left|\{k \in I_n : d(x_k, q) \geq M\}\right|\right). \end{aligned}$$

Since (x_k) is $S^f(X,d)-$bounded and $\liminf_{n\to\infty} \frac{f(\lambda_n)}{f(n)} > 0$, we get

$$\lim_{n\to\infty} \frac{1}{f(\lambda_n)} f\left(\left|\{k \in I_n : d(x_k, q) \geq M\}\right|\right) = 0.$$

Therefore (x_k) is $S_\lambda^f(X,d)-$bounded. □

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References

1. A. Aizpuru, M. C. Listàn-Garcìa, F. Rambla-Barreno, Density by moduli and statistical convergence. *Quaest. Math.* **37** (2014), no. 4, 525–530.
2. Y. Altin, M. Et, Generalized difference sequence spaces defined by a modulus function in a locally convex space. *Soochow J. Math.* **31** (2005), no. 2, 233–243.
3. M. Ayman-Mursaleen, S. Serra-Capizzano, Statistical convergence via q-calculus and a Korovkin's type approximation theorem. *Axioms* **11** (2022), no. 2, 70 pp.
4. M. Candan, Some new sequence spaces defined by a modulus function and an infinite matrix in a seminormed space. *J. Math. Anal.* **3** (2012), no. 2, 1–9.

5. K. Demirci, F. Dirik, S. Yildiz, Approximation via equi-statistical convergence in the sense of power series method. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **116** (2022), no. 2, Paper no. 65, 13 pp.
6. H. Fast, Sur la convergence statistique. (French) *Colloq. Math.* **2** (1951), 241–244 (1952).
7. D. Ghosh, P. D. Srivastava, On some vector valued sequence spaces defined using a modulus function. *Indian J. Pure Appl. Math.* **30** (1999), no. 8, 819–826.
8. I. S. Ibrahim, R. Colak, On strong lacunary summability of order α with respect to modulus functions. *An. Univ. Craiova Ser. Mat. Inform.* **48** (2021), no. 1, 127–136.
9. O. Kişi, Lacunary statistical convergence for sequence of sets in intuitionistic fuzzy metric space. *J. Appl. Math. Inform.* **40** (2022), no. 1-2, 69–83.
10. S. A. Mohiuddine, A. Asiri, B. Hazarika, Weighted statistical convergence through difference operator of sequences of fuzzy numbers with application to fuzzy approximation theorems. *Int. J. Gen. Syst.* **48** (2019), no. 5, 492–506.
11. M. Mursaleen, λ-statistical convergence. *Math. Slovaca* **50** (2000), no. 1, 111–115.
12. M. Mursaleen, P. Baliarsingh, On the convergence and statistical convergence of difference sequences of fractional order. *J. Anal.* **30** (2022), no. 2, 469–481.
13. H. Nakano, Concave modulars. *J. Math. Soc. Japan* **5** (1953), 29–49.
14. F. Nuray, E. Dündar, U. Ulusu, Wijsman statistical convergence of double sequences of set. *Iran. J. Math. Sci. Inform.* **16** (2021), no. 1, 55–64.
15. S. T. V. G Prakash, M. Chandramouleeswaran, N. Subramanian, The strongly generalized triple difference Γ^3 sequence spaces defined by a modulus. *Math. Morav.* **20** (2016), no. 1, 115–123.
16. W. H. Ruckle, FK spaces in which the sequence of coordinate vectors is bounded. *Canadian J. Math.* **25** (1973), 973–978.
17. T. Šalát, On statistically convergent sequences of real numbers. *Math. Slovaca* **30** (1980), no. 2, 139–150.
18. E. Savas, R. Patterson, Double sequence spaces defined by a modulus. *Math. Slovaca* **61** (2011), no. 2, 245–256.
19. I. J. Schoenberg, The integrability of certain functions and related summability methods. *Amer. Math. Monthly* **66** (1959), 361–375.
20. H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique. In: *Colloq. math.* **2** (1951), no. 1, 73–74.
21. F. Temizsu, M. Et, Some results on generalizations of statistical boundedness. *Math. Methods Appl. Sci.* **44** (2021), no. 9, 7471–7478.
22. E. Yilmaz, T. Gulsen, Y. Altin, H. Koyunbakan, $\lambda-$Wijsman statistical convergence on time scales. *Commun. Stat. Methods*, **30** (2021), 1–15, 5364–5378.
23. K. Zygmund, *Trigonometric Series.* Cambridge University Press, London, 1979.

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