

DYNAMICAL THERMOSTABILITY OF PRESTRESSED ORTHOTROPIC RIGIDLY FIXED SHELLS OF REVOLUTION

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Abstract. The present paper is a generalization of the results obtained in [5] for the case where a shell material is orthotropic, characterizing a number of new plastic type materials. The influence of orthotropy and boundary conditions of rigid fixing on the boundaries of regions of dynamical instability of closed shells of revolution, similar to cylindrical ones, with an elastic filler, is investigated. The shell is assumed to be under the action of external pressure (depending on time) and temperature. We consider the shells of average length in which the form of generatrix of a middle surface is a parabolic function. The shells of positive and negative Gaussian curvature are also considered. Formulas for finding eigenfrequencies and boundaries of regions of dynamical instability depending on the orthotropy parameters, boundary conditions, Gaussian curvature, initial stressed state, temperature and the amplitude of shell deviation from cylindrical shape are obtained. The main attention is paid to identifying the most dangerous area of dynamical instability and to the lowest eigenfrequencies that are practically most important.

1. The present work deals with the problem of influence of the orthotropy, temperature and boundary conditions of rigidly fixed shells of revolution, similar to cylindrical ones, with an elastic filler, on the boundaries of regions of dynamical instability. It is assumed that the shell is under the action of external pressure depending on time by the periodic law. The shell is assumed to be thin and elastic. We consider a light filler for which the influence of shearing stresses on the contact surface and the inertia forces can be neglected. Temperature in a shell body is uniformly distributed. An elastic filler is simulated by Winkler's base, its extension when heated is not taken into account. We investigate the shells of average length whose form of midsurface generatrix is a parabolic function. We study the shells of positive and negative Gaussian curvature.

We consider a shell whose midsurface is formed by the rotation of a square parabola around the z -axis of a rectangular coordinate system x, y, z with the origin at the middle of the revolution axis segment. It is assumed that the radius R of the midsurface cross-section is defined by the equality $R = r + \delta_0 [1 - \xi^2 (\frac{r}{l})^2]$, where r is the radius of the curved cross-section, δ_0 is a maximal deviation from a cylindrical shape (the shell is convex for $\delta_0 > 0$, and concave for $\delta_0 < 0$), $L = 2l$ is the shell length, $\xi = \frac{z}{r}$.

We investigate the shells of average length [2, 10]

$$\varepsilon^{1/2} \max(K, K^{-1}) \ll \left(\frac{\pi r}{L}\right)^2 \ll \min(K, K^{-1}) \varepsilon^{-1/2},$$

$$K = \sqrt{\frac{E_1}{E_2}}, \quad \varepsilon = \frac{h^2}{12r^2(1 - \nu_1\nu_2)},$$

and it is assumed that

$$\left(\frac{\delta_0}{r}\right)^2, \left(\frac{\delta_0}{l}\right)^2 \ll 1. \quad (1.1)$$

For the shells of average length, the oscillation modes corresponding to lower frequencies vary weakly in the longitudinal direction compared to the circumferential direction, therefore the relation

$$\frac{\partial^2 f}{\partial \xi^2} \ll \frac{\partial^2 f}{\partial \varphi^2} \quad (f = u, v, w) \quad (1.2)$$

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is valid, where u , v and w are, respectively, meridional, circumferential and radial components of a displacement characterizing oscillation modes. Thus, according to V. Novozhilov's statement [6], the fundamental equations of oscillations can be taken to be the equations corresponding to V. Vlasov's semi-momentless theory [1]. As a result of simplification, the system of equations takes the following form (by the assumption, temperature terms are equal to zero [7]):

$$\begin{aligned} \frac{\partial^2 u}{\partial \varphi^2} &= -\left[1 - 2\left(\frac{E_1}{G} - \nu_1\right)\delta\right] \frac{\partial w}{\partial \xi}, \\ \frac{\partial^2 v}{\partial \varphi^2} &= (1 + 2\nu_1\delta) \frac{\partial w}{\partial \varphi}, \\ \varepsilon \frac{\partial^8 w}{\partial \varphi^8} + \frac{E_1}{E_2} \left(\frac{\partial^4 w}{\partial \xi^4} + 4\delta \frac{\partial^4 w}{\partial \xi^2 \partial \varphi^2} + 4\delta^2 \frac{\partial^4 w}{\partial \varphi^4} \right) - t_1^0 \frac{\partial^6 w}{\partial \xi^2 \partial \varphi^4} \\ &\quad - t_2^0 \frac{\partial^6 w}{\partial \varphi^6} - 2s^0 \frac{\partial^6 w}{\partial \xi \partial \varphi^5} + \gamma_0 \frac{\partial^4 w}{\partial \varphi^4} + \frac{\rho r^2}{E_2} \frac{\partial^2}{\partial t^2} \left(\frac{\partial^4 w}{\partial \varphi^4} \right) = 0, \\ \delta &= \frac{\delta_0 r}{l^2}, \quad t_i^0 = \frac{T_i^0}{E_2 h} \quad (i = 1, 2), \quad s^0 = \frac{S^0}{E_2 h}, \quad \gamma_0 = \frac{\beta_0 r}{E_2 h}, \end{aligned} \quad (1.3)$$

