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FRACTIONAL SERIES OPERATORS ON $\mathbb{Z}^n$

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Abstract. For $0 \leq \alpha < n$ and $m \in \mathbb{N} \cap (1 - \frac{\alpha}{n}, \infty)$, we introduce a class of fractional series operators $T_{\alpha,m}$ defined on $\mathbb{Z}^n$ which are generated by certain m-invertible matrices with integer coefficients. In this note, we prove that $T_{\alpha,m}$ is a bounded operator $H^p(\mathbb{Z}^n) \to \ell^q(\mathbb{Z}^n)$ for $0 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. This allows us to generalize the results obtained by the author in [Acta Math. Hungar., 168 (1) (2022), 202–216].

1. Introduction

Given $0 \leq \alpha < n$, let $m \in \mathbb{N} \cap (1 - \frac{\alpha}{n}, \infty)$ and let $\alpha_1, \ldots, \alpha_m$ be m positive constants such that $\alpha_1 + \cdots + \alpha_m = n - \alpha$. We define the discrete operator $T_{\alpha,m}$ on $\mathbb{Z}^n$ by

$$(T_{\alpha,m}b)(j) = \sum_{i \neq A_k j \,:\, k=1,\ldots,m} \frac{b(i)}{|i - A_1 j|^{\alpha_1} \cdots |i - A_m j|^{\alpha_m}}, \quad j \in \mathbb{Z}^n, \tag{1.1}$$

where the A_k's are invertible matrices of $M_n(\mathbb{Z})$ (= the set of all matrices of degree n with coefficients in $\mathbb{Z}$). For the case $\alpha = 0$, we also assume that $A_k - A_l$ is invertible if $1 \leq k \neq l \leq m$.

The case for $n = 1$, $0 \leq \alpha < 1$, $m = 2$, $A_1 = 1$ and $A_2 = -1$ in (1.1) was studied by the author in [7]. Using the atomic decomposition for $H^p(\mathbb{Z})$ given in [1], we proved that such operator is bounded from $H^p(\mathbb{Z})$ into $\ell^q(\mathbb{Z})$ for $0 < p < \frac{1}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \alpha$ (see Theorems 7 and 9 in [7]).

We also observe that the operator (1.1) is a generalization of the discrete Riesz potential on $\mathbb{Z}^n$. Indeed, for $0 < \alpha < n$, $m = 1$ and $A_1 = Id$, we have $T_{\alpha,1} = I_\alpha$, where

$$(I_\alpha b)(j) = \sum_{i \in \mathbb{Z}^n \setminus \{j\}} \frac{b(i)}{|i - j|^{n-\alpha}}, \quad j \in \mathbb{Z}^n. \tag{1.2}$$

Y. Kanjin and M. Satake in [5] studied the discrete Riesz potential I_α for the case $n = 1$ and proved the $H^p(\mathbb{Z}) \to H^q(\mathbb{Z})$ boundedness of I_α for $0 < p < \frac{1}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \alpha$. To achieve this result, they furnished a molecular decomposition for elements of $H^p(\mathbb{Z})$.

Recently, by means of the atomic decomposition for $H^p(\mathbb{Z}^n)$ given in [2], the author in [10] and [9] studied the behavior of discrete Riesz potential on $H^p(\mathbb{Z}^n)$. More precisely, in [10] we proved the $H^p(\mathbb{Z}^n) \to \ell^q(\mathbb{Z}^n)$ boundedness of I_α for $0 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$; in [9], on the range $\frac{n-1}{n} < p \leq 1$, we furnished a molecular decomposition for $H^p(\mathbb{Z}^n)$ analogous to the ones given by Y. Kanjin and M. Satake in [5], and obtained the $H^p(\mathbb{Z}^n) \to H^q(\mathbb{Z}^n)$ boundedness of I_α for $\frac{n-1}{n} < p < q \leq 1$.

