

ON UNIVERSAL FUNCTIONS WITH RESPECT TO THE CLASSICAL SYSTEMS

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Abstract. This article considers the problem of the existence of functions that are, in a certain sense, universal for various function classes with respect to the trigonometric system and the Walsh system, as well as universal triads for these systems.

1. INTRODUCTION

We need some standard notation.

Let $|E|$ be the Lebesgue measure of a measurable set $E \subseteq [a, b]$ ($[a, b] = [-\pi, \pi]$ or $[0, 1]$).

Let $L^p(E)$, $p > 0$ be the class of all measurable functions f on E with a finite integral $\int_E |f(x)|^p dx$ and

$L^0(E)$ be the class of all almost everywhere (a.e.) finite Lebesgue measurable functions on E , with almost everywhere convergence.

We denote by $M(E)$ the class of all Lebesgue measurable functions on E with convergence understood in the almost everywhere sense.

The sequence of functions $\{f_k(x)\}_{k=1}^\infty \subset M(E)$ is called convergent to a function $f \in M(E)$ in $M(E)$, if $\{f_k(x)\}_{k=1}^\infty$ converges to $f(x)$ almost everywhere on E .

The sequence of functions $\{f_k\}_{k=1}^\infty \subset L^p(E)$ is called convergent to a function $f \in L^p(E)$ in $L^p(E)$, if it converges to f in the metric $L^p(E)$, that is,

$$\lim_{k \rightarrow \infty} \int_E |f_k(x) - f(x)|^p dx = 0.$$

Let

$$a_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx, \quad b_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx, \quad k \geq 0,$$

be the Fourier coefficients of the function $f \in L^1[-\pi, \pi]$ in the trigonometric system and let

$$S_N(x, f) = \frac{a_0(f)}{2} + \sum_{k=1}^{N-1} a_k(f) \cos kx + b_k(f) \sin kx$$

be the N -th partial sum of the function f in the Fourier series.

Let $\Phi := \{\varphi_k(x)\}_{k=0}^\infty$ be an orthonormal system on $[a, b]$ and let $f \in L^1[a, b]$.

We denote by $c_k(f)$ the Fourier coefficients of the function $f \in L^1[a, b]$ in the system Φ , that is,

$$c_k(f) = \int_a^b f(x) \varphi_k(x) dx, \quad S_n(x, f) = \sum_{k=0}^{n-1} c_k(f) \varphi_k(x).$$

Below, S will denote one of the spaces $M(E)$ or $L^p(E)$, $p \geq 0$.

This article is a continuation of author's studies in establishing the existence and describing the structure of functions that are, in a certain sense, universal for various classes of functions in the trigonometric and Walsh systems, as well as universal triads for these systems (see [1–15]).

Definition 1.1. A series $\sum_{k=0}^{\infty} f_k(x)$ ($f_k \in S$, $k = 0, 1, 2, \dots$) is called universal in S (universal **in the usual sense**) if for each function $f \in S$ there exists an increasing sequence of natural numbers $\{N_m\}_{m=1}^{\infty} \nearrow$ such that the corresponding subsequence of partial sums $\{\sum_{k=0}^{N_m} f_k(x)\}_{m=1}^{\infty}$ converges to f in S .

The first example is due to Birkhoff [16, (1929)], who proved the existence of an entire function $f(z)$ with the property that for any entire function $g(z)$ and every $r > 0$, there exists a subsequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ such that $\{f(z + n_k)\}_{k=1}^{\infty}$ converges to $g(z)$ uniformly on the disc $\{z \in C : |z| \leq r\}$.

In 1935, Marcinkiewicz [17] proved that for any (converging to zero) sequence $h_n \rightarrow 0$ there exists a continuous function $F(x)$ on $[0, 1]$ with the property: for every measurable function $f(x)$ on $[0, 1]$, there is an increasing subsequence $n_k \nearrow^{\infty}$ of natural numbers such that

$$\frac{F(x + h_{n_k}) - F(x)}{h_{n_k}} \rightarrow g(x),$$

as $k \rightarrow \infty$ almost everywhere on $[0, 1]$ (see also [18]).

