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AN EXTENSION OF KANTOROVICH INEQUALITY

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Abstract. In this short note, we present an extension of Kantorovich inequality for two operators on a Hilbert space. Moreover, the multiple version and related inequality for positive linear maps are obtained.

1. Introduction

In mathematics, the Kantorovich inequality is a special case of the Cauchy–Schwarz inequality and a generalization of the triangle inequality. The Kantorovich inequality is helpful in numerical analysis and statistics for determining the rate of convergence of the steepest descent method. Over the past decades, several formulations, generalizations, or improvements of the Kantorovich inequality in different settings have been proposed by many mathematicians (see [1, 4–7] and references therein).

The Kantorovich inequality [2] states that for every unit vector $x \in \mathscr{H}$,

$$\langle Ax, x\rangle \langle A^{-1}x, x\rangle \leq \frac{(M+m)^2}{4Mm}, \tag{1.1}$$

provided that $0 < mI_{\mathscr{H}} \leq A \leq MI_{\mathscr{H}}$.

The operator version of (1.1) was established by Marshall and Olkin [3], who obtained

Theorem 1.1. *Let $0 < mI_{\mathscr{H}} \leq A \leq MI_{\mathscr{H}}$. Then for every positive unital linear map Φ,*

$$\Phi\left(A^{-1}\right) \leq \frac{(M+m)^2}{4Mm}\, \Phi(A)^{-1}.$$

Recall that the linear map Φ defined on $\mathscr{B}(\mathscr{H})$ (C^*-algebra of all bounded linear operators acting in a Hilbert space $\mathscr{H}$) is called positive if $\Phi(A) \geq 0$, whenever $A \geq 0$, and is called unital when $\Phi(I_{\mathscr{H}}) = I_{\mathscr{H}}$, where $I_{\mathscr{H}}$ is the identity operator on $\mathscr{H}$.

In this paper, we intend to give a new version of the Kantorovich inequality (1.1) based on the following query:

What is the Kantorovich inequality for two operators?

2. Main Results

Here, we present a version of Kantorovich inequality for two operators.

Theorem 2.1. *Let $A, B \in \mathscr{B}(\mathscr{H})$ be two positive operators, satisfying $A \leq aI_{\mathscr{H}}$, $B \leq bI_{\mathscr{H}}$, $AB \leq I_{\mathscr{H}}$ for positive scalars a, b. Then for every unit vector $x \in \mathscr{H}$,*

$$\langle Ax, x\rangle \langle Bx, x\rangle \leq \frac{(ab+1)^2}{4ab}.$$

Proof. From the assumptions on A and B, we infer that

$$(A - aI_{\mathscr{H}})(B - bI_{\mathscr{H}}) \geq 0,$$

or, equivalently,

$$AB + abI_{\mathscr{H}} \geq bA + aB.$$

This implies that

$$\langle ABx, x\rangle + ab \geq b\langle Ax, x\rangle + a\langle Bx, x\rangle,$$

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for every unit vector $x \in \mathscr{H}$.

Using the arithmetic-geometric mean inequality, we get

$$b\langle Ax,x\rangle + a\langle Bx,x\rangle \geq 2(ab\langle Ax,x\rangle\langle Bx,x\rangle)^{\frac{1}{2}}.$$

We square both sides and obtain the desired inequality. □

Remark 2.1.

(i) The conditions $A \leq aI_{\mathscr{H}},\ B \leq bI_{\mathscr{H}}$ can be replaced by $A \geq aI_{\mathscr{H}},\ B \geq bI_{\mathscr{H}}$. Note that these conditions together with $AB \leq I_{\mathscr{H}}$, imply that $AB = BA$. This guarantees the positivity of $(A - aI_{\mathscr{H}})(B - bI_{\mathscr{H}})$.

(ii) In the case $AB = I_{\mathscr{H}}$, the usual Kantorovich inequality will be obtained.

The following theorem is a multiple interpretation of the Kantorovich inequality.