In [7], we also showed that there exists $\epsilon \in (0, \frac{1}{3})$ such that for every $0 \leq \alpha < \epsilon$, the operator $T_{\alpha,m}$ given by (1.1), with $n = 1$, $m = 2$, $\alpha_1 = \alpha_2 = \frac{1-\alpha}{2}$, $A_1 = 1$ and $A_2 = -1$, is not bounded from $H^p(\mathbb{Z})$ into $H^q(\mathbb{Z})$ for $0 < p \leq \frac{1}{1+\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \alpha$. This is a significant difference with respect to the case $0 < \alpha < 1$, $n = m = 1$ and $A_1 = 1$ (i.e., $T_{\alpha,1} = I_\alpha$ is discrete Riesz potential on $\mathbb{Z}$).

For more results about discrete fractional type operators one can consult [4, 6, 14]. On the other hand, the $H^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ boundedness of the continuous counterpart of (1.1) was studied by the author and M. Urciuolo in [11] and [12] (see also [13] and [8]).

The main aim of this note is to prove the following two results. They generalize Theorems 7 and 9 in [7], respectively.

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Theorem 3.1. *For $0 \leq \alpha < n$ and $m \in \mathbb{N} \cap (1 - \frac{\alpha}{n}, \infty)$, let $T_{\alpha,m}$ be the discrete operator given by (1.1). If $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, then there exists a positive constant C such that*

$$\|T_{\alpha,m} b\|_{\ell^q(\mathbb{Z}^n)} \leq C \|b\|_{\ell^p(\mathbb{Z}^n)},$$

for all $b \in \ell^p(\mathbb{Z}^n)$.

Theorem 4.2. *For $0 \leq \alpha < n$ and $m \in \mathbb{N} \cap (1 - \frac{\alpha}{n}, \infty)$, let $T_{\alpha,m}$ be the operator given by (1.1). Then for $0 < p \leq 1$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$,*

$$\|T_{\alpha,m}\, b\|_{\ell^q(\mathbb{Z}^n)} \leq C \|b\|_{H^p(\mathbb{Z}^n)},$$

where C does not depend on b.

Notation. We set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For every $E \subset \mathbb{Z}^n$, we denote by $\#E$ and χ_E the cardinality of the set E and the characteristic sequence of E on $\mathbb{Z}^n$, respectively. Given a real number $s \geq 0$, we write $\lfloor s \rfloor$ for the integer part of s. As usual, we denote by $\mathcal{S}(\mathbb{R}^n)$ the space of smooth and rapidly decreasing functions. If $i = (i_1, \dots, i_n) \in \mathbb{Z}^n$ and β is the multi-index $\beta = (\beta_1, \dots, \beta_n)$, then $i^\beta := i_1^{\beta_1} \cdots i_n^{\beta_n}$ and $[\beta] := \beta_1 + \cdots + \beta_n$. Throughout this paper, C denotes a positive real constant not necessarily the same at each occurrence.

2. Preliminaries

This section presents three auxiliary results necessary to obtain the main results of Sections 3 and 4.

For $0 < p < \infty$ and a sequence $b = \{b(i)\}_{i \in \mathbb{Z}^n}$, we say that b belongs to $\ell^p(\mathbb{Z}^n)$ if

$$\|b\|_{\ell^p(\mathbb{Z}^n)} := \left(\sum_{i \in \mathbb{Z}^n} |b(i)|^p \right)^{1/p} < \infty.$$

For $p = \infty$, we say that b belongs to $\ell^\infty(\mathbb{Z}^n)$ if

$$\|b\|_{\ell^\infty(\mathbb{Z}^n)} := \sup_{i \in \mathbb{Z}^n} |b(i)| < \infty.$$

In the sequel, for $j = (j_1, \dots, j_n) \in \mathbb{Z}^n$, we put $|j|_\infty = \max\{|j_k| : k = 1, \dots, n\}$ and $|j| = (j_1^2 + \cdots + j_n^2)^{1/2}$.

A discrete cube Q centered at $i_0 \in \mathbb{Z}^n$ is of the form $Q = \{i \in \mathbb{Z}^n : |i - i_0|_\infty \leq N\}$, where $N \in \mathbb{N}_0$. It is clear that $\#Q = (2N+1)^n$.

Let $0 \leq \alpha < n$, given a sequence $b = \{b(i)\}_{i \in \mathbb{Z}^n}$, we define the centered fractional maximal sequence $M_\alpha b$ by

$$(M_\alpha b)(j) = \sup_{Q \ni j} \frac{1}{\# Q^{1 - \frac{\alpha}{n}}} \sum_{i \in Q} |b(i)|, \quad j \in \mathbb{Z}^n,$$

where the supremum is taken over all discrete cubes Q centered at j. We observe that if $\alpha = 0$, then $M_0 = M$, where M is the centered discrete maximal operator.

