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EXTENSIONS OF CLASSICAL INEQUALITIES VIA E-CONVEXITY AND FRACTIONAL INTEGRALS

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Abstract. In this paper, we derive novel integral inequalities similar to Hadamard's inequality for e-convex functions using fundamental analytical methods. Our findings contribute to the growing body of work in convex analysis and provide a foundation for further generalizations in functional inequalities.

1. Introduction

Hadamard's inequality [2, 12] is a fundamental result in mathematical analysis that establishes bounds on the integral of a convex function over a closed interval. This inequalty, originally formulated by Jacques Hadamard in 1893, has significant applications in optimization [17], economics [18], and functional analysis [5]. For a convex function $f : [l, m] \to \mathbb{R}$, Hadamard's inequality is given by

$$\phi\left(\frac{l+m}{2}\right) \leq \frac{1}{m-l}\int\limits_{l}^{m} \phi(x)\, dx \leq \frac{\phi(l)+\phi(m)}{2}\,.$$

This result provides a crucial relationship between the midpoint value of a function, its integral, and the arithmetic mean of its endpoint values. It is widely utilized in estimating convex functions and has led to numerous extensions and generalizations.

Hadamard's inequality has been extensively analyzed and expanded within various mathematical approaches. In 2003, Dragomir and Pearce [9] discussed different generalizations of Hadamard's inequality for convex functions, particularly under higher-order convexity conditions. Later, in 2006, Niculescu and Persson [20] examined integral inequalities within the framework of convex analysis, providing refinements to the classical Hadamard inequality.

Recent studies have explored connections between Jensen's inequality and Hadamard's theorem [7], reinforcing the deep relationship between convex analysis and integral inequalities.

Convex functions [1, 5, 22] play a pivotal role in mathematical analysis, especially in optimization and economics. A function f is convex on an interval I if for any $u, v \in I$ and $\alpha \in [0, 1]$, the following inequality:

$$\phi(\alpha u + (1-\alpha)v) \leq \alpha\phi(u) + (1-\alpha)\phi(v)$$

holds. This convexity property ensures that the secant line lies above the function's graph, leading to fundamental inequalities such as Jensen's inequality, the Hermite–Hadamard inequality, and inequalities of convex combinations.

Extensions of Hadamard's inequality have been formulated for various types of convexity, including strongly convex [23], quasi-convex [8], and harmonically convex functions [13]. These developments highlight the continued significance and flexibility of the Hermite–Hadamard inequality [6, 10, 14, 15, 21, 24, 26] in modern mathematical research, demonstrating its extensive influence across diverse fields. The incorporation of fractional and quantum calculus, along with its applications to stochastic processes and optimization, underscores its effectiveness in solving intricate mathematical and practical problems. Additionally, recent investigations into weighted, integral, and multidimensional extensions have expanded its theoretical scope, reinforcing its importance in disciplines such as

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financial modeling, machine learning, and mathematical physics. As novel convexity concepts emerge and computational methods advance, the Hermite–Hadamard inequality remains a vital link between classical mathematical analysis and contemporary applied mathematics.

2. e-convex Functions and Some Integral Inequlities

We begin this section by recalling the concept of an e-convex function, which generalizes classical convexity using an exponential weighting scheme.

Definition 2.1 ([16])**.** Let $\phi : \mathcal{I} \to \mathbb{R}$ be a non-negative function. The function ϕ is said to be e-convex if, for any $u, v \in \mathcal{I}$ and for all $\alpha \in [0, 1]$, the following inequality:

$$\phi(\alpha u + (1-\alpha)v) \leq e^{\alpha}\phi(u) + e^{1-\alpha}\phi(v)$$

holds. This definition extends the classical notion of convexity by incorporating an exponential factor, which influences the way how the function interpolates between two points. The exponential weights e^{α} and $e^{1-\alpha}$ provide a distinctive scaling property that may lead to unique applications in optimization, functional analysis, and mathematical economics. It is also worth noting that the concept of e-convexity explored in our paper is more general than those considered in [3, 25].

Example 2.2 ([16])**.** A real-valued function $\phi : [0, 1] \to \mathbb{R}$, defined by $\phi(x) = x^{1/p}$, $p \in \mathbb{N} \setminus \{1\}$ is e-convex, but not convex.

In this work, we extend and generalize the integral inequalities established in [21, 26] to a broader class of e-convex functions. We derive new inequalities that serve as analogous to Hadamard's inequality within this generalized framework. We begin by giving the following

Lemma 2.3. *The following statements are equivalent for the mapping $\phi : [l, m] \to \mathbb{R}$:*

(1) *ϕ is e-convex on $[l, m]$.*
(2) *For all $x, y \in [l, m]$, the mapping $\psi : [0, 1] \to \mathbb{R}$ defined by $\psi(t) = \phi(tx + (1-t)y)$ is e-convex on $[0, 1]$.*

Proof. (1) $\Rightarrow$ (2).

