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ON THE EXISTENCE OF BOUNDED SOLUTIONS OF NONLINEAR SYSTEMS OF A CLASS OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS AND NONLINEAR IMPULSIVE DIFFERENTIAL SYSTEMS ON INFINITY AXES

MALKHAZ ASHORDIA

Abstract. Effective sufficient conditions are established for the existence of bounded solutions to systems of nonlinear generalized ordinary differential equations with a nondecreasing matrix function on the real axis. The sufficient conditions for the existence of a unique solution are also established. The results are realized for a nonlinear system of impulsive differential equations.

1. Statement of the Problem. Basic Notation and Definitions

For the system of the nonlinear generalized ordinary differential equations

$$dx = dA(t) \cdot f(t,x) \quad \text{for} \quad t \in \mathbb{R} \tag{1.1}$$

we consider the problem on the existence of solutions satisfying one of the two conditions

$$\sup\{\|x(t)\| : t \in \mathbb{R}\} < +\infty, \tag{1.2}$$

or

$$\sup\{\|x(t)\| : t \in \mathbb{R}_+\} < +\infty, \tag{1.3}$$

where $A = (a_{ik})_{i,k=1}^{n}$; $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ is a nondecreasing matrix function, and $f = (f_k)_{k=1}^{n} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is the restriction of a vector function that belongs on every closed interval $[a,b] \subset \mathbb{R}$ to the Carathéodory class $\mathrm{Car}([a,b] \times \mathbb{R}^n, \mathbb{R}^n; A)$.

Generalized ordinary differential equations have been introduced by J. Kurzweil [15]. To a considerable extent, the interest in the theory has also been stimulated by the fact that this theory enabled one to investigate ordinary differential, impulsive differential and difference equations from a unified point of view (see [1, 3, 5–7, 18, 20] and references therein).

Therefore, we can consider ordinary differential, impulsive differential and difference equations as equations of the same type.

The problem analogous to the problem considered in the present paper, for linear and nonlinear cases, is investigated in [13, 14] (see also references therein) for systems of ordinary differential equations. As for the generalized case, it is considered in [7] only for linear systems.

As is known, the question related to the nonlinear case has not been studied in earlier works. So, the problem under consideration is essentially new one.

Some general questions in the theory of impulsive differential equations are studied in [9,10,16,17,19] (see also references therein).

Throughout the paper, the use will be made of the following notation and definitions:

$\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$; $[a,b]$ and $]a,b[$ $(a, b \in \mathbb{R})$ are, respectively, closed and open intervals. I is an arbitrary finite or infinite interval in $\mathbb{R}$.

$\mathbb{R}^{n\times m}$ is the space of all real $n \times m$ matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm $\|X\| = \max\limits_{j=1,\dots,m} \sum\limits_{i=1}^{n} |x_{ij}|$.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^{n}$.

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If $X \in \mathbb{R}^{n\times n}$, then X^{-1} and $\det(X)$ are, respectively, the matrix inverse to X and the determinant of X. I_n is the identity $n\times n$-matrix.

$\operatorname{diag}(h_1,,\dots,h_n)$ is a diagonal matrix-function with diagonal elements $h_1.\dots,h_n$.

Inequalities between matrices are understood componentwise.

$\overset{b}{\underset{a}{\mathrm{V}}}(X)$ is the total variation of the components of the matrix function $X:[a,b]\to\mathbb{R}^{n\times m}$; $X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of X at the point $t\in[a,b]$ (we assume that $X(a-)=X(a)$, $X(b+)=X(b)$, and $X(t)$ is defined by continuity outside $[a,b]$). $d_1X(t)=X(t)-X(t-)$, $d_2X(t)=X(t+)-X(t)$

If $X=(x_{ij})_{i,j=1}^{n,m}:[a,b]\to\mathbb{R}^{n\times m}$, then $V(X)(t)=\big(\overset{t}{\underset{a}{\mathrm{V}}}(x_{ij})\big)_{i,j=1}^{n,m}$.

