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THE NUMERICAL SOLUTION OF LINEAR INTEGRO-DIFFERENTIAL EQUATIONS BY USING GENERALIZED BERNSTEIN OPERATORS

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Abstract. This paper presents a numerical method for solving linear integro-differential equations with a weakly singular kernel. The proposed method is based on approximating unknown functions with a generalized Bernstein operator. We offer an efficient method based on matrix forms of the linear integro-differential equations with a weakly singular kernel. Furthermore, we obtain an error estimation for this method.

1. Introduction

Many scientific phenomena in science and engineering can be modeled using linear and nonlinear integro-differential equations. Nonlinear integro-differential equations arising in chemistry, biology, physics, and engineering applications are modeled by means of initial value problems for a finite closed interval. The subject of integro-differential equations is one of the most useful mathematical tools in both pure and applied mathematics.

However, nonlinear integro-differential equations are usually difficult to solve analytically. Therefore, approximate solutions to nonlinear integro-differential equations can be computed by numerical methods. In recent years, several numerical methods for solving linear and nonlinear integro-differential equations have been presented by many authors. For example, some linear and nonlinear integro-differential equations have been solved using the Taylor and polynomial methods [1,2,6,8,12, 14,23,28,30,32], wavelet method [7,21,25], collocation method [20,24,34–36], homotopy perturbation method [10,18,27,29,33], direct method [5,22], and related methods [3,4,11,15–17,19,37].

In this study, we consider a class of linear integro-differential equations with a weakly singular kernel of the form

$$\sum_{i=0}^{J} P_i(x)y^{(i)}(x) = g(x) + \lambda_1 \int_0^x \frac{y^{(k)}(t)}{\sqrt{x-t}}dt + \lambda_2 \int_0^x K(x,t)y^{(l)}(t)dt, \tag{1.1}$$

under the initial conditions

$$y^{(i)}(c) = y_i, \quad 0 \leqslant c \leqslant R, \quad i = 0,1,\dots,\max(J-1, l-1, k-1) = m, \tag{1.2}$$

where$P_i(x)$, $K(x,t)$ and $g(x)$ are continuous functions on $[0,1]$, and $y^{(i)}(x)$ stands for the ith-order derivative of $y(x)$; c, λ_i and y_i are real constants.

For $\lambda_1 = 0$, equation (1.1) becomes Volterra integro-differential equation (VIDE). For $\lambda_2 = 0$, equation (1.1) becomes VIDE with a weakly singular kernel.

In [13], the authors modified and developed the method for solving equation (1.1) using the matrix relations between the Bernstein polynomials and their derivatives. In this paper, we use the generalized Bernstein operator (the Bernstein α-operator).

The rest of the paper is organized as follows: In Section 2, we introduce Bernstein operators and their elementary properties. In Section 3, we derive matrix relations between the generalized Bernstein polynomials and their derivatives. In Section 4, we find an error estimate for approximation using the generalized Bernstein operators. Section 5 offers three examples of integro-differential equations to illustrate the properties of our method, and finally, Section 6 concludes the paper.

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2. New Generalized Bernstein Operators

The Bernstein approximation $B_n(f)$ to a function $f : [0,1] \to R$ is the polynomial

$$B_n(f(x)) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) p_{n,i}(x), \tag{2.1}$$

where $p_{n,i}$ is the polynomial of degree n,

$$p_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \quad i = 0, \ldots, n.$$

Bernstein in [26] used this approximation to obtain the first constructive proof of the Weierstrass theorem.

Theorem 2.1. *For all functions f in $C[0,1]$, the sequence $\{B_n(f); n = 1, 2, 3, \ldots\}$ converges uniformly to f, where B_n is defined by equation (2.1).*

Chen [31] definition of a new family of generalized Bernstein operators and their certain elementary properties, which play an important role in the theory of uniform approximation of functions. We introduce a new family of operators as follows.

Definition 2.1. Given a function $f(x)$ on $[0,1]$, for each positive integer n and any fixed real α, we define α-Bernstein operator for $f(x)$ as follows:

$$T_{n,\alpha}(f;x) = \sum_{i=0}^{n} f_i p_{n,i}^{(\alpha)}(x), \tag{2.2}$$

where $f_i = f\left(\frac{i}{n}\right)$. For $i = 0, 1, \cdots, n$, the α-Bernstein polynomial $p_{n,i}^{(\alpha)}(x)$ of degree n is defined by $p_{1.0}^{(\alpha)}(x) = 1 - x, p_{1,1}^{(\alpha)}(x) = x$, and

$$\begin{aligned} p_{n,i}^{(\alpha)}(x) = \Bigg[& \binom{n-2}{i}(1-\alpha)x + \binom{n-2}{i-2}(1-\alpha)(1-x) \\ & + \binom{n}{i} \alpha x(1-x) \Bigg] x^{i-1}(1-x)^{n-i-1} \end{aligned} \tag{2.3}$$

where $n \geq 2$, $x \in [0,1]$, and the binomial coefficients $\binom{k}{l}$ are given by the formulas

$$\binom{k}{l} = \begin{cases} \frac{k!}{l!(k-l)!}, & \text{if } 0 \leq l \leq k, \\ 0, & \text{else.} \end{cases}$$