Assume that ϕ is e-convex on $[l, m]$ and let $x, y \in [l, m]$. We show that the mapping $\psi : [0, 1] \to \mathbb{R}$ is e-convex on $[0, 1]$. Then, for $\alpha, p, q \in [0, 1]$, we have

$$\begin{aligned}
\psi(\alpha p + (1-\alpha)q) &= \phi((\alpha p + (1-\alpha)q)x + (1 - \alpha p - (1-\alpha)q)y) \\
&= \phi(\alpha px + (1-\alpha)qx + y - \alpha py - qy + \alpha qy + \alpha y - \alpha y) \\
&= \phi\left(\alpha(px + (1-p)y) + (1-\alpha)(qx + (1-q)y)\right) \\
&\leq e^{\alpha}\phi(px + (1-p)y) + e^{1-\alpha}\phi(qx + (1-q)y) \\
&= e^{\alpha}\psi(p) + e^{1-\alpha}\psi(q).
\end{aligned}$$

Thus we conclude that

$$\psi(\alpha p + (1-\alpha)q) \leq e^{\alpha}\psi(p) + e^{1-\alpha}\psi(q),$$

which implies that ψ is e-convex on $[0, 1]$.

(2) $\Rightarrow$ (1).

We show that $\phi : [l, m] \to \mathbb{R}$ is e-convex.

Let $x, y \in [l, m]$. Then there exist $p, q \in [0, 1]$ such that

$$\begin{aligned}
x &= pl + (1-p)m, \\
y &= ql + (1-q)m.
\end{aligned}$$

Define the function $\psi : [0, 1] \to \mathbb{R}$ by

$$\psi(\lambda) = \phi(\lambda l + (1-\lambda)m).$$

Then we have

$$\psi(p) = \phi(x), \quad \psi(q) = \phi(y).$$

Since ψ is e-convex, it follows that

$$\begin{aligned}\psi(\alpha p+(1-\alpha)q) &\le e^{\alpha}\psi(p)+e^{1-\alpha}\psi(q)\\ &= e^{\alpha}\phi(x)+e^{1-\alpha}\phi(y).\end{aligned} \tag{2.1}$$

Also, we compute

$$\begin{aligned}\psi(\alpha p+(1-\alpha)q) &= \phi\left((\alpha p+(1-\alpha)q)l+(1-\alpha p-(1-\alpha)q)m\right)\\ &= \phi\left(\alpha pl+ql-\alpha ql+m-\alpha pm-qm+\alpha qm+\alpha m-\alpha m\right)\\ &= \phi\left(\alpha(pl+(1-p)m)+(1-\alpha)(lq+(1-q)m)\right)\\ &= \phi\left(\alpha x+(1-\alpha)y\right).\end{aligned} \tag{2.2}$$

Using (2.1) and (2.2), we have

$$\phi(\alpha x+(1-\alpha)y)\le e^{\alpha}\phi(x)+e^{1-\alpha}\phi(y),\quad \forall\ x,y\in[l,m],\quad \alpha\in[0,1].$$

This shows that ϕ is e-convex on $[l,m]$. □

Theorem 2.4. *Let ϕ and ψ be real-valued, integrable, and e-convex functions on $[l,m]$. Then*

$$\frac{1}{m-l}\int_l^m \phi(x)\,\psi(x)dx \le \left(\frac{e^2-1}{2}\right)U(l,m)+e\,V(l,m),$$

where

$$U(l,m)=\phi(l)\,\psi(l)+\phi(m)\,\psi(m),\quad V(l,m)=\phi(l)\,\psi(m)+\phi(m)\,\psi(l). \tag{2.3}$$

Proof. Since ϕ and ψ are e-convex on $[l,m]$, for $t\in[0,1]$, we have

$$\begin{aligned}\phi(tl+(1-t)m)&\le e^t\phi(l)+e^{1-t}\phi(m),\\ \psi(tl+(1-t)m)&\le e^t\psi(l)+e^{1-t}\psi(m).\end{aligned}$$

Thus,

$$\begin{aligned}&\phi(tl+(1-t)m)\,\psi(tl+(1-t)m)\\ &\le (e^t)^2\phi(l)\,\psi(l)+(e^{1-t})^2\phi(m)\,\psi(m)+e^te^{1-t}\left[\phi(l)\,\psi(m)+\psi(l)\,\phi(m)\right]\\ &= (e^t)^2\phi(l)\,\psi(l)+(e^{1-t})^2\phi(m)\,\psi(m)+e\,V(l,m).\end{aligned}$$

Now, integrating both sides with respect to t, we get

$$\begin{aligned}&\int_0^1 \phi(tl+(1-t)m)\,\psi(tl+(1-t)m)dt\\ &\le \int_0^1 (e^t)^2\phi(l)\,\psi(l)dt+\int_0^1 (e^{1-t})^2\phi(m)\,\psi(m)dt+e\,V(l,m)\\ &\le \phi(l)\,\psi(l)\int_0^1 (e^t)^2dt+\phi(m)\,\psi(m)\int_0^1 (e^{1-t})^2dt+e\,V(l,m)\\ &= \phi(l)\,\psi(l)\frac{(e^2-1)}{2}+\phi(m)\,\psi(m)\frac{(e^2-1)}{2}+e\,V(l,m)\\ &= \left(\frac{e^2-1}{2}\right)(\phi(l)\,\psi(l)+\phi(m)\,\psi(m))+e\,V(l,m)\\ &= \left(\frac{e^2-1}{2}\right)U(l,m)+e\,V(l,m).\end{aligned}$$

Putting $tl + (1-t)m = x$, then $(l-m)dt = dx$, we get

$$\frac{1}{m-l}\int\limits_l^m \phi(x)\,\psi(x)dx \le \left(\frac{e^2-1}{2}\right) U(l,m) + e\,V(l,m). \qquad \square$$

Theorem 2.5. *Let ϕ and ψ be real-valued, integrable, and e-convex functions on $[l,m]$. Then*

$$\phi\left(\frac{l+m}{2}\right)\psi\left(\frac{l+m}{2}\right) \le \frac{2e}{m-l}\int\limits_l^m \phi(x)\,\psi(x)dx + 2e^2\,U(l,m) + (e^2-1)\,V(l,m),$$

where $U(l,m)$ and $V(l,m)$ are the same as defined in (2.3).

