Transactions of A. Razmadze
Mathematical Institute
Vol. **180** (2026), issue 2, 299–306

$\mathscr{I}_\sigma$-CONVERGENCE IN FUZZY NORMED SPACES

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Abstract. The present paper is devoted to the study of ideal invariant convergence types and ideal invariant Cauchy sequence in fuzzy normed spaces. In this sense, we first defined the notion of $\mathscr{I}_\sigma$-convergence and investigated the relationship between this new notion and invariant convergence in fuzzy normed spaces. Also, we established the relationship between the notion of $\mathscr{I}_\sigma$-convergence and the notion of q-strongly σ-convergence in fuzzy normed spaces. Then, we defined $\mathscr{I}_\sigma^*$-convergence and, using the (AP) property, investigated the relationships between $\mathscr{I}_\sigma^*$-convergence and $\mathscr{I}_\sigma$-convergence in fuzzy normed spaces. Finally, we defined the $\mathscr{I}_\sigma$-Cauchy sequence and $\mathscr{I}_\sigma$-Cauchy sequence and examined some relationships between the new notions in fuzzy normed spaces.

1. Section

Zadeh [38] was the first who proposed the concept of a fuzzy set to describe imprecise phenomena in the real world. In the sense of fuzzy, several concepts have been redefined including fuzzy number [10], fuzzy calculus [17], fuzzy metric [16, 19, 23], fuzzy norm [1, 5, 14, 20] and fuzzy topology [4, 7, 26].

The Banach limit was defined as a generalized ordinary limit [3]. Lorentz [25] defined almost convergence using the Banach limit equality. Later, as a generalization of almost convergence, invariant convergence gained its place in the literature thanks to some important studies of [8,31,34]. Moreover, statistical invariant convergence types were studied by [27] and [37].

The notion of statistical convergence is based on the studies of Fast [13] and Steinhaus [35]. Some important features of statistical convergence were presented in [6,15,32]. Kostyrko et al. [22] introduced $\mathscr{I}$-convergence, which is the generalization of statistical convergence. The Cauchy $\mathscr{I}$-sequence was investigated by several author [9, 28]. Ideal convergence was studied by Kumar, Kumar [24] on fuzzy numbers sequences, by [11] on double sequences of fuzzy numbers by Hazarika [18] on fuzzy normed spaces and by Kişi, Nuray [21]; Ulusu and Kişi [36] on sequences of sets. Also, Nuray et al. [30] and Dündar et al. [12] have taken invariant and ideal convergence together.

Now, let us look at the basic notions. The following notions of ideal, filter, $\mathscr{I}$-convergence and $\mathscr{I}^*$-convergence are considered in the same way as given in [22].

Let $U \neq \emptyset$. A class $\mathscr{I}$ of subsets of U is said to be an ideal on U if and only if:

i) $\emptyset \in \mathscr{I}$,

ii) $K, M \in \mathscr{I}$ implies $K \cup M \in \mathscr{I}$,

iii) $K \in \mathscr{I}$, $M \subset K$ implies $M \in \mathscr{I}$.

$\mathscr{I}$ is called a nontrivial ideal if $\mathscr{I} \neq \emptyset$ and $U \notin \mathscr{I}$. A nontrivial ideal $\mathscr{I}$ in U is called admissible if $\{u\} \in \mathscr{I}$ for each $u \in U$. Then, let $\mathscr{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal.

Let $U \neq \emptyset$. A nonempty class $\mathscr{F}$ of subsets of U is said to be a filter on U if:

i) $\emptyset \notin \mathscr{F}$,

ii) $K, M \in \mathscr{F}$ implies $K \cap M \in \mathscr{F}$,

iii) $K \in \mathscr{F}$, $K \subset M$ implies $M \in \mathscr{F}$.

If $\mathscr{I}$ is a nontrivial ideal in U, $U \neq \emptyset$, then the class $\mathscr{F}(\mathscr{I}) = \{F \subset U : (\exists K \in \mathscr{I})(F = U \backslash K)\}$ is a filter on U, called the filter associated with $\mathscr{I}$.

A sequence $x = (x_n)$ is said to be $\mathscr{I}$-convergent to L if and only if for every $\varepsilon > 0$ $A(\varepsilon) = \big\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\big\} \in \mathscr{I}$. In this case, we write $\mathscr{I}\text{-}\lim_{n\to\infty} x_n = L$.

2020 *Mathematics Subject Classification.* 03E72, 40A05, 40A35, 46S40.

Key words and phrases. Ideal convergence; Invariant convergence; Fuzzy normed space.