The following result was proved in [10, see Theorem 2.3 and Proposition 2.4].

Proposition 2.1. *Let $0 \leq \alpha < n$. If $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, then*

$$\|M_\alpha b\|_{\ell^q(\mathbb{Z}^n)} \leq C \|b\|_{\ell^p(\mathbb{Z}^n)}, \quad \forall\, b \in \ell^p(\mathbb{Z}^n).$$

The following lemma is crucial to get Theorem 3.1.

Lemma 2.2 ([9, Lemma 2.1]). *If $\epsilon > 0$ and $N \in \mathbb{N}$, then*

$$\sum_{|j|_\infty \geq N} \frac{1}{|j|^{n+\epsilon}} \leq 2^n n^{n+\epsilon} \left(2 + \frac{2^{\frac{\epsilon}{n}} n}{\epsilon} \right)^n N^{-\epsilon}.$$

Next, we introduce the definition of p-atom.

Definition 2.3. Let $0 < p \leq 1$ and $d_p := \lfloor n(p^{-1} - 1) \rfloor$. We say that a sequence $a = \{a(i)\}_{i \in \mathbb{Z}^n}$ is a (p, ∞, d_p)-atom centered at a discrete cube $Q \subset \mathbb{Z}^n$ if the following three conditions hold:
(a1) $\operatorname{supp} a \subset Q$,
(a2) $\|a\|_{\ell^\infty(\mathbb{Z}^n)} \leq (\#Q)^{-1/p}$,
(a3) $\sum_{i \in Q} i^\beta a(i) = 0$ for every multi-index $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$ with $[\beta] \leq d_p$.

Given a discrete cube $Q = \{i \in \mathbb{Z}^n : |i - i_0|_\infty \leq N\}$ and m-invertible matrices $A_1, \ldots, A_m$ belonging to $M_n(\mathbb{Z})$, we define that for $k = 1, \ldots, m$, $Q_k^* = \{i \in \mathbb{Z}^n : |i - A_k^{-1} i_0|_\infty \leq 4DN\}$, where $D = \max\{\|A_k^{-1}\| : k = 1, \ldots, m\}$. Then we put $R = \mathbb{Z}^n \setminus (\bigcup_{k=1}^m Q_k^*) = (\bigcup_{k=1}^m Q_k^*)^c$. Moreover, $R = \bigcup_{l=1}^m R_l$, where

$$R_l = \{j \in R : |j - A_l^{-1} i_0| \leq |j - A_k^{-1} i_0| \text{ for all } l \neq k\},$$

for every $l = 1, \ldots, m$.

By adapting the argument used in the proof of Lemma 14 in [8] to our setting, we obtain th e following result.

Lemma 2.4. *For $0 \leq \alpha < n$ and $m \in \mathbb{N} \cap (1 - \frac{\alpha}{n}, \infty)$, let $T_{\alpha,m}$ be the discrete operator given by* (1.1). *If $a = \{a(i)\}_{i \in \mathbb{Z}^n}$ is a (p, ∞, d_p)-atom centered at a discrete cube $Q \subset \mathbb{Z}^n$, then*

$$|T_{\alpha,m} a(j)| \leq C \|a\|_{\ell^\infty} \sum_{l=1}^m \chi_{R_l}(j) \left(M_{\frac{\alpha n}{n + d_p + 1}} (\chi_Q)(A_l\, j) \right)^{\frac{n + d_p + 1}{n}}, \quad \text{if } j \in R,$$

where C does not depend on a.

3. The $\ell^p(\mathbb{Z}^n) - \ell^q(\mathbb{Z}^n)$ Boundedness of $T_{\alpha,m}$

In this section, we establish the $\ell^p(\mathbb{Z}^n) - \ell^q(\mathbb{Z}^n)$ boundedness of the discrete operator $T_{\alpha,m}$ for $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

Theorem 3.1. *For $0 \leq \alpha < n$ and $m \in \mathbb{N} \cap (1 - \frac{\alpha}{n}, \infty)$, let $T_{\alpha,m}$ be the discrete operator given by* (1.1). *If $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, then there exists a positive constant C such that*

$$\|T_{\alpha,m} b\|_{\ell^q(\mathbb{Z}^n)} \leq C \|b\|_{\ell^p(\mathbb{Z}^n)},$$

for all $b \in \ell^p(\mathbb{Z}^n)$.