$\mathrm{BV}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{n\times m})$ is the set of all matrix functions $X:\mathbb{R}\to\mathbb{R}^{n\times m}$ whose restrictions on every closed interval $[a,b]$ from $\mathbb{R}$ belong to $\mathrm{BV}([a,b];\mathbb{R}^{n\times m})$.

A matrix function is said to be continuous, integrable, nondecreasing, etc., if each of its component is such.

If $g\in\mathrm{BV}([a,b];\mathbb{R})$, $f:[a,b]\to\mathbb{R}$ and $a\le s<t\le b$, then we assume

$$\int\limits_s^t x(\tau)\,dg(\tau)=(L-S)\int\limits_{]s,t[}x(\tau)\,dg(\tau)+f(t)d_1g(t)+f(s)d_2g(s),$$

where $(L-S)\int\limits_{]s,t[}f(\tau)\,dg(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$. It is known (see [18,20]) that if the integral exists, than the right side of the integral equality equals to the Kurzeil–Stieltjes integral $(K-S)\int\limits_s^t f(\tau)\,dg(\tau)$ and, hence $\int\limits_s^t f(\tau)\,dg(\tau)=(K-S)\int\limits_s^t f(\tau)\,dg(\tau)$. If $a=b$, then we assume $\int\limits_a^b x(t)\,dg(t)=0$.

We introduce the operator $\mathcal{A}(X,Y)$ in the following way:

if $X\in\mathrm{BV}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{n\times n})$, $\det(I_n-(-1)^jd_jX(t))\ne0$ $(j=1,2)$ for $t\in\mathbb{R}$ and $Y\in\mathrm{BV}_{\mathrm{loc}}(I;\mathbb{R}^{n\times m})$, then

$$\mathcal{A}(X,Y)(0)=O_{n\times m},$$
$$\mathcal{A}(X,Y)(t)-\mathcal{A}(X,Y)(s)=Y(t)-Y(s)+\sum_{s<\tau\le t}d_1X(\tau)\,(I_n-d_1X(\tau))^{-1}\,d_1Y(\tau)$$
$$-\sum_{s\le\tau<t}d_2X(\tau)\,(I_n+d_2X(\tau))^{-1}\,d_2Y(\tau)\ \text{ for }\ s<t.$$

If $G=(g_{ik})_{i,k=1}^n\in\mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ and $x=(x_k)_k^n:[a,b]\to\mathbb{R}^n$, then

$$\int\limits_a^b dG(\tau)\cdot x(\tau)=\bigg(\sum_{k=1}^n\int\limits_a^b x_k(\tau)\,dg_{ik}(\tau)\bigg)_i^n.$$

If $G=(g_{ik})_{i,k=1}^{l,n}:[a,b]\to\mathbb{R}^{l\times n}$ is a nondecreasing matrix function, $D_1\subset\mathbb{R}^{l\times n}$ and $D_2\subset\mathbb{R}^{n\times m}$, then $\mathrm{Car}([a,b]\times D_1,D_2;G)$ is the Carathéodory class, i.e., the set of all mappings $F=(f_{kj})_{k,j=1}^{n,m}:[a,b]\times D_1\to D_2$ such that for each $i\in\{1,\dots,l\}$, $j\in\{1,\dots,m\}$ and $k\in\{1,\dots,n\}$:

(a) the function $f_{kj}(\cdot,x):[a,b]\to D_2$ is integrable with respect to g_{ik} for every $x\in D_1$;

(b) the function $f_{kj}(t,\cdot\,):D_1\to D_2$ is continuous for $t\in[a,b]$, and

$$\sup\big\{|f_{kj}(\cdot,x)|:\ x\in D_0\big\}\in L([a,b],\mathbb{R};g_{ik})$$

for every compact $D_0\subset D_1$.

If $G(t)\equiv\operatorname{diag}(t,\dots,t)$, then under $\mathrm{Car}([a,b]\times D_1,D_2;G)$ we mean the classical Carathéodory class $\mathrm{Car}([a,b]\times D_1,D_2)$.

