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FRANKLIN SYSTEM IN VARIABLE EXPONENT SPACES

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Abstract. In this paper, we study the properties of the Franklin system in variable-exponent Lebesgue spaces $L^{p(\cdot)}([0,1])$. The main result of this paper is that the Franklin system is an unconditional basis in $L^{p(\cdot)}([0,1])$, $1 < p_- \le p_+ < \infty$ if and only if the exponent function $p(\cdot)$ belongs to the Muckenhoupt class $\mathcal{A}$. We establish this equivalence using characterizations based on the Paley function and associated operators.

The classical Franklin system [11], consisting of piecewise linear continuous functions with dyadic knots, is a fundamental complete orthonormal basis in $C[0,1]$. Its properties are often studied using the exponential estimates established by Ciesielski [6]. It is well known that the Franklin system is an unconditional basis in $L^p[0,1]$ for $1 < p < \infty$ [5], as well as a basis in H^p for $1/2 \le p \le 1$ (unconditional for $1/2 < p \le 1$ [13, 14]), and VMO [7, 14]. Higher-order spline generalizations [8, 12] and their applications to signal and image processing have also been extensively investigated (see, e.g., [4]).

In this paper, we extend these investigations to variable-exponent Lebesgue spaces on $I \coloneqq [0,1]$, where the constant exponent p is replaced by a measurable function $p(\cdot)$. These spaces serve as a crucial framework in modern harmonic analysis and partial differential equations (see, e.g., [10]).

Our main result establishes that the Franklin system forms an unconditional basis in these variable-exponent Lebesgue spaces, provided the variable exponent satisfies the Muckenhoupt condition $\mathcal{A}$. The study of such orthogonal systems often relies on the boundedness of the Hardy–Littlewood maximal operator M, which is non-trivial in the variable setting, since standard sufficient criteria, like the local log-Hölder continuity, are not strictly necessary.

The Muckenhoupt condition $\mathcal{A}$ is a powerful and versatile tool in harmonic analysis. For instance, interesting summability results and properties on Vilenkin groups under the condition $\mathcal{A}$ were obtained in [3]. Also, in the recent work [2], quite surprising regularity results on the minimizers of double-phase functionals were established, based solely on the class $\mathcal{A}$.

Let $I \coloneqq [0,1]$. A measurable function $p : I \to [1,\infty]$ is called a variable exponent on I. We denote the set of all such variable exponents by $\mathcal{P}(I)$. For a given $p \in \mathcal{P}(I)$, we define

$$p_- \coloneqq \operatorname{ess\,inf}_{x\in I} p(x) \quad \text{and} \quad p_+ \coloneqq \operatorname{ess\,sup}_{x\in I} p(x).$$

The conjugate exponent $p' \in \mathcal{P}(I)$ is defined pointwise by the relation $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

For a measurable function $f : I \to \mathbb{R}$, we define the modular

$$\rho_{p(\cdot)}(f) \coloneqq \int\limits_I |f(x)|^{p(x)}\,dx,$$

where we use the standard convention $t^\infty \coloneqq 0$ for $t \in [0,1]$ and $t^\infty \coloneqq \infty$ for $t > 1$. The variable exponent Lebesgue space $L^{p(\cdot)}(I)$ is defined as the set of all measurable functions f such that $\rho_{p(\cdot)}(f/\lambda) < \infty$ for some $\lambda > 0$. It becomes a Banach space when equipped with the Luxemburg norm

$$\|f\|_{p(\cdot)} \coloneqq \inf\left\{\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \le 1\right\}.$$

If the exponent p is constant, then $L^{p(\cdot)}(I)$ simply coincides with the classical Lebesgue space. Moreover, the generalized Hölder inequality holds in these spaces: for any $f \in L^{p(\cdot)}(I)$ and $g \in L^{p'(\cdot)}(I)$,

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we have

$$\int_I |f(x)g(x)|\,dx \le 2\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}.$$

In the context of harmonic analysis tools, the variable analogue of the classical Muckenhoupt condition plays a pivotal role. For a variable exponent $p \in \mathcal{P}(I)$, we say that p satisfies the $\mathcal{A}$ condition (and write $p \in \mathcal{A}$) if

$$[p]_{\mathcal{A}} \coloneqq \sup_{J\subset I} \frac{1}{|J|}\|\mathbb{1}_J\|_{p(\cdot)}\|\mathbb{1}_J\|_{p'(\cdot)} < \infty, \tag{1}$$

where the supremum is taken over all subintervals $J \subset I$, $|J|$ denotes the Lebesgue measure of J, and $\mathbb{1}_J$ is the indicator function of J. This condition is obviously stable under duality, i.e., $[p]_{\mathcal{A}} = [p']_{\mathcal{A}}$. Note that in case $1 < p_- \le p_+ \le \infty$, this condition is necessary and sufficient for the boundedness of the Hardy–Littlewood maximal operator in $L^{p(\cdot)}([0,1])$ (see [1, 9, 10]).

The Franklin system $\mathcal{F} = \{f_n : n = 0, 1, \dots\}$ is obtained from the standard Schauder system via the Gram–Schmidt orthonormalization process. Thus, the first two Franklin functions are given by the formulas

$$f_0(t) = 1, \quad f_1(t) = \sqrt{3}(2t-1), \quad \text{for } t \in [0,1],$$

and, for $n = 2^m + r$ with $m = 0, 1, \dots$ and $r = 1, \dots, 2^m$, each f_n is both continuous on $[0,1]$ and linear on each dyadic interval $[t^n_{i-1}, t^n_i]$, where

$$t^n_i = i/2^{m+1}, \quad i = 0, \dots, 2r, \quad \text{and} \quad t^n_i = (i-r)/2^m, \quad i = 2r+1, \dots, 2^m + r.$$

A fundamental role in the study of the unconditional basis property of $\mathcal{F}$ is played by the Paley function

$$P(f) = P(f,x) = \left\{\sum_{n=0}^{\infty}[\langle f, f_n\rangle f_n(x)]^2\right\}^{1/2}, \quad f \in L^1(I),$$

and the function

$$T_\varepsilon(f) = T_\varepsilon(f,x) = \sum_{n=0}^{\infty}\varepsilon_n\langle f, f_n\rangle f_n(x), \quad f \in L^1(I),$$

where $\varepsilon_n = \pm 1$ for $n = 0, 1, \dots$ is a sequence of "signs".

The main results of this article are the following theorems.

Theorem 1. *Let $1 < p_- \le p_+ < \infty$. The following statements are equivalent:*

(1) $p(\cdot) \in \mathcal{A}$;
(2) *The Franklin system forms an unconditional basis in $L^{p(\cdot)}(I)$;*
(3) *There exist the constants $C_1, C_2 > 0$ such that*

$$C_1\|f\|_{p(\cdot)} \le \|P(f)\|_{p(\cdot)} \le C_2\|f\|_{p(\cdot)}$$

for every $f \in L^{p(\cdot)}(I)$;

(4) *There exist the constants $C_1, C_2 > 0$ such that*

$$C_1\|f\|_{p(\cdot)} \le \|T_\varepsilon(f)\|_{p(\cdot)} \le C_2\|f\|_{p(\cdot)}$$

for every $f \in L^{p(\cdot)}(I)$ and every sequence $\varepsilon = \{\varepsilon_n\}_{n=0}^{\infty}$ $\varepsilon_n = \pm 1$.

Theorem 2. *Let $1 \le p_- \le p_+ < \infty$. The following statements are equivalent:*

(1) $p(\cdot) \in \mathcal{A}$;
(2) *The Franklin system forms a basis in $L^{p(\cdot)}(I)$.*

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