Proof. For $\vartheta \in [0,1]$, we have

$$\begin{aligned}
&\phi\left(\frac{l+m}{2}\right)\,\psi\left(\frac{l+m}{2}\right)\\
&= \phi\left(\frac{\vartheta l + (1-\vartheta)l + \vartheta m + (1-\vartheta)m}{2}\right)\,\psi\left(\frac{\vartheta l + (1-\vartheta)l + \vartheta m + (1-\vartheta)m}{2}\right)\\
&\le \left[e^{\frac{1}{2}}\phi\left(\vartheta l + (1-\vartheta)m\right) + e^{\frac{1}{2}}\,\phi\left((1-\vartheta)l + \vartheta m\right)\right]\\
&\quad \times \left[e^{\frac{1}{2}}\,\psi\left(\vartheta l + (1-\vartheta)m\right) + e^{\frac{1}{2}}\,\psi\left((1-\vartheta)l + \vartheta m\right)\right]\\
&\le e\left[\phi(\vartheta l + (1-\vartheta)m)\,\psi(\vartheta l + (1-\vartheta)m) + \phi((1-\vartheta)l + \vartheta m)\,\psi((1-\vartheta)l + \vartheta m)\right]\\
&\quad + e\,\left[(e^{\vartheta}\,\phi(l) + e^{1-\vartheta}\,\phi(m))\,\left(e^{1-\vartheta}\,\psi(l) + e^{\vartheta}\,\psi(m)\right)\right.\\
&\quad \left. + \left(e^{1-\vartheta}\,\phi(l) + e^{\vartheta}\,\phi(m)\right)\,\left(e^{\vartheta}\,\psi(l) + e^{1-\vartheta}\,\psi(m)\right)\right]\\
&= e\left[\phi(\vartheta l + (1-\vartheta)m)\,\psi(\vartheta l + (1-\vartheta)m) + \phi((1-\vartheta)l + \vartheta m)\,\psi((1-\vartheta)l + \vartheta m)\right]\\
&\quad + e\,\left[2e^{\vartheta}e^{1-\vartheta}\phi(l)\,\psi(l) + 2e^{\vartheta}\,e^{1-\vartheta}\,\phi(m)\,\psi(m)\right]\\
&\quad + \left((e^{1-\vartheta})^2 + (e^{\vartheta})^2\right)^2\,\left(\phi(l)\,\psi(m) + \phi(m)\,\psi(l)\right)\\
&= e\left[\phi(\vartheta l + (1-\vartheta)m)\,\psi(\vartheta l + (1-\vartheta)m) + \phi((1-\vartheta)l + \vartheta m)\,\psi((1-\vartheta)l + \vartheta m)\right]\\
&\quad + e\,\left[2e\left(\phi(l)\,\psi(l) + \phi(m)\,\psi(m)\right)\right.\\
&\quad \left. + (e^{\vartheta})^2 + (e^{1-\vartheta})^2\,\left(\phi(l)\,\psi(m) + \phi(m)\,\psi(l)\right)\right].
\end{aligned}$$

This yields

$$\begin{aligned}
&\phi\left(\frac{l+m}{2}\right)\,\psi\left(\frac{l+m}{2}\right)\\
&\le e\left[\phi(\vartheta l + (1-\vartheta)m)\,\psi(\vartheta l + (1-\vartheta)m) + \phi((1-\vartheta)l + \vartheta m)\,\psi((1-\vartheta)l + \vartheta m)\right]\\
&\quad + 2e^2U(l,m) + \left(e^{2\vartheta} + e^{2(1-\vartheta)}\right)V(l,m).
\end{aligned}$$

Now, integrating both sides with respect to ϑ, we get

$$\begin{aligned}
&\phi\left(\frac{l+m}{2}\right)\,\psi\left(\frac{l+m}{2}\right)\\
&\le e\int\limits_0^1 \left[\phi(\vartheta l + (1-\vartheta)m)\,\psi(\vartheta l + (1-\vartheta)m) + \phi((1-\vartheta)l + \vartheta m)\,\psi((1-\vartheta)l + \vartheta m)\right]d\vartheta\\
&\quad + 2e^2U(l,m) + V(l,m)\int\limits_0^1 \left(e^{2\vartheta} + e^{2(1-\vartheta)}\right)\,d\vartheta
\end{aligned}$$

$$= e\int_0^1 \left[\phi(\vartheta l+(1-\vartheta)m)\,\psi(\vartheta l+(1-\vartheta)m)+\phi((1-\vartheta)l+\vartheta m)\,\psi((1-\vartheta)l+\vartheta m)\right]d\vartheta$$
$$+\,2e^2\,U(l,m)+(e^2-1)\,V(l,m)$$
$$= 2e\int_0^1 \phi(\vartheta l+(1-\vartheta)m)\,\psi(\vartheta l+(1-\vartheta)m)d\vartheta+2e^2\,U(l,m)+(e^2-1)\,V(l,m).$$

