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ON THE APPROXIMATION BY VILENKIN–FEJÉR MEANS IN LEBESGUE SPACES

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Abstract. In this paper, using the modulus of continuity, we study the approximation properties of Fejér means for bounded Vilenkin groups in Lebesgue spaces for any $1 \leq p < \infty$.

1. Introduction

Let $\mathbb{N}_+$ denote the set of positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, \dots)$ denote a sequence of positive integers not less than 2. Denote by

$$Z_{m_k} := \{0, 1, \dots, m_k - 1\}$$

the additive group of integers modulo m_k.

We define the group G_m as a complete direct product of the group Z_{m_j} with the product of discrete topologies of Z_{m_j}. The direct product μ of the measures

$$\mu_k(\{j\}) := 1/m_k \qquad (j \in Z_{m_k})$$

is the Haar measure on G_m with $\mu(G_m) = 1$. In this paper, we discuss only the bounded Vilenkin groups, that is $R := \sup_{n \in \mathbb{N}} m_n < \infty$.

The elements of G_m are represented by the sequences

$$x := (x_0, x_1, \dots, x_k, \dots), \qquad (x_k \in Z_{m_k}).$$

It is easy to describe a neighborhood of the base of G_m, namely:

$$\begin{aligned} &I_0(x) := G_m, \\ &I_n(x) := \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}\} \ (x \in G_m, \ n \in \mathbb{N}). \end{aligned}$$

The intervals $I_n(x)$ $(n \in \mathbb{N}, x \in G_m)$ are called Vilenkin intervals. Denote $I_n := I_n(0)$ for $n \in \mathbb{N}$ and $\overline{I_n} := G_m \setminus I_n$. Let

$$e_n := (0, \dots, 0, 1, 0, \dots) \in G_m \ \ (n \in \mathbb{N}),$$

where the n-th coordinate is 1, the others are 0.

The norms (or quasi-norms) of the Lebesgue spaces $L^p(G_m)$ are defined by

$$\|f\|_p := \left(\int_{G_m} |f|^p \, d\mu \right)^{1/p}.$$

The modulus of continuity of a function $f \in L^p(G_m), \ 1 \leq p \leq \infty$ is defined as follows:

$$\omega_p(\delta, f) = \sup_{|t| < \delta} \|f(x+t) - f(x)\|, \quad \delta > 0,$$

where

$$|t| := \sum_{i=0}^{\infty} \frac{t_i}{M_{i+1}}.$$

If we define the so-called generalized number system based on m in the following way:

$$M_0 := 1, \quad M_{k+1} := m_k M_k \ \ (k \in \mathbb{N}),$$

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then every $n \in \mathbb{N}$ can be uniquely expressed as

$$n = \sum_{j=0}^{\infty} n_j M_j, \quad \text{where} \quad n_j \in Z_{m_j} \quad (j \in \mathbb{N})$$

and only a finite number of n_j differs from zero.

Let

$$|n| := \max\{j \in \mathbb{N}, \quad n_j \neq 0\}.$$

Next, we introduce on G_m an orthonormal system, which is called the Vilenkin system. First, we define the complex-valued function $r_k(x) : G_m \to \mathbb{C}$ and the generalized Rademacher functions as follows:

$$r_k(x) := \exp(2\pi \imath x_k / m_k) \qquad \left(\imath^2 = -1, \ x \in G_m, \ k \in \mathbb{N}\right).$$

It is cleat that

$$|r_k| = 1, \quad \text{for all} \ \ x \in G_m.$$

We define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \qquad (n \in \mathbb{N}).$$

In particular, this system is called the Walsh–Paley system if $m \equiv 2$ (for more details, see [12] and [25]). The Vilenkin system is orthonormal and complete in $L^2(G_m)$ (for details see, e.g., [1, 25, 31]).