E_1 , E_2 , ν_1 , ν_2 are the moduli of elasticity and Poisson coefficients in meridional and circumferential directions ($E_1\nu_2 = E_2\nu_1$), G is a shear modulus; T_1^0 , T_2^0 are meridional and circumferential forces of initial state, S^0 is shearing force of the initial state; ρ is density of a shell material; h is a shell thickness; β_0 is the coefficient of the "bed" of the elastic filler (characterizing elastic rigidity of the filler); φ is the angular coordinate, t is time.

Consider first the case for $q = q_0$. Assuming that the initial state is momentless, we calculate interior forces. When the shell edges are rigidly fixed, they are free from meridional displacements. Relying on the corresponding solution and taking into account the filler reaction, temperature and also inequalities (1.1), we obtain the following approximate expressions:

$$\begin{aligned} T_1^0 &= -q_0 r \left\{ \nu_1 + \frac{\delta_0}{r} \left[\frac{1 + \nu_1}{3} + 2(1 - 2\nu_1\nu_2) \left(\frac{r}{l} \right)^2 - (1 - \nu_1)\xi^2 \left(\frac{r}{l} \right)^2 \right] \right\} - \frac{(\alpha_1 + \nu_2\alpha_2)}{1 - \nu_1\nu_2} T E_1 h, \\ T_2^0 &= -q_0 r \left[1 - 2\nu_2 \frac{\delta_0}{r} \left(\frac{r}{l} \right)^2 \right] + w_0 \beta_0 r, \quad S^0 = 0, \end{aligned} \quad (1.4)$$

where w_0 and β_0 are, respectively, the deflection and coefficient of the filler "bed" in the initial state, α_1 , α_2 are the coefficients of linear expansion in meridional and circumferential directions; T is temperature, q_0 is external pressure ($q_0 > 0$).

Taking relations (1.1) and (1.2) into account, we obtain

$$\frac{\delta_0}{r} \left[\frac{1 + \nu_1}{3} + 2(1 - 2\nu_1\nu_2) \left(\frac{r}{l} \right)^2 - (1 - \nu_1)\xi^2 \left(\frac{r}{l} \right)^2 \right] \frac{\partial^2 w}{\partial \xi^2} \ll \frac{\partial^2 w}{\partial \varphi^2}, \quad \nu_1 \frac{\partial^2 w}{\partial \xi^2} \ll \frac{\partial^2 w}{\partial \varphi^2}.$$

Therefore, expressions (1.4), when substituted into equation (1.3), can be simplified, and in this case, they take the form

$$T_1^0 = -\frac{(\alpha_1 + \nu_2\alpha_2)}{(1 - \nu_1\nu_2)} T E_1 h, \quad T_2^0 = -q_0 r \left[1 - 2\nu_2 \frac{\delta_0}{r} \left(\frac{r}{l} \right)^2 \right] + w_0 \beta_0 r, \quad T_i^0 = \sigma_i^0 h \quad (i = 1, 2). \quad (1.4')$$

Taking into account the fact that in the initial state the shell deformation in the circumferential direction ε_φ^0 is defined by the equalities

$$\varepsilon_\varphi^0 = \frac{\sigma_2^0 - \nu_2\sigma_1^0}{E_2} + \alpha_2 T, \quad \varepsilon_\varphi^0 = -\frac{w_0}{r},$$

we get

$$w_0 = \left(-\sigma_2^0 + \nu_2\sigma_1^0 \right) \frac{r}{E_2} - \alpha_2 T r. \quad (1.5)$$

