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FUNCTIONAL (CO)HOMOLOGICAL GROUPS OF COMPLETELY REGULAR SPACES AND THEIR APPLICATIONS

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Abstract. Using the set of functionally open finite covers of completely regular spaces, we construct a Čech-type functional homology functor $\check{H}_n^F(-,-;G) : \mathbf{Top^2_{ce}} \to \mathbf{Ab}$ and functional cohomology functor $\hat{H}^n_F(-,-;G) : \mathbf{Top^2_{ce}} \to \mathbf{Ab}$ from the category of pairs of completely regular spaces and their CE-subspaces to the category of abelian groups. We also define the Bokshtein–Nowak functional cyclicity coefficient η^F_G and the functional cohomological dimension $\dim^F_G$ of completely regular space with values in the set $N \cup \{0, \infty\}$, and prove the equalities $\check{H}_n^F(X, A; G) = \check{H}_n(\beta X, \overline{A}^{\beta X}; G)$, $\hat{H}^n_F(X, A; G) = \hat{H}^n(\beta X, \overline{A}^{\beta X}; G)$, $\eta^F_G(X) = \eta_G(\beta X)$, $\dim^F_G(X) = \dim_G(\beta X)$, where $N, \check{H}_n(\beta X, \overline{A}^{\beta X}; G)$, $\hat{H}^n(\beta X, \overline{A}^{\beta X}; G), \eta_G(\beta X)$ and $\dim_G(\beta X)$ are the set of natural numbers, Čech homology group, Čech cohomology group, Bokshtein–Nowak cyclicity coefficient and cohomological dimension of the Stone–Čech compactification of the pair $(X, A) \in \mathrm{ob}(\mathbf{Top^2_{ce}})$ and the space $X \in ob(\mathbf{Top_{ce}})$, respectively.

Introduction

Problems in the theory of compactification of topological spaces also lead to the necessity of the creation and development of such new (co)homological theories, whose methods can be effectively used in the study of specific problems in algebraic topology.

The research presented in this paper focuses on the following **Problem: Find the necessary and sufficient conditions under which the spaces of the given class possess the compactifications with the given properties.**

This problem has been studied for various compactifications and properties by several authors (cf., [1, 5, 7, 11–15, 19–21, 23, 24, 26–28, 31–36]).

The present paper is devoted to the study of this problem for the following properties:

the n-dimensional Čech (co)homology group [19], the cyclicity coefficient [13, 28] and the cohomological dimension of the Stone–Čech compactification of a completely regular space of a given group and an integer, respectively.

A special aspect of this topic was considered in [19], where it was proved that if $(\tilde{X}, \tilde{A})$ is the Tychonoff compactification of a closed pair (X, A) of normal spaces, then

$$\check{H}_n^f(X, A; G) = \check{H}_n(\tilde{X}, \tilde{A}; G)$$

and

$$\hat{H}^n_f(X, A; G) = \hat{H}^n(\tilde{X}, \tilde{A}; G),$$

where $\check{H}_n^f(X, A; G)$ and $\hat{H}^n_f(X, A; G)$, $\check{H}_n(\tilde{X}, \tilde{A}; G)$ and $\hat{H}^n(\tilde{X}, \tilde{A}; G)$ are, respectively, the n-dimensional Čech homology groups and the n-dimensional Čech cohomology groups of pairs (X, A) and $(\tilde{X}, \tilde{A})$, based on finite open covers. Note also that in [5, 11] and [27], the obtained results include the characterizations of other homology and cohomology groups of extensions of spaces from some classes of spaces.

We say that a subspace A of X has the CE-property, if every continuous map $f : A \to [0, 1]$ is continuously extendable over X (cf., [20]). The sat A we also call a CE-subspace.

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By $\mathbf{Top^2_{ce}}$ we denote the category of pairs consisting of completely regular spaces and their CE-subspaces.

In this paper, we define the Čech type functional homology and cohomology ∂-functor and δ-cofunctor (see [19]) $\check{H}^F_n(-,-;G) : \mathbf{Top^2_{ce}} \to \mathbf{Ab}$ and $\widehat{H}^n_F(-,-;G) : \mathbf{Top^2_{ce}} \to \mathbf{Ab}$ from the category $\mathbf{Top^2_{ce}}$ to the category $\mathbf{Ab}$ of abelian groups. The definitions of these functors are based on the set of functionally open finite covers of pairs $(X,A) \in ob\left(\mathbf{Top^2_{ce}}\right)$.

The main results of the paper are the following theorems.

Theorem 3.1. *For every pair $(X,A) \in ob(\mathbf{Top^2_{ce}})$, one has*

$$\check{H}^F_n(X,A;G) = \check{H}_n(\beta X, \overline{A}^{\beta X};G)$$

and

$$\hat{H}^n_F(X,A;G) = \hat{H}^n(\beta X, \overline{A}^{\beta X};G).$$

The paper also defines the functional cyclicity coefficient $\eta^F_G(X)$ and functional cohomological dimension $\dim^F_G(X)$ of completely regular space X, and proves the following theorems.

Theorem 3.4. *For every completely regular space X, the equality*

$$\eta^F_G(X) = \eta_G(\beta X)$$

holds.

Theorem 3.7. *For every completely regular space X, the relation*

$$\dim^F_G(X) = \dim_G(\beta X)$$

holds.

Note that, here η_G and $\dim_G$ are the classical cyclicity and cohomological dimension coefficients studied by several authors.

Thus, the functional (co)homology groups, intrinsically, in terms of functionally open sets of completely regular spaces describe the Čech (co)homology groups, the cyclicity coefficients and the cohomological dimensions of the Stone–Čech compactifications of completely regular spaces. In particular, the pair $(X,A) \in ob(\mathbf{Top^2_{ce}})$ has the Stone–Čech compactification with the Čech (co)homology group isomorphic to the abelian group M if and only if (X,A) has a functional (co)homology group isomorphic to abelian group M. Moreover, a completely regular space X has the Stone–Čech compactification with cyclicity coefficient or cohomological dimension equal to the integer $n \geq 0$ if and only if X has a functional cyclicity coefficient or cohomological dimension equal to the integer $n \geq 0$.

In view of these results, the relations $\check{H}^F_n(X,A;G) = M$, $\hat{H}^n_F(X,A;G) = M$, $\eta^F_G(X) = n$ and $\dim^F_G(X) = n$ are the internal necessary and sufficient conditions for (X,A) and X, so that (X,A) and X have the compactification with $\check{H}_n(\beta X, \overline{A}^{\beta X};G) = M$, $\hat{H}^n(\beta X, \overline{A}^{\beta X};G) = M$, $\eta_G(\beta X) = n$ and $\dim_G(\beta X) = n$.