For example,

$$p_{2,0}^{(\alpha)}(x) = (1-\alpha x)(1-x), \quad p_{2,1}^{(\alpha)}(x) = 2\alpha x(1-x), \quad p_{2,2}^{(\alpha)}(x) = (1-\alpha+\alpha x)x,$$
$$p_{3,0}^{(\alpha)}(x) = (1-\alpha x)(1-x)^2, \quad p_{3,1}^{(\alpha)}(x) = (1+2\alpha-3\alpha x)x(1-x),$$
$$p_{3,2}^{(\alpha)}(x) = (1-\alpha+3\alpha x)x(1-x), \quad p_{3,3}^{(\alpha)}(x) = (1-\alpha+\alpha x)x^2,$$
$$p_{4,0}^{(\alpha)}(x) = (1-\alpha x)(1-x)^3, \quad p_{4,1}^{(\alpha)}(x) = (2+2\alpha-4\alpha x)x(1-x)^2,$$
$$p_{4,2}^{(\alpha)}(x) = (1-\alpha+6\alpha x-6\alpha x^2)x(1-x), \quad p_{4,3}^{(\alpha)}(x) = (2-2\alpha+4\alpha x)x^2(1-x),$$
$$p_{4,4}^{(\alpha)}(x) = (1-\alpha+\alpha x)x^3.$$

When $\alpha = 1$, the α-Bernstein polynomial reduces to the classical Bernstein polynomial, i.e.,

$$p_{n,i}^{(1)}(x) = \binom{n}{i} x^i (1-x)^{n-i}.$$

So, the α-Bernstein operator has the following identity:

$$T_{n,1}(f;x) = \sum_{i=0}^{n} f_i \binom{n}{i} x^i (1-x)^{n-i} = B_n(f;x),$$

which means that the class of α-Bernstein operators contains the classical Bernstein ones. For α-Bernstein operators, we present here some of their properties and results.

Theorem 2.2. *If the function $f(x)$ is continuous on $[0,1]$, for any $\alpha \in [0,1]$, then the α-Bernstein operators converge uniformly to $f(x)$ on the interval $[0,1]$.*

Proof. See [31]. □

Theorem 2.3. *Let $f(x)$ be bounded on $[0,1]$. Then for any $x \in [0,1]$ at which $f''(x)$ exists, we have*

$$\lim_{n\to\infty} n\left[T_{n,\alpha}(f;x) - f(x)\right] = \frac{1}{2}x(1-x)f''(x),$$

where $0 < \alpha < 1$.

Proof. See [31]. □

Before giving the following theorems, we have two definitions:

Definition 2.2 ([9])**.** Let $f(x)$ be defined on $[a,b]$. The modulus of continuity of $f(x)$ on $[a,b]$, $\omega(\delta)$, is defined for $\delta > 0$ by

$$\omega(\delta) = \sup_{\substack{x_1,x_2\in[a,b]\\ |x_1-x_2|\le\delta}} \left| f(x_1) - f(x_2) \right|.$$

Definition 2.3. $\|\cdot\|$ is the uniform norm over the interval $[0,1]$.

$$\|f(x) - T_{n,\alpha}(f;x)\| = \max_{0\le x\le 1} \left| f(x) - T_{n,\alpha}(f;x) \right|.$$

Theorem 2.4. *If $f(x)$ is bounded on $[0,1]$, then for $0 \le \alpha \le 1$, we have*

$$\|f(x) - T_{n,\alpha}(f;x)\| \le \frac{3}{2}\omega\left(\frac{\sqrt{n+2(1-\alpha)}}{n}\right),$$

where $\omega(\delta)$ is the modulus of continuity of $f(x)$.

The α-Bernstein operator $T_{n,\alpha}(f;x)$ has the shape-preserving property as shown below.

Theorem 2.5 ([31])**.** *Let $f \in C[0,1]$. If $f(x)$ is monotonically increasing (or decreasing) on $[0,1]$ for $0 \le \alpha \le 1$, the same applies to all its α-Bernstein operators.*

Theorem 2.6 ([31])**.** *Let $f \in C[0,1]$. If $f(x)$ is convex on $[0,1]$ for $0 \le \alpha \le 1$, the same applies to all its α-Bernstein operators.*

3. Fundamental Relations

Let us consider the integro-differential equation (1.1) and use α-Bernstein operators to approximate the exact solution. Here, we find the matrix forms of each term in the equation.