We assume that $A(0) = O_{n\times n}$ without loss of generality for every system of type (1.1). Let, moreover,

$$\det(I_n - (-1)^j d_j A(t)) \neq 0 \ \text{ for } \ t \in \mathbb{R} \ \ (j = 1, 2). \tag{1.4}$$

Inequalities (1.4) guarantee the unique solvability of the Cauchy problem for the corresponding linear systems (see [18, 20]).

By a solution of system (1.1) we understand a vector function $x \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$ such that

$$x(t) = x(s) + \int_s^t dA(\tau) \cdot f(\tau, x(\tau)) d\tau \ \text{ for } \ s < t, \ s, t \in \mathbb{R}.$$

Let $\alpha \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R})$ and $t_0 \in \mathbb{R}$ be such that $1 + (-1)^j d_j \alpha(t) \neq 0$ for $(-1)^j(t - t_0) < 0$, $t \neq t_0$ $(j = 1, 2)$. Then it is known that (see [11, 12] the initial problem

$$d\xi = \xi d\alpha(t), \ \ \xi(t_0) = 1$$

has the unique solution ξ_α defined by

$$\xi_\alpha(t) = \begin{cases} \exp(s_c(\alpha)(t) - s_c(\alpha)(t_0)) \prod\limits_{t_0<\tau\le t} (1 - d_1\alpha(\tau))^{-1} \prod\limits_{t_0\le\tau<t} (1 + d_2\alpha(\tau)) \\ \hfill \text{for } \ t > t_0, \\ \exp(s_c(\alpha)(t) - s_c(\alpha)(t_0)) \prod\limits_{t<\tau\le t_0} (1 - d_1\alpha(\tau)) \prod\limits_{t\le\tau<t_0} (1 + d_2\alpha(\tau))^{-1} \\ \hfill \text{for } \ t < t_0. \end{cases}$$

Note that the following equality:

$$d\xi_\alpha^{-1}(t) \equiv -\xi_\alpha^{-1}(t) d\mathcal{A}(\alpha, \alpha)(t)$$

holds (see [5, 6]).

If $\sigma = (\sigma_i)_{i=1}^n$, where $\sigma_i \in \{-1; 1\}$, then by $\mathcal{N}_+(\sigma)$ (by $\mathcal{N}_-(\sigma)$) we denote the set of $i \in \{1, \dots, n\}$ such that $\sigma_i = 1$ ($\sigma_i = -1$). The sets have been introduced by I. Kiguradze for the ordinary differential equations (see [13, 14]).

Everywhere below, under $t_1, \dots, t_n$ we assume that

$$t_i = a \ \text{ for } \ i \in \mathcal{N}_+(\sigma) \quad \text{and} \quad t_i = b \ \text{ for } \ i \in \mathcal{N}_-(\sigma).$$

2. Formulation of the Results

The results given in this section follow immediately from similar results for the matrix function A with a bounded variation.

Theorem 2.1. *Let the matrix function A and $\sigma_i \in \{-1; 1\}$ $(i = 1, \dots, n)$, vector functions $\alpha_l = (\alpha_{li})_{i=1}^n \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^n)$ $(l = 1, 2)$, and matrix functions $(\beta_{lik})_{i,k=1}^n$, $\beta_{lik} \in L_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}; a_{ik})$ $(l = 1, 2)$ be such that*

$$\alpha_{li}(t) - \alpha_{li}(\tau) = \sum_{k=1}^n \int_\tau^t \beta_{lik}(\tau) da_{ik}(\tau) \ \textit{ for } \ t, \tau \in \mathbb{R} \ \ (l = 1.2; \ i = 1, \dots, n), \tag{2.1}$$

$$\sup\{|\alpha_{li}(t)| : t \in \mathbb{R}\} < +\infty \ \ (l = 1.2; \ i = 1, \dots, n), \tag{2.2}$$

