Transactions of A. Razmadze
Mathematical Institute
Vol. **180** (2026), issue 2, 207–210

A SHORT NOTE ON STAR-COMPACT SPACE

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Abstract. In this paper, we intend to show that in a regular space with a cellular open base, a star compact subspace and a closed subset, disjoint from each other, are strongly separable. We also find an alternative representation of star-compactness in terms of a family of closed sets by introducing some modifications to the finite intersection property.

1. Introduction

In a topological space (X, τ), the star of a subset A, with respect to a collection of subsets $\mathcal{U}$ is denoted and defined as

$$St(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}.$$

The star operator was used to generalize the concepts of compactness and Lindelöfness by Douwen, Reed, Roscoe and Tree [15] in 1991. Since then, several generalizations of star-compactness have been made and studied by various authors. Some recent advances in this direction can be found in the works of Xuan and Song [16], Kočinac et al. [12,13], Bonanzinga and Maesano [10] and Bal et al. [2,3,5–9]. A topological space (X, τ) is called a star-compact space if for every open cover $\mathcal{U}$, we can find a finite subset F such that $St(F,\mathcal{U}) = X$ [15]. By the definition itself, it follows that every compact space is a star-compact space. Imposing the finite intersection property on a family of closed sets leads to an alternative interpretation of compact sets [11]. Following the same path, we established an alternative definition for the star-compact space. On the other hand, if a regular space has two disjoint subsets, one of which is compact and the other is closed, then they are strongly separable [11]. In this article, we discuss what happens if the one of the disjoint subsets is star-compact.

2. Preliminaries

Definition 2.1 ([11])**.** A family $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ (where Λ is an index set) is said to have the finite intersection property if for every finite subset $\{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n\}$ of Λ, $\bigcap_{i=1}^{n} F_{\alpha_i} \neq \emptyset$.

Definition 2.2 ([1])**.** A family $\mathcal{F}$ is said to be cellular if the elements of $\mathcal{F}$ are pairwise disjoint.

An open base $\mathcal{B}$ of a topological space (X, τ) is called a cellular open base if the elements of $\mathcal{B}$ are pairwise disjoint.

3. Main Results

Theorem 3.1. *Let (X, τ) be a regular space with a cellular open base, and let (A, τ_A) be a star-compact subspace of (X, τ). Then for every closed set B disjoint from A, there exist open sets $U,V \subseteq X$ such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.*

Proof. Since (X, τ) is a regular space, for every $x \in A$, there exists open sets $U_x, V_x \subseteq X$ such that $x \in U_x$; $B \subseteq V_x$ and $U_x \cap V_x = \emptyset$.

Let $\mathcal{B}$ be a cellular open base for (X, τ), then there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U_x$. Clearly, $x \in B_x$ and $B \subseteq V_x$ are such that $B_x \cap V_x = \emptyset$. Then $A \subseteq \cup_{x \in A} B_x$. Let $\mathcal{U} = \{B_x : x \in A\}$. So, $\mathcal{U}_A = \{A \cap B_x : x \in A\}$ is a cover for the subspace (A, τ_A). But (A, τ_A) is a star-compact space. Therefore, there exists a finite set $F \subseteq A$ such that

$$St(F,\mathcal{U}_A) = A.$$

2020 *Mathematics Subject Classification.* 54D20, 54D30, 54D40.

Key words and phrases. Star compactness; Cellular base; Finite intersection property.

But every element of $\mathcal{U}_A$ is a subset of the corresponding element of $\mathcal{U}$, therefore

$$A \subseteq St(F,\mathcal{U}).$$

We take $U = St(F,\mathcal{U})$ and $V = \cap_{x\in F} V_x = V$. Obviously, U and V are open sets with $A \subseteq U$ and $B \subseteq V$. Now.we have to show that $U \cap V = \emptyset$.