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A sequence $x = (x_n)$ is said to be $\mathscr{I}^*$-convergent to L if and only if there exists a set $M \in \mathscr{F}$ (i.e., $\mathbb{N}\backslash M \in \mathscr{I}$), $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that $\lim\limits_{k\to\infty} x_{m_k} = L$.

An admissible ideal $\mathscr{I} \subset 2^{\mathbb{N}}$ satisfies condition (AP) if for every countable family of mutually disjoint sets $\{K_1, K_2, \dots\}$ belonging to $\mathscr{I}$, there exists a countable family of sets $\{M_1, M_2, \dots\}$ such that $K_j \Delta M_j$ is a finite set for $j \in \mathbb{N}$ and $M = \bigcup_{j=1}^{\infty} M_j \in \mathscr{I}$ [22].

A sequence $x = (x_n)$ is said to be the Cauchy $\mathscr{I}$-sequence if for every $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that $A(\varepsilon) = \big\{n \in \mathbb{N} : |x_n - x_N| \geq \varepsilon\big\} \in \mathscr{I}$ [9].

A sequence $x = (x_n)$ is said to be the Cauchy $\mathscr{I}^*$-sequence if there exists a set $M \in \mathscr{F}$ (i.e., $\mathbb{N}\backslash M \in \mathscr{I}$), $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that the subsequence $x_M = (x_{m_k})$ is an ordinary Cauchy sequence, i.e., $\lim\limits_{k,p\to\infty} |x_{m_k} - x_{m_p}| = 0$ [28].

Let σ be a mapping of the positive integers into itself. A continuous linear functional ϕ on the space of real bounded sequences (ℓ_∞) is an invariant mean (σ-mean) if the following axioms are satisfied:

(i) $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,

(ii) $\phi(e) = 1$, where $e = (1, 1, 1, \dots)$,

(iii) $\phi(x_{\sigma(n)}) = \phi(x)$, for all $x \in \ell_\infty$.

The mapping σ is one-to-one and satisfies the condition $\sigma^m(n) \neq n$ for any positive integers n and m, where $\sigma^m(n)$ denotes the m-th iteration of the mapping σ at n. The invariant mean, ϕ, is a generalization of the limit functional on the space of convergent sequences. The sequence is called invariant convergent when its invariant means have the same value. If $\sigma(n) = n+1$, then the σ-mean is a Banach limit, and invariant convergent sequence is almost convergent [34].

A bounded sequence $x = (x_n)$ is said to be invariant convergent to the number ℓ if and only if $\lim\limits_{m\to\infty} t_{mn} = \ell$ uniformly in n, where $t_{mn} = \frac{x_{\sigma(n)} + x_{\sigma^2(n)} + \cdots + x_{\sigma^m(n)}}{m}$ [34].

Let $K \subseteq \mathbb{N}$,

$$s_m = \min_n \left|K \cap \{\sigma(n), \sigma^2(n), \dots, \sigma^m(n)\}\right| \text{ and } S_m = \max_n \left|K \cap \{\sigma(n), \sigma^2(n), \dots, \sigma^m(n)\}\right|.$$

If the limits $\underline{\mathscr{V}}(K) = \lim\limits_{m\to\infty} \frac{s_m}{m}$ and $\overline{\mathscr{V}}(K) = \lim\limits_{m\to\infty} \frac{S_m}{m}$ exist, then they are called the lower and upper $\sigma-$uniform density of K, respectively. If $\underline{\mathscr{V}}(K) = \overline{\mathscr{V}}(K)$, then $\mathscr{V}(K) = \underline{\mathscr{V}}(K) = \overline{\mathscr{V}}(K)$ is called the $\sigma-$uniform density of K [30].

In the case $\sigma(n) = n+1$, the density $\mathscr{V}$ becomes the definition of the uniform density u in [2].

Denote by $\mathscr{I}_\sigma$ the class of all $K \subset \mathbb{N}$ with $\mathscr{V}(K) = 0$.

Let $\mathscr{I}_\sigma \subset 2^{\mathbb{N}}$ be an admissible ideal. A sequence $x = (x_k)$ is said to be $\mathscr{I}_\sigma$-convergent to the number L if for every $\varepsilon > 0$, $A_\varepsilon = \{k : |x_k - L| \geq \varepsilon\} \in \mathscr{I}_\sigma$; i.e., $V(A_\varepsilon) = 0$. In this case, we write $\mathscr{I}_\sigma - \lim_k = L$.