Substituting (1.5) into (1.4'), we have

$$\begin{aligned}\frac{T_1^0}{E_2 h} &= -\frac{(\alpha_1 + \nu_2 \alpha_2) E_1 T}{(1 - \nu_1 \nu_2) E_2}, \\ \frac{T_2^0}{E_2 h} &= -\frac{q_0 r}{E_2 h} \left[1 - 2\nu_2 \frac{\sigma_0}{r} \left(\frac{r}{l} \right)^2 \right] + \frac{\beta_0 r}{E_2 h} \left[(-\sigma_2^0 + \nu_2 \sigma_1^0) \frac{r}{E_2} - \alpha_2 T r \right].\end{aligned}\quad (1.6)$$

Introduce the notation: $E_1 = e_1 E$, $E_2 = e_2 E$,

$$\bar{q} = \frac{q_0 r}{E h}, \quad \gamma_0 = \frac{\beta_0 r}{E h}, \quad \delta = \frac{\delta_0}{r} \left(\frac{r}{l} \right)^2, \quad g = 1 + \gamma_0 e_2^{-1}. \quad (1.6')$$

Then expressions (1.6) take the form

$$\begin{aligned}\frac{T_1^0}{E_2 h} &= \frac{\sigma_1^0}{E_2} = -\frac{(\alpha_1 + \nu_2 \alpha_2) e_1 T}{(1 - \nu_1 \nu_2) e_2}, \\ \frac{T_2^0}{E_2 h} &= \frac{\sigma_2^0}{E_2} = -\bar{q} e_2^{-1} (1 - \nu_2 \sigma) + \left(-\frac{\sigma_0^0}{E_2} + \nu_2 \frac{\sigma_1^0}{E_2} \right) \gamma_0 e_2^{-1} - \gamma_2 T \gamma_0 e_2^{-1}\end{aligned}$$

and hence, we obtain

$$-\frac{\sigma_1^0}{E_2} = \frac{(\alpha_1 + \nu_2 \alpha_2) e_1}{(1 - \nu_1 \nu_2) e_2} T, \quad -\frac{\sigma_2^0}{E_2} = \left\{ \bar{q} (1 - 2\nu_2 \delta) + \left[\frac{\nu_2 (\alpha_1 + \nu_2 \alpha_2) e_1}{(1 - \nu_1 \nu_2) e_2} + \alpha_2 \right] T \gamma_0 \right\} (g e_2)^{-1}.$$

Consequently,

$$-\frac{\sigma_1^0}{E_2} = \frac{(\alpha_1 + \nu_2 l_2) e_1}{1 - \nu_1 \nu_2 e_2} T, \quad -\frac{\sigma_2}{E_2} = \left\{ \bar{q} (1 - 2\nu_2 \delta) + \left[\frac{\nu_2 (\alpha_1 + \nu_2 \alpha_2 e_1)}{(1 - \nu_1 \nu_2) e_2} + \alpha_2 \right] T \gamma_0 \right\} (g e_2)^{-1}.$$

As a result, the third equation of system (1.3) takes the form

$$\begin{aligned}& \varepsilon \frac{\partial^8 w}{\partial \varphi^8} + \frac{e_1}{e_2} \left(\frac{\partial^4 w}{\partial \xi^4} + 4\delta \frac{\partial^4 w}{\partial \xi^2 \partial \varphi^2} + 4\delta^2 \frac{\partial^4 w}{\partial \varphi^4} \right) \\ & + \left\{ \bar{q} (1 - 2\nu_2 \delta) + \left[\frac{\nu_2 (\alpha_1 + \nu_2 \alpha_2) e_1}{(1 - \nu_1 \nu_2) e_2} + \alpha_2 \right] T \gamma_0 \right\} (g e_2)^{-1} \frac{\partial^6 w}{\partial \varphi^6} \\ & + \frac{(\alpha_1 + \nu_2 \alpha_2) e_1}{(1 - \nu_1 \nu_2) e_2} T \frac{\partial^6 w}{\partial \xi^2 \partial \varphi^4} + \gamma_0 e_2^{-1} \frac{\partial^4 w}{\partial \varphi^4} + \frac{\rho r^2}{E} e_2^{-1} \frac{\partial^2}{\partial t^2} \left(\frac{\partial^4 w}{\partial \varphi^4} \right) = 0.\end{aligned}\quad (1.7)$$

A solution of system (1.3) for harmonic oscillations will be sought in the form

$$\begin{aligned}u &= U(\xi) \sin n\varphi \cos wt, \\ v &= V(\xi) \cos n\varphi \cos wt, \\ w &= W(\xi) \sin n\varphi \cos wt.\end{aligned}$$

From the first two equations of system (1.3), we obtain

$$n^2 U = \left[1 - 2 \left(\frac{E_1}{G} - \nu_1 \right) \right] W', \quad (1.8)$$

$$nV = (1 + 2\nu\delta)W. \quad (1.9)$$

Some simplifications of the boundary conditions of rigid fixing for the shells (for $\xi = \text{const}$) of the form

$$u = v = w = w'_\xi = 0 \quad (1.10)$$

are worth mentioning. Owing to equality (1.9), we find that the fulfillment of the condition $w = 0$ results in that of the condition $v = 0$, whereas owing to (1.8), the fulfillment of the condition $w' = 0$ results in that of the condition $u = 0$. Consequently, if the conditions $w = w' = 0$ are fulfilled, then all conditions (1.10) are likewise fulfilled.

