Transactions of A. Razmadze
Mathematical Institute
Vol. **180** (2026), issue 1, 93–103

ADVANCED ANALYSIS OF MEROMORPHIC BI-UNIVALENT FUNCTIONS AND QUASI-SUBORDINATION VIA JACKSON'S (p,q)-DERIVATIVE

FETHİYE MÜGE SAKAR[1*] AND SEHER MELİKE AYDOĞAN[2]

Abstract. In this work, we introduce a new subclass of meromorphic bi-quasi-subordinate functions by utilizing the (p,q)-Jackson derivative, specifically defined in the exterior of the unit disc. We derive coefficient bounds for these functions and explore several special consequences of our findings. By providing some new results in this area, this work extends the understanding of meromorphic function classes and their behavior under generalized derivatives, offering new insights and potential applications in complex analysis.

1. Introduction

We start by letting $\mho = \{\xi : \xi \in \mathbb{R} \text{ and } 1 < |\xi| < \infty\}$, and Υ be the class of meromorphic functions of the style

$$s(\xi) = \xi + a_0 + \sum_{m=1}^{\infty} \frac{a_m}{\xi^m}\,, \tag{1.1}$$

which are univalent in $\mho$. It is well known that every function $s \in \Upsilon$ has an inverse s^{-1} defined by

$$s^{-1}(s(\xi)) = \xi, \quad \xi \in \mho \text{ and } s(s^{-1}(\varpi)) = \varpi, \quad \eta < |\varpi| < \infty, \ \eta > 0.$$

The inverse of the function s is represented by the following series:

$$s^{-1}(\varpi) = \vartheta(\varpi) = \varpi + b_0 + \sum_{m=1}^{\infty} \frac{b_m}{\xi^m}, \quad (\eta < |\varpi| < \infty, \ \eta > 1)\,. \tag{1.2}$$

Substituting $\varpi = s(\xi)$ in the above series (1.2), the coefficients b_m of $s^{-1}(\xi)$ can be expressed in terms of the coefficients a_m of $s(\xi)$. Thus, for some initial values of m, we obtain

$$b_0 = -a_0, \ b_1 = -a_1, \ b_2 = -(a_2 + a_0 a_1), \ b_3 = -(a_3 + 2a_0 a_2 + a_0^2 a_1 + a_1^2), \text{ and so on.}$$

For a brief history in the class Υ, see [1, 9, 13, 24, 25, 27, 28, 30]. A univalent function in $\mho$ is said to be bi-univalent if its inverse map is also univalent there.

The function $s \in \Upsilon$ is said to be bi-univalent and meromorphic if $s^{-1} \in \Upsilon$. The family of these functions is denoted by Υ_M.

Springer [30] proved $|a_3| \leq 1, \quad |a_3 + \frac{1}{2}a_1^2| \leq \dfrac{1}{2}$ and conjectured that $|a_{2m-1}| \leq \dfrac{(2m-1)!}{m!(m-1)!}$ for $(m = 1, 2, \ldots)$.

With a view to recalling the principle of subordination between analytic functions, let the functions s and ϑ be analytic in $\mho$. Then we say that the function s is subordinate to ϑ if there exists a Schwarz function $\varpi(\xi)$, analytic in $\mho$ with

$$\varpi(0) = 0, \quad |\varpi(\xi)| < 1, \quad \xi \in \mho$$

such that

$$s(\xi) = \vartheta(\varpi(\xi)), \quad \xi \in \mho.$$

We denote this subordination by

$$s \prec \vartheta \text{ or } s(\xi) \prec \vartheta(\xi), \ \xi \in \mho.$$

2020 *Mathematics Subject Classification.* 30C45, 30C50.
Key words and phrases. Analytic functions; Meromorphic functions; Quasi-subordinate; q-calculus.
*Corresponding author.

