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ON THE CONVOLUTION PRODUCT OF AN INTEGRABLE FUNCTION WITH A FUNCTION OF GENERALIZED BOUNDED VARIATION

HARDEEPBHAI J. KHACHAR[1*] AND RAJENDRA G. VYAS[2]

Abstract. The convolution property of the class of integrable functions with the class of functions of generalized bounded variation is studied for functions of one variable and the result is extended to functions of several variables.

1. Introduction

In 2002, Akhobadze [1] studied various properties of the $B\Lambda(p(n) \uparrow p, \varphi)$ class, a generalization of the class of functions of bounded variation. In 2006, it was shown that various generalized classes of bounded variation can be regarded as non-unital modules over the ring of integrable functions under convolution [3]. Here, we generalize this property to the Akhobadze class and extend the result to functions of several variables.

In the sequel, we will assume that:

- φ is an increasing function defined on $\mathbb{N}$ such that $\varphi \geq 2$ and $\lim_{n\to\infty} \varphi(n) = +\infty$.
- $\mathbb{T} := [0, 2\pi)$ and $\overline{\mathbb{T}}^N = \overline{\mathbb{T}} \times \overline{\mathbb{T}} \times \cdots \overline{\mathbb{T}}$ (N times), $N \geq 2$.
- If f is a function of one variable, then
$$||f||_1 = \int\limits_{\overline{\mathbb{T}}} |f(x)|dx.$$
- If f is a function of two variables, then
$$||f||_1 = \iint\limits_{\overline{\mathbb{T}}^2} |f(x,y)|dxdy.$$

2. Result for Functions of One Variable

Definition 2.1 ([1, Definition 1])**.** Let f be a 2π periodic measurable function. Let $p(n)$ be an increasing sequence such that $1 \leq p(n) \uparrow p$ for $1 \leq p \leq \infty$. Then $f \in B\Lambda(p(n) \uparrow p, \varphi, \overline{\mathbb{T}})$ if

$$\Lambda(f, p(n) \uparrow p, \varphi, \overline{\mathbb{T}}) := \sup_{m\geq 1} \sup_{h \geq \frac{1}{\varphi(m)}} \left\{ \frac{1}{h} \int\limits_{\overline{\mathbb{T}}} \mid f(x+h) - f(x) \mid^{p(m)} dx \right\}^{\frac{1}{p(m)}} < \infty.$$

Note that if $f \in B\Lambda(p(n) \uparrow p, \varphi, \overline{\mathbb{T}})$, then f is an essentially bounded function over $\overline{\mathbb{T}}$ [1, Corollary 1].

Definition 2.2 ([4, p. 36])**.** If f and g are two integrable functions on $\overline{\mathbb{T}}$, then convolution product of f and g, denoted by $f * g$, is given by

$$f * g(x) = \frac{1}{2\pi} \int\limits_{\overline{\mathbb{T}}} f(t)g(x-t)dt; \ \forall x \in \overline{\mathbb{T}}.$$

Theorem 2.1. *If $f \in L^1(\overline{\mathbb{T}})$ and $g \in B\Lambda(p(n) \uparrow p, \varphi, \overline{\mathbb{T}})$, then $f * g \in B\Lambda(p(n) \uparrow p, \varphi, \overline{\mathbb{T}})$.*

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*Corresponding author.

Proof. Let $m \in \mathbb{N}$ and $h > 0$ be given. Also, let $p(n)$ be an increasing sequence such that $1 \leq p(n) \uparrow p$ for $1 \leq p \leq \infty$, then applying Hölder's inequality, we have