In 1987, Grosse-Erdman [22] proved the existence of an infinitely differentiable function $h(x)$ with universal Taylor expansion. Namely, there exists an infinitely differentiable function $h(x)$ on $(-\infty, \infty)$ with $h(0) = 0$, such that for every function $f(x) \in C(-\infty, \infty)$ with $f(0) = 0$ and for any number $r > 0$, there exists an increasing subsequence $\{m_q\} \nearrow \infty$ of natural numbers such that the corresponding subsequence of partial sums

$$\sum_{k=1}^{m_q} \frac{h^{(k)}(0)}{k!} x^k$$

converges to $f(x)$ uniformly on the interval $(-r, r)$.

From this it follows that the Taylor series of infinitely differentiable function $h(x)$ also is universal series in $M[-\pi, \pi]$.

The notion of a universal series in $M[-\pi, \pi]$ both in the trigonometric system and in general orthonormal systems, is due to Menshov [23] and Talalyan [24] (see also [25–29]).

Important results in this direction were obtained by these authors and their students.

The first construction of a universal trigonometric series in the class of all measurable functions, in a sense of almost everywhere convergence, was given by Menshov. He proved the following

Theorem 1.1. *There is a trigonometric series*

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx, \quad |a_k| + |b_k| \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty$$

with the following property: for every measurable function $f(x)$ on $[-\pi, \pi]$, there is an increasing sequence $\{m_q\}$ of natural numbers such that the subsequence of partial sums

$$\frac{a_0}{2} + \sum_{k=1}^{m_q} a_k \cos kx + b_k \sin kx,$$

converges to $f(x)$ as $q \rightarrow \infty$ almost everywhere on $[-\pi, \pi]$ (that is, this series is universal in $M[-\pi, \pi]$).

Remark 1.1. As noted above, there is an infinitely differentiable function with universal Taylor series, **but there is no function** $U \in L^1[-\pi, \pi]$, whose Fourier series in the trigonometric system is universal in $M[-\pi, \pi]$.

Otherwise, if there were a function $U \in L^1[-\pi, \pi]$, whose Fourier series in the trigonometric system is universal in $M[-\pi, \pi]$, then for the function $f(x) = 2U(x)$ one could find a growing subsequence of natural numbers $\{N_m\} \nearrow \infty$ such that

$$S_{N_m}(x, U) = \frac{a_0(U)}{2} + \sum_{k=1}^{N_m-1} a_k(U) \cos kx + b_k(U) \sin kx$$

converges to $2U(x)$ as $q \rightarrow \infty$ almost everywhere on $[-\pi, \pi]$.

On the other hand, from the **well-known Kolmogorov theorem [30]** (that the Fourier series in the trigonometric system of any integrable function converges to it in $L^p[-\pi, \pi]$, $p \in (0, 1)$), it follows that $\lim_{m \rightarrow \infty} \int_{-\pi}^{\pi} |S_{N_m}(x, U) - U(x)|^p dx = 0$, and therefore for some subsequence $\{N_{m_q}\}_{q=1}^{\infty}$ of the sequence $\{N_m\}_{m=1}^{\infty}$ we find that

$$S_{N_{m_q}}(x, U) = \frac{a_0(U)}{2} + \sum_{k=1}^{N_{m_q}-1} a_k(U) \cos kx + b_k(U) \sin kx$$

converges to $U(x)$ as $q \rightarrow \infty$ almost everywhere on $[-\pi, \pi]$.

Hence we see that $U(x) = 2U(x)$ almost everywhere on $[-\pi, \pi]$.

This contradiction shows that there does not exist a function $U \in L^1[-\pi, \pi]$ whose Fourier series is universal in the trigonometric system for the class $M[-\pi, \pi]$.

Remark 1.2. A similar analysis shows that there is no integrable function U , whose Fourier series, in the (Walsh, Haar, Franklin and Vilenkin) system is universal in $M[0, 1]$.

The above considerations suggest the following question, the answer to which is currently unknown.

