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ENCHANTED TOPOLOGICAL SPACES

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Abstract. In this paper, we introduce and study, in particular, a new topological invariant–enchantedness; the enchanted spaces include any Euclidean space of dimension at least two. Via enchantedness one may gain some insight into Euclidean topologies, and local enchantedness, in particular, provides a new, simple, and intrinsic way to "topologically differentiate" topological 1-manifolds from higher-dimensional manifolds. A generic construction of enchanted spaces, with some additional nice properties, is given. This construction also induces a notion of multi-point connectification, generalizing the existing, well-developed notion of one-point connectification, and a dual result related to multi-point connectification.

Like compactness, connectedness is not hereditary. However, compactness is at least closed hereditary.

Connectedness, on the other hand, seems less favorable; here, we have certainly omitted trivial cases such as singleton sets. Although there is an abstract, nontrivial characterization of connected subspaces (e.g., Theorem 6.1.7 in Engelking [2]), it is unclear whether there is a simple, easily recognizable property, "comparable" to the closedness in the compactness case, such that every subset, with this property, of a connected space inherits connectedness; this is perhaps an old, unsolved "problem".

In this paper, we develop some fundamental properties of some new concepts, initially motivated by this problem, which, however, turn out to be worthy of study independently of the initially motivating problem. Among their fruits, these new topological "infrastructures", revolving around a new topological invariant, enchantedness, appear to occupy, respectively, an unexpected and significant position; and they also serve as tools complementing the connectedness theory and capable of facilitating the construction of counterexamples.

Throughout the text, a topological space is assumed to possess a property if and only if the corresponding statement is made.

To begin with a preliminary observation is that one can easily construct topological spaces with rich subspace properties where there is some open proper subset, receiving a nontrivial subspace topology, such that every superset of this set is connected.

Theorem 1. *Let X be a topological space; let F be a finite nonempty set with $X \cap F = \varnothing$. If $Y := X \cup F$, then there exists some connected topology of Y such that* i) *every superset of F in Y is connected and* ii) *the set F is open in Y, and receives a nontrivial subspace topology, from Y whenever F has at least two elements; moreover, the given space X is embeddable into Y as a closed subspace.*

Proof. Without loss of clarity, we present a proof slightly more general with respect to the purpose.

Represent $F \equiv \{p_1, \dots, p_n\}$, and let $\mathscr{T}_X$ be the given topology of X.

Give Y the topology

$$\mathscr{T}_Y := \{\varnothing\} \cup \bigcup_{1 \le i \le n} \{G \cup \{p_1, \dots, p_i\} \mid G \in \mathscr{T}_X\}.$$

Then, since every nonempty open set in Y contains p_1, the topology $\mathscr{T}_Y$ is connected.

Likewise, since every nonempty open set in any given subspace $\{p_1, \dots, p_i\}$ of Y contains p_1 by construction, it follows that every $\{p_1, \dots, p_i\}$ is connected. Moreover, every $\{p_1, \dots, p_i\}$ is dense in Y; because, the only closed superset of $\{p_1, \dots, p_i\}$ in Y is, by construction, the space Y itself.

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For any $1 \le i \le n$, fix any $A \supset \{p_1, \ldots, p_i\}$ in Y. Since $\{p_1, \ldots, p_i\}$ is connected and dense in Y, it follows that A is connected. This proves i).

Upon remarking that every set $\{p_1, \ldots, p_i\}$ is open in Y and, as a subspace of Y, receives the topology $\{\varnothing\} \cup \{\{p_1, \ldots, p_j\} \mid 1 \le j \le i\}$, and upon referring to the definition of $\mathscr{T}_Y$, we have completed the proof. □

Remark 1. Theorem 1 evidently allows for nearly arbitrary choices of X and F to obtain the desired conclusion, and thus the collection $\{A \mid F \subset A \subset Y\} = \{S \cup F \mid S \subset X\}$ of sets in Y can, in particular, be made nontrivial in a wild sense. Moreover, the set X is closed and not open in Y, and the given topology of X is precisely the subspace topology of X obtained from Y.

The setting in Theorem 1 resembles the extension topologies, but essentially differs in that F, as a subspace of Y, receives a connected topology.

We add that, although the space Y in Theorem 1 is clearly T_0 whenever X is T_0, it cannot be T_1 (and hence not Hausdorff) as long as F has at least two elements.

The subspace topology that F, when it has at least two elements, receives in Theorem 1, does not play a main role in this paper; but, in view of its conceptual importance, it may deserve a name.