Let $\mathscr{I}_\sigma \subset 2^{\mathbb{N}}$ be an admissible ideal. A sequence $x = (x_k)$ is said to be $\mathscr{I}_\sigma^*$-convergent to the number L if there exists a set $M = \{m_1 < m_2 < \cdots\} \in \mathscr{F}(\mathscr{I}_\sigma)$ such that $\lim\limits_{k\to\infty} x_{m_k} = L$. In this case, we write $\mathscr{I}_\sigma^* - \lim_k = L$ [30].

Now, the concepts and definitions given below are available in references [19, 29].

A fuzzy number μ is a fuzzy set provided that:

(i) μ is normal, i.e., there exists a $t_0 \in \mathbb{R}$ such that $\mu(t_0) = 1$;

(ii) μ is fuzzy convex, i.e., $\mu(ct + (1-c)s) \geq \min\{\mu(t), \mu(s)\}$ for any $t, s \in \mathbb{R}$ and $0 \leq c \leq 1$,

(iii) μ is upper semi-continuous,

(iv) $cl\{t \in \mathbb{R} : \mu(t) > 0\}$ is a compact set.

The set of all fuzzy numbers is denoted by $L(\mathbb{R})$. $\mathbb{R}$ is included in $L(\mathbb{R})$, since each $r \in \mathbb{R}$ can be written as a fuzzy real number $\tilde{r}$ with $\tilde{r}(k) = 1$ if $k = r$ and $\tilde{r}(k) = 0$ if $k \neq r$.

For $\mu \in L(\mathbb{R})$, the α-level set of μ is defined by the formula

$$[\mu]_\alpha = \begin{cases} \{t \in \mathbb{R} : \mu(t) \geq \alpha\}, & \alpha \in (0,1], \\ cl\{t \in \mathbb{R} : \mu(t) > \alpha\}, & \alpha = 0. \end{cases}$$

The α-level set of a fuzzy number denoted by $[\mu]_\alpha = [\mu_\alpha^-, \mu_\alpha^+]$ is a non-empty, bounded and closed interval for each $\alpha \in [0,1]$. Also, $\mu_\alpha^- = -\infty$ and $\mu_\alpha^+ = \infty$ are admissible.

If $\mu \in L(\mathbb{R})$ and $\mu(t) = 0$ for $t < 0$, then μ is called a non-negative fuzzy number. Let $L^*(\mathbb{R})$ denote the set of all non-negative fuzzy numbers. Clearly, $\tilde{0} \in L^*(\mathbb{R})$.

The partial ordering $\preceq$ on $L(\mathbb{R})$ is defined by $\mu \preceq \eta$ iff $\mu_\alpha^- \leq \eta_\alpha^-$ and $\mu_\alpha^+ \leq \eta_\alpha^+$, for all $\alpha \in [0,1]$ and all $\mu, \eta \in L(\mathbb{R})$.

Let $\mu, \eta \in L(\mathbb{R})$. Arithmetic equations are defined as follows:

(i) $(\mu \oplus \eta)(t) = \sup_{s \in \mathbb{R}} \{\mu(s) \wedge \eta(t-s)\},\ t \in \mathbb{R}$,

(ii) $(\mu \odot \eta)(t) = \sup_{s \in \mathbb{R} s \neq 0} \{\mu(s) \wedge \eta(t/s)\},\ t \in \mathbb{R}$,

(iii) For $c \in \mathbb{R}^+$, $c\mu$ is defined as $c\mu(t) = \mu(t/c)$ and $0\mu(t) = \tilde{0}$, $t \in \mathbb{R}$.

Let $\mu, \eta \in L(\mathbb{R})$ and $[\mu]_\alpha = [\mu_\alpha^-, \mu_\alpha^+]$, $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$. Arithmetic equations in terms of α-level sets are defined by

(i) $[\mu \oplus \eta]_\alpha = [\mu_\alpha^- + \eta_\alpha^-, \mu_\alpha^+ + \eta_\alpha^+]$,

(ii) $[\mu \odot \eta]_\alpha = [\mu_\alpha^- . \eta_\alpha^-, \mu_\alpha^+ . \eta_\alpha^+]$, $\mu, \eta \in L^*(\mathbb{R})$,

(iii) $[c\mu]_\alpha = c[\mu]_\alpha = \begin{cases} [c\mu_\alpha^-, c\mu_\alpha^+], & c \geq 0, \\ [c\mu_\alpha^+, c\mu_\alpha^-], & c < 0. \end{cases}$

The supremum metric on $L(\mathbb{R})$ is defined by $D(\mu, \eta) = \sup\limits_{0 \leq \alpha \leq 1} \max\{|\mu_\alpha^- - \eta_\alpha^-|, |\mu_\alpha^+ - \eta_\alpha^+|\}$. $D(\mu, \tilde{0}) = \sup\limits_{0 \leq \alpha \leq 1} \max\{|\mu_\alpha^-|, |\mu_\alpha^+|\} = \max\{|\mu_0^-|, |\mu_0^+|\}$. $D(\mu, \tilde{0}) = \mu_\alpha^+$ if $\mu \in L^*(\mathbb{R})$.