Let now the shell edges be rigidly fixed. In this case, the solution must satisfy both the condition of periodicity with respect to φ and the following boundary conditions:

$$w = 0 \quad \left(\xi = \pm \frac{l}{r} \right), \quad w'_\xi = 0 \quad \left(\xi = \pm \frac{l}{r} \right), \quad (1.11)$$

with respect to the coordinate ξ .

A solution w of equation (1.7) for harmonic oscillations is sought in the form

$$w = W(\xi) \sin n\varphi \cos \omega t. \quad (1.12)$$

From (1.7) and (1.12), after certain simplifications, we get

$$\begin{aligned} & W^{(4)} - \left(4\delta n^2 - \frac{(\alpha_1 + \nu_2 \alpha_2) T n^4}{(1 - \nu_1 \nu_2)} \right) W^{(2)} \\ & - n^4 \left\{ \Omega \omega^2 + \left[\bar{q}(1 - 2\nu_2 \delta) + \left(\alpha_2 + \frac{\nu_2(\alpha_1 + \nu_2 \alpha_2) e_1}{(1 - \nu_1 \nu_2) e_2} \right) T \gamma_0 \right] g^{-1} n^2 - e_2 \varepsilon n^4 - e_1 4\bar{\delta}^2 \right\} e_1^{-1} = 0, \\ & \bar{\delta}^2 = \delta^2 + \left(\frac{\gamma_0}{4e_2} \right), \quad \Omega = \frac{\rho r^2}{E}. \end{aligned} \quad (1.13)$$

Putting $W = C e^{\beta \xi}$, we obtain the characteristic equation which can be represented as follows:

$$p^2 - ap - b = 0 \quad p = \beta^2, \quad a = 4\delta n^2 - \frac{(\alpha_1 + \nu_2 \alpha_2) T n^4}{(1 - \nu_1 \nu_2)}, \quad (1.14)$$

$$b = n^4 \left\{ \Omega \omega^2 + \left[\bar{q}(1 - 2\nu_2 \delta) + \left(\alpha_2 + \frac{\nu_2(\alpha_1 + \nu_2 \alpha_2) e_1}{(1 - \nu_1 \nu_2) e_2} \right) T \gamma_0 \right] g^{-1} n^2 - e_2 \varepsilon n^4 - e_1 4\bar{\delta}^2 \right\} e_1^{-1}. \quad (1.15)$$

We proceed from the condition $b > 0$. Then it follows from conditions (1.14) and (1.15) that

$$\begin{aligned} & \beta_{1,2} = \pm \sqrt{p_1}, \quad \beta_{3,4} = \pm i \sqrt{-p_2}, \\ & p_1 = \frac{a}{2} + \sqrt{\frac{a^2}{4} + b} > 0, \quad p_2 = \frac{a}{2} - \sqrt{\frac{a^2}{4} + b} < 0. \end{aligned} \quad (1.16)$$

A general solution of equation (1.13) takes the form

$$\begin{aligned} W &= A \operatorname{ch} k_1 \xi + B \operatorname{sh} k_1 \xi + C \cos k_2 \xi + D \sin k_2 \xi, \\ k_1 &= \sqrt{p_1}, \quad k_2 = \sqrt{-p_2}. \end{aligned}$$

Satisfying the boundary conditions (1.11), we obtain the system of four homogeneous equations. From the condition that the determinant of this system is equal to zero, we obtain

$$\operatorname{th} k_1 \bar{l} = \frac{k_1}{k_2} \operatorname{tg} k_2 \bar{l} = -\frac{k_2}{k_1} \operatorname{tg} k_2 \bar{l}, \quad \bar{l} = \frac{l}{r}. \quad (1.17)$$

Consequently, this system decomposes into two independent systems and a solution decomposes, respectively, into odd and even functions.