If $f \in L^1(G_m)$, we can define the Fourier coefficients, partial sums of Fourier series, Fejér means, Dirichlet and Fejér kernels with respect to the Vilenkin system in the usual manner:

$$\widehat{f}(k) := \int_{G_m} f \overline{\psi}_k d\mu \qquad (k \in \mathbb{N}),$$

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k \quad (n \in \mathbb{N}_+, \ S_0 f := 0),$$

$$\sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f \qquad (n \in \mathbb{N}_+),$$

$$D_n := \sum_{k=0}^{n-1} \psi_k \qquad (n \in \mathbb{N}_+),$$

$$K_n := \frac{1}{n} \sum_{k=1}^{n} D_k \qquad (n \in \mathbb{N}_+).$$

In [29], Toledo (see also [5]) showed that

$$\sup_n \|K_n\|_1 = \frac{17}{15},$$

where K_n is the Fejér kernel with respect to the Walsh system. For one-dimensional bounded Vilenkin-Fourier series, it was proved in [20] (see also [5, 23]) that

$$\sup_n \|K_n\|_1 \leq C,$$

where C is an absolute constant. But the exact value of the supremum, depending on the sequence m, remains still an open problem for the bounded Vilenkin systems. It follows that (see, e.g., [1] and [6]) there exists an absolute constant C_p, depending only on p such that

$$\|\sigma_n f\|_p \leq C_p \|f\|_p \quad (1 \leq p \leq \infty).$$

If we define the maximal operator σ^* of Fejér means by

$$\sigma^* f \quad := \quad \sup_{n \in \mathbb{N}_+} |\sigma_n f|,$$

then the weak type inequality

$$\mu\left(\sigma^{*}f>\lambda\right)\leq\frac{c}{\lambda}\left\|f\right\|_{1}\quad(\lambda>0)$$

holds for any integrable function. For example, this result can be found in Zygmund [35] for trigonometric series, in Schipp [24] for Walsh series and in Pál, Simon [20] (see also [22, 32–34]) for bounded Vilenkin series. It follows that the Fejér means with respect to trigonometric and Vilenkin systems of any integrable function converge a.e. to this function.

The a.e. convergence and summability of Fejér means, as well as some more general summability methods have been studied by several authors. We mention Anakidze, Areshidze and Baramidze [2] (see also [4]), Blahota, Persson and Tephnadze [10], Memiæ [13], Tephnadze [27,28], Fridli, Manchanda, Siddiqi [11], Móricz and Siddiqi [14], Nagy [15], (see also [9, 16–19] and [21]) and Tutberidze [30].

In [23], it was proved that if $1\leq p<\infty$, $f\in L^{p}(G_{m})$ and $n\in\mathbb{N}$, then there exists an absolute constant c_{p}, depending only on p such that the following inequality:

$$\left\|\sigma_{n}f-f\right\|_{p}\leq c_{p}\omega_{p}\left(1/M_{N},f\right)+c_{p}\sum_{s=0}^{N-1}\frac{M_{s}}{M_{N}}\omega_{p}\left(1/M_{s},f\right)\tag{1.1}$$

holds. In this paper, we estimate the L^{1} norms of the Vilenkin–Fejér kernels. Moreover, we use a new approach and estimate the constants in (1.1) for any $1\leq p<\infty$. This new approach is very important for generalizing the problems investigated in [2] and [3] for the Vilenkin systems.

It should be noted that inequality (1.1) was proved by Skvortsov [26] for the Walsh system, and the rate of approximation of the Vilenkin matrix transform means is discussed in [7] (see also [8]). The Fejér mean is a special matrix transform mean, so in this special case we were able to obtain a more exact estimate by using our new approach.

2. Auxiliary Lemmas

The next lemmas (see Lemmas 1, 2 and 3) can be found in [23].

Lemma 2.1. *Let $n\in\mathbb{N}$, $1\leq s\leq m_{n}-1$. Then*

$$D_{M_{n}}\left(x\right)=\begin{cases}M_{n}, & \text{if } x\in I_{n},\\ 0, & \text{if } x\notin I_{n}\end{cases}\tag{2.1}$$

and

$$D_{sM_{n}}=D_{M_{n}}\sum_{k=0}^{s-1}r_{n}^{k}.\tag{2.2}$$

Lemma 2.2. *Let $n>t$, $t,n\in\mathbb{N}$, then*

$$K_{M_{n}}\left(x\right)=\begin{cases}\frac{M_{t}}{1-r_{t}(x)}, & x\in I_{t}\backslash I_{t+1},\quad x-x_{t}e_{t}\in I_{n},\\ \frac{M_{n}}{2}, & x\in I_{n},\\ 0, & \text{otherwise.}\end{cases}\tag{2.3}$$