First, we can transform the solution of the α-Bernstein series $y(x) = T_{n,\alpha}(x)$ defined by equation (2.2) and its derivatives $y^{(k)}(x)$ into matrix forms

$$y(x) = \mathbf{T}_{n,\alpha}(x)\mathbf{A} \quad \text{and} \quad y^{(k)}(x) = \mathbf{T}^{(k)}_{n,\alpha}(x)\mathbf{A}, \tag{3.1}$$

where

$$\mathbf{T}_{n,\alpha}(x) = \begin{bmatrix} P^{(\alpha)}_{n,0}(x) & P^{(\alpha)}_{n,1}(x) & \cdots & P^{(\alpha)}_{n,n}(x) \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} a_0 & a_1 & \dots & a_n \end{bmatrix}^T.$$

We write $y(x)$ in the form of a matrix product as follows:

$$[\mathbf{T}_{n,\alpha}(x)]^T = \begin{bmatrix} P^{(\alpha)}_{n,0}(x) \\ P^{(\alpha)}_{n,1}(x) \\ \vdots \\ P^{(\alpha)}_{n,n}(x) \end{bmatrix} = \mathbf{P}(\mathbf{X}(x))^T,$$

where

$$\mathbf{P} = \begin{bmatrix} p_{00} & p_{01} & \cdots & p_{0n} \\ p_{10} & p_{11} & \cdots & p_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n0} & p_{n1} & \cdots & p_{nn} \end{bmatrix}, \quad \mathbf{X}(x) = [1 \quad x \quad \cdots \quad x^n].$$

From (2.4), we get

$$\begin{aligned} P_{n,i}^{(\alpha)}(x) &= \left[\binom{n-2}{i}(1-\alpha)x + \binom{n-2}{i-2}(1-\alpha)(1-x) + \binom{n}{i}\alpha x(1-x) \right] x^{i-1}(1-x)^{n-i-1} \\ &= \binom{n-2}{i}(1-\alpha)x^i(1-x)^{n-i-1} + \binom{n-2}{i-2}(1-\alpha)x^{i-1}(1-x)^{n-i} + \binom{n}{i}\alpha x^i(1-x)^{n-i} \\ &= \binom{n-2}{i}(1-\alpha)x^i\left(\sum_{j_1=0}^{n-i-1}\binom{n-i-1}{j_1}(-1)^{n-i-j_1-1}x^{n-i-j_1-1}\right) \\ &\quad + \binom{n-2}{i-2}(1-\alpha)x^{i-1}\left(\sum_{j_2=0}^{n-i}\binom{n-i}{j_2}(-1)^{n-i-j_2}x^{n-i-j_2}\right) \\ &\quad + \binom{n}{i}\alpha x^i\left(\sum_{j_3=0}^{n-i}\binom{n-i}{j_3}(-1)^{n-i-j_3}x^{n-i-j_3}\right) \\ &= \sum_{j_1=0}^{n-i-1}(-1)^{n-i-j_1-1}(1-\alpha)\binom{n-2}{i}\binom{n-i-1}{j_1}x^{n-j_1-1} \\ &\quad + \sum_{j_2=0}^{n-i}(-1)^{n-i-j_2}(1-\alpha)\binom{n-2}{i-2}\binom{n-i}{j_2}x^{n-j_2-1} \\ &\quad + \sum_{j_3=0}^{n-i}(-1)^{n-i-j_3}\alpha\binom{n}{i}\binom{n-i}{j_3}x^{n-j_3}. \end{aligned} \tag{3.2}$$

So, we obtain the coefficient of x^j, the first of (3.2): $j_1 := n-j-1$, $j_2 := n-j-1$, and $j_3 := n-j$

$$\begin{aligned} p_{ij} &= (-1)^{j-i}(1-\alpha)\binom{n-2}{i}\binom{n-i-1}{n-j-1} + (-1)^{j-i+1}(1-\alpha)\binom{n-2}{i-2}\binom{n-i}{n-j-1} \\ &\quad + (-1)^{j-i}\alpha\binom{n}{i}\binom{n-i}{n-j}. \end{aligned}$$

Similarly, we present the equation of the matrix $\mathbf{X}(x)$ and its derivative $\mathbf{X}^{(1)}(x)$,

$$\mathbf{X}^{(k)}(x) = \mathbf{X}^{(k-1)}(x)\mathbf{B} = \cdots = \mathbf{X}(x)\mathbf{B}^k,$$

where

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Consequently, we have the matrix relation

$$y^{(k)}(x) = \mathbf{X}(x)\mathbf{B}^k\mathbf{P}^T\mathbf{A} \tag{3.3}$$

$$\int_0^x \frac{t^n}{\sqrt{x-t}}dt = \frac{\sqrt{\pi}x^{\left(\frac{1}{2}+n\right)}\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)}. \tag{3.4}$$

Analogously, we get

$$\lambda_1 \int_0^x \frac{y^{(k)}(t)}{\sqrt{x-t}} dt = \lambda_1 \Bigg(\int_0^x \frac{\mathbf{X}(t)}{\sqrt{x-t}} dt \Bigg) \mathbf{B}^k \mathbf{P}^T \mathbf{A} = \lambda_1 \mathbf{Q}_x \mathbf{B}^k \mathbf{P}^T \mathbf{A},$$

and

$$\lambda_2 \int_0^x K(x,t) y^{(l)}(t) dt = \lambda_2 \Bigg(\int_0^x K(x,t) \mathbf{X}(t) dt \Bigg) \mathbf{B}^l \mathbf{P}^T \mathbf{A} = \lambda_2 \mathbf{V}_x \mathbf{B}^l \mathbf{P}^T \mathbf{A},$$

where

$$\mathbf{Q}_x = \begin{bmatrix} \frac{\sqrt{\pi} x^{\left(\frac{1}{2}\right)} \Gamma(1)}{\Gamma\left(\frac{3}{2}\right)} & \frac{\sqrt{\pi} x^{\left(\frac{3}{2}\right)} \Gamma(2)}{\Gamma\left(\frac{5}{2}\right)} & \dots & \frac{\sqrt{\pi} x^{\left(\frac{1}{2}+n\right)} \Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)} \end{bmatrix},$$

$$\mathbf{V}_x = \begin{bmatrix} \int_0^x K(x,t) dt & \int_0^x K(x,t) t dt & \dots & \int_0^x K(x,t) t^n dt \end{bmatrix}.$$