Definition 1. A topological space is called a *generalized Sierpiński space* if and only if it is a finite set $\{p_1, \ldots, p_n\}$ having at least two elements and receiving a topology of the form $\{\varnothing\} \cup \bigcup_{1 \le i \le n} \{\{q_1, \ldots, q_i\}\}$ with $\{q_1, \ldots, q_n\} = \{p_1, \ldots, p_n\}$.

The position of a generalized Sierpiński space is seen in the following

Proposition 2. i) *Every generalized Sierpiński topology is nontrivial and nondiscrete, and is well-ordered by the inclusion relation.* ii) *Every set in any given generalized Sierpiński space is connected.*

Proof. Statement i) is evident.

For ii), we fix a generalized Sierpiński space $S \equiv \{p_1, \ldots, p_n\}$, and, without loss of generality, let $G_i := \{p_1, \ldots, p_i\}$ be a nonempty open set in the given topology of S for all $1 \le i \le n$.

If $A \subset S$ is disconnected, we choose some open sets G_i, G_j in S such that $G_i \cap A$ and $G_j \cap A$ are nonempty and form a partition of A. Since then $G_i \neq G_j$, assume that $G_i \subsetneq G_j$ without loss of generality. It follows that $A \subset G_j \setminus G_i$; for, if $p \in A$ is also contained in G_i, then $p \in G_i \cap G_j \cap A$.

In turn, we have $A \subset \{p_{i+1}, \ldots, p_j\}$. If k is the smallest element of $\{i+1, \ldots, j\}$ with $p_k \in A$, then every nonempty set open in A contains p_k; and so A receives a connected subspace topology, that leads to a contradiction. □

Remark 2. For the notion of a generalized Sierpiński space, requiring the finiteness is not essential; with an additional notation, one can easily extend the notion to countable sets (with at least two elements) without losing Proposition 2.

However, for our purposes, it suffices to consider finite sets.

Convention. By the Sierpiński space, we will always mean the topological space $(\{0,1\}, \{\varnothing, \{0\}, \{0,1\}\})$.

We exemplify Theorem 1 by constructing strictly increasing sequences of connected subspaces in certain spaces with "increasing complexity" of their connected components and topological structures.

Example 1. i) Consider the discrete space $X := \mathbb{N} := \omega \setminus \{\varnothing\}$ of positive integers, and, given any $n \in \mathbb{N}$, let $F := \{1/(n+1), \ldots, n/(n+1)\}$. Upon giving $Y := X \cup F$ the specific topology as in Theorem 1, it follows by Theorem 1 that each of the subspaces $\{1, \ldots, m\} \cup F$ of Y is connected.

ii) For each $n \in \mathbb{N}$, let K_n be the remaining closed set in $[0,1]$ obtained from the nth removal stage of the usual Cantor ternary operation, so that $X := \cap_{n \in \mathbb{N}} K_n$ is the classical Cantor set. Then X is totally disconnected, nondiscrete and nontrivial.

For each $n \in \mathbb{N}$, choose some $x_n \in \cap_{m \ge n} K_m$. If F is a finite set of positive prime numbers, then the subspaces $\{x_1, \ldots, x_n\} \cup F$ of $Y := X \cup F$ topologized as in Theorem 1 are all connected by Theorem 1.

iii) As another choice of X as a nondiscrete, nontrivial, and totally disconnected space, we can consider an uncountable ordinal number taken as an ordinal space. Upon choosing a finite set F that does not meet X, e.g., the choice of F in subexample i), the union of F and each of the initial segments included in X, is by Theorem 1 connected as a subspace of $Y := X \cup F$ topologized as in Theorem 1.

iv) We now consider a disconnected space X with infinitely many connected components each of which has continuum many elements. Let $X := \cup_{n\in\omega}]n, n+1[$ be a subspace of the Euclidean space $\mathbb{R}$. Then X is a desired choice. If F is a finite set of negative even integers, and if $Y := X \cup F$ is topologized in the same way as in Theorem 1, then the subspaces $(\cup_{n=0}^{m}]n, n+1[) \cup F$ of Y are all connected by Theorem 1.

Owing to Theorem 1, we can introduce a new class of topological spaces.

Definition 2. Let Y be a topological space; let $E \subset Y$. The space Y is said to be *enchanted by* E if and only if i) the set E is a proper subset of Y and ii) every proper subset A of Y with $A \supset E$ is connected. In this case, the set E is called an *enchanting set* in Y.

The space Y is said to be an *enchanted space* if and only if there is some enchanting set in Y.