$$\begin{aligned}
&\frac{1}{h} \int_{\overline{\mathbb{T}}} |f * g(x+h) - f * g(x)|^{p(m)}\, dx \\
&\quad = \frac{1}{h} \int_{\overline{\mathbb{T}}} \left| \frac{1}{2\pi} \int_{\overline{\mathbb{T}}} f(y)(g(x+h-y) - g(x-y)) dy \right|^{p(m)} dx \\
&\quad \leq \frac{1}{(2\pi)^{p(m)} h} \int_{\overline{\mathbb{T}}} \left(\int_{\overline{\mathbb{T}}} |f(y)|^{1-\frac{1}{p(m)}} |f(y)|^{\frac{1}{p(m)}} |g(x+h-y) - g(x-y)|\, dy \right)^{p(m)} dx \\
&\quad \leq \frac{||f||_1^{p(m)-1}}{(2\pi)^{p(m)} h} \int_{\overline{\mathbb{T}}} \left(|f(y)| \int_{\overline{\mathbb{T}}} |g(x+h-y) - g(x-y)|^{p(m)}\, dx \right) dy.
\end{aligned}$$

Now, taking supremum over $h \geq \frac{1}{\varphi(m)}$, we have

$$\sup_{h \geq \frac{1}{\varphi(m)}} \frac{1}{h} \int_{\overline{\mathbb{T}}} |f * g(x+h) - f * g(x)|^{p(m)}\, dx \leq \left(\frac{||f||_1}{2\pi} \Lambda(g, p(n) \uparrow p, \varphi, \overline{\mathbb{T}}) \right)^{p(m)}.$$

Thus, taking the $p(m)$th root and supremum over $m \geq 1$, we get $f * g \in B\Lambda(p(n) \uparrow p, \varphi, \overline{\mathbb{T}})$. □

Remark 2.1. Since $L^1(\overline{\mathbb{T}})$ is a ring under the convolution product and in view of the previous result, $B\Lambda(p(n) \uparrow p, \varphi, \overline{\mathbb{T}})$ can be regarded as a non-unital module over the ring $L^1(\overline{\mathbb{T}})$.

3. Result for Functions of Two Variables

Definition 3.1 ([2, Definition 3.1])**.** Let f be a measurable function and 2π periodic in both variables. Let $p(n)$ and $q(n)$ be increasing sequences such that $p(n) \leq q(n)$, $1 \leq p(n) \uparrow p$ for $1 \leq p \leq \infty$ and $1 \leq q(n) \uparrow q$ for $1 \leq q \leq \infty$. Then $f \in B\Lambda\left(p(n) \uparrow p, q(n) \uparrow q, \varphi, \overline{\mathbb{T}}^2\right)$ if

$$\begin{aligned}
&\Lambda(f, p(n) \uparrow p, q(n) \uparrow q, \varphi, \overline{\mathbb{T}}^2) \\
&\qquad := \sup_{m \geq 1} \sup_{hk \geq \frac{1}{\varphi(m)^2}} \left\{ \frac{1}{k} \int_{\overline{\mathbb{T}}} \left(\frac{1}{h} \int_{\overline{\mathbb{T}}} |\, \Delta f(x, y; h, k)\, |^{p(m)}\, dx \right)^{\frac{q(m)}{p(m)}} dy \right\}^{\frac{1}{q(m)}} < \infty,
\end{aligned}$$

where

$$\Delta f(x, y; h, k) = f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x, y).$$

Definition 3.2. If f and g are integrable functions on $\overline{\mathbb{T}}^2$, then the convolution product of f and g, denoted by $f * g(x, y)$, is defined by

$$f * g(x, y) = \frac{1}{4\pi^2} \iint_{\overline{\mathbb{T}}^2} f(u, v) g(x-u, y-v) du dv.$$

Theorem 3.1. *If* $f \in L^1\left(\overline{\mathbb{T}}^2\right)$ *and* $g \in B\Lambda\left(p(n) \uparrow p, q(n) \uparrow q, \varphi, \overline{\mathbb{T}}^2\right) \cap L^1\left(\overline{\mathbb{T}}^2\right)$, *then*

$$f * g \in B\Lambda\left(p(n) \uparrow p, q(n) \uparrow q, \varphi, \overline{\mathbb{T}}^2\right) \cap L^1\left(\overline{\mathbb{T}}^2\right).$$