A sequence (μ_n) in $L(\mathbb{R})$ is called convergent to $\mu \in L(\mathbb{R})$ denoted by $(D) - \lim\limits_{n \to \infty} \mu_n = \mu$ if $\lim\limits_{n \to \infty} D(\mu_n, \mu) = 0$, i.e., for all given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$, $D(\mu_n, \mu) < \varepsilon$.

Felbin defined the following fuzzy norm [14], which is the basis of this study, using Kaleva and Seikkala's metric [19].

Let U be a vector space over $\mathbb{R}$, $\|.\| : U \to L^*(\mathbb{R})$ and the mappings $L, R : [0,1] \times [0,1] \to [0,1]$ be symmetric, nondecreasing in both arguments and satisfy $L(0,0) = 0$ and $R(1,1) = 1$. The quadruple $(U, \|.\|, L, R)$ is called fuzzy normed linear space (FNS) and $\|.\|$ is a fuzzy norm if the following axioms are satisfied:

(i) $\|u\| = \widetilde{0}$ iff $u = \theta$,

(ii) $\|cu\| = |c| \, \|u\|$ for $u \in U$, $c \in \mathbb{R}$,

(iii) For all $u, v \in U$,

(a) $\|u+v\|(s+t) \geq L(\|u\|(s), \|v\|(t))$, whenever $s \leq \|u\|_1^-$, $t \leq \|v\|_1^-$ and $s + t \leq \|u+v\|_1^-$,

(b) $\|u+v\|(s+t) \leq R(\|u\|(s), \|v\|(t))$, whenever $s \geq \|u\|_1^-$, $t \geq \|v\|_1^-$ and $s + t \geq \|u+v\|_1^-$.

Taking $L = \min$ and $R = \max$ in above (iii), the triangle inequalities become $\|u+v\|_\alpha^- \leq \|u\|_\alpha^- + \|v\|_\alpha^-$ and $\|u+v\|_\alpha^+ \leq \|u\|_\alpha^+ + \|v\|_\alpha^+$ for all $\alpha \in (0,1]$ and $u, v \in U$. Both $\|u\|_\alpha^-$ and $\|u\|_\alpha^+$ have the properties of ordinary norms on U. From now on, let $(U, \|.\|)$ be a fuzzy normed linear space.

Example. Let $(U, \|.\|_C)$ be an ordinary normed linear space. Then the fuzzy norm $\|.\|$ on U can be obtained as follows:

$$\|u\|(t) = \begin{cases} 0, & \text{if } \ 0 \leq t \leq a\|u\|_C \ \text{ or } \ t \geq b\|u\|_C, \\ \frac{t}{(1-a)\|u\|_C} - \frac{a}{1-a}, & \text{if } \ a\|u\|_C \leq t \leq \|u\|_C, \\ \frac{-t}{(b-1)\|u\|_C} + \frac{b}{b-1}, & \text{if } \ \|u\|_C \leq t \leq b\|u\|_C, \end{cases}$$

where $\|u\|_C$ is the ordinary norm of $u\ (\neq \theta)$, $0 < a < 1$ and $1 < b < \infty$. For $u = \theta$, define $\|u\| = \widetilde{0}$. Hence, $(U, \|.\|)$ is a fuzzy normed linear space.

Let us consider the topological structure of an FNS $(U, \|.\|)$. For any $\varepsilon > 0, \alpha \in [0,1]$ and $u \in U$, the $(\varepsilon, \alpha)-$ neighborhood of u is the set $\mathscr{N}_u(\varepsilon, \alpha) := \{v \in U : \|u - v\|_\alpha^+ < \varepsilon\}$.

A sequence $u = (u_n)$ in U is said to be convergent to $\ell \in U$ with respect to the fuzzy norm, and we denote it by $u_n \overset{FN}{\to} \ell$, provided that $(D) - \lim\limits_{n \to \infty} \|u_n - \ell\| = \tilde{0}$, i.e., for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $D(\|u_n - \ell\|, \tilde{0}) < \varepsilon$. This means that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$, $\sup_{\alpha\in[0,1]} \|u_n - \ell\|^+_\alpha = \|u_n - \ell\|^+_0 < \varepsilon$. In terms of neighborhoods, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $u_n \in \mathscr{N}_\ell(\varepsilon, 0)$.