In particular, if the function ϑ is univalent in $\mho$, the above subordination is equivalent to

$$s(0) = \vartheta(0), \quad s(\mho) \subset \vartheta(\mho).$$

In 1970, Robertson [23] introduced the concept of quasi-subordination. For two analytic function s and ϑ, the function s is said to be quasi-subordinate to ϑ in $\mho$ and is written as

$$s(\xi) \prec_\rho \vartheta(\xi), \ \ \xi \in \mho,$$

if there exists an analytic function $|h(\xi)| \leq 1$ such that $\dfrac{s(\xi)}{h(\xi)}$ is analytic in $\mho$ and

$$\frac{s(\xi)}{h(\xi)} \prec \vartheta(\xi), \ \ \xi \in \mho,$$

that is, there exists a Schwarz function $\varpi(\xi)$ such that $s(\xi) = h(\xi)\vartheta(\varpi(\xi))$. Observe that if $h(\xi) = 1$, then $s(\xi) = \vartheta(\varpi(\xi))$, so $s(\xi) \prec \vartheta(\xi)$ in $\mho$. Also note that if $\varpi(\xi) = \xi$, then $s(\xi) = h(\xi)\vartheta(\xi)$ and it is majorized by ϑ and written as $s(\xi) \ll \vartheta(\xi)$ in $\mho$. Hence it is obvious that quasi-subordination is a generalization of both subordination and majorization (see related works on quasi-subordination, for example, [18, 21, 23, 26]).

We shortly recall the concept of *q-operators,* i.e., *q-difference operators*, that play an important role in hypergeometric series, quantum physics and special functions. The implementation of the *q-calculus* was started by Jackson [15] (see [5, 6, 17, 20, 31]).

The q-calculus has attracted the attention of researchers due to its numerous applications in various branches of mathematics, especially in geometric function theory. Jackson [15, 16] initiated and developed the application of the q-calculus. Chakrabarti and Jagannathan defined the Jackson (p, q)-derivative as a generalization of the q-derivative (see [8]). Al-Hawary et al. [2] introduced a new differential operator defined by the Jackson (p, q)-derivative. Some applications of (p, q)-differential operators are studied by Altınkaya and Yalçın [4] and Aracı et al. [5]. We believe that one can find more deeper understanding of the idea by carefully studying some of the works on the fractional q-functions (see, for example, [7, 10–12, 22, 29]).

For the expedience, we present some definitions and concepts of (p, q)-calculus that were used in this article by assuming p and q are fixed numbers such that $0 < p < q \leq 1$. The Jackson's *q-derivative operator* (or *q-difference operator*) is presented by

$$\partial_{p,q} s(\xi) = \begin{cases} \dfrac{s(p\xi) - s(q\xi)}{(p-q)\xi}, & \xi \neq 0 \\ \partial_{p,q} s(0) = s'(0), & \xi = 0 \end{cases}$$

provided $s'(0)$ exist, where the symbol $[m]_{p,q}$ denotes the basic twin number by

$$[m]_{p,q} = \frac{p^m - q^m}{p - q}, \quad [0]_{p,q} = 0, \quad [1]_{p,q} = 1.$$

Note that for $0 < q < 1$ and $\xi \neq 0$, we have

$$\partial_{1,q} s(\xi) = \partial_q s(\xi) = \frac{s(q\xi) - s(\xi)}{q\xi - 1},$$

(for more details, see [15]) and

$$[m]_{1,q} = [m]_q = \frac{1 - q^m}{1 - q} = \sum_{i=0}^{m-1} q^i.$$

It is clear that for the function s of the form (1.1), we have

$$\partial_{p,q} s(\xi) = 1 + \sum_{m=1}^{\infty} \frac{[m]_{p,q}}{(pq)^m} \, \frac{a_m}{\xi^{m+1}} \, .$$

2. Definitions and Lemma

For $0 \leq \nu < 1$, $\chi \geq 1$ and $s \in \Upsilon$, we present a subclass of meromorphic bi-univalent functions denoted by $B\Upsilon(\nu, \chi; p, q)$ as

Definition 2.1 ([3])**.** A function s is presented by (1.1), is said to be in the class $B\Upsilon(\nu, \chi; p, q)$ if the following conditions:

$$\Re\{(1-\chi)\frac{s(\xi)}{\xi} + \chi \partial_{p,q} s(\xi)\} > \nu, \quad (\xi \in \mho),$$

and

$$\Re\{(1-\chi)\frac{\vartheta(\varpi)}{\varpi} + \chi \partial_{p,q} \vartheta(\varpi)\} > \nu, \quad (\varpi \in \mho),$$

hold true, where $0 \leq \nu < 1$, $\chi \geq 1$ and $\vartheta = s^{-1}$.