$$\alpha_1(t) \le \alpha_2(t) \ \textit{ for } \ t \in \mathbb{R}; \tag{2.3}$$

and the inequalities

$$(-1)^l \sigma_i \big(f_k(t, x_1, \dots, x_{i-1}, \alpha_{ji}(t), x_{i+1}, \dots, x_n) - \beta_{lik}(t)\big) \le 0 \\ (l, j = 1, 2; \ i, k = 1, \dots, n), \tag{2.4}$$

$$(-1)^l \left(x_i + (-1)^j \sum_{k=1}^n f_k(t, x_1, \dots, x_n) d_j \alpha_{ik}(t) - \alpha_{li}(t) - (-1)^j d_j \alpha_{li}(t) \right) \le 0 \\ \textit{for } \ (-1)^j \sigma_i \ge 0 \ \ (l, j = 1, 2; \ i = 1, \dots, n) \tag{2.5}$$

be fulfilled on the set

$$\{(t, x_1, \dots, x_n) : t \in \mathbb{R},\ \ \alpha_{1i}(t) \le x_i \le \alpha_{2i}(t)\ \ (i = 1, \dots, n)\}.$$

Then problem (1.1), (1.2) *is solvable.*

Theorem 2.1$_1$. *Let the matrix function A and $\sigma_i \in \{-1; 1\}$ $(i = 1, \dots, n)$, vector functions $\alpha_l = (\alpha_{li})_{i=1}^n \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^n)$ $(l = 1, 2)$ and the matrix functions $(\beta_{lik})_{i,k=1}^n$, $\beta_{lik} \in L_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}; a_{ik})$ $(l = 1, 2)$ be such that* (2.1) *holds for $t, \tau \in \mathbb{R}_+$,*

$$\sup\{|\alpha_{li}(t)| : t \in \mathbb{R}_+\} < +\infty\ \ (l = 1, 2;\ i = 1, \dots, n),$$
$$\alpha_1(t) \le \alpha_2(t)\ \ \textit{for}\ \ t \in \mathbb{R}_+;$$

and inequalities (2.4) *and* (2.5) *are fulfilled on the set*

$$\{(t, x_1, \dots, x_n) : t \in \mathbb{R}_+,\ \ \alpha_{1i}(t) \le x_i \le \alpha_{2i}(t)\ \ (i = 1, \dots, n)\}.$$

Then system (1.1) *for every $c_i \in [\alpha_{1i}(0), \alpha_{2i}(0)]$ $(i \in \mathcal{N}_+(\sigma))$, where $\sigma = (\sigma_i)_{i=1}^n$, has a solution satisfying conditions* (1.3) *and*

$$x_i(0) = c_i\ \ \textit{if}\ \ i \in \mathcal{N}_+(\sigma). \tag{2.6}$$

(If $\mathcal{N}_+(\sigma) = \emptyset$, then condition (2.6) is eliminated).

In Theorems 2.2 and 2.3, we assume that

$$b_{ii}(t) \equiv \sum_{k=1}^n a_{ik}(t)\ \ (i = 1, \dots, n)$$

and

$$\zeta_i(t) \equiv \sigma_i \eta_{ii} b_{ii}(t)\ \ (i = 1, \dots, n).$$

It is evident that b_{ii} $(i = 1, \dots, n)$ are nondecreasing functions.

Theorem 2.2. *Let the matrix function A and $\sigma_i \in \{-1, 1\}$ $(i = 1. \dots, n)$ be such that*

$$\sigma_i f_i(t, x_1, \dots, x_n) \operatorname{sgn} x_i \le \sum_{l=1}^n \eta_{il} |x_l| + q_i(t)\ \ (i = 1, \dots, n) \tag{2.7}$$

and

$$\Big(|f_i(t, x_1, \dots, x_n)| - \sum_{l=1}^n \eta_{il} |x_l| - q_i(t)\Big) d_j a_{ii}(t) \le 0\ \ (j = 1, 2;\ \ i = 1, \dots, n) \tag{2.8}$$