Proof. Given a sequence $b = \{b(i)\}_{i \in \mathbb{Z}^n}$, we put $|b| = \{|b(i)|\}_{i \in \mathbb{Z}^n}$. We study the cases $0 < \alpha < n$ and $\alpha = 0$ separately. For $0 < \alpha < n$, it is easy to check that

$$|(T_{\alpha,m}\, b)(j)| \leq \sum_{k=1}^m (I_\alpha |b|)(A_k j), \quad \forall j \in \mathbb{Z}^n.$$

We observe that $\sum_{j \in \mathbb{Z}^n} |(I_\alpha |b|)(Aj)|^q \leq \sum_{j \in \mathbb{Z}^n} |(I_\alpha |b|)(j)|^q$ for every invertible matrix $A \in M_n(\mathbb{Z})$, since $A(\mathbb{Z}^n) \subset \mathbb{Z}^n$. Thus, the $\ell^p(\mathbb{Z}^n) - \ell^q(\mathbb{Z}^n)$ boundedness of $T_{\alpha,m}$ $(0 < \alpha < n)$ follows from Theorem [14, Proposition (a)] or [10, Theorem 3.1].

For $\alpha = 0$, we have that $m \geq 2$ and, without loss of generality, we may consider the case for $m = 2$. We point out that this case is entirely representative for the general case $m \geq 2$. Now, we introduce the auxiliary operator $\widetilde{T}$ defined by $(\widetilde{T} b)(j) = (T_{0,2} b)(j)$ if $j \neq \mathbf{0}$ and $(\widetilde{T} b)(\mathbf{0}) = 0$. From the Hölder inequality and Lemma 2.2 applied with $N = 1$ and $\epsilon = n(p' - 1)$, we have that

$$|(T_{0,2} b)(\mathbf{0})| \leq \|\{|i|^{-n}\}\|_{\ell^{p'}(\mathbb{Z}^n \setminus \{\mathbf{0}\})} \|b\|_{\ell^p(\mathbb{Z}^n)} < \infty, \quad \text{for all } 1 \leq p < \infty.$$

So, it suffices to show that $\widetilde{T}$ is bounded on $\ell^p(\mathbb{Z}^n)$, $1 < p < +\infty$, for which we put $d = \min\{|A_1 x - A_2 x| : |x| = 1\}$ and $D = \max\{\|A_1\|, \|A_2\|\}$. Since the matrices A_1, A_2 and $A_1 - A_2$ are invertible with integer coefficients, we have that $d > 0$ and $D \geq 1$. For $j_0 \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$, we write $\mathbb{Z}^n \setminus \{A_1 j_0, A_2 j_0\} = I_1 \cup I_2 \cup I_3 \cup I_4$, where

$$I_k = \left\{ i \in \mathbb{Z}^n : 0 < |i - A_k j_0| \leq \frac{d}{2}|j_0| \right\}, \quad \text{for } \ k = 1, 2,$$

$$I_3 = \{i \in \mathbb{Z}^n : |i| < 2\sqrt{n}D|j_0|\} \cap (I_1^c \cap I_2^c), \quad \text{and} \quad I_4 = \{i \in \mathbb{Z}^n : |i| \geq 2\sqrt{n}D|j_0|\} \cap (I_1^c \cap I_2^c).$$

Then

$$|(\widetilde{T}b)(j_0)| = |(T_{\alpha,2}\, b)(j_0)| \leq \left(\sum_{i \in I_1} + \sum_{i \in I_2} + \sum_{i \in I_3} + \sum_{i \in I_4} \right) \frac{|b(i)|}{|i - A_1 j_0|^{\alpha_1} |i - A_2 j_0|^{\alpha_2}} \, .$$