Question 1.1. Does there exist an orthonormal system $\{\varphi_k(x)\}_{k=1}^{\infty}$ of bounded functions and a function $U \in L^1[0, 1]$ whose Fourier series in the system $\{\varphi_k(x)\}_{k=1}^{\infty}$ is universal in $M[0, 1]$?

Despite the fact that, as Menshov proved, there exists a trigonometric series that is universal in $M[-\pi, \pi]$ and (as noted above) there is no function $U \in L^1[-\pi, \pi]$ whose Fourier series in the trigonometric system is universal in $M[-\pi, \pi]$, we have nevertheless succeeded in constructing an integrable function U and proving that after an appropriate choice of signs $\{\delta_k; \delta_k = \pm 1\}_{k=0}^{\infty}$ for its Fourier coefficients, the resulting series $\frac{a_0(U)}{2} + \sum_{k=1}^{\infty} \delta_k(a_k(U) \cos kx + b_k(U) \sin kx)$ is universal in $M[-\pi, \pi]$.

Before formulating the main result of the paper, we provide the necessary definitions.

2. DEFINITIONS AND AUXILIARY STATEMENTS

Let $\Phi := \{\varphi_k(x)\}_{k=0}^{\infty}$ be an orthonormal system on $[a, b]$.

Definition 2.1. We say that a function $U \in L^1[a, b]$ and a sequence of signs $\delta = \{\delta_k = \pm 1, k = 0, 1, 2, \dots\}$ form **universal pairs: (U, δ)** for the space S with respect to this system $\{\varphi_k(x)$ in sense of universal series, if the series $\sum_{k=0}^{\infty} \delta_k c_k(f) \varphi_k(x)$ is universal in S .

Definition 2.2. We say that a function $U \in L^1[a, b]$ and a measurable set $E \subset [a, b]$ form **universal pairs: (U, E)** with respect to this system $\{\varphi_k(x)\}_{k=0}^{\infty}$ in sense of modification, if for each function $f \in L^1[a, b]$, there exists a function $g \in L^1[a, b]$ that coincides with f on E , and such that

$$|c_k(g)| = |c_k(U)|, \quad k = 0, 1, 2, \dots$$

Definition 2.3. We say that a function $U \in L^1[a, b]$, a sequence of signs $\delta = \{\delta_k = \pm 1, k = 0, 1, 2, \dots\}$ and a measurable set $E \subset [a, b]$ form **universal triads: (U, δ, E)** for the space S with respect to this system $\{\varphi_k(x)\}_{k=0}^{\infty}$ if the series $\sum_{k=0}^{\infty} \delta_k c_k(f) \varphi_k(x)$ is universal in S , and for each function $f \in L^1[a, b]$, there exists a function $g \in L^1[a, b]$ that coincides with f on E , and such that

$$|c_k(g)| = |c_k(U)| \quad k = 1, 2, \dots$$

Definition 2.4. We say that a function $U \in L^1[a, b]$ is:

1) **universal** for the space S with respect to a system $\{\varphi_k(x)\}_{k=0}^{\infty}$ if the Fourier series of the function U with respect to this system is universal in S ;

2) **conditional universal** for the space S with respect to a system $\{\varphi_k(x)\}_{k=0}^{\infty}$ if there exists a sequence of signs $\delta = \{\delta_k = \pm 1, k = 0, 1, 2, \dots\}$ such that the series $\sum_{k=0}^{\infty} \delta_k c_k(U) \varphi_k(x)$ is universal in S ;

3) **quasiuniversal** for the space S with respect to a system $\{\varphi_k(x)\}$ if there exists a sequence of signs $\delta = \{\delta_k = \pm 1, k = 0, 1, 2, \dots\}$ with $d_{\Lambda}(\Omega^+) = 1$ such that the series $\sum_{k=0}^{\infty} \delta_k c_k(U) \varphi_k(x)$ is

universal in S , where

$$d_{\Lambda}(\Omega^+) := \lim_{n \rightarrow \infty} \sup \frac{\#(\Omega^+ \cap [0, n])}{\#(\Lambda \cap [0, n])} = \lim_{n \rightarrow \infty} \sup \frac{\#(k \in [0, n], \delta_k = 1)}{\#(\text{spec}(U) \cap [0, n])}$$

is the upper density of the subset $\Omega^+ = \{k \in N, \delta_k = 1\}$ with respect to the set $\Lambda = \text{spec}(U) = \{k \in N, c_k(U) \neq 0\}$,

and $\#(*)$ is the cardinality of the finite set, where $\#(E)$ is a number of elements of a finite set E ;