Thus every enchanting set in an enchanted space is connected.

Remark 3. In other words, Theorem 1 asserts (in particular) that for any finite nonempty set F, one can always construct "many" connected spaces enchanted by F with F being an open nontrivial subspace; and so, Theorem 1 indeed provides nontrivial examples of enchanted spaces that are connected.

But there are natural and simple enchanted (and connected) spaces: the Sierpiński space $\{0, 1\}$ (receiving the topology $\{\varnothing, \{0\}, \{0, 1\}\}$) is enchanted by $\{0\}$. More generally, we can also consider the generalized Sierpiński spaces: By Proposition 2, every proper subset, empty or not, of any given generalized Sierpiński space S is an enchanting set in S; in particular, every generalized Sierpiński space is an enchanted space.

In passing, every given indiscrete (i.e., trivial) space is enchanted by each of its proper subsets.

Remark 4. It follows immediately from the definition of an enchanting set that, in any given enchanted space, every union of enchanting sets is an enchanting set provided that the union is a proper subset of the given space.

However, a union of enchanted subspaces, even with one point in common, is not necessarily an enchanted subspace. The interval $]0, 2]$ as a subspace of the Euclidean space $\mathbb{R}$ is enchanted by $]0, 2[$, and the interval $[1, 3[$ as a subspace of $\mathbb{R}$ is enchanted by $]1, 3[$; but, since every connected subspace of $]0, 3[$ evidently has some disconnected superset in $]0, 3[$, the interval $]0, 3[=]0, 2] \cup [1, 3[$ is not an enchanted subspace of $\mathbb{R}$.

In addition, the notion of an enchanting set and that of an enchanted subspace, although they sometimes agree, are logically independent. In the subspace $[0, 1]$ of the Euclidean space $\mathbb{R}$, the set $]0, 1]$ is both an enchanting set in and an enchanted subspace of $[0, 1]$; the set $]0, 1[$ is an enchanting set in $[0, 1]$, but not an enchanted subspace of $[0, 1]$; and the singleton $\{0\}$ is an enchanted subspace (by the empty set) of $[0, 1]$, but not an enchanting set in $[0, 1]$.

Enchantedness is, in particular, a topological invariant:

Theorem 3. *Every continuous bijective image of an enchanted space is enchanted.*

Proof. Let Y be an enchanted space; fix a topological space Z and a continuous bijection $f : Y \to Z$.

Choose by assumption some enchanting set E in Y; then the f-image $f^{1)}(E)$ of E is, by assumption. a proper subset of Z. If A is a proper subset of Z that includes $f^{1)}(E)$, then, since the f-preimage $f^{-1)}(A)$ of A is a proper subset of Y that includes E, the set $f^{-1)}(A)$ is connected by assumption. But then, as f is by assumption continuous, the set $A = f^{1)} \circ f^{-1)}(A)$ is connected.

Now, since Z is enchanted by $f^{1)}(E)$, the proof is complete. □

Enchantedness is not hereditary and does not carry over to Cartesian products.

Proposition 4. i) *A subspace of an enchanted space need not be enchanted.* ii) *A Cartesian product of enchanted spaces need not be enchanted.*

Proof. i) Take any topological space X that is not enchanted (e.g., any discrete space with at least three elements) and consider the space $X \cup F$ enchanted by some chosen finite nonempty F that does not meet X in the way described in Theorem 1; but then X is by construction a subspace of $X \cup F$.

ii) The discrete space $\{0,1\}$ is enchanted by, e.g., $\{0\}$; but the Cartesian product $\{0,1\} \times \{0,1\}$, also being a discrete space, is evidently not enchanted. □

Nevertheless, for enchantedness there is a relatively easily recognizable simple property such that every subset of an enchanted space with this property inherits enchantedness.

Theorem 5. *If a subset Z of an enchanted space Y properly includes some enchanting set E in Y, then Z is an enchanted subspace of Y with E being an enchanting set in Z.*

Proof. The enchanting set E in Y is, by assumption, a proper subset of Z and, hence, connected in Z. But every proper subset A of Z with $A \supset E$ is, by assumption, connected in Y, and hence, connected in Z; so E enchants Z. □

Regarding enchanted subspaces, we also have

Proposition 6. *If A, Z are subsets of a topological space Y with $A \supset Z$, it follows that Z is an enchanted subspace of Y if and only if Z is an enchanted subspace of the subspace A of Y.*

Proof. By assumption, the topology of the subspace Z received from Y coincides with that of Z received from A. □

Theorem 7. *Let Y be a topological space; let Θ be a nonempty set; let Z_θ be an enchanted subspace of Y for all $\theta \in \Theta$. If E_θ is an enchanting set in Z_θ for all $\theta \in \Theta$, and if $\cap_{\theta\in\Theta} E_\theta \neq \varnothing$, then $\cup_{\theta\in\Theta} Z_\theta$ is an enchanted subspace of Y with $\cup_{\theta\in\Theta} E_\theta$ being an enchanting set in $\cup_{\theta\in\Theta} Z_\theta$.*

Proof. By assumption, every E_θ is also connected in Y, so that $\cup_\theta E_\theta$ is connected in Y and hence in $\cup_\theta Z_\theta$.