First, we estimate the sum on I_1. If $i \in I_1$, then $|i - A_2 j_0| = |A_1 j_0 - A_2 j_0 + i - A_1 j_0| \geq \frac{d}{2}|j_0|$. So,

$$\sum_{i \in I_1} \frac{|b(i)|}{|i - A_1 j_0|^{\alpha_1} |i - A_2 j_0|^{\alpha_2}} \leq \frac{2^{\alpha_2}}{d^{\alpha_2}|j_0|^{\alpha_2}} \sum_{0 < |i - A_1 j_0| \leq \frac{d}{2}|j_0|} \frac{|b(i)|}{|i - A_1 j_0|^{\alpha_1}} =: \sum\nolimits_1 .$$

Now, we take $k_0 \in \mathbb{N}_0$ such that $2^{k_0} \leq \frac{d}{2}|j_0| < 2^{k_0+1}$, thus

$$\begin{aligned}
\sum\nolimits_1 &= \sum_{k=0}^{k_0} \frac{2^{\alpha_2}}{d^{\alpha_2}|j_0|^{\alpha_2}} \sum_{2^{-(k+2)}d|j_0| < |i - A_1 j_0| \leq 2^{-(k+1)}d|j_0|} \frac{|b(i)|}{|i - A_1 j_0|^{\alpha_1}} \\
&\leq 2^{\alpha_2} \sum_{k=0}^{k_0} \frac{2^{(k+2)\alpha_1}}{d^n |j_0|^n} \sum_{|i - A_1 j_0| \leq \lfloor 2^{-(k+1)}d|j_0| \rfloor} |b(i)| \\
&= 2^{\alpha_2 + 2\alpha_1} \sum_{k=0}^{k_0} \frac{2^{-\alpha_2 k}}{(2 \cdot 2^{-(k+1)}d|j_0|)^n} \sum_{|i - A_1 j_0| \leq \lfloor 2^{-(k+1)}d|j_0| \rfloor} |b(i)| \\
&\leq 2^{\alpha_2 + 2\alpha_1} \sum_{k=0}^{k_0} 2^{-\alpha_2 k} \frac{1}{(2 \cdot \lfloor 2^{-(k+1)}d|j_0| \rfloor + 1)^n} \sum_{|i - A_1 j_0| \leq \lfloor 2^{-(k+1)}d|j_0| \rfloor} |b(i)| \, ,
\end{aligned}$$

this last inequality follows from the fact that $\lfloor 2^{-(k+1)}d|j_0| \rfloor \leq 2^{-(k+1)}d|j_0|$ and that $\frac{2 \cdot \lfloor 2^{-(k+1)}d|j_0| \rfloor + 1}{2 \cdot \lfloor 2^{-(k+1)}d|j_0| \rfloor} \leq 2$ for each $k = 0, \ldots, k_0$. Thus

$$\sum_{i \in I_1} \frac{|b(i)|}{|i - A_1 j_0|^{\alpha_1} |i - A_2 j_0|^{\alpha_2}} \leq 2^{\alpha_2 + 2\alpha_1} \left(\sum_{k=0}^{\infty} 2^{-\alpha_2 k} \right) (Mb)(A_1 j_0). \tag{3.1}$$

Similarly, it is seen that

$$\sum_{i \in I_2} \frac{|b(i)|}{|i - A_1 j_0|^{\alpha_1} |i - A_2 j_0|^{\alpha_2}} \leq C(Mb)(A_2 j_0). \tag{3.2}$$

On I_3, we obtain

$$\begin{aligned}
\sum_{i \in I_3} \frac{|b(i)|}{|i - A_1 j_0|^{\alpha_1} |i - A_2 j_0|^{\alpha_2}} &\leq \frac{2^n}{d^n}|j_0|^{-n} \sum_{|i| < 2\sqrt{n}D|j_0|} |b(i)| \\
&\leq \frac{2^n}{d^n}|j_0|^{-n} \sum_{|i - j_0| \leq (2\sqrt{n}D + 1)|j_0|} |b(i)| \leq C(Mb)(j_0).
\end{aligned} \tag{3.3}$$