Now, putting $\vartheta l+(1-\vartheta)m=x$, we get

$$\phi\left(\frac{l+m}{2}\right)\psi\left(\frac{l+m}{2}\right)$$
$$\le \frac{2e}{m-l}\int_l^m \phi(x)\,\psi(x)\,dx+2e^2\,U(l,m)+(e^2-1)\,V(l,m).$$ □

Theorem 2.6. *Let ϕ and ψ be real-valued, integrable, and e-convex functions on [l, m]. Then*

$$\frac{1}{(m-l)^2}\int_l^m\int_l^m\int_0^1 \phi(tx+(1-t)y)\,\psi(tx+(1-t)y)\,dt\,dy\,dx$$
$$\le \frac{e^2-1}{(m-l)}\int_l^m \phi(x)\,\psi(x)\,dx+2e\,(e-1)^2\left[U(l,m)+V(l,m)\right],$$

where $U(l,m)$ and $V(l,m)$ are the same as defined in (2.3).

Proof. Since ϕ and ψ are e-convex on $[l,m]$, for $x,y\in[l,m]$ and $t\in[0,1]$, we obtain

$$\phi(tx+(1-t)y)\le e^t\,\phi(x)+e^{1-t}\,\phi(y)$$

and

$$\psi(tx+(1-t)y)\le e^t\,\psi(x)+e^{1-t}\,\psi(y).$$

We have

$$\phi(tx+(1-t)y)\,\psi(tx+(1-t)y)$$
$$\le e^{2t}\,\phi(x)\,\psi(x)+e^{2(1-t)}\,\phi(y)\,\psi(y)+e\,\left[\phi(x)\,\psi(y)+\phi(y)\,\psi(x)\right].$$

This gives

$$\int_0^1 \phi(tx+(1-t)y)\,\psi(tx+(1-t)y)\,dt$$
$$\le \left(\frac{e^2-1}{2}\right)\left[\phi(x)\,\psi(x)+\phi(y)\,\psi(y)\right]+e\,\left[\phi(x)\,\psi(y)+\phi(y)\,\psi(x)\right].$$

Integrating both sides over $[l,m]\times[l,m]$ and using Hermite–Hadamard type integral inequality for e-convex functions [16, Theorem 3.7], we get

$$\int_l^m\int_l^m\int_0^1 \phi(tx+(1-t)y)\,\psi(tx+(1-t)y)\,dt\,dy\,dx$$
$$\le \frac{e^2-1}{2}(m-l)\,\left[\int_l^m \phi(x)\,\psi(x)\,dx+\int_l^m \phi(y)\,\psi(y)\,dy\right]$$
$$+\,e\,\left[\int_l^m \phi(x)\,dx\int_l^m \psi(y)\,dy+\int_l^m \phi(y)\,dy\int_l^m \psi(x)\,dx\right]$$

$$
\begin{aligned}
\leq & (e^2-1)\,(m-l)\int_l^m \phi(x)\,\psi(x)\,dx \\
& \quad + 2e\,(m-l)^2\,(e-1)^2\,\left[(\phi(l)+\phi(m))\,(\psi(l)+\psi(m))\right] \\
& = (e^2-1)\,(m-l)\int_l^m \phi(x)\,\psi(x)\,dx + 2e\,(m-l)^2\,(e-1)^2\,\left[U(l,m)+V(l,m)\right].
\end{aligned}
$$

Thus

$$
\begin{aligned}
& \frac{1}{(m-l)^2}\int_l^m\int_l^m\int_0^1 \phi(tx+(1-t)y)\,\psi(tx+(1-t)y)\,dt\,dy\,dx \\
& \leq \frac{e^2-1}{(m-l)}\int_l^m \phi(x)\,\psi(x)\,dx + 2e\,(e-1)^2\,\left[U(l,m)+V(l,m)\right]. \qquad \square
\end{aligned}
$$

Theorem 2.7. *Let ϕ and ψ be real-valued, integrable, and e-convex functions on $[l,m]$. Then*

$$
\begin{aligned}
& \frac{1}{m-l}\int_l^m\int_0^1 \phi\left(tx+(1-t)\frac{l+m}{2}\right)\,\psi\left(tx+(1-t)\frac{l+m}{2}\right)dt\,dx \\
& \leq \frac{e^2-1}{2(m-l)}\int_l^m \phi(x)\,\psi(x)\,dx + e(e-1)\,\left(\frac{e+1+4\sqrt{e}}{2}\right)(U(l,m)+V(l,m)),
\end{aligned}
$$

where $U(l,m)$ and $V(l,m)$ are the same as defined in (2.3).