4) **almost universal** for the space S with respect to the system $\{\varphi_k(x)\}$ if there exists a sequence of signs $\delta = \{\delta_k = \pm 1, k = 0, 1, 2, \dots\}$ with $\rho_{\Lambda}(\Omega) = 0$ such that the series $\sum_{k=0}^{\infty} \delta_k c_k(U) \varphi_k(x)$ is universal in S , where

$$\rho_{\Lambda}(\Omega) := \lim_{n \rightarrow \infty} \frac{\#(\Omega \cap [0, n])}{\#(\Lambda \cap [0, n])} = \lim_{n \rightarrow \infty} \frac{\#(k \in [0, n], \delta_k = -1)}{\#(\text{spec}(U) \cap [0, n])}$$

is the density of the subset $\Omega = \{k \in N, \delta_k = -1\}$ with respect to the set $\Lambda = \text{spec}(U) = \{k \in N, c_k(U) \neq 0\}$;

5) **universal in a sense of signs** if its Fourier series $\sum_{k=0}^{\infty} c_k(U) \varphi_k(x)$ in the system $\{\varphi_k(x)\}_{k=0}^{\infty}$ is universal in S in a sense of signs: that is, for each function $f \in S$ one can find a sequence of signs $\delta = \{\delta_k = \pm 1, k = 0, 1, 2, \dots\}$ such that the series $\sum_{k=1}^{\infty} \delta_k c_k(U) \varphi_k(x)$ converges to f in S ;

6) **universal in a sense of rearrangements** if its Fourier series $\sum_{k=0}^{\infty} c_k(U) \varphi_k(x)$ in the system $\{\varphi_k(x)\}_{k=0}^{\infty}$ is universal in S in a sense of rearrangements: that is, for each function $f \in S$, one can find $\{\sigma(k)\}_{k=1}^{\infty}$, some permutation of the natural numbers, such that the series $\sum_{k=1}^{\infty} f_{\sigma(k)}(x)$ converges to f in S .

The Walsh system, which is an extension of the Rademacher system, can be obtained as follows. Let r be the periodic function with period at least 1, defined on $[0, 1)$ by

$$r(x) = \chi_{[0, 1/2)}(x) - \chi_{[1/2, 1)}(x),$$

where $\chi_E(x)$ is the characteristic function of the set E .

The Rademacher system, $R = r_n : n = 0, 1, \dots$, is defined by the conditions

$$r_n(x) = r(2^n x), \quad \forall x \in R, \quad n = 0, 1, \dots,$$

and, in the ordering employed by Payley (see [31, 32]), the n -th element of the Walsh system $\{W_n(x)\}_{n=0}^{\infty}$ is given by the formula

$$W_n(x) = \prod_{k=0}^{\infty} (r_k(x))^{\theta_k(n)},$$

where $\sum_{k=0}^{\infty} \theta_k(n) 2^k$ is the unique binary expansion of n , with each $\theta_k(n)$ either 0 or 1.

Note that (see [33]) the Walsh system is a basis for all $L^p[0, 1]$ $p \in (1, \infty)$, i.e., every function $f(x)$ is $L^p[0, 1]$ $p \in (1, \infty)$ uniquely representable by the series $\sum_{k=0}^{\infty} b_k W_k(x)$ in the Walsh system in the norm $L^p[0, 1]$ convergent to $f(x)$. It is easy to see that for all $k \in \mathbb{N} \cup \{0\}$,

$$b_k = c_k(f) = \int_0^1 f(x) W_k(x) dx.$$

We will use the following properties:

for all $k \in \mathbb{N} \cup \{0\}$,

$$\sum_{j=0}^{2^k-1} W_j(x) = \begin{cases} 2^k, & \text{if } x \in [0, \frac{1}{2^k}), \\ 0, & \text{if } x \in [\frac{1}{2^k}, 1) \end{cases}$$

and therefore,

$$\int_0^1 \left| \sum_{j=0}^{2^k-1} V_j(x) \right| dx = 1, \quad \forall k \in \mathbb{N} \cup \{0\}.$$

We also use (in the proof of Theorem 3.7) the following lemma, previously proved in [4].