be fulfilled on $\mathbb{R} \times \mathbb{R}^n$ and let, in addition,

$$\rho_i = \sup\bigg\{\bigg| \bigvee_t^{\zeta_i(t)+1} \big(\mathcal{A}(\zeta_i, \sigma_i g_i)\big)\bigg|\ \ t \in \mathbb{R}\bigg\} << \infty\ \ (i = 1, \dots, n), \tag{2.9}$$

$$1 + (-1)^j d_j \zeta_i(t) > 0\ \ \textit{for}\ \ t \in \mathbb{R}\ \ (j = 1, 2;\ i = 1, \dots, n). \tag{2.10}$$

and

$$\sum_{k=1}^n |\eta_{ki}| d_j b_{kk}(t) < 1\ \ \textit{for}\ \ t \in \mathbb{R}\ (j = 1, 2;\ \ i = 1, \dots, n), \tag{2.11}$$

where $\eta_{il} \in \mathbb{R}_+$ $(i \ne l)$, $\eta_{ii} < 0$, $q_i \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R})$ $(i, l = 1, \dots, n)$, $g_i(t) \equiv \int\limits_{t_i}^{t} q_i(\tau) db_{ii}(\tau)$ $(i, l = 1, \dots, n)$ and the real part of every characteristic value of the matrix $(\eta_{il})_{i,l=1}^n$ be negative. Then problem (1.1), (1.2) *is solvable.*

Note that if $d_j a_{ii}(t) = 0$, then the corresponding inequality is eliminated in (2.8).

Theorem 2.3. *Let the matrix function A and $\sigma_i \in \{-1,1\}$ $(i=1\dots,n)$ be such that*

$$\sigma_i\big(f_i(t,x_1,\dots,x_n)-f_i(t,y_1,\dots,y_n)\big)\operatorname{sgn}(x_i-y_i)\le\sum_{l=1}^n\eta_{il}|x_l-y_l|\ \ (i=1,\dots,n) \tag{2.12}$$

and

$$\bigg(|f_i(t,x_1,\dots,x_n)-f_i(t,y_1,\dots,y_n)|-\sum_{l=1}^n\eta_{il}|x_l-y_l|\bigg)d_ja_{ii}(t)\le 0$$
$$(j=1,2;\ \ i=1,\dots,n)$$

be fulfilled on $\mathbb{R}\times\mathbb{R}^n$ and let, in addition, conditions (2.7), (2.9) *and* (2.10) *hold, where $\eta_{il}\in\mathbb{R}_+$ $(i\ne l)$, $\eta_{ii}<0$, $g_i(t)\equiv\int\limits_{t_i}^t f_i(\tau,0,\dots,0)db_{ii}(\tau)$ $(i,l=1,\dots,n)$ and the real part of every characteristic value of the matrix $(\eta_{il})_{i,l=1}^n$ is negative. Then problem* (1.1), (1.2) *has the unique solution.*

3. Formulation of the Results for Impulsive Problems

The obtained results we use for nonlinear impulsive differential system

$$\frac{dx}{dt}=f(t,x)\ \text{ for a.a. }\ t\in\mathbb{R}\setminus T, \tag{3.1}$$

$$x(\tau_l+)-x(\tau_l-)=h(\tau_l,x(\tau_l))\ \ (l=1,2,\dots), \tag{3.2}$$

where $f=(f_k)_{k=1}^n:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ is a vector function restriction that on the every closed interval $[a,b]\subset\mathbb{R}$ belongs to the Carathéodory class $\mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^n)$, and the vector function $h(\tau_l,.)=(h_k(\tau_l,.))_{k=1}^n:\mathbb{R}^n\to\mathbb{R}^n$ $(l=1,2,\dots)$ is continuous, $T=\{\tau_1,\tau_2,\dots\}$, $\tau_l\in\mathbb{R}$, $\tau_l<\tau_{l+1}$ $(l=1,2,\dots)$.