Let us consider first the case $\delta = 0$, $q = \gamma = T = 0$ with $p_1 = -p_2 = \sqrt{b}$, $k_1 = k_2 = \sqrt[4]{b} = k$. Equation (1.17), corresponding to the skew-symmetric forms of oscillation, looks here as

$$\operatorname{th} k \bar{l} = \operatorname{tg} k \bar{l}.$$

The lowest root of this equation corresponds to the value

$$k = 3,927 \frac{r}{l},$$

whereas equation (1.17), corresponding to the symmetrical forms of oscillation, takes for $\delta = q = \gamma = T = 0$ the form

$$\operatorname{th} k \bar{l} = -\operatorname{tg} k \bar{l}.$$

The lowest root of this equation corresponds to

$$k = 2,365 \frac{r}{l} = 0,75 \frac{\pi r}{l}, \quad (1.18)$$

that is, the smallest value k corresponds to symmetrical forms of oscillation. Therefore, in the sequel, we will consider the oscillations with symmetrical forms of deflection with respect to ξ . Taking into account that for $q = \gamma = T = 0$,

$$-p_1 p_2 = b, \quad b = \left(\Omega \omega^2 - \frac{e_2}{e_1} \varepsilon n^4 \right) e_1^{-1} n^4,$$

we obtain

$$k^4 = \left(\Omega \omega^2 - \frac{e_2}{e_1} \varepsilon n^4 \right) e_1^{-1} n^4.$$

This implies that the lowest root (1.18) for fixed n corresponds to the smallest value of eigenfrequency which is determined by the following expression:

$$\bar{\Omega} \omega^2 = e_2 \varepsilon n^4 + e_1 (d_1 \lambda_1)^4 n^{-4}, \quad d_1 = 1, 55, \quad \lambda_1 = \frac{\pi r}{2l}. \quad (1.19)$$

The lowest frequency value depending on n is realized for

$$n_0^2 = d_1 \left(\frac{e_1}{e_2} \right)^{-1/4} \lambda_1 \varepsilon^{-1/4}. \quad (1.20)$$

For $n = n_0$, from (1.19), we obtain the known formula for the lowest frequency of an orthotropic cylindrical shell of average length with rigidly fixed edges [3],

$$\Omega \omega_{01}^2 = 2d_1^2 \sqrt{e_1 e_2} \lambda_1^2 \varepsilon^{1/2}. \quad (1.21)$$

For freely supported edges of a cylindrical orthotropic shell, the lowest frequency is defined, as is known, by the formula

$$\Omega \omega_0^2 = 2 \sqrt{e_1 e_2} \lambda_1^2 \varepsilon^{1/2}.$$

Let us now turn to a more general case and investigate symmetrical in axial direction forms of oscillations corresponding to the lowest frequencies. On the basis of (1.16), we have

$$-p_2 = p_1 - a, \quad a = \left(4\delta^2 - \frac{(\alpha_1 + \nu_2 \alpha_2)}{1 - \nu_1 \nu_2} T n^2 \right) n^2,$$

whence by putting $x = \bar{l} \sqrt{p_1}$, we get

$$-p_2 \bar{l}^2 = x^2 - \bar{\beta}, \quad \bar{\beta} = 4n^2 \frac{\delta_0}{r} - \frac{(\alpha_1 + \nu_2 \alpha_2)}{1 - \nu_1 \nu_2} T \bar{l}^2 n^4. \quad (1.22)$$

Then equation (1.17), corresponding to the symmetrical forms of oscillation, can be represented as

$$x \operatorname{th} x = -\sqrt{x^2 - \bar{\beta}} \operatorname{tg} \sqrt{x^2 - \bar{\beta}}. \quad (1.23)$$

Relying on the first equality of (1.16), it follows that $p_1(p_1 - a) = b$, whence we get

$$\begin{aligned} \Omega \omega^2 &= e_2 \varepsilon n^4 + x^2 (x^2 - \bar{\beta}) \left(\frac{r}{\bar{l}} \right)^4 n^{-4} e_1 + 4\bar{\delta}^2 e_1 \\ &+ \left[\bar{q} (1 - 2\nu_2 \delta) + \left(\alpha_2 + \frac{\nu_2 (\alpha_1 + \nu_2 \alpha_2) e_1}{(1 - \nu_1 \nu_2) e_2} \right) T \gamma_0 \right] g^{-1} n^2. \end{aligned} \quad (1.24)$$

Consequently, in a general case, for the shells under consideration, the eigenfrequencies ω are defined by formula (1.24), where x is any root of equation (1.23). The lowest frequency ω is obtained as a result of minimizing the right-hand side of (1.24) with respect to n , when x is taken to be the lowest root of equation (1.23) which we denote by x_ω . In view of (1.22) and (1.23), it is not difficult to see that x_ω depends on the orthotropy parameters e_1 , e_2 , as well as on δ , γ_0 , T , n . Such a minimization is realized by enumeration of natural values n in the vicinity of n_0 defined by equality (1.20).