A sequence $u = (u_n)$ in U is said to be invariant convergent to ℓ with respect to the fuzzy norm and denoted by $u_n \overset{\sigma-FN}{\longrightarrow} \ell$ if $(D) - \lim_{m\to\infty} \|t_{mn} - \ell\| = \tilde{0}$, uniformly in n. Namely, since $D(\|t_{mn} - \ell\|, \tilde{0}) = \sup_{\alpha\in[0,1]} \|t_{mn} - \ell\|^+_\alpha = \|t_{mn} - \ell\|^+_0$, $\lim_{m\to\infty} \|t_{mn} - \ell\|^+_0 = 0$, uniformly in n. So, for every $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that for all $m > m_0$ and every $n \in \mathbb{N}$, $\|t_{mn} - \ell\|^+_0 < \varepsilon$. In terms of neighborhood, for every $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that for all $m > m_0$, $t_{mn} \in \mathscr{N}_\ell(\varepsilon, 0)$, uniformly in n [37].

The sequence $u = (u_n)$ in U is said to be q-strongly invariant convergent to ℓ with respect to the fuzzy norm $(0 < q < \infty)$ if

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} \left(D\left(\|u_{\sigma^k(n)} - \ell\|, \tilde{0}\right)\right)^q = \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} \left(\|u_{\sigma^k(n)} - \ell\|^+_0\right)^q = 0,$$

uniformly in n. This convergence is denoted by $u_n \overset{[\sigma-FN]_q}{\longrightarrow} \ell$.

2. Main Results

Definition 2.1. The sequence $u = (u_n)$ in U is said to be $\mathscr{I}_\sigma$-convergent to ℓ with respect to the fuzzy norm if for every $\varepsilon > 0$,

$$K(\varepsilon) = \left\{n \in \mathbb{N} : \|u_n - \ell\|^+_0 \geq \varepsilon\right\} \in \mathscr{I}_\sigma,$$

that is, $\mathscr{V}(K(\varepsilon)) = 0$. This convergence can be written as

$$\mathscr{I}_\sigma - \lim_{n\to\infty} \|u_n - \ell\|^+_0 = 0$$

and denoted by $u_n \overset{\mathscr{I}_\sigma-FN}{\longrightarrow} \ell$.

Theorem 2.1. *Let $u = (u_n)$ be a bounded sequence in U. If u is $\mathscr{I}_\sigma$-convergent to ℓ with respect to fuzzy norm, then u is invariant convergent to ℓ with respect to the fuzzy norm.*

Proof. Assume that the bounded sequence $u = (u_n)$ is $\mathscr{I}_\sigma$-convergent to ℓ with respect to the fuzzy norm. We can write

$$\begin{aligned}
\left\|\frac{1}{m}\sum_{k=1}^{m} u_{\sigma^k(n)} - \ell\right\|^+_0 &\leq \frac{1}{m}\sum_{k=1}^{m} \left\|u_{\sigma^k(n)} - \ell\right\|^+_0 \\
&= \frac{1}{m} \sum_{\substack{k=1 \\ \|u_{\sigma^k(n)}-\ell\|^+_0 \geq \varepsilon}}^{m} \left\|u_{\sigma^k(n)} - \ell\right\|^+_0 + \frac{1}{m} \sum_{\substack{k=1 \\ \|u_{\sigma^k(n)}-\ell\|^+_0 < \varepsilon}}^{m} \left\|u_{\sigma^k(n)} - \ell\right\|^+_0 \\
&\leq \frac{1}{m} M \max_n \left|\left\{1 \leq k \leq m : \left\|u_{\sigma^k(n)} - \ell\right\|^+_0 \geqslant \varepsilon\right\}\right| + \varepsilon \\
&= M\frac{S_m}{m} + \varepsilon
\end{aligned}$$

for some $M > 0$, because of the boundedness. Since u is $\mathscr{I}_\sigma$-convergent to ℓ with respect to the fuzzy norm, i.e.,

$$\lim_{m\to\infty} \frac{S_m}{m} = 0,$$

we have

$$\lim_{m\to\infty} \|t_{mn} - \ell\|^+_0 = \lim_{m\to\infty} \left\|\frac{1}{m}\sum_{k=1}^{m} u_{\sigma^k(n)} - \ell\right\|^+_0 = 0,$$

uniformly in n. Hence u is invariant convergent to ℓ with respect to the fuzzy norm. ∎

The converse of the previous theorem does not hold. For example, the sequence $u = (u_n)$, defined by

$$u_n = \begin{cases} 1, & \text{if } n \text{ is even}, \\ 0, & \text{if } n \text{ is odd}, \end{cases}$$

is invariant convergent to $\frac{1}{2}$, but it is not $\mathscr{I}_\sigma$-convergent with respect to the fuzzy norm when $\sigma(n) = n+1$.