Proof. Clearly, $f * g \in L^1\left(\overline{\mathbb{T}}^2\right)$. Let $m \in \mathbb{N}$ and $h, k > 0$ be given. Let $p(n)$ and $q(n)$ be increasing sequences such that $1 \leq p(n) \uparrow p$ for $1 \leq p \leq \infty, 1 \leq q(n) \uparrow q$ for $1 \leq q \leq \infty$ and $p(n) \leq q(n)$. Also, denote $\Delta f(x,y;h,k)=f(x+h,y+k)-f(x+h,y)-f(x,y+k)+f(x,y)$. Then applying Hölder's inequality twice, we have

$$\begin{aligned}
&\frac{1}{k}\int\limits_{\overline{\mathbb{T}}}\left(\frac{1}{h}\int\limits_{\overline{\mathbb{T}}}|\Delta\left(f*g(x,y;h,k)\right)|^{p(m)}\,dx\right)^{\frac{q(m)}{p(m)}}dy\\
&=\frac{1}{k}\int\limits_{\overline{\mathbb{T}}}\left(\frac{1}{h}\int\limits_{\overline{\mathbb{T}}}\left|\frac{1}{4\pi^2}\iint\limits_{\overline{\mathbb{T}}^2}f(u,v)\Delta g(x-u,y-v;h,k)dudv\right|^{p(m)}dx\right)^{\frac{q(m)}{p(m)}}dy\\
&\le\frac{||f||_1^{q(m)\left(1-\frac{1}{p(m)}\right)}}{(4\pi^2)^{q(m)}k}\int\limits_{\overline{\mathbb{T}}}\left\{\int\int\limits_{\overline{\mathbb{T}}^2}(|f(u,v)|)\left(\frac{1}{h}\int\limits_{\overline{\mathbb{T}}}|\Delta g(x-u,y-v;h,k)|^{p(m)}\,dx\right)dudv\right\}^{\frac{q(m)}{p(m)}}dy\\
&\le\frac{||f||_1^{q(m)-1}}{(4\pi^2)^{q(m)}}\int\int\limits_{\overline{\mathbb{T}}^2}\left\{|f(u,v)|\,\frac{1}{k}\int\limits_{\overline{\mathbb{T}}}\left(\frac{1}{h}\int\limits_{\overline{\mathbb{T}}}|\Delta g(x-u,y-v;h,k)|^{p(m)}\,dx\right)^{\frac{q(m)}{p(m)}}dy\right\}dudv.
\end{aligned}$$

Now, taking supremum over $hk\ge\frac{1}{\varphi(m)^2}$, we get

$$\begin{aligned}
&\sup_{hk\ge\frac{1}{\varphi(m)^2}}\frac{1}{k}\int\limits_{\overline{\mathbb{T}}}\left(\frac{1}{h}\int\limits_{\overline{\mathbb{T}}}|\Delta\left(f*g(x,y;h,k)\right)|^{p(m)}\,dx\right)^{\frac{q(m)}{p(m)}}dy\\
&\qquad\qquad\le\left(\frac{||f||_1}{4\pi^2}\Lambda\left(g,p(n)\uparrow p,q(n)\uparrow q,\varphi,\overline{\mathbb{T}}^2\right)\right)^{q(m)}.
\end{aligned}$$

Thus, taking the $q(m)$th root and supremum over $m\ge1$, we obtain

$$f*g\in B\Lambda\left(p(n)\uparrow p,q(n)\uparrow q,\varphi,\overline{\mathbb{T}}^2\right). \qquad\square$$

Remark 3.1. Since $L^1\left(\overline{\mathbb{T}}^2\right)$ is a ring under the convolution product and in view of the previous result, the class $B\Lambda\left(p(n)\uparrow p,q(n)\uparrow q,\varphi,\overline{\mathbb{T}}^2\right)\cap L^1\left(\overline{\mathbb{T}}^2\right)$ can be regarded as a non-unital module over the ring $L^1\left(\overline{\mathbb{T}}^2\right)$.