Choose by assumption some point x of $\cap_\theta E_\theta$, and let A be a proper subset of $\cup_\theta Z_\theta$ that includes $\cup_\theta E_\theta$. Then $A = \cup_\theta (A \cap Z_\theta)$, with each $A \cap Z_\theta \supset E_\theta$ being connected in Z_θ. In particular, the set A is a union of connected sets $A \cap Z_\theta$ in Y. Since $x \in \cap_\theta (A \cap Z_\theta)$, it follows that A is connected in Y and hence in $\cup_\theta Z_\theta$.

We have proved that $\cup_\theta E_\theta$ enchants $\cup_\theta Z_\theta$. □

Moreover, regarding the Cartesian products, we do at least have

Proposition 8. *Let Θ be a nonempty set; let Y_θ be a topological space for all $\theta \in \Theta$; let $\gamma \in \Theta$. If E_γ is a proper subset of Y_γ, and if $\prod_{\theta\in\Theta} Y_\theta$ is enchanted by the natural projective preimage $\pi_\gamma^{-1)}(E_\gamma)$ in $\prod_{\theta\in\Theta} Y_\theta$, then Y_γ is enchanted by E_γ.*

Proof. We might first remark that the proof of Theorem 3 also serves continuous surjections that carry some enchanting set to a proper subset. But then the natural projection $\pi_\gamma : \prod_\theta Y_\theta \to Y_\gamma, (y_\theta)_\theta \mapsto y_\gamma$ is, from assumption, such a continuous surjection. □

It follows from Theorem 3 that the cardinal of the collection of all enchanting sets in an enchanted space is also a topological invariant.

Corollary 9. *Let Y, Z be enchanted spaces. If Y is homeomorphic to Z, then the collection of all enchanting sets in Y is equinumerous to that of all enchanting sets in Z, of which the corresponding sets are homeomorphic.*

Proof. This follows immediately from the proof of Theorem 3 by choosing a homeomorphism f acting between Y and Z; because then the f-image map $f^{1)}$ is a desired choice. □

Remark 5. We now have a new view on the topological differences between compact intervals and a bounded half-closed interval in the Euclidean space $\mathbb{R}$: The compact interval $[-1,1]$, as an enchanted subspace of $\mathbb{R}$, has more than one enchanting sets, e.g., $[-1,1[$ or $]-1,1[$; but the bounded half-closed interval $]-1,1]$, as an enchanted subspace of $\mathbb{R}$, has exactly one enchanting set, e.g., $]-1,1[$ (since every connected proper subset of $]-1,1]$, other than $]-1,1[$, has evidently some disconnected superset in $]-1,1]$); and so, Corollary 9 provides a new proof that a compact interval is never homeomorphic to a bounded half-closed interval.

Moreover, we obtain a new view on the extended real line $\overline{\mathbb{R}}$, topologized as usual: Since $[-1,1]$ as a subspace of $\mathbb{R}$ evidently has exactly three enchanting sets, and since $[-1,1]$ is homeomorphic to $\overline{\mathbb{R}}$, it holds by Theorem 3 and Corollary 9 that $\overline{\mathbb{R}}$ has exactly three enchanting sets.

Although an enchanted space is sometimes also a connected space, the two notions are logically independent.

Proposition 10. i) *There exists some enchanted space that is disconnected.* ii) *There exists some connected space that is nonenchanted.*

Proof. For a quick justification of i), we can consider once more the discrete space $\{0,1\}$.

There are also less trivial examples; we give a general construction of enchanted disconnected spaces with at least three elements. Let $(X, \mathscr{T}_X)$ be a connected space with cardinality at least two; let $p \notin X$. Topologize the set $Y := X \cup \{p\}$ by the topology $\{\varnothing\} \cup \mathscr{T}_X \cup \{G \cup \{p\} \mid G \in \mathscr{T}_X\}$. Then, since X and $\{p\}$ are clopen in Y, the space Y is disconnected.