Lemma 2.1. *Let numbers $n_0 \in N$, ($N_0 = 2^{n_0}$), $0 < p_1 < p_2 < 1$, $\varepsilon \in (0, 1)$ and the polynomial $W(x)$ in the Walsh system $\{W_k(x)\}_{k=0}^\infty$ be given. Then there exist the polynomials $U(x)$ and $B(x)$ in the Walsh system of the following form:*

$$U(x) = \sum_{k=2^{n_0}}^{2^n-1} b_k W_k(x), \quad B(x) = \sum_{k=N_0}^{N-1} \varepsilon_k b_k W_k(x), \quad N_0 = 2^{n_0}, \quad N = 2^n,$$

which satisfy the following conditions:

- 1) $0 < b_{k+1} \leq b_k < \varepsilon, \quad \varepsilon_k = \pm 1, \quad \forall k \in [2^{n_0}, 2^n) = [N_0, N);$
- 2) $B(x) = W(x) \quad \forall x \in G, \quad |G| \geq 1 - \varepsilon - 2^{-n_0};$
- 3) $U(x) \chi_{[2^{-n_0}, 1)}(x) = 0;$
- 4) $\max_{m \in [N_0, N)} \int_0^1 \left| \sum_{k=N_0}^m \varepsilon_k b_k W_k(x) \right|^p dx < 4 \int_0^1 |W(x)|^p dx, \quad \forall p \in (p_1, p_2);$
- 5) $\max_{m \in [N_0, N)} \int_0^1 \left| \sum_{k=N_0}^m \varepsilon_k b_k W_k(x) \right| dx < 5 \int_0^1 |W(x)| dx;$
- 6) $\int_0^1 |W(x) - B(x)|^p dx < \varepsilon \quad \forall p \in (p_1, p_2);$
- 7) $\max_{m \in [N_0, N)} \int_0^1 \left| \sum_{k=N_0}^m b_k W_k(x) \right| dx < \varepsilon.$

3. THEOREMS

In recent years, a number of results have been obtained concerning both the existence and structural characterization of functions (so-called universal functions) whose Fourier series, relative to a given classical system, are universal in a certain sense for various classes of functions.

In particular, in [2, 3, 11], and [15], the following results were obtained.

Theorem 3.1. *Let $\{W_k(x)\}_{k=0}^\infty$ be a Walsh system. There exists an integrable function U with monotonically decreasing Fourier–Walsh coefficients, which is universal in a sense of signs with respect to the Walsh system for the spaces $L^p[0, 1]$ for all $p \in (0, 1)$. In addition, the Fourier–Walsh series of the function U converges to it in the norm of $L^1[0, 1]$.*

Theorem 3.2. *There exist an integrable function $U \in L^1[-\pi, \pi]$ and a sequence of signs $\delta = \{\delta_k = \pm 1, k = 0, 1, 2, \dots\}$, which form **universal pairs**: (U, δ) for the space $L^p[-\pi, \pi]$, $p \in (0, 1)$ with respect to the trigonometric system.*

Theorem 3.3. *For any $p \in (0, 1)$ there exists an integrable function U that is both conditional universal and universal in a sense of signs with respect to the trigonometric system for the class $L^p[-\pi, \pi]$.*

Theorem 3.4. *There exists an integrable function U , whose Fourier series in the Walsh system converges to it in the norm of $L^1[0, 1]$ with monotonically decreasing Fourier–Walsh coefficients, which is universal in a sense of signs with respect to the Walsh system for the space $L^0[0, 1]$. In addition, the Fourier–Walsh series of the function U converges to it in the norm of $L^1[0, 1]$.*

Remark 3.1. It should be noted that there is no integrable function U that is universal in a sense of signs with respect to the Walsh system for the space $M[0, 1]$, in the case of almost everywhere convergence. However, there exists an integrable function U with monotonically decreasing Fourier–Walsh coefficients that is universal in a sense of signs with respect to the Walsh system for the space $M[0, 1]$ in the case of convergence in measure; that is, for each function, one can find a sequence of signs $\delta = \{\delta_k = \pm 1, k = 0, 1, 2, \dots\}$ such that the series $\sum_{k=0}^{\infty} \delta_k c_k(U) W_k(x)$ converges to f in measure.