If possible, let $U \cap V \neq \emptyset$ and $p \in U \cap V$.

$$\implies p \in St(F,\mathcal{U}) \text{ and } p \in \bigcap_{x\in F} V_x,$$

$$\implies \{\exists B_q \in \mathcal{U} \text{ such that } p \in B_q \text{ and } F \cap B_q \neq \emptyset\} \text{ and } \{p \in V_x \forall x \in F\}.$$

Let $r \in B_q \cap F \implies r \in B_q$ and $r \in F$.

Therefore, $B_q \in \mathcal{U}_A$ such that $p, r \in B_q$ and $p \in V_r$. p may or may not belong to A, but $q, r \in A$. Since $B_r, B_q \in \mathcal{U}$ and $\mathcal{U}$ is a cellular open base, therefore either $B_q \cap B_r = \emptyset$, or $B_r = B_q$.

Since $r \in B_r$ and $r \in B_q$, therefore $B_r \cap B_q \neq \emptyset$, so $B_r = B_q$.

Thus $p \in B_q = B_r$ also $p \in V_r$, $\therefore B_r \cap V_r \neq \emptyset$, which is a contradiction.

Therefore $U \cap V = \emptyset$. Hence we have the following □

Corollary. *Let (X, τ) be a Hausdörff space with a cell-open base and let (A, τ_A) be a star-compact subspace of (X, τ). Then for every $x \in X$ such that $x \notin A$, there exist open sets U,$V \subseteq X$ such that $A \subseteq U$, $x \in V$ and $U \cap V = \emptyset$.*

Proof. In a Hausdörff space, every singleton set is closed. So, the result follows directly from the aforementioned Theorem 3.1. □

Definition 3.1. A family $\mathcal{F}$ of subsets of a space X is said to have the modified non-finite intersection property (MNI property) if there exists a subfamily $\mathcal{E} \subseteq \mathcal{F}$ with the property $\cap\mathcal{E} = \emptyset$ for which we can find a finite subset $P \subseteq X$ such that $P \cap (X \setminus F) \neq \emptyset$ for all $F \in \mathcal{E}$.

Theorem 3.2. *For a topological space (X, τ), the following statements are equivalent:*

(1) *X is a star-compact space.*

(2) *For a given family $\mathcal{U}$ of open sets, if for every finite subset $F \subseteq X$, $St(F,\mathcal{U}) \neq X$, then $\mathcal{U}$ cannot cover X.*

(3) *For a given family $\mathcal{F}$ of closed subsets of X, if $\bigcap \mathcal{F} = \emptyset$, then the family $\mathcal{F}$ have the MNI property.*

(4) *Every family of closed subsets of X that does not have the MNI property has a non-empty intersection.*

Proof.
(1) $\implies$ **(2)**.

Let X be a star-compact space. If possible, suppose that $\mathcal{U}$ is a family of open sets that covers X. In this case, by condition (2), for every finite subset $F \subseteq X$, $St(F,\mathcal{U}) \neq X$, which contradicts the fact that X is star-compact. So, $\mathcal{U}$ cannot be a cover of X.

(2) $\implies$ **(1)**.

Let $\mathcal{U}$ be an arbitrary open cover of X and condition (2) be satisfied. Clearly, $\mathcal{U}$ is a collection of open sets. If possible, suppose that for every finite set $F \subseteq X$, $St(F,\mathcal{U}) \neq X$. So, by condition (2), $\mathcal{U}$ is not a cover of X, which is a contradiction. Thus there must exist a finite subset $F \subseteq X$ such that $St(F,\mathcal{U}) = X$.

Therefore, X is a star-compact space.

(1) $\implies$ **(3)**.