Theorem 2.2. *Let $u = (u_n)$ be a sequence in U and $0 < q < \infty$.*

(i) *If $u_n \overset{[\sigma-FN]_q}{\longrightarrow} \ell$, then $u_n \overset{\mathscr{I}_\sigma-FN}{\longrightarrow} \ell$.*

(ii) *If u is bounded and $u_n \overset{\mathscr{I}_\sigma-FN}{\longrightarrow} \ell$, then $u_n \overset{[\sigma-FN]_q}{\longrightarrow} \ell$.*

(iii) *u is bounded, then $u_n \overset{\mathscr{I}_\sigma-FN}{\longrightarrow} \ell$ if and only if $u_n \overset{[\sigma-FN]_q}{\longrightarrow} \ell$.*

Proof. **(i)** Suppose that the sequence $u = (u_n)$ is q-strongly σ-convergent to ℓ with respect to the fuzzy norm, $0 < q < \infty$ and $\varepsilon > 0$. Then we can write

$$\begin{aligned} \frac{1}{m}\sum_{k=1}^{m}\left(\left\|u_{\sigma^k(n)} - \ell\right\|_0^+\right)^q &\geq \frac{1}{m} \sum_{\substack{k=1 \\ \|u_{\sigma^k(n)}-\ell\|_0^+ \geq \varepsilon}}^{m} \left(\left\|u_{\sigma^k(n)} - \ell\right\|_0^+\right)^q \\ &\geq \frac{1}{m}\varepsilon^q \left|\left\{1 \leq k \leq m : \left\|u_{\sigma^k(n)} - \ell\right\|_0^+ \geqslant \varepsilon\right\}\right| \\ &\geq \varepsilon^q \frac{1}{m} \max_n \left|\left\{1 \leq k \leq m : \left\|u_{\sigma^k(n)} - \ell\right\|_0^+ \geqslant \varepsilon\right\}\right| \\ &= \varepsilon^q \frac{S_m}{m}, \end{aligned}$$

for every $n \in \mathbb{N}$. Since

$$\lim_{m\to\infty} \frac{1}{m}\sum_{k=1}^{m}\left(\|u_{\sigma^k(n)} - \ell\|_0^+\right)^q = 0,$$

uniformly in n, we obtain

$$\lim_{m\to\infty} \frac{S_m}{m} = 0.$$

So, u is $\mathscr{I}_\sigma$-convergent to ℓ with respect to the fuzzy norm.

(ii) Assume that the sequence $u = (u_n)$ is bounded and $\mathscr{I}_\sigma-$convergent to ℓ with respect to the fuzzy norm. So, there exists $M > 0$ such that for every $m, n \in \mathbb{N}$,

$$\|u_{\sigma^m(n)} - \ell\|_0^+ < M,$$

and we have

$$\mathscr{V}\left(\left\{m \in \mathbb{N} : \left\|u_{\sigma^m(n)} - \ell\right\|_0^+ \geq \varepsilon\right\}\right) = 0.$$

Let $0 < q < \infty$ and $\varepsilon > 0$. We know that

$$\begin{aligned} \frac{1}{m}\sum_{k=1}^{m}\left(\left\|u_{\sigma^k(n)} - \ell\right\|_0^+\right)^q &= \frac{1}{m} \sum_{\substack{k=1 \\ \|u_{\sigma^k(n)}-\ell\|_0^+ \geq \varepsilon}}^{m} \left(\left\|u_{\sigma^k(n)} - \ell\right\|_0^+\right)^q \\ &\quad + \frac{1}{m} \sum_{\substack{k=1 \\ \|u_{\sigma^k(n)}-\ell\|_0^+ < \varepsilon}}^{m} \left(\left\|u_{\sigma^k(n)} - \ell\right\|_0^+\right)^q \\ &\leq \frac{1}{m} M \max_n \left|\left\{1 \leq k \leq m : \left\|u_{\sigma^k(n)} - \ell\right\|_0^+ \geqslant \varepsilon\right\}\right| + \varepsilon^q \\ &= M\frac{S_m}{m} + \varepsilon^q, \end{aligned}$$

for every $n \in \mathbb{N}$. Since

$$\lim_{m\to\infty} \frac{S_m}{m} = 0,$$

we obtain

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} \left(\|u_{\sigma^k(n)} - \ell\|_0^+\right)^q = 0,$$

uniformly in n. Therefore, u is q-strongly invariant convergent to ℓ with respect to the fuzzy norm.
(iii) This is a consequence of **(i)** and **(ii)**. □