4. Result for Functions of Several Variables

Definition 4.1 ([2, Definition 4.1])**.** Let a measurable function f be 2π periodic in all the variables. Let $p_j(n)$ be increasing sequences such that $p_{j+1}(n)\le p_j(n)$, $1\le p_j(n)\uparrow q_j$ and $1\le q_j\le\infty$ for $j=1,2,\dots,N$, where $N\in\mathbb{N}$. Then $f\in B\Lambda(p_1(n)\uparrow q_1,\dots,p_N(n)\uparrow q_N,\varphi,\overline{\mathbb{T}}^N)$ if

$$\begin{aligned}
&\Lambda\left(f,p_1(n)\uparrow q_1,\dots,p_N(n)\uparrow q_N,\varphi,\overline{\mathbb{T}}^N\right)\\
&:=\sup_{m\ge1}\sup_{\prod_1^N h_j\ge\frac{1}{\varphi(m)^N}}\left\{\frac{1}{h_N}\int\limits_{\overline{\mathbb{T}}}\cdots\left(\frac{1}{h_1}\int\limits_{\overline{\mathbb{T}}}|\,\Delta f(x_1,\dots,x_N;h_1,\dots,h_N)\,|^{p_1(m)}dx_1\right)^{\frac{p_2(m)}{p_1(m)}}\dots dx_N\right\}^{\frac{1}{p_N(m)}}<\infty,
\end{aligned}$$

where

$$\begin{aligned}
&\Delta f(x_1;h_1)=f(x_1+h_1)-f(x_1),\\
&\Delta f(x_1,x_2;h_1,h_2)=f(x_1+h_1,x_2+h_2)-f(x_1,x_2+h_2)-f(x_1+h_1,x_2)+f(x_1,x_2),\\
&\vdots\\
&\Delta f(x_1,\dots,x_N;h_1,\dots,h_N)=\sum_{u_1=0}^{1}\cdots\sum_{u_N=0}^{1}(-1)^{u_1+u_2+\dots+u_N}f(x_1+u_1h_1,\dots,x_N+u_Nh_N).
\end{aligned}$$

Definition 4.2. If f and g are integrable functions on $\overline{\mathbb{T}}^N$, then the convolution product of f and g, denoted by $f * g(x_1, x_2 \ldots x_N)$, is defined by

$$f * g(x_1, x_2, \ldots, x_N) = \frac{1}{(2\pi)^N} \underset{\overline{\mathbb{T}}^N}{\int \cdots \int} f(u_1, \ldots, u_N) g(x_1 - u_1, \ldots, x_N - u_N) du_1 \ldots du_N.$$

The result of functions of two variables can be extended to the functions of N variables, as below.

Theorem 4.1. *If* $f \in L^1\left(\overline{\mathbb{T}}^N\right)$ *and* $g \in B\Lambda\left(p_1(n) \uparrow q_1, \ldots, p_N(n) \uparrow q_N, \varphi, \overline{\mathbb{T}}^N\right) \cap L^1\left(\overline{\mathbb{T}}^N\right)$, *then*

$$f * g \in B\Lambda\left(p_1(n) \uparrow q_1, \ldots, p_N(n) \uparrow q_N, \varphi, \overline{\mathbb{T}}^N\right) \cap L^1\left(\overline{\mathbb{T}}^N\right).$$

Remark 4.1. Since $L^1\left(\overline{\mathbb{T}}^N\right)$ is a ring under the convolution product and in view of the previous result, the class $B\Lambda\left(p_1(n) \uparrow q_1, \ldots, p_N(n) \uparrow q_N, \varphi, \overline{\mathbb{T}}^N\right) \cap L^1\left(\overline{\mathbb{T}}^N\right)$ can be regarded as a non-unital module over the ring $L^1\left(\overline{\mathbb{T}}^N\right)$.

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[1]Department of Mathematics, Government Science College, Bhilad 396105, Gujarat, India

[2]Department of Mathematics, Faculty of Science, The Maharaja Sayajirao University of Baroda, Vadodara 390002, Gujarat, India

Email address: hardeep1996k@gmail.com

Email address: drrgvyas@yahoo.com