We note from Definition 2.1 that

$$\lim_{p \to 1} B\Upsilon(\nu, \chi; p, q) = \left\{ s : s \in \Upsilon \text{ and } \begin{Bmatrix} \Re\{(1-\chi)\frac{s(\xi)}{\xi} + \chi \partial_q s(\xi)\} > \nu \\ \Re\{(1-\chi)\frac{\vartheta(\varpi)}{\varpi} + \chi \partial_q \vartheta(\varpi)\} > \nu \end{Bmatrix} \right\} = B\Upsilon(\nu, \chi; q).$$

Furthermore,

$$\lim_{q \to 1} B\Upsilon(\nu, \chi; q) = \left\{ s : s \in \Upsilon \text{ and } \begin{Bmatrix} \Re\{(1-\chi)\frac{s(\xi)}{\xi} + \chi s'(\xi)\} > \nu \\ \Re\{(1-\chi)\frac{\vartheta(\varpi)}{\varpi} + \chi \vartheta'(\varpi)\} > \nu \end{Bmatrix} \right\} = B\Upsilon(\nu, \chi),$$

where the class $B\Upsilon(\nu, \chi)$ is defined and studied by Hamidi [14].

Definition 2.2 ([3])**.** A function s is presented by (1.1), is said to be in the class $B\Upsilon(\ltimes, \chi; p, q)$ if the following conditions

$$\Re\{(1-\chi)\frac{s(\xi)}{\xi} + \chi \partial_{p,q} s(\xi)\} < \frac{\ltimes \pi}{2}, \quad (\xi \in \mho),$$

and

$$\Re\{(1-\chi)\frac{\vartheta(\varpi)}{\varpi} + \chi \partial_{p,q} \vartheta(\varpi)\} < \frac{\ltimes \pi}{2}, \quad (\varpi \in \mho),$$

hold true, where $0 < \ltimes \leq 1$, $\chi \geq 1$ and $\vartheta = s^{-1}$ in $\mho$.

We note from Definition 2.2 that

$$\lim_{p \to 1} B\Upsilon(\ltimes, \chi; p, q) = \left\{ s : s \in \Upsilon \text{ and } \begin{Bmatrix} \Re\{(1-\chi)\frac{s(\xi)}{\xi} + \chi \partial_q s(\xi)\} < \frac{\ltimes\pi}{2} \\ \Re\{(1-\chi)\frac{\vartheta(\varpi)}{\varpi} + \chi \partial_q \vartheta(\varpi)\} < \frac{\ltimes\pi}{2} \end{Bmatrix} \right\} = B\Upsilon(\ltimes, \chi; q).$$

Furthermore,

$$\lim_{q \to 1} B\Upsilon(\ltimes, \chi; q) = \left\{ s : s \in \Upsilon \text{ and } \begin{Bmatrix} \Re\{(1-\chi)\frac{s(\xi)}{\xi} + \chi s'(\xi)\} < \frac{\ltimes\pi}{2} \\ \Re\{(1-\chi)\frac{\vartheta(\varpi)}{\varpi} + \chi \vartheta'(\varpi)\} < \frac{\ltimes\pi}{2} \end{Bmatrix} \right\} = B\Upsilon(\ltimes, \chi).$$

In the sequel, it is assumed that Θ is analytic function with the positive real part in $\mho$ satisfying $\Theta(0) = 1$, $\Theta'(0) > 0$, and $\Theta(\mho)$ is symmetric with respect to the real axis. Such a function is known to be typically real with the series expansion $\Theta(\xi) = 1 + B_1\xi + B_2\xi^2 + B_3\xi^3 + \cdots$, where B_1, B_2 are real and $B_1 > 0$. We define the following class of meromorphic functions:

Definition 2.3. A function s is presented by (1.1), is said to be in the class $B\Upsilon(\Theta, \chi; p, q)$ if the following conditions:

$$\left[(1-\chi)\frac{s(\xi)}{\xi} + \chi \partial_{p,q} s(\xi)\right] \prec \Theta(\xi), \quad (\xi \in \mho),$$

and

$$\left[(1-\chi)\frac{\vartheta(\varpi)}{\varpi} + \chi \partial_{p,q} \vartheta(\varpi)\right] \prec \Theta(\varpi), \quad (\varpi \in \mho),$$

hold true, where $0 < p < q \leq 1$, $\chi \geq 1$ and $\vartheta = s^{-1}$ in $\mho$.