So, (1.1) can be written as

$$\sum_{i=0}^{J} P_i(x) \mathbf{X}(x) \mathbf{B}^i \mathbf{P}^T \mathbf{A} = g(x) + \lambda_1 \mathbf{Q}_x \mathbf{B}^k \mathbf{P}^T \mathbf{A} + \lambda_2 \mathbf{V}_x \mathbf{B}^l \mathbf{P}^T \mathbf{A}. \tag{3.5}$$

Using the nodes $\{x_i \mid i = 0, 1, \dots, n;\ 0 = x_0 < x_1 < \dots < x_n = 1\}$ in (3.5), we get the system of matrix equations

$$\sum_{i=0}^{J} P_i(x_j) \mathbf{X}(x_j) \mathbf{B}^i \mathbf{P}^T \mathbf{A} = g(x_j) + \lambda_1 \mathbf{Q}_{x_i} \mathbf{B}^k \mathbf{P}^T \mathbf{A} + \lambda_2 \mathbf{V}_{x_i} \mathbf{B}^l \mathbf{P}^T \mathbf{A}, \quad i = 0, 1, \dots, n \tag{3.6}$$

or briefly, the fundamental matrix equation

$$\left[\sum_{i=0}^{J} \mathbf{P}_i \mathbf{X} \mathbf{B}^i \mathbf{P}^T - \lambda_1 \mathbf{Q} \mathbf{B}^k \mathbf{P}^T - \lambda_2 \mathbf{V} \mathbf{B}^l \mathbf{P}^T \right] \mathbf{A} = \mathbf{G},$$

where

$$\mathbf{P}_i = \begin{bmatrix} P_i(x_0) & 0 & 0 & \cdots & 0 \\ 0 & P_i(x_1) & 0 & \cdots & 0 \\ 0 & 0 & P_i(x_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & P_i(x_n) \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ g(x_2) \\ \vdots \\ g(x_n) \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{x_0} \\ \mathbf{Q}_{x_1} \\ \vdots \\ \mathbf{Q}_{x_n} \end{bmatrix},$$

$$\mathbf{V} = \begin{bmatrix} \mathbf{V}_{x_0} \\ \mathbf{V}_{x_1} \\ \vdots \\ \mathbf{V}_{x_n} \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} \mathbf{X}(x_0) \\ \mathbf{X}(x_1) \\ \mathbf{X}(x_2) \\ \vdots \\ \mathbf{X}(x_n) \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix},$$

$$\mathbf{W} \mathbf{A} = \mathbf{G}, \tag{3.7}$$

where

$$\mathbf{W} = \sum_{i=0}^{J} \mathbf{P}_i \mathbf{X} \mathbf{B}^i \mathbf{P}^T - \lambda_1 \mathbf{Q} \mathbf{B}^k \mathbf{P}^T - \lambda_2 \mathbf{V} \mathbf{B}^l \mathbf{P}^T. \tag{3.8}$$

Here, (3.7) corresponds to a system of $(n+1)$ linear algebraic equations with unknown coefficients $a_0, a_1, \dots, a_n$.

We use the same method [27] to obtain the corresponding matrix forms for the conditions (1.2), by means of relation (3.3), as follows

$$\mathbf{X}(c)\mathbf{B}^i\mathbf{P}^T\mathbf{A} = [y_i], \quad 0 \leqslant c \leqslant R, \quad i = 0, 1, \ldots, m. \tag{3.9}$$

On the other hand, the augmented matrix form for conditions can be written as

$$\mathbf{U}_i\mathbf{A} = [y_i] \text{ or } [\mathbf{U}_i; y_i], \quad i = 0, 1, \ldots, m, \tag{3.10}$$

where

$$\mathbf{U}_i = \mathbf{X}(c)\mathbf{B}^i\mathbf{P}^T = \begin{bmatrix} u_{i0} & u_{i1} & \cdots & u_{in} \end{bmatrix}, \quad i = 0, 1, \ldots, m. \tag{3.11}$$

Replacing the row matrices (3.10) by any $m+1$ rows of the matrix (3.7), we get the new augmented matrix $[\widetilde{\mathbf{W}}; \widetilde{\mathbf{G}}]$:

$$[\widetilde{\mathbf{W}}; \widetilde{\mathbf{G}}] = \begin{bmatrix} w_{00} & w_{01} & w_{02} & \cdots & w_{0n}; & g(x_0) \\ w_{10} & w_{11} & w_{12} & \cdots & w_{1n}; & g(x_1) \\ w_{20} & w_{21} & w_{22} & \cdots & w_{2n}; & g(x_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{(n-m-1)0} & w_{(n-m-1)1} & w_{(n-m-1)2} & \cdots & w_{(n-m-1)n}; & g(x_{n-m-1}) \\ u_{00} & u_{01} & u_{02} & \cdots & u_{0n}; & y_0 \\ u_{10} & u_{11} & u_{12} & \cdots & u_{1n}; & y_1 \\ u_{20} & u_{21} & u_{22} & \cdots & u_{2n}; & y_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{m0} & u_{m1} & u_{m2} & \cdots & u_{mn}; & y_m \end{bmatrix}.$$