Proof. Since ϕ and ψ are e-convex on $[l,m]$, it follows that for $t\in[0,1]$, we have

$$
\begin{aligned}
\phi\left(tx+(1-t)\frac{l+m}{2}\right) &\leq e^t\phi(x)+e^{1-t}\phi\left(\frac{l+m}{2}\right),\\
\psi\left(tx+(1-t)\frac{l+m}{2}\right) &\leq e^t\psi(x)+e^{1-t}\psi\left(\frac{l+m}{2}\right).
\end{aligned}
$$

Note that

$$
\begin{aligned}
& \phi\left(tx+(1-t)\frac{l+m}{2}\right)\,\psi\left(tx+(1-t)\frac{l+m}{2}\right) \\
& \leq e^{2t}\,\phi(x)\,\psi(x)+e^{2(1-t)}\,\phi\left(\frac{l+m}{2}\right)\,\psi\left(\frac{l+m}{2}\right) \\
& \quad + e\left[\phi(x)\,\psi\left(\frac{l+m}{2}\right)+\phi\left(\frac{l+m}{2}\right)\,\psi(x)\right].
\end{aligned}
$$

This implies that

$$
\begin{aligned}
& \int_0^1 \phi\left(tx+(1-t)\frac{l+m}{2}\right)\,\psi\left(tx+(1-t)\frac{l+m}{2}\right)dt \\
& \leq \left(\frac{e^2-1}{2}\right)\left[\phi(x)\,\psi(x)+\phi\left(\frac{l+m}{2}\right)\,\psi\left(\frac{l+m}{2}\right)\right] \\
& \quad + e\,\left[\phi(x)\,\psi\left(\frac{l+m}{2}\right)+\phi\left(\frac{l+m}{2}\right)\,\psi(x)\right].
\end{aligned}
$$

This further gives

$$\begin{aligned}
&\int_l^m \int_0^1 \phi\left(tx+(1-t)\frac{l+m}{2}\right)\,\psi\left(tx+(1-t)\frac{l+m}{2}\right) dt\,dx \\
&\le \left(\frac{e^2-1}{2}\right)\int_l^m \phi(x)\,\psi(x)\,dx + \left(\frac{e^2-1}{2}\right)(m-l)\,\phi\left(\frac{l+m}{2}\right)\,\psi\left(\frac{l+m}{2}\right) \\
&\quad + e\left[\int_l^m \phi(x)\,\psi\left(\frac{l+m}{2}\right)\,dx + \int_l^m \phi\left(\frac{l+m}{2}\right)\,\psi(x)\,dx\right] \\
&\le \left(\frac{e^2-1}{2}\right)\int_l^m \phi(x)\,\psi(x)\,dx + \left(\frac{e^2-1}{2}\right)(m-l)\,e\,\left[(\phi(l)+\phi(m))\,(\psi(l)+\psi(m))\right] \\
&\quad + e\,\big[\sqrt{e}\,(\psi(l)+\psi(m))\,(m-l)\,(e-1)\,(\phi(l)+\phi(m)) \\
&\quad + \sqrt{e}\,(\phi(l)+\phi(m))\,(m-l)\,(e-1)\,(\psi(l)+\psi(m))\big] \\
&\le \frac{(e^2-1)}{2}\int_l^m \phi(x)\,\psi(x)\,dx + \frac{(e^2-1)e}{2}(m-l)\,(U(l,m)+V(l,m)) \\
&\quad + 2e^{3/2}\,(m-l)\,(e-1)\,\left(U(l,m)+V(l,m)\right) \\
&= \frac{(e^2-1)}{2}\int_l^m \phi(x)\,\psi(x)\,dx + (m-l)e(e-1)\,\left(U(l,m)+V(l,m)\right)\,\left(\frac{e+1}{2}+2\sqrt{e}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{1}{m-l}\int_l^m \int_0^1 \phi\left(tx+(1-t)\frac{l+m}{2}\right)\,\psi\left(tx+(1-t)\frac{l+m}{2}\right) dt\,dx \\
&\le \frac{e^2-1}{2(m-l)}\int_l^m \phi(x)\,\psi(x)\,dx + e(e-1)\,\left(\frac{e+1+4\sqrt{e}}{2}\right)(U(l,m)+V(l,m))\,. \qquad \square
\end{aligned}$$

Theorem 2.8. *Let $\phi: I\subseteq\mathbb{R}\to\mathbb{R}$ be an e-convex function on $[l,m]$. Then*

$$\frac{1}{m-l}\int_l^m \phi(x)\,\phi(l+m-x)\,dx \le Q(\phi(l),\phi(m)) + (e^2-1) + 2e\,\left(M(\phi(l),\phi(m))\right)^2,$$

where $Q(p,q)=\frac{p^2+q^2}{2}$, $M(p,q)=\sqrt{pq}$.