Theorem 2.2. *Let A_i, B_i $(i = 1, 2 \ldots n)$ be positive operators satisfying $A_i \leq aI_{\mathscr{H}},\ B_i \leq bI_{\mathscr{H}}$, $A_iB_i \leq I_{\mathscr{H}}$ for positive scalars a, b. Let $x_1, \ldots, x_n$, be any finite numbers of vectors in $\mathscr{H}$ such that $\sum\limits_{i=1}^{n} \|x_i\|^2 = 1$. Then*

$$\left(\sum_{i=1}^{n}\langle A_ix_i,x_i\rangle\right)\left(\sum_{i=1}^{n}\langle B_ix_i,x_i\rangle\right) \leq \frac{(ab+1)^2}{4ab}. \tag{2.1}$$

Proof. We set

$$\widetilde{A} := \begin{pmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_n \end{pmatrix}, \qquad \widetilde{B} := \begin{pmatrix} B_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_n \end{pmatrix}$$

and

$$\widetilde{x} := \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

which provide

$$\sum_{i=1}^{n}\langle A_ix_i,x_i\rangle = \left\langle \widetilde{A}\widetilde{x},\widetilde{x}\right\rangle$$

and

$$\sum_{i=1}^{n}\langle B_ix_i,x_i\rangle = \left\langle \widetilde{B}\widetilde{x},\widetilde{x}\right\rangle.$$

It follows from Theorem 2.1 that

$$\left\langle \widetilde{A}\widetilde{x},\widetilde{x}\right\rangle\left\langle \widetilde{B}\widetilde{x},\widetilde{x}\right\rangle \leq \frac{(ab+1)^2}{4ab}.$$

Hence, we have the desired inequality (2.1). □

Next, we present a version of Kantorovich inequality for positive linear maps. Notice that for the positive and invertible operators $A, B \in \mathscr{B}(\mathscr{H})$, the geometric mean of the operator is defined by

$$A \sharp B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}.$$

Theorem 2.3. *Let Φ be a normalized positive linear map on $\mathscr{B}(\mathscr{H})$. If A, B are positive operators on $\mathscr{H}$ satisfying $A \leq aI_{\mathscr{H}},\ B \leq bI_{\mathscr{H}},\ AB \leq I_{\mathscr{H}}$ for positive scalars a, b, then*

$$\Phi(A)\sharp\Phi(B) \leq \frac{ab+1}{2\sqrt{ab}}. \tag{2.2}$$

Proof. Obviously, we have

$$AB + abI_{\mathscr{H}} \geq bA + aB.$$

Applying Φ in both sides, we obtain

$$\Phi\left(AB\right) + ab\Phi\left(I_{\mathscr{H}}\right) \geq b\Phi\left(A\right) + a\Phi\left(B\right).$$

Since Φ is normalized, we may state that

$$\Phi\left(AB\right) + abI_{\mathscr{H}} \geq b\Phi\left(A\right) + a\Phi\left(B\right).$$

Using the operator version of the arithmetic-geometric inequality, it follows that

$$b\Phi\left(A\right) + a\Phi\left(B\right) \geq 2\sqrt{ab}\left(\Phi\left(A\right)\sharp\Phi\left(B\right)\right),$$

which is the desired inequality (2.2). ☐

As an application, we have the subsequent corollary.

Corollary 2.1. *Let $U_1, U_2, \ldots, U_n$ be contractions on $\mathscr{H}$ such that $\sum_{i=1}^n U_i^* U_i = I_{\mathscr{H}}$. If A, B are positive operators on $\mathscr{H}$ satisfying $A \leq aI_{\mathscr{H}}$, $B \leq bI_{\mathscr{H}}$, $AB \leq I_{\mathscr{H}}$ for positive scalars a, b, then*

$$\left(\sum_{i=1}^n U_i^* AU_i\right)\sharp\left(\sum_{i=1}^n U_i^* BU_i\right) \leq \frac{ab+1}{2\sqrt{ab}}. \tag{2.3}$$