The following theorems are true.

Theorem 3.5. *There exist an integrable function $U \in L^1[-\pi, \pi]$ and a sequence of signs $\delta = \{\delta_k = \pm 1, k = 0, 1, 2, \dots\}$, and a measurable set $E \subset [a, b]$ that form **universal triads**: (U, δ, E) for the space $L^0[-\pi, \pi]$ with respect to the trigonometric system.*

Theorem 3.6. *There exist an integrable function $U \in L^1[0, 1]$ and a sequence of signs $\delta = \{\delta_k = \pm 1, k = 0, 1, 2, \dots\}$, and a measurable set $E \subset [a, b]$ that form **universal triads**: (U, δ, E) for the space $L^0[0, 1]$ with respect to the Walsh system.*

Moreover, the following statement holds.

Theorem 3.7. *For any $\varepsilon \in (0, 1)$, there exists a function $U \in L^1[0, 1]$ with $\text{supp}(U) \subset [0, \varepsilon]$, which has the following properties:*

- a) *the Fourier coefficients of the function U in the Walsh system are positive and monotonically decreasing;*
- b) *the Fourier–Walsh series of the function U converges to it in the norm of $L^1[0, 1]$;*
- c) *the function U is conditional universal for the class $L^0[0, 1]$ with respect to the Walsh system;*
- d) *for any $\delta \in (0, 1)$, there exists a measurable set $E \subset [0, 1]$ with $|E| > 1 - \delta$, so for each function $g \in L^1[0, 1]$ one can find such a function $f \in L^1[0, 1]$ coinciding with $g(x)$ on E , and such that*

$$|c_k(f)| = c_k(U), \quad k = 0, 1, 2, \dots;$$

- e) *the Fourier–Walsh series of the corrected function f converges to it in the norm of $L^1[0, 1]$.*

Remark 3.2. The method used to prove Theorem 3.7 provides a novel approach to constructing universal series in the Walsh system: any measurable, almost everywhere finite function can be transformed by modifying its values on a set of arbitrarily small measure into a function such that, after choosing appropriate signs for the terms of its Fourier–Walsh series, the resulting series becomes universal in $M[0, 1]$.

Remark 3.3. It should be noted (as also follows from the above) that the existence of universal functions and universal triads depends on the type (or sense) of universality, the system, and the space S . Therefore, the questions in this direction are quite broad.

There arise the following questions, the answers to which are not yet known.

Question 3.1. Do there exist an orthonormal system $\{\varphi_k(x)\}_{k=1}^{\infty}$ of bounded functions and a function $U \in L^1[0, 1]$ that is universal for the space $L^p[0, 1]$ for some $p \in [0, 1)$ (or at least for the space $M[0, 1]$) with respect to the system $\{\varphi_k(x)\}_{k=1}^{\infty}$?

Question 3.2. Are Theorems 3.4 and 3.6 true for the trigonometric system?

Question 3.3. Are Theorems 3.1–3.7 true for the Vilenkin system?

Question 3.4. Is it possible to construct a function that is universal for the space $L^0[0, 1]$ with respect to the Walsh system in a sense of rearrangements?

Question 3.5. Does there exist a function $U \in L^1[-\pi, \pi]$ that is universal for the class $L^p[-\pi, \pi]$ with respect to the trigonometric system, in a sense of permutations?

Question 3.6. Are Theorems 3.1–3.7 true for the Franklin system (for the Haar system)?

Question 3.7. Are Theorems 3.1–3.6 true for spherical harmonics?

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