Below, we present the results of calculations for the shells of geometrical dimensions $l = r$, $h/r = 10^{-2}$ for different values δ (with $q = \gamma_0 = T = 0$) for the cases $e_1 = e_2 = 1(0)$; $e_1 = 2$, $e_2 = 1(1)$; $e_1 = 1$, $e_2 = 2(2)$.

Figure 1 shows the curves of dependence of dimensionless values of the square of the lowest frequency ω^2/ω_0^2 and critical pressure $\bar{q}_*/\bar{q}_0$ on δ , which are denoted, respectively, by A_i and B_i ($i = 0, 1, 2$), ($\omega_0^2 = 2\Omega^{-1} \lambda_1 \varepsilon^{1/2}$, is the lowest frequency of an isotropic, with freely supported edges, cylindrical shell of average length [4]); $\bar{q}_0 = 1, 755 \lambda_1 \varepsilon^{3/4}$ is the corresponding critical pressure [10].

For $\omega = 0$, equality (1.24) yields

$$\begin{aligned} \bar{q}_*(1 - 2\nu_2\delta) &= [e_2\varepsilon n^2 + e_1 4\bar{\delta}^2 n^{-2} + x^2(x^2 - \bar{\beta})\left(\frac{r}{l}\right)^4 n^{-6} e_1]g \\ &\quad - \left(\alpha_2 + \frac{\nu_2(\alpha_1 + \nu_2\alpha_2)e_1}{(1 - \nu_1\nu_2)\bar{e}_2}\right)T\gamma_0. \end{aligned} \quad (1.25)$$

The lowest value $\bar{q}_*$ is obtained as a result of minimizing the right-hand side of equality (1.25) just in the same way as for the frequency ω .

2. Consider now the case for

$$q = q_0 + q_t \cos \Omega t.$$

A solution of equation (1.3) will be sought in the form

$$w = \mathcal{F}(\xi, t) \sin n\varphi,$$

where

$$\mathcal{F}(\xi, t) = f(t)W(\xi).$$

Substituting the above solution into (1.3) and introducing notation (1.6'), we obtain the equation which can be decomposed with respect to the variables ξ and t . Introducing the parameter λ^2 , we get two equations:

$$W^{(4)} - \left(4\delta - \frac{\alpha_1 + \nu_2\alpha_2}{1 - \nu_1\nu_2}Tn^2\right)n^2W^{(2)}(\xi) - \frac{e_2}{e_1}\lambda^2W(\xi) = 0, \quad (2.1)$$

and

$$\begin{aligned} \frac{\rho r^2}{E}f^{(2)}(t) + \left\{e_2\varepsilon n^4 + e_1 4\bar{\delta}^2 - \left(\alpha_2 + \frac{\nu_2(\alpha_1 + \nu_2\alpha_2)e_1}{(1 - \nu_1\nu_2)e_2}\right)T\gamma_0 g^{-1}n^2 \right. \\ \left. - \bar{q}(t)(1 - 2\nu_2\delta)g^{-1}n^2 + \lambda^2 e_2 n^{-4}\right\}f(t) = 0. \end{aligned} \quad (2.2)$$

Let us determine the value λ by using equation (2.1). Putting $W = Ce^{\bar{\alpha}\xi}$, we obtain the corresponding characteristic equation

$$p^2 - ap - \lambda^2 \frac{e_2}{e_1} = 0, \quad p = \bar{\alpha}^2, \quad a = \left(4\delta - \frac{\alpha_1 + \nu_2\alpha_2}{1 - \nu_1\nu_2}Tn^2\right)n^2. \quad (2.3)$$

Then it follows from (2.3) that

$$\begin{aligned} \bar{\alpha}_{1,2} &= \pm\sqrt{p_1}, & \bar{\alpha}_{3,4} &= \pm i\sqrt{-p_2}, \\ p_1 &= \frac{a}{2} + \sqrt{\frac{a^2}{4} + \lambda^2 \frac{e_2}{e_1}} > 0, & p_2 &= \frac{a}{2} - \sqrt{\frac{a^2}{4} + \lambda^2 \frac{e_2}{e_1}} < 0. \end{aligned} \quad (2.4)$$