Finally, we note that without any additional references, if necessary, we will use the definitions and results from the books "General Topology" [20], "Algebraic Topology" [19], and "Dimension theory" [21, 28].

1. Čech–type Functional (co)homology Groups

In this section of the paper, using the methods developed in [19] and [20], we present the construction of the Čech-type functional homology and cohomology functors on the category $\mathbf{Top^2_{ce}}$ and deduce some of their consequences.

Let (X,A) be a pair consisting of a completely regular space X and its CE subset A. By $\mathrm{cov}_F(X)$ we denote the set of functionally open finite covers of space X. Let $\alpha = \{\alpha_v\}_{v \in V_\alpha} \in \mathrm{cov}_F(X)$, $|V_\alpha| < \chi_0$ and $s \subset V_\alpha$ be a subset of V_α. By $\mathrm{car}_\alpha(s)$ we denote the carrier of s. By the definition, $\mathrm{car}_\alpha(s) = \bigcap_{v \in s} \alpha_v$. Consider the nerve X_α of α and its subcomplex A_α consisting of all finite subsets $s \subset V_\alpha$ such that $\mathrm{car}_\alpha(s) \cap A \neq \emptyset$.

The chain and cochain groups $C_n(X_\alpha, A_\alpha; G)$ and $C^n(X_\alpha, A_\alpha; G)$ with coefficients in the abelian group G of the simplicial pair (X_α, A_α) are defined in the usual way [19].

Now, we assign to the simplicial pair (X_α, A_α) the homology and cohomology groups $H_n(X_\alpha, A_\alpha; G)$ and $H^n(X_\alpha, A_\alpha; G)$ with coefficients in the group G (see [19, Ch. VI, 3.9]).

Let $C_n(X_\alpha)$ be the free abelian group generated by all ordered n-simplices of X_α and let $C_n(A_\alpha)$ be the free abelian subgroup generated by all ordered n-simplices of A_α. By $C_n(X_\alpha, A_\alpha)$ we denote the quotient group $C_n(X_\alpha)/C_n(A_\alpha)$. For every generator $(v_0, v_1, \ldots, v_n)$, $C_n(X_\alpha)$ has its boundary defined as

$$\partial(v_0, v_1, \ldots, v_n) = \sum_{i=0}^{n} (-1)^i (v_0, v_1, \ldots, \check{v}_i, \ldots, v_n)\,.$$

Hence, there exists the boundary homomorphism

$$\partial_n : C_n(X_\alpha) \to C_{n-1}(X_\alpha)$$

with the properties

$$\partial_n(C_n(X_\alpha)) \subset C_{n-1}(X_\alpha)$$

and

$$\partial_n(C_n(X_\alpha, A_\alpha)) \subset C_{n-1}(X_\alpha, A_\alpha)\,.$$

Thus, we have the ordered chain complexes

$$C(X_\alpha) = \{C_n(X_\alpha), \partial_n\},\ C(A_\alpha) = \{C_n(A_\alpha), \partial_n\}$$

and

$$C(X_\alpha, A_\alpha) = \{C_n(X_\alpha, A_\alpha), \partial_n\}.$$

Let

$$C_n(X_\alpha; G) = C_n(X_\alpha) \otimes G,\ C_n(A_\alpha; G) = C_n(A_\alpha) \otimes G$$

and

$$C_n(X_\alpha, A_\alpha; G) = C_n(X_\alpha, A_\alpha) \otimes G,$$

where the symbol $\otimes$ denotes the tensor product of groups. Consequently, we have obtained the chain complexes

$$C(X_\alpha; G) = \{C_n(X_\alpha; G), \partial_n \otimes 1_G\},\ C(A_\alpha; G) = \{C_n(A_\alpha; G), \partial_n \otimes 1_G\},$$
$$C(X_\alpha, A_\alpha; G) = \{C_n(X_\alpha, A_\alpha; G), \partial_n \otimes 1_G\}.$$

For the sake of simplicity, the homomorphisms $\partial_n \otimes 1_G$ we again denote by ∂_n.

As in [19, Ch. 5], we can describe cochain complexes $\{C^n(X_\alpha; G), \delta^n\}$, $\{C^n(A_\alpha; G), \delta^n\}$ and $\{C^n(X_\alpha, A_\alpha; G), \delta^n\}$.

Let $H_n(X_\alpha, A_\alpha; G) = \ker \partial_n / \operatorname{im} \partial_{n+1}$ and $H^n(X_\alpha, A_\alpha; G) = \ker \delta^n / \operatorname{Im} \delta^{n-1}$. The boundary homomorphism $\partial : H_n(X_\alpha, A_\alpha; G) \to H_{n-1}(A_\alpha; G)$ and coboundary homomorphism $\delta : H^n(A_\alpha; G) \to H^{n+1}(X_\alpha, A_\alpha; G)$ are defined in the usual way. Note also that every simplicial map $f_\beta : (X_\alpha, A_\alpha) \to (Y_\beta, B_\beta)$ induces the homomorphisms $f_{\beta *} : H_n(X_\alpha, A_\alpha; G) \to H_n(Y_\beta, B_\beta; G)$ and $f^*_\beta : H^n(Y_\beta, B_\beta; G) \to H^n(X_\alpha, A_\alpha; G)$.

Let $\alpha' \in \operatorname{cov}_F(X)$ be a refinement of $\alpha \in \operatorname{cov}_F(X)$ (notation $\alpha < \alpha'$). Note that the set $\operatorname{cov}_F(X)$ is a directed set with respect to the refinement relation $<$.

Any two refinement projection maps induce contiguous simplicial maps from the pair $(X_{\alpha'}, A_{\alpha'})$ to the pair (X_α, A_α). Consequently, any two refinement projection maps $p_{\alpha\alpha'}$ and $p'_{\alpha\alpha'}$ induce the unique homomorphisms

$$p_{\alpha\alpha' *} = p'_{\alpha\alpha' *} : H_n(X_{\alpha'}, A_{\alpha'}; G) \to H_n(X_\alpha, A_\alpha; G)$$

and

$$p^*_{\alpha\alpha'} = p'^*_{\alpha\alpha'} : H^n(X_\alpha, A_\alpha; G) \to H^n(X_{\alpha'}, A_{\alpha'}; G).$$

For every $\alpha \in \operatorname{cov}_F(X)$, the homomorphisms $p_{\alpha\alpha *}$ and $p^*_{\alpha\alpha}$ are the identity homomorphisms, and for every triple $\alpha < \alpha' < \alpha''$,

$$p_{\alpha\alpha'' *} = p_{\alpha\alpha' *} \cdot p_{\alpha'\alpha'' *}$$

and

$$p^*_{\alpha\alpha''} = p^*_{\alpha'\alpha''} \cdot p^*_{\alpha\alpha'}.$$

Thus, there exist the inverse system $\{H_n(X_\alpha, A_\alpha; G), p_{\alpha\alpha'*}, \mathrm{cov}_F(X, A)\}$ and the direct system $\{H^n(X_\alpha, A_\alpha; G), p^*_{\alpha\alpha'}, \mathrm{cov}_F(X, A)\}$ for the pair $(X, A) \in ob(\mathbf{Top^2_{ce}})$.