Note that $\operatorname{rank} \widetilde{\mathbf{W}} = \operatorname{rank}[\widetilde{\mathbf{W}}; \widetilde{\mathbf{G}}] = n+1$. Thus, we can write

$$\mathbf{A} = (\widetilde{\mathbf{W}})^{-1}\widetilde{\mathbf{G}}$$

and hence the elements $a_0, a_1, \ldots, a_n$ of $\mathbf{A}$ are uniquely determined.

4. Estimation of Error Bound by Using Generalized Bernstein Operators

We give an error bound for this solution in the following theorem.

Theorem 4.1. *Consider the integral equation of* (1.1). *Assume $P_i(x)$, $K(x,t)$ and $g(x)$ are continuous functions on $[0,1]$ and $y^{(i)}(x)$ stands for the ith-order derivative of $y(x)$; c, λ_i and y_i are the real constants. We obtained the matrix $\mathbf{W}$ from relation* (3.8). *Then*

$$\sup_{x_i \in [0,1]} y(x_i) - T_{n,\alpha}(y_n(x_i)) \,| \leqslant \frac{1}{8n}\left[\|y''\| + (M + C(1+J))\|\mathbf{W^{-1}}\|\|y^{(m)}\|\right],$$

where $x_i = \frac{i}{n}$, $i = 0, \ldots, n, y(x)$ is exact solution, $M = \max(|\lambda_1 c_1|, |\lambda_2 c_2|)$, $c_1 = \frac{\sqrt{\pi}\Gamma(1)}{\Gamma(\frac{3}{2})}$, $c_2 = \sup_{x,t\in[0,1]} |k(x,t)|$, $C = \max_{x\in[0,1]}(|P_i(x)|)$, $\|y^{(m)}\| = \max\left(\|y^{(i+2)}\|\right)$ and $T_{n,\alpha}(y_n(x))$ is the solution of the proposed method.

Proof. It is clear that

$$\begin{aligned} \sup_{x_i \in [0,1]} |y(x_i) - T_{n,\alpha}(y_n(x_i))| \leq{}& \sup_{x_i \in [0,1]} |y(x_i) - T_{n,\alpha}(y(x_i))| \\ &+ \sup_{x_i \in [0,1]} |T_{n,\alpha}(y(x_i)) - T_{n,\alpha}(y_n(x_i))| = T_1 + T_2. \end{aligned} \tag{4.1}$$

From Theorem 2.3, we have the following bound:

$$\sup_{x_i \in [0,1]} |y(x) - T_{n,\alpha}(y;x)| \leq \frac{1}{2n}x(1-x)\|y''\| \leq \frac{1}{8n}\|y''\|. \tag{4.2}$$

Then it suffices to find a bound for T_2. If we substitute $T_{n,\alpha}(y^{(i)};x)$, $T_{n,\alpha}(y^{(k)};x)$, $T_{n,\alpha}(y^{(l)};x)$ instead of $y^{(i)}(x)$, $y^{(k)}(x)$, $y^{(l)}(x)$ in the integral equation

$$g(x)=\sum_{i=0}^{J}P_i(x)y^{(i)}(x)-\lambda_1\int_0^x\frac{y^{(k)}(t)}{\sqrt{x-t}}dt-\lambda_2\int_0^x K(x,t)y^{(l)}(t)dt, \tag{4.3}$$

then the left-hand side of the integral equation is replaced by a new function which we denote by $\widehat{g(x)}$. We have

$$\widehat{g(x)}=\sum_{i=0}^{J}P_i(x)T_{n,\alpha}(y^{(i)}(x))-\lambda_1\int_0^x\frac{T_{n,\alpha}(y^{(k)}(t))}{\sqrt{x-t}}dt-\lambda_2\int_0^x K(x,t)T_{n,\alpha}(y^{(l)}(t))dt \tag{4.4}$$

and, consequently, from (4.3), (4.4) and using Theorem 2.3, we have

$$\begin{aligned}\sup_{x_i\in[0,1]}\left|g(x)-\widehat{g(x)}\right| &\leq \sum_{i=0}^{J}P_i(x)\sup_{x_i\in[0,1]}\left|T_{n,\alpha}(y^{(i)};x)-y^{(i)}(x)\right|\\ &\quad-\lambda_1\int_0^x\frac{\sup_{x_i\in[0,1]}\left|T_{n,\alpha}(y^{(k)};x)-y^{(k)}(x)\right|}{\sqrt{x-t}}dt\\ &\quad-\lambda_2\int_0^x K(x,t)\sup_{x_i\in[0,1]}\left|T_{n,\alpha}(y^{(t)};x)-y^{(t)}(x)\right|dt\\ &\leq \sum_{i=0}^{J}\frac{1}{8n}P_i(x)\|y^{(i+2)}\|+\lambda_1c_1\|y^{(k+2)}\|+\lambda_2c_2\|y^{(l+2)}\|,\end{aligned}$$

where we define $M=\max\left(|\lambda_1c_1|,|\lambda_2c_2|\right)$, $C=\max_{x\in[0,1]}(|P_i(x)|)$, $\|y^{(m)}\|=\max(\|y^{(i+2)}\|,\|y^{(k+2)}\|,\|y^{(l+2)}\|)$, $i=0,\dots,J$, and

$$\sup_{x_i\in[0,1]}\left|g(x)-\widehat{g(x)}\right|\leq\frac{1}{8n}\left(M+C(J+1)\right)\|y^{(m)}\|. \tag{4.5}$$