Moreover, the given topology of X coincides with the subspace topology of X in Y; and so, the set X enchants Y.

On the other hand, for ii), the Euclidean space $\mathbb{R}$ cannot be enchanted; for, every connected proper subset of $\mathbb{R}$, which has to be a singleton or an interval, evidently has some disconnected superset in $\mathbb{R}$. This completes the proof. $\square$

Remark 6. Thus, from the proof of the statement i) in Proposition 10, an enchanting set in an enchanted space can also be closed.

In many cases, no connected component of a topological space can be an enchanting set.

Proposition 11. *If Y is a topological space with every connected component having at least two elements, then no connected component of Y is an enchanting set in Y.*

Proof. If Y is connected, then the only connected component, being Y itself, is not a proper subset of Y.

Suppose Y is disconnected. If C is a connected component of Y, then, due to the maximality of connected components, every set in Y properly including C is disconnected. But, from assumption, we can choose some element x of another connected component of Y such that the set $C \cup \{x\}$, which properly includes C, is a proper subset of Y; and therefore C cannot enchant Y. $\square$

There is a simple sufficient condition for a topological space to be an enchanted connected space.

Proposition 12. *If Y is a topological space, and if E is a connected proper subset of Y that is dense in Y, then Y is connected and is enchanted by E.*

Proof. Since $Y = \mathrm{cl}(E)$ by assumption, if $A \subset Y$ includes E, then A is connected. $\square$

Example 2. Let E be the graph of the function $x \mapsto \sin(1/x)$ on $]0,1]$, considered as a subspace of the Euclidean space $\mathbb{R}^2$, so that $Y := E \cup \{(0,z) \mid -1 \leq z \leq 1\} = \mathrm{cl}(E)$ as a subspace of the Euclidean space $\mathbb{R}^2$ is the "topologist's sinusoidal curve". Thus Y by Proposition 12 is, in particular, an enchanted space with E being an enchanting set.

Remark 7. The converse of Proposition 12 is not true: The Sierpiński space $\{0,1\}$ can be enchanted by $\{1\}$, but $\{1\}$ is closed therein.

Apart from Example 2, the assumption of Proposition 12 is, by Theorem 1, far from vacuous.

Corollary 13. *The closure of every given nonclosed connected set in a topological space Y is an enchanted subspace of Y.*

Proof. If $Q \subset Y$ is nonclosed and connected, then Q is a proper subset of the subspace $\mathrm{cl}(Q)$ of Y and is connected in $\mathrm{cl}(Q)$. The desired conclusion follows from Proposition 12. $\square$

For our purposes, we adopt the definition that an interval in the real field $\mathbb{R}$ is precisely a subset I of $\mathbb{R}$ such that if $c \in \mathbb{R}$ and if $a < c < b$ for some $a, b \in I$, then $c \in I$. Then there is a characterization for (nonempty) enchanted connected subspaces of the Euclidean space $\mathbb{R}$.

Theorem 14. *A nonempty enchanted connected subspace of the Euclidean space $\mathbb{R}$ is precisely a subspace of $\mathbb{R}$ that is an interval nonopen in $\mathbb{R}$.*

Proof. A singleton in $\mathbb{R}$ is enchanted by the empty set; and a compact interval $[a, b]$ in $\mathbb{R}$ is enchanted by, for example, the subinterval $[a, b[$. Every half-closed interval $[a, b[$ in $\mathbb{R}$, with $a, b \in \mathbb{R}$, is evidently enchanted by the subinterval $]a, b[$. The same observation shows that every half-closed interval $]a, b]$ in $\mathbb{R}$ is an enchanted subspace, and that all closed rays $[a, +\infty[$ or $]-\infty, a]$ in $\mathbb{R}$ are enchanted subspaces. This proves one of the assertions.

Suppose the converse is false, and choose one such nonempty enchanted connected subspace Y in $\mathbb{R}$. Then Y is an interval in $\mathbb{R}$ that is open in $\mathbb{R}$. Since every connected proper subset of a nonempty interval open in $\mathbb{R}$ evidently has, by the properties of the real numbers, some disconnected superset being a proper subset of the interval, it follows that Y is not enchanted; this contradiction completes the proof. □

Remark 8. From Theorems 14 and 3, although less information is really needed, a new view on the fact that every half-closed interval and every closed ray in the Euclidean space $\mathbb{R}$ are not homeomorphic to $\mathbb{R}$ is readily available.