We have the following

Proposition 1.1. *Let A be a CE-subspace of space $X \in ob(\mathbf{Top_{ce}})$. The set of covers $\{\alpha_v \cap A | \alpha_v \in \alpha \in \mathrm{cov}_F(X)\}$ is a cofinal subset of the set $\mathrm{cov}_F(A)$.*

Proof. Let $\alpha = \{\alpha_{v_i}\}_{v_i \in V_\alpha}$, $i = 1, 2, \dots, n$ be a functionally open cover of A. There exists a functionally closed shrinking covering $\beta = \{\beta_{v_i}\}_{v_i \in V_\alpha}$, $i = 1, 2, \dots, n$ of the covering α (see [20, Theorem 7.1.5]). For every pair $(A \setminus \alpha_{v_i}, \beta_{v_i})$, $v_i \in V_\alpha$, there exists a continuous map $f_{v_i} : A \to [0, 1]$ such that $f_{v_i}(A \setminus \alpha_{v_i})) = \{0\}$ and $f_{v_i}(\beta_{v_i}) = \{1\}$. By the condition of our proposition, every map f_{v_i} has a continuous extension $\tilde{f}_{v_i} : X \to [0, 1]$. The family

$$\{\tilde{f}^{-1}_{v_1}((1/2, 1]), \tilde{f}^{-1}_{v_2}((1/2, 1]), \dots, \tilde{f}^{-1}_{v_n}((1/2, 1]), \quad \bigcap_{i=1}^{n} \tilde{f}^{-1}_{v_i}([0, 1))\}$$

is a functionally open finite cover of X. Note that $A \cap \tilde{f}^{-1}_{v_i}((1/2, 1]) \subset \alpha_{v_i}$ for each $i = 1, 2, \dots, n$ and $A \cap \bigcap_{i=1}^{n} \tilde{f}^{-1}_{v_i}([0, 1)) = \emptyset$. Thus, the family

$$\{A \cap \tilde{f}^{-1}_{v_1}((1/2, 1]), A \cap \tilde{f}^{-1}_{v_2}((1/2, 1]), \dots, A \cap \tilde{f}^{-1}_{v_n}((1/2, 1])\}$$

is a functionally open finite cover of A and it is a refinement of α. □

The limit groups

$$\check{H}^F_n(X, A; G) = \varprojlim\{H_n(X_\alpha, A_\alpha; G), p_{\alpha\alpha'*}, \mathrm{cov}_F(X, A)\}$$

and

$$\hat{H}^n_F(X, A; G) = \varinjlim\{H^n(X_\alpha, A_\alpha; G), p^*_{\alpha\alpha'}, \mathrm{cov}_F(X, A)\}$$

are called the n-dimensional functional homology and cohomology groups of the pair $(X, A) \in ob(\mathbf{Top^2_{ce}})$ with coefficients in the abelian group G, respectively.

Let

$$(f : (X, A) \to (Y, B)) \in Mor_{\mathbf{Top^2_{ce}}}((X, A), (Y, B))$$

and

$$\beta = \{\beta_v\}_{v \in V_\beta} \in \mathrm{cov}_F(Y, B).$$

The family $\alpha = \{\alpha_v\}_{v \in V_\alpha}$, where $\alpha_v = f^{-1}(\beta_v)$ for every $v \in V_\alpha = V_\beta$, is a functionally open cover of X. Note that X_α is a subcomplex of Y_β and A_α is a subcomplex of B_β. Hence, there exists the inclusion simplicial map $f_\beta : (X_\alpha, A_\alpha) \to (Y_\beta, B_\beta)$. Let $\beta' \in \mathrm{cov}_F(Y)$ and $\beta < \beta'$. Then any refinement projection $p_{\beta\beta'}$ of β' into β induces a refinement projection $p_{\alpha\alpha'}$ of α' into α. From the equality $f_\beta . p_{\alpha\alpha'} = p_{\beta\beta'} . f'_\beta$ it follows that

$$f_{\beta*} \cdot p_{\alpha\alpha'*} = p_{\beta\beta'*} \cdot f_{\beta'*}$$

and

$$p^*_{\alpha\alpha'} \cdot f^*_\beta = f^*_{\beta'} \cdot p^*_{\beta\beta'}.$$

Consequently, the map $f^{-1} : \mathrm{cov}_f(Y, B) \to \mathrm{cov}_F(X, A)$ and the homomorphisms

$$f_{\beta*} : H_n(X_\alpha, A_\alpha; G) \to H_n(Y_\beta, B_\beta; G)$$

and

$$f^*_\beta : H^n(Y_\beta, B_\beta; G) \to H^n(X_\alpha, A_\alpha; G)$$

form the morphisms

$$(f_{\beta*}, f^{-1}) : \{H_n(X_\alpha, A_\alpha; G), p_{\alpha\alpha'*}, \mathrm{cov}_F(X, A)\} \to \{H_n(Y_\beta, B_\beta; G), p_{\beta\beta'*}, \mathrm{cov}_F(Y, B)\}$$

and

$$(f^*_\beta, f^{-1}) : \{H^n(Y_\beta, B_\beta; G), p^*_{\beta\beta'}, \mathrm{cov}_F(Y, B)\} \to \{H^n(X_\alpha, A_\alpha; G), p^*_{\alpha\alpha'}, \mathrm{cov}_F(X, A)\}.$$

Let

$$f_* = \varprojlim (f_{\beta *}, f^{-1}) : \check{H}_n^F(X, A; G) \to \check{H}_n^F(Y, B; G)$$

and

$$f^* = \varinjlim (f_\beta^*, f^{-1}) : \hat{H}_F^n(Y, B; G) \to \hat{H}_F^n(X, A; G).$$

The homomorphisms f_* and f^* are called the homomorphisms induced by the map $f : (X, A) \to (Y, B)$.