On the other hand, we have $WX=Y$ and $W\bar{X}=\bar{Y}$, where

$$X=\left[T_{n,\alpha}\left(y_n\left(x_j\right)\right)\right],\quad Y=\left[g\left(x_i\right)\right],\quad \bar{X}=\left[T_{n,\alpha}\left(y\left(x_j\right)\right)\right],\quad \bar{Y}=\left[\widehat{g}\left(x_i\right)\right].$$

Therefore, using (4.5), we obtain the result:

$$\begin{aligned}\sup_{x_i\in[0,1]}\left|T_{n,\alpha}\left(y\left(x_i\right)\right)-T_{n,\alpha}\left(y_n\left(x_i\right)\right)\right| &\leq\|\mathbf{W^{-1}}\|\cdot\sup_{x_i\in[0,1]}\left|g(x)-\widehat{g(x)}\right|\\ &\leq\frac{(M+C(1+J))}{8n}\|\mathbf{W^{-1}}\|\|y^{(m)}\|. \end{aligned}\tag{4.6}$$

Finally, applying (4.6) and (4.2) to (4.1), the proof is completed. $\square$

5. Illustrative Examples

This section provides several numerical examples illustrating the properties of the method. All calculations were performed in Maple 2019. Note that

$$\|e_n\|=\left(\int_0^1 e_n^2(x)dx\right)^{1/2}\approx\left(\frac{1}{n}\sum_{i=0}^{n}e_n^2\left(x_i\right)\right)^{1/2},$$

where

$$e_n\left(x_i\right)=y\left(x_i\right)-T_{n,\alpha}\left(y_n\left(x_i\right)\right),\quad i=0,\dots,n.$$

Here, $x_i=i/n$, $i=0,\dots,n$ and $T_{n,\alpha}\left(y_n(x)\right),y(x)$ are the approximate solutions of order n with the α-Bernstein approximation and exact solutions of the integral equations, respectively.

Example 5.1. We consider the Volterra integro-differential equation with a weakly singular kernel

$$\begin{cases} y''(x) + y(x) + \frac{1}{\sqrt{\pi}} \int_0^x \frac{y''(t)}{\sqrt{x-t}} dt = f(x) \\ y(0) = y'(0) = 1 \end{cases}$$

where the exact solution is $1 + x + x^2$ when $y(x)$ is $3 + x + x^2 + \frac{4\sqrt{x}}{\sqrt{4}}$.

For $n = 7$, $\alpha = 0.01$, the fundamental matrix equation of the problem is

$$\left\{ \mathbf{X}\mathbf{B}^2\mathbf{P}^T + \mathbf{X}\mathbf{P}^T + \frac{1}{\sqrt{\pi}} \mathbf{Q}\mathbf{B}^2\mathbf{P}^T \right\} \mathbf{A} = \mathbf{G},$$

where

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} 1 & -6.01 & 15.06 & -20.15 & 15.20 & -6.15 & 1.06 & -0.01 \\ 0 & 5.02 & -25.17 & 50.55 & -50.90 & 25.80 & -5.37 & 0.07 \\ 0 & 0.99 & 5.16 & -30.75 & 51.60 & -36.75 & 9.96 & -0.21 \\ 0 & 0 & 4.95 & -9.55 & -1.4 & 12 & -6.35 & 0.35 \\ 0 & 0 & 0 & 9.9 & -24.4 & 18.75 & -3.9 & -0.35 \\ 0 & 0 & 0 & 0 & 9.9 & -18.6 & 8.49 & 0.21 \\ 0 & 0 & 0 & 0 & 0 & 4.95 & -4.88 & -0.07 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.99 & 0.01 \end{bmatrix},$$