Definition 2.2. The sequence $u = (u_n)$ in U is said to be $\mathscr{I}_\sigma^*$-convergent to ℓ with respect to the fuzzy norm, if there exists a set $F = \{n_1 < n_2 < \cdots < n_k < \cdots\} \in \mathscr{F}(\mathscr{I}_\sigma)$ (i.e., $\exists H = \mathbb{N} \setminus F \in \mathscr{I}_\sigma$) such that

$$\lim_{k\to\infty} \|u_{n_k} - \ell\|_0^+ = 0.$$

This convergence is denoted by $u_n \overset{\mathscr{I}_\sigma^*-FN}{\longrightarrow} \ell$.

Theorem 2.3. *Let $\mathscr{I}_\sigma$ be an admissible ideal and $u = (u_n)$ be a sequence in U. If $u_n \overset{\mathscr{I}_\sigma^*-FN}{\longrightarrow} \ell$, then $u_n \overset{\mathscr{I}_\sigma-FN}{\longrightarrow} \ell$.*

Proof. Assume that the sequence $u = (u_n)$ is $\mathscr{I}_\sigma^*$-convergent to ℓ. Then there exists a set $F = \{n_1 < n_2 < \cdots < n_k < \cdots\} \in \mathscr{F}(\mathscr{I}_\sigma)$ (i.e., $\exists H = \mathbb{N} \setminus F \in \mathscr{I}_\sigma$) such that

$$\lim_{k\to\infty} \|u_{n_k} - \ell\|_0^+ = 0.$$

Namely, for every $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for every $k > k_0$,

$$\|u_{n_k} - \ell\|_0^+ < \varepsilon.$$

Then

$$K(\varepsilon) = \{n \in \mathbb{N} : \|u_n - \ell\|_0^+ \geq \varepsilon\} \subseteq H \cup \{n_1, n_2, \ldots, n_{k_0}\}.$$

Since $\mathscr{I}_\sigma$ is an admissible ideal,

$$H \cup \{n_1, n_2, \ldots, n_{k_0}\} \in \mathscr{I}_\sigma.$$

Therefore, by the definition of the ideal $K(\varepsilon) \in \mathscr{I}_\sigma$, u is $\mathscr{I}_\sigma$-convergent to ℓ with respect to the fuzzy norm. □

For the converse theorem to hold, condition (AP) is necessary.

Theorem 2.4. *Let $\mathscr{I}_\sigma$ be an admissible ideal with the property (AP) and $u = (u_n)$ be a sequence in U. Then $u_n \overset{\mathscr{I}_\sigma^*-FN}{\longrightarrow} \ell$ if and only if $u_n \overset{\mathscr{I}_\sigma-FN}{\longrightarrow} \ell$.*

Proof. Let $u = (u_n)$ be a sequence in U. We have that if $u_n \overset{\mathscr{I}_\sigma^*-FN}{\longrightarrow} \ell$, then $u_n \overset{\mathscr{I}_\sigma-FN}{\longrightarrow} \ell$, by Theorem 2.3.

Now, assume that the admissible ideal $\mathscr{I}_\sigma$ satisfies condition (AP) and u is $\mathscr{I}_\sigma$-convergent to ℓ with respect to the fuzzy norm. Then for every $\varepsilon > 0$,

$$K(\varepsilon) = \left\{n \in \mathbb{N} : \|u_n - \ell\|_0^+ \geq \varepsilon\right\} \in \mathscr{I}_\sigma.$$

Take the following sets:

$$K_1 = \left\{n \in \mathbb{N} : \|u_n - \ell\|_0^+ \geq 1\right\} \text{ and } K_i = \left\{n \in \mathbb{N} : \frac{1}{i-1} > \|u_n - \ell\|_0^+ \geq \frac{1}{i}\right\},$$

for $i \geq 2$. Obviously $K_i \cap K_j = \emptyset$ for $i \neq j$. By condition (AP), for the sets $\{K_1, K_2, \ldots\}$ belonging to $\mathscr{I}_\sigma$, there exist the sets $\{M_1, M_2, \ldots\}$ such that $K_j \Delta M_j$ is a finite set for any $j \in \mathbb{N}$ and