In this section, in addition, we also use the following designations:

$\mathrm{AC}([a,b];D)$ is the set of all absolutely continuous matrix functions $X:[a,b]\to D$;

$\mathrm{AC}_{\rm loc}(I;D)$ is the set of all matrix-functions $X:I\to D$ whose restrictions to an arbitrary closed interval $[a,b]$ from I belong to $AC([a,b];D)$;

$\mathrm{AC}_{\rm loc}(I\setminus T;D)$, where is the set of all matrix functions $X:I\to D$ whose restrictions to an arbitrary closed interval $[a,b]$ from $I\setminus T$ belong to $AC([a,b],D)$.

By a solution of thez impulsive differential system (3.1), (3.2) we understand a continuous from the left vector function $x\in\mathrm{BV}(\mathbb{R};\mathbb{R}^n)\cap\mathrm{AC}_{\rm loc}(\mathbb{R}\setminus T;\mathbb{R}^n)$ satisfying both the system

$$x'(t)=f(t,x(t))\ \text{ for a.a. }\ t\in\mathbb{R}\setminus T$$

and relation (3.2) for every $l\in\{1,2,\dots\}$.

We assume that the impulsive points τ_l $(l=1,2,\dots)$ are such that $0\le\tau_1<\tau_2<\dots<\tau_l<\tau_{l+1}<\cdots$ and $\lim\limits_{l\to+\infty}\tau_l=+\infty$.

Let $\alpha(t)\equiv t+\iota(t)$, where $\iota:\mathbb{R}\to\mathbb{N}$ is the function defined by $\iota(t)=\max\{l:\tau_l<t\}$. It is evident that the functions ι and α are nondecreasing, $\iota(t)=l$ for $t\in]\tau_l,\tau_{l+1}]$ and $d_2\iota(t_l)=l$ $(l=1,2,\dots)$. Then the matrix function $\widetilde{A}(t)\equiv\operatorname{diag}(t+\iota(t),\dots,t+\iota(t))$ is also nondecreasing, and $d_1\widetilde{A}(t)\equiv O_{n\times n}$, $d_2\widetilde{A}(t)=O_{n\times n}$ for $t\in\mathbb{R}\setminus T$ and $d_2\widetilde{A}(t)=I_n$ for $t\in T$. Therefore, we can rewrite system (3.1), (3.2) in the form

$$dx=d\widetilde{A}(t)\cdot\widetilde{f}(t,x)\quad\text{for}\quad t\in\mathbb{R}, \tag{3.3}$$

where

$$\widetilde{f}(t,x)=\begin{cases}f(t,x) & \text{for }\ t\in\mathbb{R}\setminus T,\\ h(\tau_l,x) & \text{for }\ t=\tau_l\ \ (l=1,2,\dots).\end{cases}$$

Note that according to (3.3), we can assume that $h(\tau_l,x)\equiv f(\tau_l,x)$ for $t=\tau_l$ $(l=1,2,\dots)$ without the loss of generallity.

Using the results for system (3.3) given above, we obtain the following

Theorem 3.1. *Let the numbers* $\sigma_i \in \{-1;1\}$ $(i=1,\dots,n)$, *the vector-functions* $\alpha_l = (\alpha_{li})_{i=1}^n \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)$, *and* $(\beta_{li})_{i=1}^n \in L_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n)$ $(l=1,2)$ *be such that conditions* (2.2), (2.3),

$$\alpha_{li}(t) - \alpha_{li}(\tau) = \int\limits_{\tau}^{t} \beta_{li}(\tau)d\tau + \sum_{\tau \le \tau_l < t} \beta_{li}(\tau_l) \ \text{ for } \ t,\tau \in \mathbb{R} \ \ (l=1.2;\ i=1,\dots,n), \tag{3.4}$$

and inequalities

$$\begin{gathered}(-1)^l \sigma_i\big(f_i(t,x_1,\dots,x_{i-1},\alpha_{ji}(t),x_{i+1},\dots,x_n) - \beta_{li}(t)\big) \le 0 \\ \text{for } \ t \in \mathbb{R}\setminus T, \ \ \alpha_{1i}(t) \le x_i \le \alpha_{2i}(t) \ \ (l,j=1,2;\ i,k=1,\dots,n),\end{gathered} \tag{3.5}$$