$$\mathbf{X} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \left(\frac{1}{7}\right) & \left(\frac{1}{7}\right)^2 & \left(\frac{1}{7}\right)^3 & \left(\frac{1}{7}\right)^4 & \left(\frac{1}{7}\right)^5 & \left(\frac{1}{7}\right)^6 & \left(\frac{1}{7}\right)^7 \\ 1 & \left(\frac{2}{7}\right) & \left(\frac{2}{7}\right)^2 & \left(\frac{2}{7}\right)^3 & \left(\frac{2}{7}\right)^4 & \left(\frac{2}{7}\right)^5 & \left(\frac{2}{7}\right)^6 & \left(\frac{2}{7}\right)^7 \\ 1 & \left(\frac{3}{7}\right) & \left(\frac{3}{7}\right)^2 & \left(\frac{3}{7}\right)^3 & \left(\frac{3}{7}\right)^4 & \left(\frac{3}{7}\right)^5 & \left(\frac{3}{7}\right)^6 & \left(\frac{3}{7}\right)^7 \\ 1 & \left(\frac{4}{7}\right) & \left(\frac{4}{7}\right)^2 & \left(\frac{4}{7}\right)^3 & \left(\frac{4}{7}\right)^4 & \left(\frac{4}{7}\right)^5 & \left(\frac{4}{7}\right)^6 & \left(\frac{4}{7}\right)^7 \\ 1 & \left(\frac{5}{7}\right) & \left(\frac{5}{7}\right)^2 & \left(\frac{5}{7}\right)^3 & \left(\frac{5}{7}\right)^4 & \left(\frac{5}{7}\right)^5 & \left(\frac{5}{7}\right)^6 & \left(\frac{5}{7}\right)^7 \\ 1 & \left(\frac{6}{7}\right) & \left(\frac{6}{7}\right)^2 & \left(\frac{6}{7}\right)^3 & \left(\frac{6}{7}\right)^4 & \left(\frac{6}{7}\right)^5 & \left(\frac{6}{7}\right)^6 & \left(\frac{6}{7}\right)^7 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 3 \\ 4.016239 \\ 4.573635 \\ 5.089641 \\ 5.603908 \\ 6.131799 \\ 6.681189 \\ 7.256758 \end{bmatrix},$$

$$\mathbf{Q} = \begin{bmatrix} 0.6325 & 0.0421 & 0.0034 & 0.2891 \times 10^{-3} & 0.2570 \times 10^{-4} & 0.2336 \times 10^{-5} & 0.2157 \times 10^{-6} & 0.2013 \times 10^{-7} \\ 0.7559 & 0.0720 & 0.0082 & 0.0010 & 0.1279 \times 10^{-3} & 0.1661 \times 10^{-4} & 0.2191 \times 10^{-5} & 0.2921 \times 10^{-6} \\ 1.0690 & 0.2036 & 0.0465 & 0.0114 & 0.0029 & 0.7519 \times 10^{-3} & 0.1983 \times 10^{-3} & 0.5288 \times 10^{-4} \\ 1.3093 & 0.3741 & 0.1283 & 0.0471 & 0.0179 & 0.0070 & 0.0028 & 0.0011 \\ 1.5119 & 0.5759 & 0.2633 & 0.1290 & 0.0655 & 0.0340 & 0.0179 & 0.0096 \\ 1.6903 & 0.8049 & 0.4599 & 0.2816 & 0.1788 & 0.1161 & 0.0765 & 0.0510 \\ 1.8516 & 1.0581 & 0.7255 & 0.5331 & 0.4061 & 0.3165 & 0.2504 & 0.2003 \\ 2 & 1.3333 & 1.0667 & 0.9143 & 0.8127 & 0.7388 & 0.6820 & 0.6365 \end{bmatrix},$$

$$[\mathbf{W}] = \begin{bmatrix} 31.12 & -50.34 & 10.32 & 9.9 & 0 & 0 & 0 & 0 \\ 25.335826 & -29.411782 & -5.857459 & 4.005563 & 4.831761 & 1.808127 & 0.34832 & 0.035586 \\ 16.269295 & -10.811952 & -9.704065 & -3.480806 & 2.459566 & 3.917923 & 1,730908 & 0.247962 \\ 10.209425 & -2.460924 & -5.759776 & -6.307126 & -2.606899 & 2.698031 & 3.990197 & 1.316604 \\ 6.732903 & 0.616962 & -0.397316 & -4.727566 & -6.334275 & -2.375886 & 4.550125 & 4.332585 \\ 4.990979 & -1.516737 & 2.297612 & 0.047102 & -5.487912 & -8.753002 & -1.543187 & 10.515137 \\ -6.01 & 5.02 & 0.99 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solving this system, we obtain the α-Bernstein coefficients matrix

$$\mathbf{A} = [1 \quad 1.135789 \quad 1.321553 \quad 1.557293 \quad 1.843007 \quad 2.178693 \quad 2.564361 \quad 3],$$

and from (3.1) the solution in the form of α-Bernstein series:

$$\begin{aligned} T_{n,\alpha}(x) =& \mathbf{X}_7(x)\mathbf{P}^T\mathbf{A} = 1 + x + x^2 + 5.3 \times 10^{-17}x^3 - 3.8 \times 10^{-16}x^4 \\ &+ 1.032 \times 10^{-15}x^5 - 1.2102 \times 10^{-15}x^6 + 5.1608 \times 10^{-16}x^7. \end{aligned}$$

The results are shown in Table 1 for $n = 2, 5, 7, 20$ and $\alpha = 0.01, 0.1$. Figure 1 also shows the exact and approximate solutions for $n = 7$ and $\alpha = 0.01$.