$$M = \bigcup_{j=1}^{\infty} M_j \in \mathscr{I}_\sigma.$$

We need to prove that for $F = \mathbb{N} \setminus M$ and all $n \in F$,

$$\lim_{n\to\infty} \|u_n - \ell\|_0^+ = 0.$$

For any $\eta > 0$, we choose $t \in \mathbb{N}$ such that $\frac{1}{t+1} < \eta$. Then

$$\left\{n \in \mathbb{N} : \|u_n - \ell\|_0^+ \geq \eta\right\} \subset \left\{n \in \mathbb{N} : \|u_n - \ell\|_0^+ \geq \frac{1}{t+1}\right\} = \bigcup_{j=1}^{t+1} K_j.$$

Since $K_j \Delta M_j$ is a finite set for $j = 1, 2, \ldots, t+1$, there exists $n_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{j=1}^{t+1} M_j\right) \cap \{n \in \mathbb{N} : n > n_0\} = \left(\bigcup_{j=1}^{t+1} K_j\right) \cap \{n \in \mathbb{N} : n > n_0\}.$$

Namely, when n exceeds some fixed n_0, we have

$$\bigcup_{j=1}^{t+1} M_j = \bigcup_{j=1}^{t+1} K_j.$$

If $n > n_0$ and $n \in F (i.e., n \notin M = \mathbb{N} \setminus F)$, then

$$n \notin \bigcup_{j=1}^{t+1} M_j \quad \text{and so} \quad n \notin \bigcup_{j=1}^{t+1} K_j.$$

Therefore, we have $\|u_n - \ell\|_0^+ < \frac{1}{t+1} < \eta$. Thus, we conclude that

$$\lim_{\substack{n\to\infty \\ n\in F}} \|u_n - \ell\|_0^+ = 0,$$

i.e., u is $\mathscr{I}_\sigma^*$-convergent to ℓ. □

Definition 2.3. Let $\mathscr{I}_\sigma$ be an admissible ideal. The sequence $u = (u_n)$ in U is said to be the $\mathscr{I}_\sigma$-Cauchy sequence with respect to the fuzzy norm if for every $\varepsilon > 0$, there exists N such that

$$K(\varepsilon) = \left\{n \in \mathbb{N} : \|u_n - u_N\|_0^+ \geq \varepsilon\right\} \in \mathscr{I}_\sigma,$$

that is, $\mathscr{V}(K(\varepsilon)) = 0$.

Definition 2.4. Let $\mathscr{I}_\sigma$ be an admissible ideal. The sequence $u = (u_n)$ in U is said to be $\mathscr{I}_\sigma$-Cauchy sequence with respect to the fuzzy norm if there exists a set $F = \{n_1 < n_2 < \cdots < n_k < \cdots\} \in \mathscr{F}(\mathscr{I}_\sigma)$ such that

$$\lim_{k,l\to\infty} \|u_{n_k} - u_{n_l}\|_0^+ = 0.$$

The following theorems show relationships between $\mathscr{I}_\sigma$-convergence and the $\mathscr{I}_\sigma$-Cauchy sequence, between the $\mathscr{I}_\sigma$-Cauchy sequence and $\mathscr{I}_\sigma$-Cauchy sequence in the fuzzy normed spaces. These theorems can be proved in the same way as in [9, 28], they are given without the proof.

Theorem 2.5. *Let $\mathscr{I}_\sigma$ be an admissible ideal. If the sequence $u = (u_n)$ in U is $\mathscr{I}_\sigma$-convergent to any value, then u is $\mathscr{I}_\sigma$-Cauchy sequence with respect to the fuzzy norm.*

Theorem 2.6. *Let $\mathscr{I}_\sigma$ be an admissible ideal. If $u = (u_n)$ in U is the Cauchy $\mathscr{I}_\sigma^*$-sequence with respect to fuzzy norm, then u is $\mathscr{I}_\sigma$-Cauchy sequence with respect to the fuzzy norm.*

Theorem 2.7. *Let $\mathscr{I}_\sigma$ be an admissible ideal with the property (AP). If $u = (u_n)$ in U is the Cauchy $\mathscr{I}_\sigma$-sequence with respect to fuzzy norm, then u is $\mathscr{I}_\sigma$-Cauchy sequence with respect to the fuzzy norm.*

3. Conclusion

This paper defines ideal invariant convergences in fuzzy normed spaces and analyzes their properties and relations. As a new field of research, fuzzy norm space and ideal invariant convergence are expected to inspire many future studies.

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(Received: 15.05.2023; Accepted: 07.07.2024; Published online: 10.02.2026)

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