Moreover, if $n,t,s\in\mathbb{N}$ and $1\leq s\leq m_{n}-1$, then

$$sM_{n}K_{sM_{n}}=\sum_{l=0}^{s-1}\left(\sum_{i=0}^{l-1}r_{n}^{i}\right)(M_{n}+1)D_{M_{n}}+\left(\sum_{l=0}^{s-1}r_{n}^{l}\right)M_{n}K_{M_{n}}.\tag{2.4}$$

Lemma 2.3. *Let $n=\sum_{i=1}^{r}s_{n_{i}}M_{n_{i}}$, where $n_{1}>n_{2}>\cdots>n_{r}\geq0$ and $1\leq s_{n_{i}}<m_{n_{i}}$ for all $1\leq i\leq r$ as well as*

$$n^{(k)}=n-\sum_{i=1}^{k}s_{n_{i}}M_{n_{i}},\tag{2.5}$$

where $0<k\leq r$. Then

$$nK_{n}=\sum_{k=1}^{r}\left(\prod_{j=1}^{k-1}r_{n_{j}}^{s_{n_{j}}}\right)s_{n_{k}}M_{n_{k}}K_{s_{n_{k}}M_{n_{k}}}+\sum_{k=1}^{r-1}\left(\prod_{j=1}^{k-1}r_{n_{j}}^{s_{n_{j}}}\right)n^{(k)}D_{s_{n_{k}}M_{n_{k}}}.\tag{2.6}$$

Next, we establish several new lemmas that may also be of independent interest:

Lemma 2.4. *Let $n \in \mathbb{N}$ and $1 \leq s \leq m_n - 1$. Then*

$$|K_{sM_n}| \leq s|K_{M_n}| \tag{2.7}$$

and

$$n|K_n| \leq 2R^2 \sum_{k=0}^{|n|} M_k \left|K_{M_k}\right|. \tag{2.8}$$

Proof. Combining (2.1) and (2.3), we obtain

$$D_{M_n} \leq 2\left|K_{M_n}\right|. \tag{2.9}$$

From (2.4) and (2.9), we find that

$$\begin{aligned} sM_n|K_{sM_n}| &\leq 2M_n\left|K_{M_n}\right|\left(\sum_{l=0}^{s-1} l\right) + sM_n\left|K_{M_n}\right| \\ &\leq 2M_n|K_{M_n}|\frac{s(s-1)}{2} + sM_n\left|K_{M_n}\right| \\ &= s^2 M_n|K_{M_n}|, \end{aligned}$$

which proves inequality (2.7).

According to (2.5), we obtain $n^{(k)} \leq M_{n_k}$. Applying (2.2), we find that

$$|D_{sM_n}| \leq s|D_{M_n}|.$$

Using (2.6) and (2.7), we have

$$\begin{aligned} n|K_n| &\leq \sum_{k=1}^{r} s_{n_k}^2 M_{n_k}\left|K_{M_{n_k}}\right| + \sum_{k=1}^{r-1} s_{n_k} M_{n_k}|D_{M_{n_k}}| \\ &\leq R^2 \sum_{k=1}^{r} M_{n_k}\left|K_{M_{n_k}}\right| + 2R\sum_{k=1}^{r-1} M_{n_k}|K_{M_{n_k}}| \\ &\leq (R^2+2R)\sum_{k=1}^{r} M_{n_k}\left|K_{M_{n_k}}\right|. \end{aligned}$$