Now, on I_4, for every $k = 1, 2$, we have $|i - A_k j_0| \geq \frac{(2\sqrt{n}D - 1)}{2\sqrt{n}D}|i|$ for all $i \in I_4$. Since $I_4 \subset \{i \in \mathbb{Z}^n : |i| \geq 2\sqrt{n}D|j_0|\} \subset \{i \in \mathbb{Z}^n : |i|_\infty \geq 2\lfloor D \rfloor |j_0|\}$, it follows that

$$\sum_{i \in I_4} \frac{|b(i)|}{|i - A_1 j_0|^{\alpha_1} |i - A_2 j_0|^{\alpha_2}} \leq C \sum_{|i|_\infty \geq 2\lfloor D \rfloor |j_0|} |i|^{-n}|b(i)| \leq C\|b\|_{\ell^p}|j_0|^{-n/p} \leq C\|b\|_{\ell^p}|j_0|_\infty^{-n/p}, \tag{3.4}$$

where the second inequality follows from Hölder's inequality and Lemma 2.2 applied with $N = 2\lfloor D\rfloor |j_0|$ and $\epsilon = n(p'-1)$. Thus (3.4) implies that

$$\#\left\{ j \neq 0 : \left| \sum_{i\in I_4} |i - A_1 j|^{-\alpha_1} |i - A_2 j|^{-\alpha_2} b(i) \right| > \lambda \right\} \leq \left(C \frac{\|b\|_{\ell^p}}{\lambda} \right)^p, \ 1 \leq p < \infty. \tag{3.5}$$

Finally, (3.1), (3.2), (3.3), (3.5) and Proposition 2.1 with $\alpha = 0$ allow us to conclude that $\widetilde{T}$ is a bounded operator $\ell^p(\mathbb{Z}^n) \to \ell^{p,\infty}(\mathbb{Z}^n)$, for every $1 \leq p < \infty$. Then, the $\ell^p(\mathbb{Z})$ boundedness of $\widetilde{T}$ follows from the Marcinkiewicz interpolation theorem (see Theorem 1.3.2 in [3]). This completes the proof. $\square$

Remark 3.2. Let $0 \leq \alpha < n$. If $\frac{n}{n-\alpha} < q < \infty$ and $0 < p \leq \frac{nq}{n+\alpha q}$, then the operator $T_{\alpha,m}$ is bounded from $\ell^p(\mathbb{Z}^n)$ into $\ell^q(\mathbb{Z}^n)$. This follows from Theorem 3.1 and the embedding $\ell^{p_1}(\mathbb{Z}^n) \hookrightarrow \ell^{p_2}(\mathbb{Z}^n)$, valid for $0 < p_1 < p_2 \leq \infty$.

4. The $H^p(\mathbb{Z}^n) - \ell^q(\mathbb{Z}^n)$ Boundedness of $T_{\alpha,m}$

First, we recall the definition of $H^p(\mathbb{Z}^n)$ spaces and state the atomic decomposition given by S. Boza and M. Carro in [2].

Let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \Phi = 1$, and let Φ^d denote the restriction of Φ on $\mathbb{Z}^n$. Now, for $t > 0$, we consider $\Phi^d_t(j) = t^{-n}\Phi(j/t)$ if $j \neq \mathbf{0}$ and $\Phi^d_t(\mathbf{0}) = 0$. Then, by [2, Theorem 2.7], we define

$$H^p(\mathbb{Z}^n) = \left\{ b \in \ell^p(\mathbb{Z}^n) : \sup_{t>0} |(\Phi^d_t *_{\mathbb{Z}^n} b)| \in \ell^p(\mathbb{Z}^n) \right\}, \ \ 0 < p \leq 1,$$

with the "$H^p(\mathbb{Z}^n)$-norm" given by

$$\|b\|_{H^p(\mathbb{Z}^n)} := \|b\|_{\ell^p(\mathbb{Z}^n)} + \|(\Phi^d_t *_{\mathbb{Z}^n} b)\|_{\ell^p(\mathbb{Z}^n)}.$$

From Definition 2.3, we find that if $a = \{a(j)\}_{j\in\mathbb{Z}^n}$ is a (p,∞,d_p)-atom, then $a = \{a(j)\}_{j\in\mathbb{Z}^n} \in H^p(\mathbb{Z}^n)$. The atomic decomposition for $H^p(\mathbb{Z}^n)$, $0 < p \leq 1$, developed in [2], is as follows.