The general solution of equation (2.1) takes the form

$$\begin{aligned} W &= A \operatorname{ch} k_1 \xi + B \operatorname{sh} k_1 \xi + C \cos k_2 \xi + D \sin k_2 \xi, \\ k_1 &= \sqrt{p_1}, \quad k_2 = \sqrt{-p_2}. \end{aligned} \quad (2.5)$$

Satisfying the boundary conditions (1.11), we obtain the system of four homogeneous algebraic equations. From the condition that the determinant of this system is equal to zero, we have

$$\operatorname{th} k_1 \bar{l} = \frac{k_1}{k_2} \operatorname{tg} k_2 \bar{l} = -\frac{k_2}{k_1} \operatorname{tg} k_2 \bar{l}, \quad \bar{l} = \frac{l}{r}. \quad (2.6)$$

Equality (2.6) coincides by its form with (1.17), however, the values k_1 and k_2 have another meaning: the corresponding expressions (2.4) and (2.5) involve the parameter λ^2 .

According to the above-said, let us consider the solution corresponding to the symmetrical form of deflection with respect to ξ . In this case,

$$\operatorname{th} k_1 \bar{l} = -\frac{k_1}{k_2} \operatorname{tg} k_2 \bar{l}. \quad (2.7)$$

By virtue of (2.4), we have

$$-p_2 = p_1 - a, \quad a = 4\delta n^2 - \frac{\alpha_1 + \nu_2\alpha_2}{1 - \nu_1\nu_2} T n^4,$$

whence, by putting $x = \bar{l}\sqrt{p_1}$, we get

$$-p_2\bar{l}^2 = x^2 - \bar{\beta}, \quad \bar{\beta} = a\bar{l}^2 = 4\frac{\delta_0}{r}n^2 - \frac{\alpha_1 + \nu_2\alpha_2}{1 - \nu_1\nu_2} \left(\frac{l}{r}\right)^2 T n^4.$$

Then equation (2.7) can be represented in the form

$$x \operatorname{th} x = -\sqrt{x^2 - \bar{\beta}} \operatorname{tg} \sqrt{x^2 - \bar{\beta}}, \quad (2.8)$$

and according to (2.4),

$$x^2\bar{l}^{-2} - \frac{a}{2} = \sqrt{\frac{a^2}{4} + \lambda^2 \frac{e_2}{e_1}},$$

whence it follows that

$$\lambda^2 = x^2\bar{l}^{-2}(x^2\bar{l}^{-2} - a)\frac{e_1}{e_2} = x^2(x^2 - \bar{\beta})\bar{l}^{-4} \frac{e_1}{e_2}, \quad (2.9)$$

where x is any root of equation (2.8). Next, substituting the value of λ^2 according to (2.9), into equation (2.2), we obtain

$$\begin{aligned} f^{(2)}(t) + \frac{E}{\rho r^2} \left\{ e_2 \varepsilon n^4 + 4\bar{\delta}^2 e_1 + x^2(x^2 - \bar{\beta})\bar{l}^{-4} n^{-4} e_1 - \left(\alpha_2 + \frac{\nu_2(\alpha_1 + \nu_2\alpha_2)e_1}{(1 - \nu_1\nu_2)e_2} \right) T_{\gamma_0} g^{-1} n^2 \right. \\ \left. - \bar{q}(t)(1 - 2\nu_2\delta)g^{-1}n^2 \right\} f(t) = 0, \end{aligned} \quad (2.10)$$

where $\bar{q}(t) = \bar{q}_0 + \bar{q}_t \cos \Omega t$.

It is not difficult to reduce equation (2.10) to the Matheieu equation [8]

$$\frac{d^2 f}{d\tau^2} + \frac{4\omega^2}{\Omega^2} (1 - 2\varepsilon \cos 2\tau) f = 0, \quad (2.11')$$

where

$$\tau = \frac{\Omega t}{2}, \quad \varepsilon = \frac{\bar{q}_t}{2(\bar{q}_* - \bar{q}_0)}; \quad (2.11)$$

ω and $\bar{q}_*$ are, respectively, the frequency of eigenoscillations and the critical pressure (their expressions are given above).

The frequency of eigenoscillations of the shells under consideration (when $\bar{q} = \bar{q}_0$) is determined by equation (2.10) (if $f(t) = C \sin \omega t$) and expressed by formula (1.24).

Critical pressure $\bar{q}_*$ is determined by equation (2.10) (if $f(t) = \text{const}$) and expressed by formula (1.25). The results obtained for $e_1 = e_2 = 1$ coincide completely with those obtained in [5].