For (nonempty) open sets in the Euclidean space $\mathbb{R}$, we have

Proposition 15. *Every nonempty open subset of the Euclidean space $\mathbb{R}$ is nonenchanted.*

Proof. Let $G \subset \mathbb{R}$ be an open set. If G is connected, then by Theorem 14, we are through. If G is disconnected, we fix a connected proper subset Q of G. Since every open set in $\mathbb{R}$ is a disjoint union of intervals, open in $\mathbb{R}$, we choose some partition $\mathscr{I}$ of G by intervals open in $\mathbb{R}$. Then $\mathscr{I}$ has at least two elements, and we can choose some $I \in \mathscr{I}$ such that $Q \subset I$.

But, upon choosing some point x from some $J \in \mathscr{I}$ with $J \neq I$, we see that $Q \cup \{x\}$ is a disconnected superset of Q that is a proper subset of G. □

Even though the Euclidean space $\mathbb{R}$ is not enchanted, for higher-dimensional Euclidean spaces we have

Theorem 16. *Every Euclidean space $\mathbb{R}^n$ with $n \geq 2$ is enchanted.*

Proof. If $x \in \mathbb{R}^n$, then the set $E := \mathbb{R}^n \setminus \{x\}$, being a co-countable set in $\mathbb{R}^n$, is a connected subspace of $\mathbb{R}^n$; so E enchants $\mathbb{R}^n$. □

Remark 9. The proof of Theorem 16 implies also that every Euclidean space of dimension at least two has at least continuum-many enchanting sets.

In addition, a new aspect of the fact that every Euclidean space $\mathbb{R}^n$ with $n \geq 2$ is not homeomorphic to $\mathbb{R}$ is obtained immediately from Proposition 15, Theorem 16, and Theorem 3.

Enchanted open subspaces are "abundant" in the Euclidean spaces $\mathbb{R}^n$ with $n \geq 2$:

Theorem 17. *Let $n \geq 2$ be an integer; let Θ be a nonempty set. If $\{Z_\theta\}_{\theta \in \Theta}$ is a collection of subsets of $\mathbb{R}^n$ homeomorphic to $\mathbb{R}^n$ with nonempty intersection, then $\cup_{\theta \in \Theta} Z_\theta$ is an enchanted connected open subspace of $\mathbb{R}^n$.*

Proof. That the union $\cup_\theta Z_\theta$ is connected is evident from assumption.

By the invariance of domain, every Z_θ is open in $\mathbb{R}^n$; so, $\cup_\theta Z_\theta$ is open in $\mathbb{R}^n$.

To show the enchantedness of $\cup_\theta Z_\theta$, we choose by assumption some $x \in \cap_\theta Z_\theta$. For every θ, we choose some $z_\theta \in Z_\theta$ such that $z_\theta \neq x$. Then, from assumption, the set $Z_\theta \setminus \{z_\theta\}$ enchants Z_θ for all θ. Since $x \in \cap_\theta (Z_\theta \setminus \{z_\theta\})$, it follows by Theorem 7 that $\cup_\theta Z_\theta$ is enchanted. This completes the proof. □

Example 3. Thus the union of a collection of open balls in the Euclidean space $\mathbb{R}^2$ containing the origin is an enchanted subspace of $\mathbb{R}^2$.

It would be natural to consider the following "local version" of enchantedness.

Definition 3. Let Y be a topological space. The space Y is called *locally enchanted* if and only if there exists some basis of Y consisting of enchanted subspaces.

It is easy to show that local enchantedness is also a topological invariant.

Theorem 18. *Every homeomorphic copy of a locally enchanted space is locally enchanted.*

Proof. Let Y be a locally enchanted space; let Z be a topological space; let $f : Y \to Z$ be a homeomorphism.

If G is open in Z, we choose by assumption some collection $\mathscr{V}$ of enchanted open subspaces of Y such that $\cup\mathscr{V} = f^{-1)}(G)$. Then, since $f^{1)}(V)$, which is open in Z for all $V \in \mathscr{V}$ by assumption, for all $V \in \mathscr{V}$ it is enchanted as a subspace of Z by Theorem 3, and since $G = \cup_{V\in\mathscr{V}} f^{1)}(V)$, we are finished. □

Like enchantedness, local enchantedness is not hereditary and does not carry over to the Cartesian products.

Proposition 19. i) *A subspace of a locally enchanted space is not necessarily locally enchanted.* ii) *The Cartesian product of locally enchanted spaces is not necessarily locally enchanted.*

Proof. i) By Theorem 16, every open ball in the Euclidean space $\mathbb{R}^2$, being homeomorphic to $\mathbb{R}^2$, is an enchanted open subspace of $\mathbb{R}^2$; so $\mathbb{R}^2$ is locally enchanted.

