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APPLICATION OF THE DEEP NEURAL NETWORK FOR NUMERICAL SOLUTION OF A FOURTH-ORDER NONLINEAR INTEGRO-DIFFERENTIAL MODEL

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Abstract. The paper considers the numerical solution of an initial-boundary value problem for a fourth-order nonlinear partial integro-differential equation. The model under consideration is a generalization of the second-order nonlinear partial integro-differential equation based on the Maxwell system. Algorithms for finding approximate solutions applying deep neural networks are used. The results of numerical experiments with graphical illustrations and their analysis are presented.

Integro-differential equations occur in many applications. Numerous scientific works, monographs and textbooks are devoted to the study of integro-differential models. Integro-differential equations, often encountered in physics and mathematics, contain derivatives of various variables; therefore, these equations are called integro-differential equations with partial derivatives or partial integro-differential equations (PIDEs).

The purpose of the present note is to study a fourth-order nonlinear PIDE. The process of electromagnetic field penetration into the substance is described by the well-known system of Maxwell equations, which show the process of propagation of an electromagnetic field into a medium [18]. The literature dealing with the questions of the existence, uniqueness, regularity, asymptotic behavior of solutions and numerical resolutions of the initial-boundary value problems for models of these types is quite extensive (see, e.g., [1, 5, 7, 12, 14] and references therein).

Note that some of such integro-differential models in the scalar case have the following form:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left[a\left(\int\limits_0^t \left(\frac{\partial U}{\partial x}\right)^2 d\tau\right)\frac{\partial U}{\partial x}\right]. \tag{1}$$

Models of such a type have arisen for the first time in [8]. In [5,6,10], the unique solvability of the initial-boundary value problems for equations of type (1) was presented under more general assumptions on the function $a = a(S)$, than in [8]. The existence theorems, proven in the above-mentioned works, are based on the a priori estimates, modified version of the Galerkin method, and on compactness arguments as in [23, 29] for nonlinear parabolic equations.

Based on the works [5, 6, 8, 10], the models of the type

$$\frac{\partial U}{\partial t} = a\left(\int\limits_0^t \int\limits_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx d\tau\right)\frac{\partial^2 U}{\partial x^2}, \tag{2}$$

appeared in [20], the author called the averaged integro-differential equations (AIDEs). Here, $a = a(S)$, as in (1), is a given function defined for $S \in [0,\infty)$. In [20], the author mentioned that the study of models of type (2) requires a somewhat different approach than that of the Volterra-type models (1). The existence and uniqueness of the solutions to initial-boundary value problems for the AIDEs of type (2) were first studied in [11].

The literature on the questions of the existence, uniqueness, regularity, asymptotic behavior of the solutions and numerical resolutions of the initial-boundary value problems for (1) and (2) type models are very rich. The first publication on models of type (1) quickly attracted the attention of scientists,

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and in a short period, following [5,6,8,10], many scientific papers were published. Numerous scientific works are devoted to the construction and justification of algorithms of the numerical resolution of initial-boundary value problems for the above-stated models (1) and (2). The possibility of finite-difference approximations, the application of Galerkin methods and the method of finite elements, as well as the algorithms for their realization are discussed, for example, in [2,5,6,8,14,19–22,28,30]. For relatively complete citations on this study, up to 2019, we refer the reader to [14] and [12]. Since then, this interest has not diminished, but rather increased. Many works are devoted to multidimensional and to higher order integro-differential equations of types (1) and (2) (see, e.g., [5,6,11,19,20]).

The purpose of the present work is to continue our study, to formulate the problem rigorously, discuss the analytical challenges, and provide computational strategies based solely on deep learning methods, in particular, fully connected feedforward neural networks, for its approximate solution (see, for example, [3,25–27]).

The presented work discusses a natural mathematical generalization of the integro-differential models (1) and (2). Specifically, in the domain $[0,1]\times[0,T]$, where T is a fixed positive number, the following initial-boundary value problem is studied for the corresponding fourth-order integro-differential equation:

$$\frac{\partial U}{\partial t}+\frac{\partial^2}{\partial x^2}\left\{\left[1+\int\limits_0^t\left(\frac{\partial U}{\partial x}\right)^2 d\tau\right]\frac{\partial^2 U}{\partial x^2}\right\}=f(x,t), \tag{3}$$

$$U(0,t)=U(1,t)=0, \tag{4}$$

$$\frac{\partial^2 U}{\partial x^2}(0,t)=\frac{\partial^2 U}{\partial x^2}(1,t)=0, \tag{5}$$

$$U(x,0)=U_0(x), \tag{6}$$

where $U=U(x,t)$ is an unknown function, and in equation (3) and in the initial condition (6), $f(x,t)$ and $U_0(x)$ are the given functions of their arguments, respectively.

The stability and uniqueness of the solution to problem of type (3)–(6) for the Volterra-type models (1) were studied in [4]. Analogous results for the same type of nonlinearity are also true for the AIDE problem of type (3)–(6).