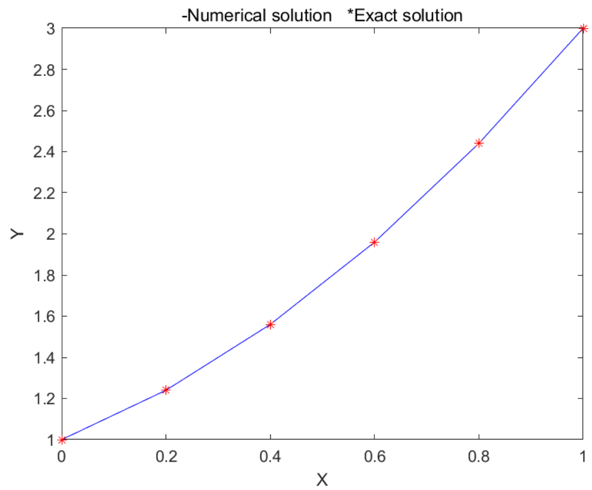

FIGURE 1. Results for Example 5.1 with $n = 7$, $\alpha = 0.01$.

Example 5.2. Let us consider the Volterra integro-differential equation

$$\begin{cases} y''(x) + y(x) + \int_0^x xy(t)dt + \int_0^x \frac{y(t)}{\sqrt{x-t}}dt = 2 + x^2 + \frac{16}{15}x^{\frac{5}{2}} + \frac{x^4}{3} \\ y(0) = y'(0) = 0 \end{cases}$$

with the exact solution $y(x) = x^2$.

The results are shown in Table 1 for $n = 2, 5, 7, 20$ and $\alpha = 0.01, 0.5$. Figure 2 also shows the exact and approximate solutions for $n = 7$ and $\alpha = 0.5$.

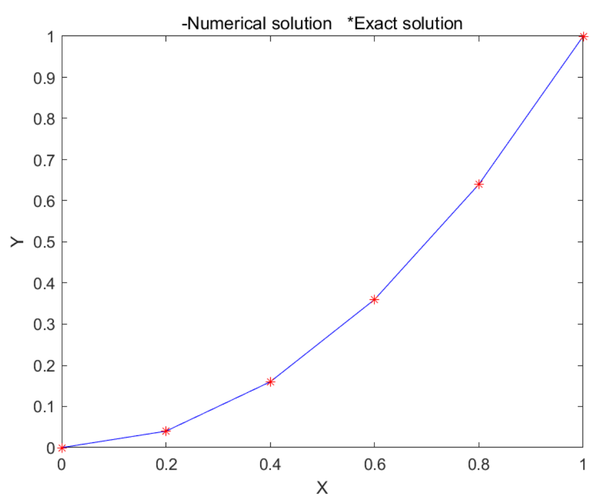

FIGURE 2. Results for Example 5.2 with $n = 7$, $\alpha = 0.5$.

Example 5.3. Consider the Volterra integral equation

$$y(x) = \cos(x) - e^{x}\sin(x) + \int_0^x e^{x} y(t)dt, \quad 0 \le x \le 1$$

where the exact solution is $y(x) = \cos(x)$.

Table 1 shows the results for $n = 2, 5, 7, 20$ and $\alpha = 0.01, 0.8$. Figure 3 also shows the exact and approximate solutions for $n = 7$ and $\alpha = 0.01$.

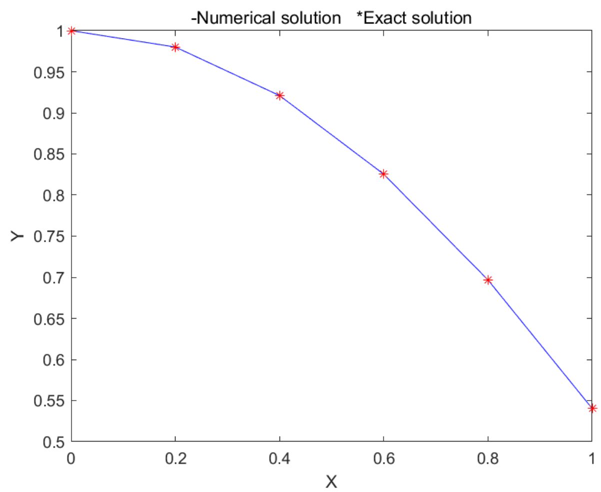

FIGURE 3. Results for Example 5.3 with $n = 7$, $\alpha = 0.01$.

TABLE 1. Computed errors for various Examples 5.1–5.3

	Example 5.1		Example 5.2		Example 5.3	
n	α=0.01	α=0.1	α=0.01	α=0.5	α=0.01	α=0.8
2	0	0	0	0	0.122E-2	0.132E-2
5	0.3821E-18	0.9483E-19	0.2015E-19	0.1061E-19	0.1839E-4	0.1839E-4
7	0.3626E-17	0.1813E-16	0.4947E-19	0.3388E-19	0.526E-7	0.526E-7
20	0.9069E-10	0.4586E-10	0.9602E-12	0.5527E-11	0.6211E-10	0.4011E-9

6. CONCLUSIONS

This paper presents a numerical method for solving linear integro-differential equations with a weakly singular kernel using a generalized Bernstein operator. The generalized Bernstein operators possess the same elementary properties as the operators of the Bernstein class. Especially, the α-Bernstein operator has the monotony-preserving and convexity-preserving properties. It is observed from the examples that the proposed method is a good approximation for integro-differential equations. Furthermore, we obtain an error estimate for this method.

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