Proof. Since ϕ is e-convex on $[l,m]$, it follows that for $t\in[0,1]$, we have

$$\begin{aligned}
&\frac{1}{m-l}\int_l^m \phi(x)\,\phi(l+m-x)\,dx \\
&= \int_0^1 \phi(tl+(1-t)m)\,\phi((1-t)l+tm)\,dt \\
&\le \frac{1}{2}\int_0^1 \left[(\phi(tl+(1-t)m))^2 + (\phi((1-t)l+tm))^2\right]\,dt
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}\int_0^1 \left[\left(e^t\,\phi(l)+e^{1-t}\,\phi(m)\right)^2+\left(e^{1-t}\,\phi(l)+e^t\,\phi(m)\right)^2\right]dt \\
&= \frac{1}{2}\int_0^1 \Big[e^{2t}\left((\phi(l))^2+(\phi(m))^2\right)+e^{2(1-t)}\left((\phi(l))^2+(\phi(m))^2\right) \\
&\qquad\quad +4e\,\phi(l)\,\phi(m)\Big]\,dt \\
&= \frac{1}{2}\int_0^1 \left(e^{2t}+e^{2(1-t)}\right)\left((\phi(l))^2+(\phi(m))^2\right)\,dt+2e\,\phi(l)\,\phi(m) \\
&= \frac{1}{2}\left((\phi(l))^2+(\phi(m))^2\right)\int_0^1 \left(e^{2t}+e^{2(1-t)}\right)\,dt+2e\,\phi(l)\,\phi(m) \\
&= \left((\phi(l))^2+(\phi(m))^2\right)\int_0^1 e^{2t}\,dt+2e\,\phi(l)\,\phi(m) \\
&= \left(\phi(l)^2+\phi(m)^2\right)\cdot\frac{(e^2-1)}{2}+2e\,\phi(l)\,\phi(m) \\
&= \left(\frac{\phi(l)^2+\phi(m)^2}{2}\right)(e^2-1)+2e\,\phi(l)\,\phi(m) \\
&= Q(\phi(l),\phi(m))\,(e^2-1)+2e\,M(\phi(l),\phi(m))^2. \qquad \square
\end{aligned}
$$

From [4, 11, 19], we recall that for an integrable function ϕ on the interval $[l,m]$, the Riemann-Liouville fractional integrals $F^{\alpha}_{l+}\phi$ and $F^{\alpha}_{m-}\phi$ of order $\alpha>0$, with $l\geq 0$, are defined as follows:

$$
F^{\alpha}_{l+}\phi(x)=\frac{1}{\Gamma(\alpha)}\int_l^x (x-t)^{\alpha-1}\,\phi(t)\,dt,\quad x>l,
$$

$$
F^{\alpha}_{m-}\phi(x)=\frac{1}{\Gamma(\alpha)}\int_x^m (t-x)^{\alpha-1}\,\phi(t)\,dt,\quad m>x,
$$

where $\Gamma(\alpha)=\int_0^\infty e^{-s}s^{\alpha-1}\,ds$ denotes the Gamma function. It is also understood that $F^0_{l+}\phi(x)=F^0_{m-}\phi(x)=\phi(x)$.

Next, we give Hermite–Hadamard type fractional integral inequalities for e-convex functions.

Theorem 2.9. *Let $\phi:[l,m]\to\mathbb{R}$ be an e-convex, integrable function on $[l,m]$. Then for the fractional integrals, the following inequalities hold:*

$$
\frac{1}{2\sqrt{e}}\phi\left(\frac{l+m}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(m-l)^{\alpha}}\left(F^{\alpha}_{l+}f(m)+F^{\alpha}_{m-}f(l)\right)\leq\frac{e^{\alpha}}{2}(\phi(l)+\phi(m))\int_0^1 t^{\alpha}(e^t+e^{1-t})\,dt
$$

for $\alpha>0$.

Proof. Since ϕ is e-convex on $[l,m]$, we have

$$
\phi\left(\frac{x+y}{2}\right)\leq\sqrt{e}\,(\phi(x)+\phi(y))\quad\text{for } x,y\in[l,m].
$$

Let $x=tl+(1-t)m$ and $y=(1-t)l+tm$, where $t\in(0,1)$. Then

$$
\frac{1}{\sqrt{e}}\phi\left(\frac{l+m}{2}\right)\leq\phi(tl+(1-t)m)+\phi((1-t)l+tm),\quad\forall\, t\in(0,1). \tag{2.4}
$$

Multiplying both sides of (2.4) by $t^{\alpha-1}$ and integrating with respect to t, we get

$$\begin{aligned}
\frac{1}{\alpha\sqrt{e}}\phi\left(\frac{l+m}{2}\right) &\leq \int_0^1 t^{\alpha-1}\,\phi(tl+(1-t)m)\,dt + \int_0^1 t^{\alpha-1}\,\phi((1-t)l+tm)\,dt \\
&= \frac{1}{(m-l)^{\alpha}}\int_l^m (m-l)^{\alpha-1}\,\phi(u)\,du + \frac{1}{(m-l)^{\alpha}}\int_l^m (v-l)^{\alpha-1}\,\phi(v)\,dv \\
&= \frac{\Gamma(\alpha)}{(m-l)^{\alpha}}\left(F^{\alpha}_{l^+}\phi(m) + F^{\alpha}_{m^-}\phi(l)\right) \\
\Rightarrow \frac{1}{\sqrt{e}}\phi\left(\frac{l+m}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{(m-l)^{\alpha}}\left(F^{\alpha}_{l^+}\phi(m) + F^{\alpha}_{m^-}\phi(l)\right). \qquad (2.5)
\end{aligned}$$

Using the e-convexity of ϕ, we have

$$\phi(tl+(1-t)m) \leq e^{t}\,\phi(l) + e^{1-t}\,\phi(m), \qquad (2.6)$$

$$\phi((1-t)l+tm) \leq e^{1-t}\,\phi(l) + e^{t}\,\phi(m). \qquad (2.7)$$

Adding (2.6) and (2.7), we have

$$\phi(tl+(1-t)m) + \phi((1-t)l+tm) \leq (\phi(l)+\phi(m))\,\left(e^{t}+e^{1-t}\right). \qquad (2.8)$$