2. Čech Type Functional (co)homology Functors

In this section, we construct the Čech-type functional homology and cohomology ∂ and δ founctors

$$\check{H}_n^F(-, -; G) : \mathbf{Top^2_{ce}} \to \mathbf{Ab}$$

and

$$\hat{H}_F^n(-, -; G) : \mathbf{Top^2_{ce}} \to \mathbf{Ab}.$$

By the definition,

$$\check{H}_n^F(-, -; G)((X, A)) = \check{H}_n^F(X, A; G), \quad (X, A) \in ob(\mathbf{Top^2_{ce}}),$$
$$\check{H}_n^F(f) = f_*, \quad f \in Mor_{\mathbf{Top^2_{ce}}}((X, A), (Y, B)),$$
$$\hat{H}_F^n(-, -; G)((X, A)) = \hat{H}_F^n(X, A; G), \quad (X, A) \in ob(\mathbf{Top^2_{ce}}),$$
$$\hat{H}_F^n(f) = f^*, \quad f \in Mor_{\mathbf{Top^2_{ce}}}((X, A), (Y, B)).$$

This implies that for every continuous maps

$$f \in Mor_{\mathbf{Top^2_{ce}}}((X, A), (Y, B)), \quad g \in Mor_{\mathbf{Top^2_{ce}}}((Y, B), (Z, C))$$

and the identity map

$$1_{(X,A)} \in Mor_{\mathbf{Top^2_{ce}}}((X, A), (X, A)),$$

the following equalities

$$\check{H}_n^F(g \circ f) = \check{H}_n^F(g) \circ \check{H}_n^F(f),$$
$$\hat{H}_F^n(g \circ f) = \hat{H}_F^n(f) \circ \hat{H}_F^n(g),$$
$$\hat{H}_n^F(1_{(X,A)}) = 1_{\check{H}_n^F(X,A;G)},$$
$$\hat{H}_F^n(1_{(X,A)}) = 1_{\hat{H}_F^n(X,A;G)}$$

hold.

Thus we have the following

Theorem 2.1. *There exist a functional homology functor and a functional cohomology functor*

$$\check{H}_n^F(-, -; G) : \mathbf{Top^2_{ce}} \to \mathbf{Ab}$$

and

$$\hat{H}_F^n(-, -; G) : \mathbf{Top^2_{ce}} \to \mathbf{Ab},$$

respectively.

Let $(X, A) \in ob(\mathbf{Top^2_{ce}})$. Consider a cover $\alpha \in \mathrm{cov}_F(X, A)$, indexed by the pair (V_α, V_α^A).

Let $V_\alpha^A = \{v \in V_\alpha | \alpha_v \cap A \neq \varnothing\}$. Now, define a cover $\alpha' = \{\alpha_v \cap A\}_{v \in V_\alpha^A}$ of A and an order preserving map $\varphi : \mathrm{cov}_F(X, A) \to \mathrm{cov}_F(A, \varnothing)$. By the definition, $\varphi(\alpha) = \alpha'$. The correspondence $\alpha_v \to \varphi(\alpha)_v$ induces the identification of simplicial complexes A_α and $A_{\varphi(\alpha)}$, i.e., $A_\alpha = A_{\varphi(\alpha)}$. Note that the following equalities: $p_{\alpha\alpha'*} \cdot \varphi_{\alpha'*} = \varphi_{\alpha*} \cdot p_{\varphi(\alpha)\varphi(\alpha')*}$ and $\varphi_{\alpha'}^* \cdot p_{\alpha\alpha'}^* = p_{\varphi(\alpha)\varphi(\alpha')}^* \cdot \varphi_\alpha^*$ hold for every pair $\alpha < \alpha'$ of $\mathrm{cov}_F(X, A)$.

Now, consider the map $\psi : \mathrm{cov}_F(X, A) \to \mathrm{cov}(X, \varnothing)$ given by the formula $\psi(\alpha)_v = \alpha_v$. Note also that the nerves X_α and $X_{\psi(\alpha)}$ are isomorphic simplicial complex, i.e., $X_\alpha = X_{\psi(\alpha)}$. There exist the maps of the inverse systems:

$$\{H_n(A_\alpha; G), p_{\alpha\alpha'*}, \mathrm{cov}_F(A, \varnothing)\} \to \{H_n(A_\alpha; G), p_{\alpha\alpha'*}, \mathrm{cov}_F(X, A)\}$$

and

$$\{H_n(X_\alpha;G), p_{\alpha\alpha'*}, \mathrm{cov}_F(X,\varnothing)\} \to \{H_n(X_\alpha;G), p_{\alpha\alpha'*}, \mathrm{cov}_F(X,A)\}.$$

It is clear that the limit homomorphisms

$$\varphi_\infty : \check{H}_n^F(A;G) \to \check{H}_n^F(A;G)_{(X,A)}$$

and

$$\psi_\infty : \check{H}_n^F(X;G) \to \check{H}_n^F(X;G)_{(X,A)}$$

are isomorphisms.

For cohomology, we have the isomorphisms

$$\varphi^\infty : \hat{H}_F^n(A;G)_{(X,A)} \to \hat{H}_F^n(A;G)$$

and

$$\psi^\infty : \hat{H}_F^n(X;G)_{(X,A)} \to \hat{H}_F^n(X;G).$$

Let $\alpha \in \mathrm{cov}_F(X,A)$. Consider the homology and cohomology sequences of the pair (X_α, A_α),

$$\cdots \leftarrow H_n(X_\alpha,A_\alpha;G) \xleftarrow{j_{\alpha*}} H_n(X_\alpha;G) \xleftarrow{i_{\alpha*}} H_n(A_\alpha;G) \xleftarrow{\partial_\alpha} H_{n+1}(X_\alpha,A_\alpha;G) \leftarrow \cdots$$

and

$$\cdots \to H^n(X_\alpha,A_\alpha;G) \xrightarrow{j_\alpha^*} H^n(X_\alpha;G) \xrightarrow{i_\alpha^*} H^n(A_\alpha;G) \xrightarrow{\delta_\alpha} H^{n+1}(X_\alpha,A_\alpha;G) \to \cdots .$$

There exist the following limit sequences:

$$\cdots \leftarrow \check{H}_n^F(X,A) \xleftarrow{j'_*} \check{H}_n^F(X)_{(X,A)} \xleftarrow{i'_*} \check{H}_n^F(A)_{(X,A)} \xleftarrow{\partial'} \check{H}_{n+1}^F(X,A) \leftarrow \cdots$$

and

$$\cdots \to \hat{H}_F^n(X,A) \xrightarrow{j'^*} \hat{H}_F^n(X)_{(X,A)} \xrightarrow{i'^*} \hat{H}_F^n(A)_{(X,A)} \xrightarrow{\delta'} \hat{H}_F^{n+1}(X,A) \to \cdots ,$$

respectively.

Let $\partial : \check{H}_n^F(X,A;G) \to \check{H}_{n-1}^F(A;G)$ and $\delta : \hat{H}_F^n(A;G) \to \hat{H}_F^{n+1}(X,A;G)$ be the homomorphisms given by the formulas $\partial = (\varphi_\infty)^{-1} \cdot \partial'$ and $\delta = \delta' \cdot (\varphi^\infty)^{-1}$.