Theorem 4.1 ([2, Theorem 3.7])**.** *Let $0 < p \leq 1$, $d_p = \lfloor n(p^{-1} - 1)\rfloor$ and $b \in H^p(\mathbb{Z}^n)$. Then there exist a sequence of (p,∞,d_p)-atoms $\{a_k\}_{k=0}^{+\infty}$, a sequence of scalars $\{\lambda_k\}_{k=0}^{+\infty}$ and a positive constant C depending only on p and n, with $\sum_{k=0}^{+\infty} |\lambda_k|^p \leq C\|b\|^p_{H^p(\mathbb{Z}^n)}$ such that $b = \sum_{k=0}^{+\infty} \lambda_k a_k$, where the series converges in $H^p(\mathbb{Z}^n)$.*

Now, we are in a position to prove our main result.

Theorem 4.2. *For $0 \leq \alpha < n$ and $m \in \mathbb{N} \cap (1 - \frac{\alpha}{n}, \infty)$, let $T_{\alpha,m}$ be the operator given by (1.1). Then for $0 < p \leq 1$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$,*

$$\|T_{\alpha,m}\, b\|_{\ell^q(\mathbb{Z}^n)} \leq C\|b\|_{H^p(\mathbb{Z}^n)},$$

where C does not depend on b.

Proof. For $0 < p \leq 1$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, we prove that there exists a universal positive constant C such that

$$\|T_{\alpha,m} a\|_{\ell^q} \leq C, \tag{4.1}$$

for all (p,∞,d_p)-atoms $a = \{a(i)\}_{i\in\mathbb{Z}^n}$ for which we consider a (p,∞,d_p)-atom $a = \{a(i)\}_{i\in\mathbb{Z}^n}$ supported on the discrete cube $Q = \{i \in \mathbb{Z}^n : |i - i_0|_\infty \leq N\}$. For every $k = 1,\dots,m$, let $Q^*_k = \{i \in \mathbb{Z}^n : |i - A_k^{-1} i_0|_\infty \leq 4DN\}$, where $D = \max\{\|A_k^{-1}\| : k = 1,\dots,m\}$. Now, we decompose $\mathbb{Z}^n = (\bigcup_{k=1}^m Q^*_k) \cup R$, where $R = (\bigcup_{k=1}^m Q^*_k)^c$. So,

$$\sum_{j\in\mathbb{Z}^n} |(T_{\alpha,m} a)(j)|^q \leq \sum_{k=1}^m \sum_{j\in Q^*_k} |(T_{\alpha,m} a)(j)|^q + \sum_{j\in R} |(T_{\alpha,m} a)(j)|^q = I_1 + I_2.$$

To estimate I_1, we take $\frac{n}{n-\alpha} < q_0 < \infty$ and put $\frac{1}{p_0} = \frac{1}{q_0} + \frac{\alpha}{n}$. By the Hölder inequality with q_0/q and Theorem 3.1, we obtain

$$\begin{aligned} I_1 &\le \|T_{\alpha,m} a\|_{\ell^{q_0}}^q \sum_{k=1}^m (\#Q_k^*)^{1-q/q_0} \le C\|a\|_{\ell^{p_0}}^q \sum_{k=1}^m (\#Q_k^*)^{1-q/q_0} \\ &\le C(\#Q)^{-q/p}(\#Q)^{q/p_0} \sum_{k=1}^m (\#Q_k^*)^{1-q/q_0} \le C, \end{aligned} \tag{4.2}$$

where the constant C does not depend on the p-atom a.

Now, we proceed to estimate I_2. By Lemma 2.4, we have

$$I_2 \le C\|a\|_{\ell^\infty}^q \sum_{l=1}^m \sum_{j\in\mathbb{Z}^n} \left(M_{\frac{\alpha n}{n+d_p+1}}(\chi_Q)(A_l\, j) \right)^{q\frac{n+d_p+1}{n}}. \tag{4.3}$$

Since $A_l(\mathbb{Z}^n) \subset \mathbb{Z}^n$ for every $l = 1, \dots, m$, it follows that

$$\sum_{j\in\mathbb{Z}^n} \left(M_{\frac{\alpha n}{n+d_p+1}}(\chi_Q)(A_l\, j) \right)^{q\frac{n+d_p+1}{n}} \le \sum_{j\in\mathbb{Z}^n} \left(M_{\frac{\alpha n}{n+d_p+1}}(\chi_Q)(j) \right)^{q\frac{n+d_p+1}{n}}. \tag{4.4}$$