Hence,

$$n|K_n| \leq 2R^2 \sum_{k=0}^{|n|} M_k \left|K_{M_k}\right|,$$

which proves (2.8). Thus the proof is complete. □

Lemma 2.5. *Let $n \in \mathbb{N}$. Then*

$$\int_{G_m} |K_n(x)|\, d\mu(x) \leq 4R^3.$$

Proof. Since

$$\int_{G_m} |K_n(x)|\, d\mu(x) \quad\leq\quad \frac{2R^2}{n}\sum_{k=0}^{|n|} M_k \int_{G_m} |K_{M_k}(x)|\, d\mu(x), \tag{2.10}$$

we obtain

$$\begin{aligned}
&\int\limits_{G_m} \left|K_{M_k}(x)\right| d\mu(x) \\
=&\int\limits_{I_k} \left|K_{M_k}(x)\right| d\mu(x) + \sum_{s=0}^{k-1}\sum_{a_s=1}^{m_s-1} \int\limits_{I_k(a_s e_s)} \left|K_{M_k}(x)\right| d\mu(x) \\
=&\frac{M_k+1}{2}\frac{1}{M_k} + \sum_{s=0}^{k-1}\sum_{a_s=1}^{m_s-1} \int\limits_{I_k(a_s e_s)} \left|K_{M_k}(x)\right| d\mu(x),
\end{aligned} \tag{2.11}$$

and on the set $I_k(a_s e_s)$, the modulus of K_{M_k} equals

$$\left|K_{M_k}\right| = \left|\frac{M_s}{1-r_s(x)}\right|. \tag{2.12}$$

Since

$$\left|\frac{1}{r_s(x)-1}\right| = \left|\frac{1}{2\sin\frac{\pi x_s}{m_s}}\right| \leq \frac{1}{\left|2\sin\frac{\pi}{m_s}\right|},$$

applying the well-known inequality

$$\frac{1}{2\sin\frac{t}{2}} \leq \frac{\pi}{2t}, \quad \text{for} \quad 0 < t \leq \pi,$$

we get

$$\left|\frac{1}{r_s(x)-1}\right| \leq \frac{\pi}{2\frac{2\pi}{m_s}} = \frac{m_s}{4} \tag{2.13}$$

and, finally, we find that

$$\left|K_{M_k}(x)\right| \leq \frac{M_{s+1}}{4}, \quad \text{for} \quad x \in I_k(a_s e_s).$$

Therefore, using (2.11), we obtain

$$\begin{aligned}
\int\limits_{G_m} \left|K_{M_k}(x)\right| d\mu(x) &\leq \frac{M_k+1}{2}\frac{1}{M_k} + \frac{1}{4M_k}\sum_{s=0}^{k-1} m_s M_{s+1} \\
&\leq \frac{M_k+1}{2}\frac{1}{M_k} + \frac{R}{4M_k}\cdot 2M_k \leq 1 + \frac{R}{2} \leq R.
\end{aligned}$$

It follows from (2.10) that

$$\int\limits_{G_m} \left|K_n(x)\right| d\mu(x) \quad \leq \quad R\frac{2R^2}{n}\sum_{k=0}^{|n|} M_k \leq 4R^3. \tag{2.14}$$

The proof is complete. □

3. Approximation of Vilenkin–Fejér Means

Our main result reads:

Theorem 3.1. *Let $1 \leq p < \infty$, $f \in L^p(G_m)$, $n \in \mathbb{N}$ and $M_N \leq n < M_{N+1}$. Then*

$$\left\|\sigma_n f - f\right\|_p \leq (4R^3+2)\omega_p\left(1/M_N, f\right) + \frac{R^2}{4}\sum_{s=0}^{N-1}\frac{M_s}{M_N}\omega_p\left(1/M_s, f\right).$$

Proof. Let $f \in L^p(G_m)$, $1 \leq p < \infty$ and $M_N \leq n < M_{N+1}$. Using Young's convolution inequality

$$\|\sigma_n f\|_p = \|f * K_n\|_p \leq \|K_n\|_1 \|f\|_p$$

and the inequality $\|S_{M_n} f - f\|_p \leq \omega_p(1/M_n, f)$, we get

$$\begin{aligned} & \|\sigma_n f - f\|_p \\ \leq & \|\sigma_n f - \sigma_n S_{M_N} f\|_p + \|\sigma_n S_{M_N} f - S_{M_N} f\|_p + \|S_{M_N} f - f\|_p \\ = & \|\sigma_n (S_{M_N} f - f)\|_p + \|S_{M_N} f - f\|_p + \|\sigma_n S_{M_N} f - S_{M_N} f\|_p \\ \leq & (4R^3 + 1)\omega_p(1/M_N, f) + \|\sigma_n S_{M_N} f - S_{M_N} f\|_p . \end{aligned} \tag{3.1}$$