The solution of equation (2.11') was investigated in a number of works, where it was noted that for certain relations between ε , Ω , ω and $t \rightarrow \infty$ the solution of equation (2.11') will increase indefinitely in the regions of instability. Generalizing the results of [8] and [1] to the considered shell, we will present below the following formulas. To elucidate the influence of the boundary conditions of rigid fixing, temperature and orthotropy parameters on the location of regions of dynamical instability, we consider first the case for $q_t \rightarrow 0$ ($\varepsilon \rightarrow 0$). In this case, we find that these regions of dynamical instability locate in the vicinity of exiting frequencies $\Omega_* = 2\omega/k$. Depending on k , we distinguish the first, second, third etc. regions of dynamical instability. The region of instability ($k = 1$) lying in the vicinity of $\Omega_* = 2\omega$, when ω is expressed by formula (1.24) and takes the lowest value, is the most dangerous and hence of the greatest practical significance. This region is called the principal region of dynamical instability.

When q_t is other than zero, we obtain the formula for the boundaries of the principal region of instability bounded by the following values of critical frequencies:

$$\Omega_* = 2\omega\sqrt{1 \pm \varepsilon}.$$

Taking into account the resistance forces, proportional to the first derivative of the displacement with respect to time (with the coefficient of damping θ), the formula for determining the boundaries of the principal region of instability takes the form

$$\Omega_* = 2\omega \sqrt{1 \pm \sqrt{\varepsilon^2 - \left(\frac{\Delta}{\pi}\right)^2}}, \quad \Delta = \frac{2\pi\theta}{\omega}, \quad (2.12)$$

where the terms containing higher powers of $\frac{\Delta}{\pi}$ are discarded and the damping decrement Δ is usually very small as compared with unity. The values of the expressions ω , $\bar{q}_*$, ε are defined by virtue of formulas (1.24), (1.25), (2.11), where x_ω and n correspond to the lowest value of ω .

For the boundary of the second region of instability ($k = 2$), the formula

$$\Omega_* = \omega \sqrt{1 + \varepsilon^2 \pm \sqrt{\varepsilon^4 - \left(\frac{\Delta}{\pi}\right)^2 (1 - \varepsilon^2)}}$$

holds. In this case, the critical value of the exciting coefficient is defined approximately by the equality $\varepsilon_{*2} = \left(\frac{\Delta}{\pi}\right)^{1/2}$. Analogously, generalizing the results of [1], we can deduce formulas for the boundaries of the third region of dynamical instability which is practically realized very seldom.

On the basis of the given formulas, it is not difficult to find the intervals of critical frequencies Ω_* (depending on e_1 , e_2 , δ , q_0 , q_t , T) falling into the regions of dynamical instability when the edges of orthotropic shells are rigidly fixed.

The given formulas allow one to quite easily determine how significantly the orthotropy parameters, rigid fixing of the shell edges, temperature and acting loading can affect the boundaries of the regions of dynamical instability. The issues under consideration are of significant importance for dynamical calculations in aircraft and rocket engineering.

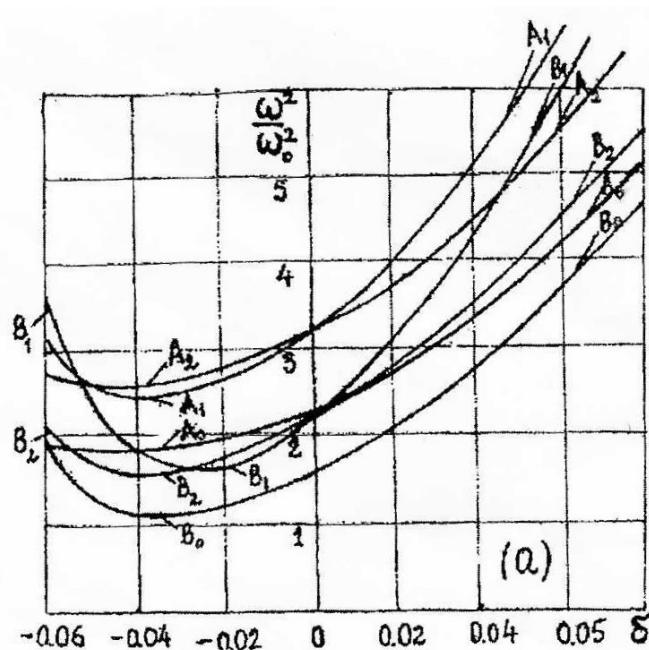


FIGURE 1

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