Thus, we obtain the following

Theorem 2.2. *For every map $f : (X,A) \to (Y,B)$, an integer n and abelian group G, the following rectangles:*

$$\begin{tikzcd}
\check{H}_n^F(X,A;G) \arrow[r, "\partial"] \arrow[d, "f_*"'] & \check{H}_{n-1}^F(A;G) \arrow[d, "(f_{|A})_*"] \\
\check{H}_n^F(Y,B;G) \arrow[r, "\partial"'] & \check{H}_{n-1}^F(B;G)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\hat{H}_F^{n-1}(B;G) \arrow[r, "\delta"] \arrow[d, "(f_{|A})^*"'] & \hat{H}_F^n(Y,B;G) \arrow[d, "f^*"] \\
\hat{H}_F^{n-1}(A;G) \arrow[r, "\delta"'] & \hat{H}_F^n(X,A;G)
\end{tikzcd}$$

are commutative.

Note that $\varphi_\infty \cdot \partial = \varphi_\infty \cdot \varphi_\infty^{-1} \cdot \partial' = \partial'$. Now, we prove that the equality $i'_* \cdot \varphi_\infty = \psi_\infty \cdot i_*$ holds. Consider the maps $\tau, \eta : \operatorname{cov}_F(X, A) \to \operatorname{cov}_F(A, \varnothing)$ defined by the formulas

$$\tau\alpha = \varphi\alpha, \quad (\tau\alpha)_v = A \cap \alpha_v, \ \ \alpha \in \operatorname{cov}_F(X, A),$$
$$\eta\alpha = i^{-1}\psi\alpha, \ \ (\eta\alpha)_v = A \cap \alpha_v, \ \ \alpha \in \operatorname{cov}_F(X, A).$$

Let $\tau_\alpha : H_n(A_{\tau\alpha}; G) \to H_n(X_\alpha; G)$ and $\eta_\alpha : H_n(A_{\eta\alpha}; G) \to H_n(X_\alpha; G)$ be the homomorphisms induced by the simplicial embedding maps of complex $A_{\tau\alpha}$ into complex X_α, and complex $A_{\eta\alpha}$ into complex X_α, respectively.

For every finite functionally open cover $\alpha \in \operatorname{cov}_F(X, A)$, we have $\eta\alpha < \tau\alpha$. There exists a homomorphism $p_{\eta\alpha,\tau\alpha *} : H_n(A_{\tau\alpha}; G) \to H_n(A_{\eta\alpha}; G)$ such that $\tau_\alpha = \eta_\alpha \cdot p_{\eta\alpha,\tau\alpha *}$.

The homomorphisms $i'_* \cdot \varphi_\infty$ and $\psi_\infty \cdot i_*$ are the limits of the morphisms

$$(\tau_\alpha, \tau), (\eta_\alpha, \eta) : \{H_n(A_\alpha; G), p_{\alpha\alpha' *}, \operatorname{cov}_F(A, \varnothing)\} \to \{H_n(X_\alpha; G), p_{\alpha\alpha' *}, \operatorname{cov}_F(X, A)\}.$$

Since $\tau_\infty = \eta_\infty$, we have $i'_* \cdot \varphi_\infty = \psi_\infty \cdot i_*$.

Let $\sigma, \rho : \operatorname{cov}_F(X, A) \to \operatorname{cov}_F(X, \varnothing)$ be the functions given by the formulas

$$\sigma\alpha = \alpha$$

and

$$\rho\alpha = \alpha.$$

Note that $X_{\sigma\alpha} = X_{\rho\alpha} = X_\alpha$. Let the homomorphisms $\sigma_\alpha, \rho_\alpha : H_n(X_\alpha; G) \to H_n(X_\alpha, A_\alpha; G)$ be induced by the inclusion map $X_\alpha \to (X_\alpha, A_\alpha)$. We have the morphisms

$$(\sigma_\alpha, \sigma), (\rho_\alpha, \rho) : \{H_n(X_\alpha; G), p_{\alpha\alpha' *}, \operatorname{cov}_F(X, \varnothing)\} \to \{H_n(X_\alpha, A_\alpha; G), p_{\alpha\alpha' *}, \operatorname{cov}_F(X, A)\}.$$

Note that for every $\alpha \in \operatorname{cov}_F(X, A)$, $\rho\alpha < \sigma\alpha$, $\sigma\alpha = \rho_\alpha \cdot p_{\rho\alpha,\sigma\alpha *}$. In addition, $\sigma_\infty = \rho_\infty$.

Homomorphisms $j'_* \cdot \psi_\infty$ and j_* are the limits of morphisms (σ_α, σ) and (ρ_α, ρ), respectively. Hence, $j'_* \cdot \psi_\infty = j_*$.

Consequently, the diagram below satisfies the commutativity relations:

$$\begin{tikzcd}
\check{H}^F_{n+1}(X, A; G) \arrow[r, "\partial"] \arrow[d, equal] & \check{H}^F_n(A; G) \arrow[r, "i_*"] \arrow[d, "\varphi_\infty"'] & \check{H}^F_n(X; G) \arrow[r, "j_*"] \arrow[d, "\psi_\infty"'] & \check{H}^F_n(X, A; G) \arrow[d, equal] \\
\check{H}^F_{n+1}(X, A; G) \arrow[r, "\partial'"] & \check{H}^F_n(A; G)_{(X,A)} \arrow[r, "i'_*"] & \check{H}^F_n(X; G)_{(X,A)} \arrow[r, "j'_*"] & \check{H}^F_n(X, A; G)
\end{tikzcd}$$

For cohomology there exists a similar diagram.

Therefore, we have the following

Theorem 2.3. *For any pair $(X, A) \in ob(\mathbf{Top^2_{ce}})$, the homology sequence*

$$\cdots \to \check{H}^F_n(A; G) \to \check{H}^F_n(X; G) \to \check{H}^F_n(X, A; G) \to \check{H}^F_{n-1}(A; G) \to \cdots$$

is of order 2, and the cohomology sequence

$$\cdots \to \hat{H}^{n-1}_F(A; G) \to \hat{H}^n_F(X, A; G) \to \hat{H}^n_F(X; G) \to \hat{H}^n_F(A; G) \to \cdots$$

is exact.

Therefore, the functional homology and cohomology functors

$$\check{H}^F_n(-, -; G) : \mathbf{Top^2_{cr}} \to \mathbf{Ab}$$

and

$$\hat{H}^n_F(-, -; G) : \mathbf{Top^2_{cr}} \to \mathbf{Ab}$$

are ∂-functor and δ-cofunctor in the sense of [E − S], respectively. This is a direct consequence of Theorems 2.1, 2.2 and 2.3.