Now, multiplying both sides of (2.8) by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t, we obtain

$$\begin{aligned}
&\int_0^1 t^{\alpha-1}\phi(tl+(1-t)m)\,dt + \int_0^1 t^{\alpha-1}\phi((1-t)l+tm)\,dt \\
&\leq (\phi(l)+\phi(m))\int_0^1 t^{\alpha-1}\left(e^{t}+e^{1-t}\right)dt \\
\Rightarrow\quad & \frac{1}{(m-l)^{\alpha}}\int_l^m (m-u)^{\alpha-1}\phi(u)\,du + \frac{1}{(m-l)^{\alpha}}\int_l^m (v-l)^{\alpha-1}\phi(v)\,dv \\
&\leq (\phi(l)+\phi(m))\int_0^1 t^{\alpha-1}\left(e^{t}+e^{1-t}\right)dt \\
\Rightarrow\quad & \frac{\Gamma(\alpha)}{(m-l)^{\alpha}}\left(F^{\alpha}_{l^+}\,\phi(m) + F^{\alpha}_{m^-}\,\phi(l)\right) \leq (\phi(l)+\phi(m))\int_0^1 t^{\alpha-1}\left(e^{t}+e^{1-t}\right)dt \\
\Rightarrow\quad & \frac{\Gamma(\alpha+1)}{2(m-l)^{\alpha}}\left(F^{\alpha}_{l^+}\,\phi(m) + F^{\alpha}_{m^-}\,\phi(l)]\right) \leq \frac{\alpha}{2}\left(\phi(l)+\phi(m)\right)\int_0^1 t^{\alpha-1}\left(e^{t}+e^{1-t}\right)dt. \qquad (2.9)
\end{aligned}$$

Therefore, by (2.5) and (2.9), we obtain

$$\begin{aligned}
\frac{1}{2\sqrt{e}}\phi\left(\frac{l+m}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{2(m-l)^{\alpha}}\left(F^{\alpha}_{l^+}\,\phi(m) + F^{\alpha}_{m^-}\,\phi(l)\right) \\
&\leq \frac{\alpha}{2}\left(\phi(l)+\phi(m)\right)\int_0^1 t^{\alpha-1}\left(e^{t}+e^{1-t}\right)dt. \qquad \square
\end{aligned}$$

Remark 2.10. If we take $\alpha = 1$, then the above inequality reduces to the Hermite–Hadamard-type inequality for e-convex functions [16, Theorem 3.7].

Theorem 2.11. *Let ϕ and ψ be real-valued, e-convex functions on $[l,m]$. Then for all $l,m,\alpha>0$, we have*

$$\frac{F^{\alpha}_{l^+}[\phi(m)\,\psi(m)]}{(m-l)^{\alpha}} \le \frac{e}{\Gamma(\alpha+1)}V(l,m)+\frac{\phi(l)\,\psi(l)}{\Gamma(\alpha)}\int\limits_0^1 t^{\alpha-1}\,e^{2t}\,dt+\frac{\phi(m)\,\psi(m)}{\Gamma(\alpha)}\int\limits_0^1 t^{\alpha-1}\,e^{2(1-t)}\,dt.$$

Proof. Since ϕ and ψ are e-convex functions on $[l,m]$, it follows that for $t\in[0,1]$, we have

$$\phi(tl+(1-t)m)\le e^{t}\phi(l)+e^{1-t}\phi(m),$$
$$\psi(tl+(1-t)m)\le e^{t}\psi(l)+e^{1-t}\psi(m).$$

Then

$$\begin{aligned}
&\phi(tl+(1-t)m)\cdot\psi(tl+(1-t)m)\\
&\le \left(e^{2t}\phi(l)\,\psi(l)+e^{2(1-t)}\,\phi(m)\,\psi(m)\right)+e\,(\phi(l)\,\psi(m)+\psi(l)\,\phi(m))\\
&= e^{2t}\,\phi(l)\,\psi(l)+e^{2(1-t)}\,\phi(m)\,\psi(m)+e[V(l,m)].
\end{aligned}$$

Now, multiplying both sides of the above inequality by $\frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and integrating with respect to t, we get

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int\limits_0^1 t^{\alpha-1}\,\phi(tl+(1-t)m)\,\psi(tl+(1-t)m)dt\\
&\le \frac{e}{\Gamma(\alpha+1)}V(l,m)+\frac{\phi(l)\,\psi(m)}{\Gamma(\alpha)}\int\limits_0^1 t^{\alpha-1}\,e^{2t}\,dt+\frac{\phi(m)\,\psi(m)}{\Gamma(\alpha)}\int\limits_0^1 t^{\alpha-1}\,e^{2(1-t)}\,dt.
\end{aligned}$$

Substituting $u=tl+(1-t)m\Rightarrow t=\frac{m-u}{m-l}$, we obtain

$$\begin{aligned}
&\frac{F^{\alpha}_{l^+}[\phi(m)\,\psi(m)]}{(m-l)^{\alpha}}\\
&\le \frac{e}{\Gamma(\alpha+1)}V(l,m)+\frac{\phi(l)\,\psi(l)}{\Gamma(\alpha)}\int\limits_0^1 t^{\alpha-1}\,e^{2t}\,dt+\frac{\phi(m)\,\psi(m)}{\Gamma(\alpha)}\int\limits_0^1 t^{\alpha-1}\,e^{2(1-t)}\,dt. \qquad \square
\end{aligned}$$

Remark 2.12. If $\alpha=1$, the above inequality reduces to Theorem 2.4.