However, the Euclidean space $\mathbb{R}$ cannot be locally enchanted; indeed, every nonempty enchanted subspace of $\mathbb{R}$ is not open by Proposition 15. But then, By Theorem 18, the subspace $\mathbb{R} \times \{(0, 0)\}$ of $\mathbb{R}^2$ is not locally enchanted.

ii) Consider an infinite collection of discrete spaces Y_θ, each of which has at least two elements. Then no set in $\prod_\theta Y_\theta$ with at most two elements is open in $\prod_\theta Y_\theta$.

Fix any θ and any nonempty open G_θ in Y_θ. If $V \subset \pi_\theta^{-1)}(G_\theta)$ is an enchanted subspace of $\prod_\theta Y_\theta$, then, since $\prod_\theta Y_\theta$ is totally disconnected, the set V has at most two elements; but then V is not open in $\prod_\theta Y_\theta$. Thus $\prod_\theta Y_\theta$ is not locally enchanted. □

However, local enchantedness is hereditary open.

Theorem 20. *Every open set in a locally enchanted space Y is a locally enchanted subspace of Y.*

Proof. Fix any open set G in Y. Since every open set in G is open in Y, every open set in G is a union of enchanted open subspaces of Y.

Fix any nonempty $V \subset G$ such that V is open in G. Then, for every point of V, we can choose some enchanted open subspace W of Y such that W contains this point and $W \subset V$, and W is also open in G.

But since $W \subset G$, the set W is also an enchanted subspace of G by Proposition 6. The proof is complete. □

For reference, given any $n \in \mathbb{N}$, by a topological n-manifold we mean precisely a (not necessarily second-countable or Hausdorff) locally Euclidean n-space, i.e., a topological space where every point has some neighborhood homeomorphic to some open subset of $\mathbb{R}^n$. To justify the introduction and show the position of local enchantedness, we have the following results:

Theorem 21. i) *Every topological n-manifold with $n \geq 2$ is locally enchanted.* ii) *Every topological 1-manifold is not locally enchanted.*

Proof. Let $n \in \mathbb{N}$, and fix a topological n-manifold M.

To prove i), we suppose that $n \geq 2$. If G is open in M, and if $x \in G$, then we can choose some coordinate neighborhood V of x, and choose in turn a neighborhood W of x that is included in $V \cap G$ and homeomorphic to $\mathbb{R}^n$. Since W is then by Theorems 3 and 16 also enchanted, the desired conclusion follows.

For ii), suppose $n = 1$, and assume that M, now being a topological 1-manifold, is locally enchanted. Fix a point x of M, and choose some chart (G, φ) of M with respect to x. And by assumption, we can further choose some enchanted neighborhood V of x such that $V \subset G$.

But then $\varphi^{1)}(V)$ is by Theorem 3 an enchanted open subspace of $\mathbb{R}$, which contradicts Proposition 15. This completes the proof. □

Proposition 22. i) *There exists a topological space that is enchanted and locally enchanted.* ii) *There exists a topological space that is neither enchanted nor locally enchanted.* iii) *There exists an enchanted space that is not locally enchanted.* iv) *There exits a locally enchanted space that is not enchanted.*

Proof. i) The Sierpiński space $\{0, 1\}$ is enchanted. But the only choice of basis of this space is its topology; and both $\{0\}$ and $\{0, 1\}$ are enchanted open subspaces. In view of Theorems 16 and 21, we can also consider any Euclidean space $\mathbb{R}^n$ with $n \geq 2$.

ii) The Euclidean space $\mathbb{R}$ is (by Proposition 15, for concreteness) not enchanted, and is (by Theorem 21, for concreteness) not locally enchanted.

iii) Choose a topological space X that is connected and not locally enchanted (e.g., the Euclidean space $\mathbb{R}$), and take a point $p \notin X$. Topologize the set $Y \coloneqq X \cup \{p\}$ in the same way as in the proof of the first statement in Proposition 10, so Y is an enchanted space. Now, since X is open in Y, and since a subset of X open in Y is precisely a subset of X open in the given topology of X, for Y to be locally enchanted, it is necessary that X be locally enchanted. This establishes the desired conclusion.

iv) Every discrete space with cardinality at least three is not enchanted; but, since every singleton in such a space is an enchanted open subspace of the space, the basis of the space, consisting of all singletons in the space, is a desired choice. □

Remark 10. Evidently, from the proof of Theorem 21, Theorem 21 holds for locally Euclidean n-spaces: Every locally Euclidean n-space with $n \geq 2$ (resp., $n = 1$) is locally enchanted (resp., not locally enchanted). In particular, from Theorems 21 and 18, we have obtained a new and simple proof that every topological n-manifold with $n \geq 2$ is not homeomorphic to any given topological 1-manifold.