Let α be the cover of X consisting of a single point. The inverse system of nerves of functionally open covers of X has a single term $X_\alpha = N(\alpha)$ consisting of a single vertex. Consequently, $\check{H}^F_n(X; G) = H_n(N(\alpha); G)$ and $\hat{H}^n_F(X; G) = H^n(N(\alpha); G)$. Hence, $\check{H}^F_0(X; G) \approx G$ and $\check{H}^F_n(X; G) = 0$ for $n \neq 0$. Analogously, $\hat{H}^0_F(X; G) \approx G$ and $\hat{H}^n_F(X; G) = 0$ for $n \neq 0$.

Thus we have the following

Theorem 2.4. *For the distinguished singleton space* $X=\{*\}$,

$$\check{H}_n^F(X;G)=\begin{cases}0, & n\neq 0\\ G, & n=0\end{cases}$$

and

$$\hat{H}_F^n(X;G)=\begin{cases}0, & n\neq 0\\ G, & n=0.\end{cases}$$

3. Characterizations of Čech (co)homology Groups, Cyclicity Coefficients and Cohomological Dimensions of Stone–Čech Compactifications

Now, we are mainly interested in the following question: How can the Čech (co)homology groups, cyclicity coefficients and cohomological dimensions of the Stone–Čech compactifications of completely regular spaces be characterized intrinsically, in terms of functionally open subsets of completely regular spaces.

Consider the inclusion maps $i: A\to \bar{A}^{\beta X}$, $j: A\to \beta X$, $k:\bar{A}^{\beta X}\to\beta X$ and the homeomorphism $\varphi:\bar{A}^{\beta X}\to\beta A$ (see [En$_1$]) for which the equalities $\varphi\cdot i=j$ and $\beta(f)\cdot j=f$ hold, where $\beta(f)$ is an extension of the map f. For an arbitrary extension $\psi:\bar{A}^{\beta X}\to[0,1]$ of the map f, we have $\beta(f)\cdot\varphi_{|A}=\psi_{|A}$. It is clear that $\psi=\beta(f)\cdot\varphi$.

Let $l=k\cdot i$. Moreover, there exists an extension $\tilde{f}:\beta X\to[0,1]$ of the composition $\beta(f)\cdot\varphi$, i.e., $\tilde{f}\cdot k=\beta(f)\cdot\varphi$.

Note that

$$\tilde{f}\cdot l=\tilde{f}\cdot(k\cdot i)=(\tilde{f}\cdot k)\cdot i=(\beta(f)\cdot\varphi)\cdot i=\beta(f)\cdot(\varphi\cdot i)=\beta(f)\cdot j=f.$$

Now, we list a number of facts about the CE-subspaces, which are simple consequences of general results in the compactification theory. Note that if A is the CE-subspace of X and X is the CE-subspace of Y, then A is the CE-subspace of Y. Furthermore, if A is the CE-subspace of completely regular space X, then A and $\overline{A}^{\beta X}$ are the CE-subspaces of βX.

The main result on the connection between the Čech (co)homology groups of compactifications of completely regular spaces and the Čech functional (co)homology groups of completely regular spaces is the following

Theorem 3.1. *For every pair* $(X,A)\in ob(\mathbf{Top^2_{ce}})$, *one has*

$$\check{H}_n^F(X,A;G)=\check{H}_n(\beta X,\overline{A}^{\beta X};G)$$

and

$$\hat{H}_F^n(X,A;G)=\hat{H}^n(\beta X,\overline{A}^{\beta X};G).$$

Proof. Let X be a completely regular space and A be a subspace with the CE-property. For every finite open cover $\alpha=\{\alpha_{u_i}\}_{u_i\in V_\alpha}$, $i=1,2,\dots,k$ of βX, there exists a functionally open shrinking covering $\beta=\{\beta_{v_i}\}_{v_i\in V_\beta}$, $i=1,2,\dots,k$ of α such that $\bar{\beta}_{v_i}\subset\alpha_{u_i}$, $i=1,2,\dots,k$. For the cover $\beta\wedge X=\{\beta_{v_i}\cap X\}_{v_i\in V_{\beta\wedge X}}$, $i=1,2,\dots,k$, there exist a functionally open shrinking covering $\gamma=\{\gamma_{w_i}\}_{w_i\in V_\gamma}$, $i=1,2,\dots,k$.

Let $E_X(\gamma_{w_i})=\beta X\setminus\overline{(X\setminus\gamma_{w_i})}$, $i=1,2,\dots,k$. From the relation $E_X(\gamma_{w_i})\subset\overline{\gamma_{w_i}}\subset\overline{\beta_{v_i}}\subset\alpha_{u_i}$, it follows that the family $E_X(\gamma)=\{E_X(\gamma_{w_i})\}_{i=1}^k$ is an open shrinking of $\alpha=\{\alpha_{u_i}\}_{u_i\in V_\alpha}$. Note that $E_X(\gamma)\wedge X=\gamma$ and $E_X(\gamma)>\alpha$.

Let $\alpha=\{\alpha_{u_i}\}_{u_i\in(V_\alpha,V_\alpha^A)}$ be a cover of $(X,A)\in ob(\mathbf{Top^2_{ce}})$. The closure $\overline{A}^{\beta X}$ of A in βX is equivalent to the Stone–Čech compactification βA of A. For finite functionally open cover α,

$$E_X(\alpha_{u_1})\cap E_X(\alpha_{u_2})\cap\cdots E_X(\alpha_{u_n})\cap\overline{A}^{\beta X}\neq\varnothing$$

if and only if

$$\alpha_{u_1}\cap\alpha_{u_2}\cap\cdots\cap\alpha_{u_n}\cap A\neq\varnothing.$$

The above arguments show that the nerves of functionally open cover of α and $E_X(\alpha)$ of the pairs (X, A) and $(\beta X, \overline{A}^{\beta X})$ are isomorphic simplicial complexes.

The set $\{\alpha = \beta \wedge X | \beta \in \mathrm{cov}_F(\beta X)\}$ is a cofinal subset of the set $\mathrm{cov}_F(X)$, and the set $\{E_X(\alpha) | \alpha = \beta \wedge X,\ \beta \in \mathrm{cov}(\beta X)\}$ is a cofinal subset of $\mathrm{cov}(\beta X)$. The simplicial isomorphisms yield the isomorphisms of the groups

$$H_n(X_\alpha, A_\alpha; G) \approx H_n((\beta X)_{E_X(\alpha)}, (\overline{A}^{\beta X})_{E_X(\alpha)}; G)$$

for each $\alpha = \beta \wedge X_1$, $\beta \in cov_F\,(\beta X)$. Hence, the limit groups

$$\check{H}_n(\beta X, \overline{A}^{\beta X}; G) = \varprojlim (H_n((\beta X)_\beta, (\overline{A}^{\beta X})_\beta; G), p_{\beta\beta'_*}, \mathrm{cov}_F(\beta X))$$

and

$$\check{H}^F_n(X, A; G) = \varprojlim \{H_n(X_\alpha, A_\alpha; G), p_{\alpha,\alpha' *}, \mathrm{cov}_F(\beta X)\}$$

are isomorfic.