As mentioned above, many studies are devoted to classical methods of approximate solving initial-boundary value problems for models of type (1) and (2).

As we already said, our goal is to explore an alternative method for solving problem (3)–(6) using Machine Learning techniques. Machine learning, specifically neural networks, is utilized to create surrogate models that predict the solutions under study at any point within the domain. Neural networks, consisting of input, hidden, and output layers, provide flexibility in terms of architecture and the number of neurons per layer (see, e.g., [3]). In this approach, the solution to the problem is approximated by a neural network output, and the network parameters are optimized during the training process. A significant advantage of using deep neural networks in solving problem is their ability to incorporate physical laws into the learning process, reducing the volume of required training data (as discussed in [3,25–27]).

The above-mentioned differential and integro-differential models using Machine Learning are also studied in some other papers too (see, e.g., [4,15–17] and references therein).

In our experiment, we have chosen the following exact solution to problem (3)–(6) with the corresponding right-hand side function $f(x,t)$ and the initial condition $U_0(x)$:

$$U(x,t)=x^3(1-x)^3(1+t^2).$$

The numerical simulation was carried out using a neural network model implemented in the TensorFlow framework. Development and training of such a model can be implemented in Python environment, for example, with commonly used libraries such as TensorFlow or PyTorch, as well as scientific computing tools like NumPy, SciPy, and Matplotlib for visualization. Although training can be performed on a standard computer, the use of a GPU (graphics processing unit) is recommended for solving more complex problems or reducing training time.

Table 1. Comparison between exact and predicted solutions

x	t	**Exact** $U(x,t)$	**Predicted** $U(x,t)$	**Absolute Error**
0.2	0.7	0.012713	0.01315	0.000438
0,4	0.7	0.028603	0.029312	0.000708
0.6	0.7	0.028603	0.029312	0.000708
0.8	0.7	0.012713	0.01315	0.000438

A convenient development environment is Jupyter Notebook hosted on Google Colab, which provides interactive coding capabilities and access to cloud-based computational resources. Google Colab is especially useful, as it supports all required Python libraries for deep learning and offers free GPU access.

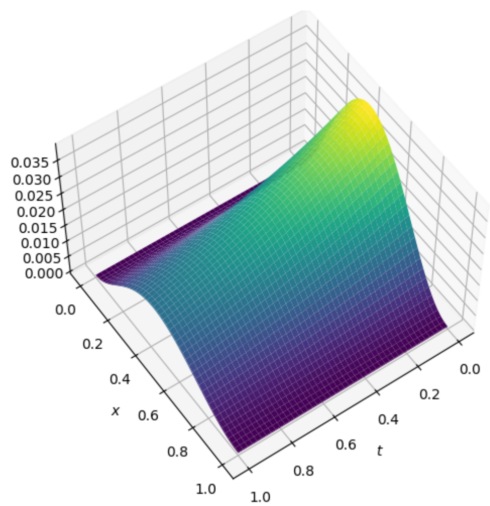

Figure 1. Exact solutions

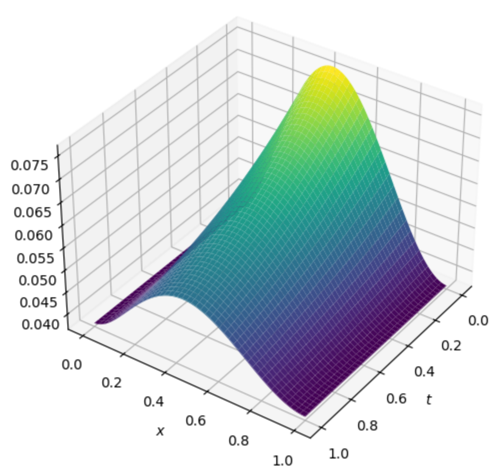

Figure 2. Predicted solutions

To further illustrate the accuracy of model, Table presents numerical values comparing the exact and predicted solutions for selected values of x and t.

To visually evaluate the accuracy of the neural network's prediction, Figure 1 presents the approximate solutions, and Figure 2 shows the exact solutions in the form of 3D surface plots. These visualizations clearly indicate that the neural network's output closely matches the true solution across the spatio-temporal domain. Although small discrepancies are noticeable, they remain within acceptable limits, confirming the reliability and robustness of the model.

This study proposes a deep learning-based method for solving fourth-order nonlinear parabolic-type integro-differential equations. A fully connected feedforward neural network is used to approximate the solution by embedding the equation, initial, and boundary conditions into a unified loss function.

Numerical results show that the model accurately describes the solution behavior with low error and reliable generalization. Both visual and quantitative analysis confirm the robustness of the method.

Overall, the findings support deep learning as an effective alternative to traditional mesh-based approaches, especially for complex problems with nonlocal and nonlinear features.

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