Theorem 2.13. *Let ϕ and ψ be real-valued, integrable e-convex functions on $[l,m]$. Then for all $\alpha,l,m>0$, we have*

$$\phi\left(\frac{l+m}{2}\right)\psi\left(\frac{l+m}{2}\right)\le\frac{e\,\Gamma(\alpha+1)}{(m-l)^{\alpha}}\left(F^{\alpha}_{l^+}(\phi(m)\,\psi(m)+\phi(m)\,\psi(l))+F^{\alpha}_{m^-}(\phi(l)\,\psi(l)+\phi(l)\,\psi(m))\right).$$

Proof. We have

$$\begin{aligned}
&\phi\left(\frac{l+m}{2}\right)\psi\left(\frac{l+m}{2}\right)\\
&=\phi\left(\frac{tl+(1-t)m}{2}+\frac{(1-t)l+tm}{2}\right)g\left(\frac{tl+(1-t)m}{2}+\frac{(1-t)l+tm}{2}\right)\\
&\le e\,\left(\phi(tl+(1-t)m)+\phi((1-t)l+tm)\right)\cdot\left(\psi(tl+(1-t)m)+\psi((1-t)l+tm)\right)\\
&=e\,\big(\phi(tl+(1-t)m)\,\psi(tl+(1-t)m)+\phi((1-t)l+tm)\,\psi((1-t)l+tm)\\
&\quad+\phi((1-t)l+tm)\,\psi(tl+(1-t)m)+\phi(tl+(1-t)m)\,\psi((1-t)l+tm)\big).
\end{aligned}$$

Now, multiplying both sides of the above inequality by $\frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and integrating the resulting inequality with respect to t, we obtain

$$
\begin{aligned}
&\frac{1}{\Gamma(\alpha+1)}\phi\left(\frac{l+m}{2}\right)\,\psi\left(\frac{l+m}{2}\right)\\
&\le \frac{e}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}\,\phi(tl+(1-t)m)\,\psi(tl+(1-t)m)\,dt\\
&\quad+\frac{e}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}\,\phi((1-t)l+tm)\,\psi((1-t)l+tm)\,dt\\
&\quad+\frac{e}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}\,\phi((1-t)l+tm)\,\psi(tl+(1-t)m)\,dt\\
&\quad+\frac{e}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}\,\phi(tl+(1-t)m)\,\psi((1-t)l+tm)\,dt\\
&=\frac{e}{\Gamma(\alpha)\,(m-l)^{\alpha}}\int_l^m (m-l)^{\alpha-1}\phi(u)\,\psi(u)\,du\\
&\quad+\frac{e}{\Gamma(\alpha)\,(m-l)^{\alpha}}\int_l^m (m-u)^{\alpha-1}\phi(l+m-u)\,\psi(l+m-u)\,du\\
&\quad+\frac{e}{\Gamma(\alpha)\,(m-l)^{\alpha}}\int_0^1 (m-u)^{\alpha-1}\phi(l+m-u)\,\psi(u)\,du\\
&\quad+\frac{e}{\Gamma(\alpha)\,(m-l)^{\alpha}}\int_0^1 (m-u)^{\alpha-1}\phi(u)\,\psi(l+m-u)\,du\\
&=\frac{e}{\Gamma(\alpha)\,(m-l)^{\alpha}}\int_l^m (m-u)^{\alpha}\phi(u)\,\psi(u)\,du\\
&\quad+\frac{e}{\Gamma(\alpha)\,(m-l)^{\alpha}}\int_l^m (v-l)^{\alpha-1}\phi(v)\,\psi(v)\,dv\\
&\quad+\frac{e}{\Gamma(\alpha)\,(m-l)^{\alpha}}\int_l^m (v-l)^{\alpha-1}\phi(v)\,\psi(l+m-v)\,dv\\
&\quad+\frac{e}{\Gamma(\alpha)\,(m-l)^{\alpha}}\int_l^m (m-u)^{\alpha-1}\phi(u)\,\psi(l+m-u)\,du\\
&=\frac{e}{(m-l)^{\alpha}}\Big(F^{\alpha}_{l^+}\phi(m)\,\psi(m)+F^{\alpha}_{m^-}\phi(l)\,\psi(l)+F^{\alpha}_{m^-}\phi(l)\,\psi(m)+F^{\alpha}_{l^+}\phi(m)\,\psi(l)\Big).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\phi\left(\frac{l+m}{2}\right)\,\psi\left(\frac{l+m}{2}\right)\\
&\le \frac{e\,\Gamma(\alpha+1)}{(m-l)^{\alpha}}\left(F^{\alpha}_{l^+}(\phi(m)\,\psi(m)+\phi(m)\,\psi(l))+F^{\alpha}_{m^-}(\phi(l)\,\psi(l)+\phi(l)\,\psi(m))\right).
\end{aligned}
$$
□

Remark 2.14. For $\alpha=1$, the above inequality reduces to Theorem 2.5.

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