Applying the last isomorphisms, we conclude that $\check{H}^F_n(X, A; G) = \check{H}_n(\beta X, \overline{A}^{\beta X}; G)$.

Analogously, we can verify that $\hat{H}^n_F(X, A; G) = \hat{H}^n(\beta X, \overline{A}^{\beta X}; G)$. □

Corollary 3.2. *The compactification $(\beta X, \overline{A}^{\beta X})$ of the pair $(X, A) \in ob\left(\mathbf{Top^2_{ce}}\right)$ has a Čech (co)homology group isomorphic to an abelian group M if and only if (X, A) has a Čech functional (co)homology group isomorphic to M.*

Now, we present the following

Definition 3.3. Let G be an abelian group. A functional cyclicity coefficient $\eta^F_G(X)$ of $X \in ob(\mathbf{Top_{ce}})$ with coefficients group G is equal to n, if $\hat{H}^m_F(X; G) = 0$ for every $m > n$ and $\hat{H}^n_F(X; G) \neq 0$. We put $\eta^F_G(X) = \infty$ if $X \neq \varnothing$ and for every m there is an $n \geq m$ with $\hat{H}^n_F(X; G) \neq 0$.

Thus, the functional cyclicity coefficient of X with the coefficients group G is a function $\eta^F_G : \mathbf{Top_{ce}} \to \boldsymbol{N} \cup \{\mathbf{0}, \infty\} : \mathbf{X} \to \mathbf{n}$ such that $\eta^F_G(X) = n$.

The following theorem is a direct consequence of Theorem 3.1.

Theorem 3.4. *For every completely regular space X, the equality*

$$\eta^F_G\,(X) = \eta_G\,(\beta X)$$

holds.

Proof. Assume that $\eta_G(\beta X) = n$. Then for every $m > n$: $\hat{H}^m(\beta X; G) = 0$ and $\hat{H}^n(\beta X; G) \neq 0$. From the relation $\hat{H}^k(\beta X; G) = \hat{H}^k_F(X; G)$ it follows that $\hat{H}^m_F(X; G) = 0$ for every $m > n$, and $\hat{H}^n_F(X; G) \neq 0$. Thus, $\eta^F_G(X) = n$. Consequently, $\eta^F_G(X) = \eta_G(\beta X)$. □

Corollary 3.5. *For every completely regular space X, the cyclicity coefficient $\eta_G(\beta X)$ is equal to n if and only if the functional cyclicity coefficient $\eta^F_G(X)$ is equal to n.*

The construction of meaningfull cohomological dimension theory for general topological spaces is one of the main problems of modern topology. It is well-known that the theorems of classical cohomological dimension theory ([2, 13, 16–18, 22, 24–30, 32, 33]) of "good" (metrically compact or Hausdorff compact) spaces often cannot be generalized to completely regular spaces. In this paper, we also investigate the given problem by using the Čech functional cohomology theory. For this aim, we define a functional cohomological dimension and establish some of its main properties.

Definition 3.6. The functional cohomological dimension $\dim^F_G X$ of a completely regular space X is defined as the largest integer $n \geq 0$ such that $\hat{H}^n_F(X, A; G) \neq 0$ for the CE-subspace A of X.

The functional cohomological dimension of X is the function $\dim^F_G : \mathbf{Top_{ce}} \to N \cup \{0, \infty\} : X \to n$, where $\dim^F_G X = n$.

Theorem 3.7. *For every completely regular space X, the relation*

$$\dim^F_G X = \dim_G \beta X$$

holds.

This Theorem is a consequence of Theorem 3.1.

Corollary 3.8. *For the functional covering dimension of a completely regular space X,* $\dim X = 0$ [20] *if and only if* $\dim_G^F X = 0$.

Theorem 3.9. *The following propositions are equivalent:*

i) $\dim_G^F X \leq n$;

ii) *For every $m \geq n$ and CE-subspace A of a completely regular space X, the induced homomorphism $\check{H}_F^n(X : G) \to H_F^n(A; G)$ is an epimorphism.*

Proof. The proof follows from Theorem 3.1 and Skliarenko's Theorem 37.7 (see [28]). □

Theorem 3.10. *Let A be a CE-subspace of a completely regular space X. Then*

$$\dim_G^F A \leq \dim_G^F X.$$

Proof. Note that

$$\dim_G^F A = \dim_G \beta A = \dim_G \overline{A}^{\beta X} \leq \dim_G \beta X = \dim_G^F X.$$

Hence, $\dim_G^F A \leq \dim_G^F X$. □

Theorem 3.11. *Let a completely regular space X be the union of CE-subspaces X_1 and X_2. If $\dim_G^F X_1 \leqslant m$ and $\dim_G^F X_2 \leqslant n$, then $\dim_G^F X \leq \max\{m, n\}$.*

Proof. Note that $\beta X = \overline{X} = \overline{X_1 \cup X_2} = \overline{X}_1 \cup \overline{X}_2$. Since $\overline{X}_1$ and βX_1, $\overline{X}_2$ and βX_2 are homeomorphic spaces and Theorem 3.7 holds, we have the following relations:

$$\dim_G \overline{X}_1 = \dim_G \beta X_1 = \dim_G^F X_1 \leq m$$

and

$$\dim_G \overline{X}_2 = \dim_G \beta X_2 = \dim_G^F X_2 \leq n.$$

From Kodama's Theorem 38.3 (see [28]), it follows that

$$\dim_G \beta X = \dim_G(\overline{X}_1 \cup \overline{X}_2) \leq \max\{m, n\}.$$

By Theorem 3.7, we have $\dim_G^F X \leq \max\{m, n\}$. □

The propositions below follow from the above-obtained results.