Proof. Put

$$\Phi\left(x\right) = \sum_{i=1}^n U_i^* XU_i.$$

Clearly, Φ is a normalized positive linear map on $\mathscr{B}\left(\mathscr{H}\right)$. So, we have

$$\Phi\left(I_{\mathscr{H}}\right) = \sum_{i=1}^n U_i^* U_i = I_{\mathscr{H}}.$$

Now, applying Theorem 2.3 for Φ, we obtain the desired result (2.3). ☐

Theorem 2.4. *Let A, B be self-adjoint operators in a Hilbert space $\mathscr{H}$ such that $0 < m_1 I_{\mathscr{H}} \leq A \leq M_1 I_{\mathscr{H}}$ and $0 < m_2 I_{\mathscr{H}} \leq B \leq M_2 I_{\mathscr{H}}$, and let Φ be a positive lieanr map with the condition $\Phi\left(AB\right) = \Phi\left(BA\right)$. Then for any unit vectors x in $\mathscr{H}$,*

$$\begin{aligned}
&\left\langle \Phi\left(A^2\right)x, x\right\rangle \left\langle \Phi\left(B^2\right)x, x\right\rangle - \left\langle \Phi\left(AB\right)x, x\right\rangle^2 \\
&\leq \left(\left\langle \Phi\left(AB\right)x, x\right\rangle - m\right)\left(M - \left\langle \Phi\left(AB\right)x, x\right\rangle\right) \\
&\leq \frac{\left(M-m\right)^2}{4},
\end{aligned} \tag{2.4}$$

where

$$m = \frac{m_1}{M_2}\left\langle \Phi^2\left(B\right)x, x\right\rangle, \qquad M = \frac{M_1}{m_2}\left\langle \Phi^2\left(B\right)x, x\right\rangle. \tag{2.5}$$

Proof. For $0 < m_1 I_{\mathscr{H}} \leq A \leq M_1 I_{\mathscr{H}}$ and $0 < m_2 I_{\mathscr{H}} \leq B \leq M_2 I_{\mathscr{H}}$, we have

$$\frac{m_1}{M_2}B \leq A \leq \frac{M_1}{m_2}B.$$

Therefore

$$\left(A - \frac{m_1}{M_2}B\right)\left(\frac{M_1}{m_2}B - A\right) \geq 0,$$

i.e.,

$$A^2 \leq \left(\frac{M_1}{m_2} + \frac{m_1}{M_2}\right)AB - \frac{m_1 M_1}{M_2 m_2}B^2.$$

Since Φ is a positive linear map, we get

$$\Phi\left(A^2\right) \leq \left(\frac{M_1}{m_2} + \frac{m_1}{M_2}\right)\Phi\left(AB\right) - \frac{m_1 M_1}{M_2 m_2}\Phi\left(B^2\right)$$

or, equivalently,

$$\left\langle \Phi\left(A^2\right)x, x\right\rangle \leq \left(\frac{M_1}{m_2}+\frac{m_1}{M_2}\right)\left\langle \Phi\left(AB\right)x, x\right\rangle - \frac{m_1 M_1}{M_2 m_2}\left\langle \Phi\left(B^2\right)x, x\right\rangle$$

for each $x \in \mathscr{H}$ with $\|x\| = 1$. Using (2.5), we can get the left-hand side of inequality (2.4).

The right-hand side of inequality (2.4) is immediate. Apply the arithmetic-geometric mean inequality

$$\left(\left\langle \Phi\left(AB\right)x, x\right\rangle - m\right)\left(M - \left\langle \Phi\left(AB\right)x, x\right\rangle\right) \leq \left(\frac{M-m}{2}\right)^2. \qquad \square$$

Remark 2.2.

(i) Choosing $B = A^{-1}$ in (2.4), we get

$$\begin{aligned}\left\langle \Phi\left(A\right)x, x\right\rangle \left\langle \Phi\left(A^{-2}\right)x, x\right\rangle - I_{\mathscr{H}} &\leq \left(I_{\mathscr{H}} - m\right)\left(M - I_{\mathscr{H}}\right)\\ &\leq \frac{\left(M-m\right)^2}{4}.\end{aligned}$$

(ii) Choosing $\Phi\left(A\right) = tr\left(A\right) I_{\mathscr{H}}$ in (2.4), we have

$$tr\left(A^2\right) tr\left(B^2\right) - tr^2\left(AB\right) \leq \left(tr\left(AB\right) - m\right)\left(M - tr\left(AB\right)\right) \leq \frac{\left(M-m\right)^2}{4}.$$

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