The straightforward calculations results in:

$$\begin{aligned} & \sigma_n S_{M_N} f - S_{M_N} f \\ = & \frac{1}{n} \sum_{k=1}^{M_N} S_k S_{M_N} f + \frac{1}{n} \sum_{k=M_N+1}^{n} S_k S_{M_N} f - S_{M_N} f \\ = & \frac{1}{n} \sum_{k=1}^{M_N} S_k f + \frac{1}{n} \sum_{k=M_N+1}^{n} S_{M_N} f - S_{M_N} f \\ = & \frac{1}{n} \sum_{k=1}^{M_N} S_k f + \frac{n - M_N}{n} S_{M_N} f - S_{M_N} f \\ = & \frac{M_N}{n} \sigma_{M_N} f - \frac{M_N}{n} S_{M_N} f \\ = & \frac{M_N}{n} (S_{M_N} \sigma_{M_N} f - S_{M_N} f) \\ = & \frac{M_N}{n} S_{M_N} (\sigma_{M_N} f - f) . \end{aligned} \tag{3.2}$$

Using (3.2) and the fact that

$$\|S_{M_N} f\|_p \leq \|f\|_p , \qquad f \in L_p(G_m), \quad 1 \leq p < \infty,$$

we find that

$$\begin{aligned} \|\sigma_n S_{M_N} f - S_{M_N} f\|_p &= \frac{M_N}{n} \|S_{M_N} (\sigma_{M_N} f - f)\|_p \\ &\leq \|S_{M_N} (\sigma_{M_N} f - f)\|_p \leq \|\sigma_{M_N} f - f\|_p . \end{aligned} \tag{3.3}$$

Moreover,

$$\begin{aligned} & \sigma_{M_N} f(x) - f(x) \\ = & \int_{G_m} (f(x-t) - f(x)) K_{M_N}(t) \, d\mu(t) \qquad (3.4) \\ = & \int_{I_N} (f(x-t) - f(x)) K_{M_N}(t) \, d\mu(t) \\ + & \sum_{s=0}^{N-1} \sum_{n_s=1}^{m_s-1} \int_{I_N(n_s e_s)} (f(x-t) - f(x)) K_{M_N}(t) \, d\mu(t) \\ := & I + II. \end{aligned} \tag{3.5}$$

Applying (2.3) and generalized Minkowski's inequality, we arrive at

$$\begin{aligned}\|I\|_p &\leq \int_{I_N} \|f(x-t)-f(x)\|_p \frac{M_N+1}{2} d\mu(t)\\ &\leq \omega_p(1/M_N, f) \int_{I_N} \frac{M_N+1}{2} d\mu(t) \leq \omega_p(1/M_N, f) \end{aligned} \tag{3.6}$$

and

$$\|II\|_p = \sum_{s=0}^{N-1} \sum_{n_s=1}^{m_s-1} \frac{M_s}{2\sin((\pi n_s)/m_s)} \int_{I_N(n_s e_s)} \|f(x-t)-f(x)\|_p \, d\mu(t)$$

$$\begin{aligned} &\leq \sum_{s=0}^{N-1} \sum_{n_s=1}^{m_s-1} \frac{M_s}{2\sin(\pi/m_s)} \int_{I_N(n_s e_s)} \omega_p(1/M_s, f)\, d\mu(t)\\ &= \frac{1}{2} \sum_{s=0}^{N-1} \sum_{n_s=1}^{m_s-1} \frac{M_s}{\sin(\pi/m_s)} \omega_p(1/M_s, f) \frac{1}{M_N}\\ &\leq \frac{R^2}{4} \sum_{s=0}^{N-1} \frac{M_s}{M_N} \omega_p(1/M_s, f). \end{aligned} \tag{3.7}$$

The proof is complete by combining (3.1)–(3.7). □

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