$$\begin{gathered}(-1)^l \sigma_i\big(g_i(\tau_m,x_1,\dots,x_{i-1},\alpha_{ji}(\tau_m),x_{i+1},\dots,x_n) - \beta_{li}(\tau_m)\big) \le 0 \\ \text{for } \ \alpha_{1i}(\tau_m) \le x_i \le \alpha_{2i}(\tau_m) \ \ (l,j=1,2;\ i,k=1,\dots,n;\ m=1,2,\dots),\end{gathered} \tag{3.6}$$

$$\begin{gathered}(-1)^l \big(x_i + h_i(\tau_m,x_1,\dots,x_n) - \alpha_{li}(\tau_m+)\big) \le 0 \\ \text{for } \ (-1)^j\sigma_i \ge 0, \ \ \alpha_{1i}(\tau_m) \le x_i \le \alpha_{2i}(\tau_m), \ \ (l=1,2;\ i=1,\dots,n\ m=1,2,\dots)\end{gathered} \tag{3.7}$$

are fulfilled. Then problem (3.1), (3.2), (1.2) *is solvable.*

Theorem 3.1$_1$. *Let the numbers* $\sigma_i \in \{-1;1\}$ $(i=1,\dots,n)$, *the vector-functions* $\alpha_l = (\alpha_{li})_{i=1}^n \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^n)$ $(l=1,2)$, *and* $(\beta_{li})_{i=1}^n \in L_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n)$ $(l=1,2)$ *be such that*

$$\begin{gathered}\sup\{|\alpha_{li}(t)| : t \in \mathbb{R}_+\} < +\infty \ \ (l=1,2;\ i=1,\dots,n), \\ \alpha_1(t) \le \alpha_2(t) \ \text{ for } \ t \in \mathbb{R}_+;\end{gathered}$$

and conditions (3.4), (3.5), (3.6) *and* (3.7) *are fulfilled on the set*

$$\{(t,x_1,\dots,x_n) : t \in \mathbb{R}_+, \ \ \alpha_{1i}(t) \le x_i \le \alpha_{1i}(t) \ \ (i=1,\dots,n)\}.$$

Then system (3.1), (3.2), *for every* $c_i \in [\alpha_{1i}(0), \alpha_{2i}0)]$ $(i \in \mathcal{N}_+(\sigma))$, *where* $\sigma = (\sigma_i)_{i=1}^n$, *has a solution satisfying conditions* (1.3) *and* (2.6).

In Theorems 3.2 and 3.3, for the given case, we use the equalities

$$b_{ii}(t) \equiv \alpha(t), \ \ d_1 b_{ii}(t) \equiv 0, \ \ d_2 b_{ii}(t) = 0 \ \text{ for } \ t \in \mathbb{R}, \ \ d_2 b_{ii}(t) = 1 \ \text{ for } \ t \in T,$$

and

$$\begin{gathered}\zeta_i(t) \equiv \sigma_i \eta_{ii} \alpha(t), \ \ d_1\zeta_i(t) \equiv 0, \ \ d_2\zeta_i(t) = 0 \ \text{ for } \ t \in \mathbb{R}, \ \ d_2\zeta_i(t) = \sigma_i\eta_{ii} \ \text{ for } \ t \in T \\ (i=1,\dots,n);\end{gathered}$$

$$\begin{gathered}g_i(t) \equiv \int\limits_0^t q_i(\tau)d\tau + \sum_{0 \le \tau_l < t} q_i(\tau_l), \ \ d_1 g_i(t) \equiv 0, \\ d_2 g_i(t) \equiv 0 \ \text{ for } \ t \in \mathbb{R}\setminus T, \ \ d_2 g_i(t) = q_i(t) \ \text{ for } \ t \in T \ \ (i=1,\dots,n); \\ \mathcal{A}(\zeta_i, \sigma_i g_i)(t) \equiv \sigma_i \int\limits_0^t q_i(\tau)d\tau + \sum_{0 \le \tau_l < t} \eta_{ii}(1+\sigma_i\eta_{ii})^{-1} q_i(\tau_l) \ \ (i=1,\dots,n).\end{gathered}$$