As summarized in the proof of Proposition 22, it holds that: i) every Euclidean n-space with $n \geq 2$ is both enchanted and locally enchanted, and ii) the Euclidean 1-space is neither enchanted nor locally enchanted. Thus we have obtained a new perspective regarding the intrinsic topological differences between the Euclidean 1-space and the higher-dimensional Euclidean spaces.

Last, but not least, Theorem 21 also implies that not every locally connected space is locally enchanted: We might consider the Euclidean space $\mathbb{R}$.

There is an interesting result admitting a very simple proof:

Theorem 23. *No open set in a topological* 1*-manifold is homeomorphic to any given open set in a locally enchanted space.*

Proof. Let M be a topological 1-manifold; let Y be a locally enchanted space. If a nonempty open set G in M is homeomorphic to a nonempty open set W in Y via a homeomorphism $f : G \to W$, choose in M some coordinate neighborhood U of a point of G. Then $G \cap U$ is homeomorphic to $f^{1)}(G \cap U)$.

Since $f^{1)}(G \cap U) \subset W$ is open in Y, we can choose by assumption some nonempty enchanted open subspace V of Y with $V \subset f^{1)}(G \cap U)$, and so, since $G \cap U$ is by assumption also homeomorphic to some open subset of $\mathbb{R}$, the set $f^{-1)}(V) \subset G \cap U$ is homeomorphic to some open subset of $\mathbb{R}$.

But $f^{-1)}(V)$, being by Theorem 3 (and Proposition 6, for concreteness) a nonempty enchanted open subspace of M, cannot be homeomorphic to any given open subset of $\mathbb{R}$ by Theorem 3 and Proposition 15; this contradiction establishes the proof. □

For reference, a topological space is said to be locally homeomorphic to a topological space if and only if the former space has some open cover with each element being (topologically) embeddable into the latter space as an open subset; thus for every homeomorphic pair of topological spaces Y, Z, the space Y (resp., Z) is locally homeomorphic to Z (resp., Y). Then we have

Corollary 24. *No topological* 1*-manifold is locally homeomorphic to any given locally enchanted space; no locally enchanted space is locally homeomorphic to any given topological* 1*-manifold.*

Proof. This follows plainly from Theorem 23. □

Remark 11. Since every topological n-manifold with $n \geq 2$ is by Theorem 21 locally enchanted, Theorem 23 offers a new intrinsic perspective to further topologically distinguishing topological 1-manifolds from higher-dimensional menifolds.

We make some additional remarks. The construction introduced in the proof of Theorem 1 is motivated by the special one-point connectification given in Chou [1], which admits other independent focal points; one-point connectifications have a substantial history and rich developments in topology; one can refer, for example, to Koushesh [3], devoted to the involved technical and nontechnical accounts.

Theorem 1 also inspires the following notion, which generalizes that of one-point connectification.

Definition 4. Let X, Y be topological spaces. The space Y is called a *multi-point connectificaction* of X if and only if Y is a connected space including X as a subspace, and $Y \setminus X$ is a finite nonsingleton nonempty set.

Here, the density relaxation, which is usually required in the context of space extension, in Definition 4 is not always standard.

Theorem 1 immediately admits a dual result regarding multi-point connectifications.

Corollary 25. *Let X be a topological space; let ∞ be a finite non-singleton nonempty set that does not meet X. Then there exists some multi-point connectification X_∞ of X, with $X_\infty \setminus X = \infty$ such that* i) *the space X_∞ is enchanted by ∞ and* ii) *the set ∞ is a generalized Sierpiński space as a subspace of X_∞.*

Proof. Upon choosing the specific topology for X_∞ as in Theorem 1, the desired conclusion follows plainly from Theorem 1. □

Remark 12. Although it is not a focus of the present work, we wish to point out that the topology introduced in Theorem 1, apart from its simplicity, would be manageable and convenient in some ways. For example, some important properties such as being second-countable or being an Alexandroff space can easily be seen to be transferable.

Moreover, we indicate once more that the topologies considered in Theorem 1 can be useful for constructing counterexamples in many contexts.

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