Proposition 3.12. *Let $\{G_\alpha, p_{\alpha\alpha'}, A\}$ be a direct system of abelian groups and $G = \varinjlim\{G_\alpha, p_{\alpha\alpha'}, A\}$. If for every $\alpha \in A$ and completely regular space X $\dim_{G_\alpha}^F X \leq n$, then $\dim_G^F X \leq n$.*

Proposition 3.13. *If $G = \bigoplus\limits_{\alpha \in A} G_\alpha$, then for every completely regular space X,*

$$\dim_G^F X = \max\{\dim_{G_\alpha}^F X | \alpha \in A\}.$$

Using Theorem 3.7 and the results of papers [17] and [27], we can easily prove the following

Theorem 3.14. *Let*

$$0 \to G' \to G \to G'' \to 0$$

be a short exact sequence of abelian groups. Then the equalities $\dim_G^F X \leq \max\{\dim_{G'}^F X, \dim_{G''}^F X\}$, $\dim_{G'}^F X \leqslant \max\{\dim_G^F X, \dim_{G''}^F X + 1\}$ and $\dim_{G''}^F X \leq \max\{\dim_G^F X, \dim_{G'}^F X - 1\}$ hold.

References

1. J. M. Aarts, T. Nishiura, *Dimension and Extensions*. North-Holland Mathematical Library, 48. North-Holland Publishing Co., Amsterdam, 1993.
2. P. S. Aleksandrov, On the concept of space in topology. (Russian) *Uspehi Matem. Nauk (N.S.)* **2** (1947), no. 1(17), 5–57.
3. V. Baladze, Approximation theorem for a map between spaces. Interim Reports of the, Prague Topological symposium, *Math. Inst. Of Čech. Academy of sci.* **1** (1987), 16.
4. V. Baladze, Fiber shape theory. *Proc. A. Razmadze Math. Inst.* **132** (2003), 1–70.
5. V. Baladze, Intrinsic characterization of Alexander-Spanier cohomology groups of compactifications. *Topology Appl.* **156** (2009), no. 14, 2346–2356.
6. V. Baladze, On coshape invariant extensions of functors. *Proc. A. Razmadze Math. Inst.* **150** (2009), 1–50.
7. V. Baladze, The coshape invariant and continuous extensions of functors. *Topology Appl.* **158** (2011), no. 12, 1396–1404.
8. V. Baladze, On homology and shape theories of compact Hausdorff spaces. In: *III International Conference of the Georgian Mathematical Union. Book of Abstracts*, 82–83, 2012.
9. V. Baladze, The (co)shape and (co)homological properties of continuous maps. *Mat. Vesnik* **66** (2014), no. 3, 235–247.
10. V. Baladze, F. Dumbadze, An inverse system approach of map and its application in (co)homology theory. *J. Math. Sci. (N.Y.)* **289** (2025), no. 6, 785–795.
11. V. Baladze, L. Turmanidze, Čech's type functors and completions of spaces. *Proc. A. Razmadze Math. Inst.* **165** (2014), 1–11.
12. B. J. Ball, Geometric topology and shape theory: a survey of problems and results. *Bull. Amer. Math. Soc.* **82** (1976), no. 6, 791–804.
13. M. F. Bokshtein, Homology invariants of topological spaces. (Russian) *Trudy Moskov. Mat. Obšč.* **5** (1956), 3–80.
14. A. Chigogidze, On the dimension of increments of Tychonoff spaces. *Fund. Math.* **111** (1981), no. 1, 25–36.
15. H. de Vries, *Compact spaces and compactifications*. Doct. Diss, Amsterdam, 1962.
16. A. N. Dranishnikov, Homological dimension theory. (Russian) *translated from Uspekhi Mat. Nauk* **43** (1988), no. 4(262), 11–55, 255; *Russian Math. Surveys* **43** (1988), no. 4, 11–63.
17. A. N. Dranishnikov, Cohomological dimension theory of compact metric spaces. *Topology Atlas Invited Contributions* **6** (2001), no. 3 61 pp.
18. J. Dydak, Compactifications and cohomological dimension. *Topology Appl.* **50** (1993), no. 1, 1–10.
19. S. Eilenberg, N. E. Steenrod, *Foundations of Algebraic Topology*. Princeton University Press, Princeton, NJ, 1952.
20. R. Engelking, *General Topology*. Translated from the Polish by the author. Monografie Matematyczne, PWN—Polish Scientific Publishers, Warsaw, 1977.
21. R. Engelking, *Theory of Dimensions Finite and Infinite*. Sigma Series in Pure Mathematics, 10. Heldermann Verlag, Lemgo, 1995.
22. W. Hurewicz, H. Wallman, *Dimension Theory*. Princeton Mathematical Series, vol. 4. Princeton University Press, Princeton, NJ, 1941.
23. H. N. Inasaridze, Universal functors. (Russian) *Soobšč. Akad. Nauk Gruzin. SSR* **38** (1965), 513–520.
24. J. Keesling, The Stone-Čech compactification and shape dimension. *Topology Proc.* **2** (1977), no. 2, 483–508 (1978).
25. V. I. Kuz'minov, Homological dimension theory. *Russian Math. Surveys* **23** (1968), no. 5, 1–45.
26. S. Mardešić, J. Segal, *Shape Theory*. The inverse system approach. North-Holland Mathematical Library, 26. North-Holland Publishing Co., Amsterdam-New York, 1982.
27. T. Miyata, Cohomological dimension of uniform spaces. *Quaestiones Math.* **19** (1996), no. 1-2, 137–162.
28. K. Nagami, *Dimension Theory*. With an appendix by Yukihiro Kodama. Pure and Applied Mathematics, vol. 37. Academic Press, New York-London, 1970.
29. J. Nagata, *Modern Dimension Theory*. Edited with the cooperation of the "Mathematisch Centrum" and the "Wiskundig Genootschap" at Amsterdam. Bibliotheca Mathematica, vol. VI. Interscience Publishers John Wiley & Sons, Inc., New York, 1965.
30. S. Nowak, Algebraic theory of fundamental dimension. In: *Instytut Matematyczny Polskiej Akademi Nauk (Warszawa)*, 1–59, 1981.
31. B. A. Pasinkov, On universal bicompacta and metric spaces of given dimension. *Fund. Math.* **60** (1967), 285–308.
32. E. G. Skliarenko, On embedding of normal spaces into bicompacta of the same weight and dimension. (Russian) *Doklady AN SSSR* **123** (1958), 36–39.
33. E. E. Skurihin, Dimension of completely regular spaces. (Russian) *Uspekhi Mat. Nauk* **35** (1980), no. 3(213), 224–226.
34. Ju. M. Smirnov, On the dimension of proximity spaces. (Russian) *Mat. Sb. (N.S.)* **38(80)** (1956), 283–302.
35. Ju. M. Smirnov, Proximity and construction of compactifications with given properties. In: *General Topology and its Relations to Modern Analysis and Algebra. Proceedings of the second Prague topological symposium*, 1966. Academia Publishing House of the Czechoslovak Acad. of Sciences, Prague, 1967, 332–340.

36. A. V. Zarelua, A universal bicompactum of given weight and dimension. English translation. *Soviet Math. Dokl.* **5** (1964), 214–218.

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