In this case, we find that conditions (2.9), (2.10), (2.11) are equivalent, respectively, to the following conditions:

$$\sup\left\{\left|\int\limits_t^{\zeta_i(t)+1}|q_i(\tau)|d\tau+\sum_{t\le\tau_l<\zeta_i(t)+1}\eta_{ii}(1+\sigma_i\eta_{ii})^{-1}q_i(\tau_l)\right|:t\in\mathbb{R}\right\}<+\infty$$
$$(i=1,\dots,n); \tag{3.8}$$

$$\eta_{ii}>-1 \ \text{ if } \ \sigma_i=1 \ \ (i=1,\dots,n), \tag{3.9}$$

$$\sum_{k=1}^n|\eta_{ki}|<1 \ \ (i=1,\dots,n). \tag{3.10}$$

Note that if $h(i,x)\equiv 0_n$ for some $i\in\{1,\dots,n\}$, then the corresponding inequality is eliminated in (2.8).

Theorem 3.2. *Let the matrix function A and $\sigma_i\in\{-1,1\}$ $(i=1\dots,n)$ be such that inequalities* (2.7) *are fulfilled for $t\in\mathbb{R}\setminus T$, $(x_i)_{i=1}^n\mathbb{R}^n$, and*

$$|h_i(\tau_m,x_1,\dots,x_n)|\le\sum_{l=1}^n\eta_{il}|x_l|+q_i(\tau_m) \ \ \textit{for} \ \ (x_l)_{l=1}^n\in\mathbb{R}^n$$
$$(i=1,\dots,n,\ m=1,2,\dots).$$

Let, in addition, conditions (3.8), (3.9) *and* (3.10) *be fulfilled, where $\eta_{il}\in\mathbb{R}_+$ $(i\neq l)$, $\eta_{ii}<0$, $\zeta_i(t)\equiv\sigma_i\eta_{ii}\alpha(t)$, $q_i\in\mathrm{BV}_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$ $(i,l=1,\dots,n)$. and the real part of every characteristic value of the matrix $(\eta_{il})_{i,l=1}^n$ be negative. Then problem* (3.1), (3.2), (1.2) *is solvable.*

Theorem 3.3. *Let the matrix function A and $\sigma_i\in\{-1,1\}$ $(i=1\dots,n)$ be such that inequalities* (2.12) *are fulfilled for $t\in\mathbb{R}\setminus T$, $(x_i)_{i=1}^n\mathbb{R}^n$, and*

$$\left(|h_i(\tau_m,x_1,\dots,x_n)-h_i(\tau_m,y_1,\dots,y_n)|-\sum_{l=1}^n\eta_{il}|x_l-y_l|\right)\le 0$$
$$\textit{for} \ \ (x_l)_{l=1}^n,\ (y_l)_{l=1}^n\in\mathbb{R}^n \ \ (i=1,\dots,n,\ m=1,2,\dots).$$

Let, in addition, conditions (3.8), (3.9) *and* (3.10) *be fulfilled, where $\eta_{il}\in\mathbb{R}_+$ $(i\neq l)$, $\eta_{ii}<0$, $\zeta_i(t)\equiv\sigma_i\eta_{ii}\alpha(t)$, $q_i(t)\equiv f_i(t,0,\dots,0)$ $(i,l=1,\dots,n)$ and the real part of every characteristic value of the matrix $(\eta_{il})_{i,l=1}^n$ be negative. Then problem* (3.1), (3.2), (1.2) *has a unique solution.*

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A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 2 Merab Aleksidze II Lane, Tbilisi 0193, Georgia

Sukhumi State University, 61 A. Politkovskaia Str., Tbilisi 0186, Georgia

N. Muskhelishvili Institute of Computational Mathematics, 4 Grigol Peradze Str., Tbilisi 0159, Georegia
Email address: malkhaz.ashordia@tsu.ge; ashord@rmi.ge