Taking into account that $d_p = \lfloor n(\frac{1}{p} - 1) \rfloor$, we have $q\frac{n+d+1}{n} > p\frac{n+d+1}{n} > 1$. Then, we write $\widetilde{q} = q\frac{n+d+1}{n}$ and let $\frac{1}{\widetilde{p}} = \frac{1}{\widetilde{q}} + \frac{\alpha}{n+d+1}$, so $1 < \widetilde{p} < \widetilde{q} < \infty$ and $\widetilde{p}/\widetilde{q} = p/q$. Then Proposition 2.1 leads to

$$\sum_{j\in\mathbb{Z}^n} \left(M_{\frac{\alpha n}{n+d_p+1}}(\chi_Q)(j) \right)^{q\frac{n+d_p+1}{n}} \le C\left(\sum_{j\in\mathbb{Z}^n} \chi_Q(j) \right)^{q/p} = C(\#Q)^{q/p}.$$

This inequality, together with (4.4) and (4.3), give

$$I_2 \le C\|a\|_{\ell^\infty}^q (\#Q)^{q/p} = C, \tag{4.5}$$

where C is independent of the p-atom a. Now, (4.2) and (4.5) allow us to obtain (4.1).

Given $b \in H^p(\mathbb{Z}^n)$, by Theorem 4.1, we can write $b = \sum \lambda_k a_k$, where the a_k's are discrete (p, ∞, d_p) atoms and the scalars λ_k satisfy $\sum_k |\lambda_k|^p \le C\|b\|_{H^p(\mathbb{Z}^n)}$. By Theorem 3.1 applied with $\frac{n}{n-\alpha} < q_0 < \infty$ and $\frac{1}{p_0} = \frac{1}{q_0} + \frac{\alpha}{n}$ and since $b = \sum_k \lambda_k a_k$ converges in $\ell^{p_0}(\mathbb{Z}^n)$, we have

$$|(T_{\alpha,m} b)(j)| \le \sum_{k=1}^\infty |\lambda_k| |(T_{\alpha,m} a_k)(j)|, \quad \text{for all } j \in \mathbb{Z}^n. \tag{4.6}$$

Finally, inequalities (4.1) and (4.6) allow us to obtain

$$\|T_{\alpha,m} b\|_{\ell^q(\mathbb{Z}^n)} \le C\left(\sum_k |\lambda_k|^{\min\{1,q\}} \right)^{\frac{1}{\min\{1,q\}}} \le C\left(\sum_k |\lambda_k|^p \right)^{1/p} \le C\|b\|_{H^p(\mathbb{Z}^n)}.$$

Thus the proof is concluded. □

In the following corollary we recover Theorem 3.3 obtained in [10].

Corollary 4.3. *For $0 < \alpha < n$, let I_α be the discrete Riesz potential given by* (1.2). *Then for $0 < p \le 1$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$,*

$$\|I_\alpha\, b\|_{\ell^q(\mathbb{Z}^n)} \le C\|b\|_{H^p(\mathbb{Z}^n)},$$

where C does not depend on b.

Proof. To apply Theorem 4.2 with $0 < \alpha < n$, $m = 1$ and $A_1 = Id$. □

Remark 4.4. Let $0 \le \alpha < n$. If $0 < q \le \frac{n}{n-\alpha}$ and $0 < p \le \frac{nq}{n+\alpha q}$, then the operator $T_{\alpha,m}$ is bounded from $H^p(\mathbb{Z}^n)$ into $\ell^q(\mathbb{Z}^n)$. This follows from Theorem 4.2 and the embedding $H^{p_1}(\mathbb{Z}^n) \hookrightarrow H^{p_2}(\mathbb{Z}^n)$ valid for $0 < p_1 < p_2 \le 1$. In particular, for $0 < \alpha < n$, $0 < q \le \frac{n}{n-\gamma}$ and $0 < p \le \frac{nq}{n+\alpha q}$, the discrete Riesz potential I_α is bounded from $H^p(\mathbb{Z}^n)$ into $\ell^q(\mathbb{Z}^n)$.

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