Let X be a star compact space. Suppose $\mathcal{F}$ be a family of closed sets such that $\bigcap \mathcal{F} = \emptyset$.

$$\therefore X \setminus \left(\bigcap \mathcal{F}\right) = X \setminus \emptyset,$$

$$\implies \bigcup_{F\in\mathcal{F}} (X \setminus F) = X.$$

Thus $\mathcal{G} = \{G = (X \setminus F) : F \in \mathcal{F}\}$ is an open cover of X. But X is a star compact space. So, there exists a finite set $P \subseteq X$ such that $St(P, \mathcal{G}) = X$.

$$\implies \bigcup\{G \in \mathcal{G} : P \cap G \neq \emptyset\} = X,$$
$$\implies \bigcup\{(X \setminus F) \in \mathcal{G} : P \cap (X \setminus F) \neq \emptyset\} = X,$$
$$\implies \bigcup\{(X \setminus F) : F \in \mathcal{F} \text{ and } P \cap (X \setminus F) \neq \emptyset\} = X,$$
$$\implies X \setminus \bigcap\{F \in \mathcal{F} : P \cap (X \setminus F) \neq \emptyset\} = X,$$
$$\implies \bigcap\{F \in \mathcal{F} : P \cap (X \setminus F) \neq \emptyset\} = \emptyset.$$

Suppose that $\mathcal{E} = \{F \in \mathcal{F} : P \cap (X \setminus F) \neq \emptyset\}$. Therefore, $\mathcal{E} \subseteq \mathcal{F}$ is such that $\bigcap \mathcal{E} = \emptyset$ and $P \subseteq X$ is finite such that $P \cap (X \setminus F) \neq \emptyset$ for all $F \in \mathcal{E}$. Therefore, $\mathcal{F}$ has the MNI property.

$(\mathbf{3}) \implies (\mathbf{1})$.

Let condition (3) be satisfied and $\mathcal{U}$ be an arbitrary open cover of X. Therefore, $\bigcup \mathcal{U} = X$

$$\implies X \setminus \left(\bigcup \mathcal{U}\right) = X \setminus X = \emptyset,$$
$$\implies \bigcap\{X \setminus U : U \in \mathcal{U}\} = \emptyset.$$

Suppose that $\mathcal{F} = \{F = X \setminus U : U \in \mathcal{U}\}$. So, $\mathcal{F}$ is a family of closed sets with $\bigcap \mathcal{F} = \emptyset$. Now, by condition (3), the family $\mathcal{F}$ has the MNI property. Thus there exists a sub-family $\mathcal{E} \subseteq \mathcal{F}$ with $\bigcap \mathcal{E} = \emptyset$ for which there exists a finite subset $P \subseteq X$ such that $P \cap (X \setminus F) \neq \emptyset$ for all $F \in \mathcal{E}$.

Now,

$$\mathcal{E} \subseteq \mathcal{F} \implies \{X \setminus F : F \in \mathcal{E}\} \subseteq \{X \setminus F : F \in \mathcal{F}\},$$
$$\implies \mathcal{U}' = \{U = X \setminus F : F \in \mathcal{E}\} \subseteq \mathcal{U}.$$

Also,

$$\bigcup \mathcal{U}' = X \setminus \left(\bigcap\{F : F \in \mathcal{E}\}\right) = X \setminus \left(\bigcap \mathcal{E}\right) = X \setminus \emptyset = X.$$

But $P \cap (X \setminus F) \neq \emptyset$ for all $F \in \mathcal{E}$.

$$\text{So, } St(P,\mathcal{U}') = \bigcup\{U = X \setminus F \in \mathcal{U}' : P \cap (X \setminus F) \neq \emptyset\} = \bigcup \mathcal{U}' = X,$$
$$\implies St(P,\mathcal{U}) = X(\text{ Since, } \mathcal{U}' \subseteq \mathcal{U}).$$

Therefore, X is a star-compact space.

$(\mathbf{3}) \iff (\mathbf{4})$.

Statements (3) and (4) are contrapositive to each other, hence they are equivalent. □

Acknowledgement

We are grateful to the reviewers for their informative suggestions, which have significantly enhanced paper's quality and representation.

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(Received: 16.06.2023; Accepted: 04.12.2023; Published online: 10.02.2026)

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