We note from Definition 2.2 that

$$\lim_{p\to 1} B\Upsilon(\Theta,\chi;p,q)=\left\{s: s\in\Upsilon \text{ and } \begin{Bmatrix}\{(1-\chi)\frac{s(\xi)}{\xi}+\chi\partial_q s(\xi)\}\prec\Theta(\xi)\\ \{(1-\chi)\frac{\vartheta(\varpi)}{\varpi}+\chi\partial_q\vartheta(\varpi)\}\prec\Theta(\varpi)\end{Bmatrix}\right\}=B\Upsilon(\Theta,\chi;q).$$

Furthermore,

$$\lim_{q\to 1} B\Upsilon(\Theta,\chi;q)=\left\{s: s\in\Upsilon \text{ and } \begin{Bmatrix}\{(1-\chi)\frac{s(\xi)}{\xi}+\chi s'(\xi)\}\prec\Theta(\xi)\\ \{(1-\chi)\frac{\vartheta(\varpi)}{\varpi}+\chi\vartheta'(\varpi)\}\prec\Theta(\varpi)\end{Bmatrix}\right\}=B\Upsilon(\Theta,\chi).$$

Definition 2.4. A function s is presented by (1.1), is said to be in the class $B\Upsilon_\rho(\Theta,\chi;p,q)$ if the following conditions:

$$\left[(1-\chi)\frac{s(\xi)}{\xi}+\chi\partial_{p,q}s(\xi)-1\right]\prec_\rho(\Theta(\xi)-1),\quad(\xi\in\mho),\tag{2.1}$$

and

$$\left[(1-\chi)\frac{\vartheta(\varpi)}{\varpi}+\chi\partial_{p,q}\vartheta(\varpi)-1\right]\prec_\rho(\Theta(\varpi)-1),\quad(\varpi\in\mho),\tag{2.2}$$

hold true, where $0<p<q\le 1$, $\chi\ge 1$ and $\vartheta=s^{-1}$ in $\mho$.

We note from Definition 2.3 that

$$\lim_{p\to 1} B\Upsilon_\rho(\Theta,\chi;p,q)$$

$$=\left\{s: s\in\Upsilon \text{ and } \begin{Bmatrix}\{(1-\chi)\frac{s(\xi)}{\xi}+\chi\partial_q s(\xi)-1\}\prec_\rho(\Theta(\xi)-1)\\ \{(1-\chi)\frac{\vartheta(\varpi)}{\varpi}+\chi\partial_q\vartheta(\varpi)-1\}\prec_\rho(\Theta(\varpi)-1)\end{Bmatrix}\right\}=B\Upsilon_\rho(\Theta,\chi;q).$$

Furthermore,

$$\lim_{q\to 1} B\Upsilon_\rho(\Theta,\chi;q)$$

$$=\left\{s: s\in\Upsilon \text{ and } \begin{Bmatrix}\{(1-\chi)\frac{s(\xi)}{\xi}+\chi s'(\xi)-1\}\prec_\rho(\Theta(\xi)-1)\\ \{(1-\chi)\frac{\vartheta(\varpi)}{\varpi}+\chi\vartheta'((\varpi)-1)\}\prec_\rho(\Theta(\varpi)-1)\end{Bmatrix}\right\}=B\Upsilon_\rho(\Theta,\chi).$$

Lemma 2.1 ([19]). *If $\Bbbk\in\wp$, the class of all functions with $\Re(\Bbbk(\xi))>0$, $\xi\in\mho$, then $|\Bbbk_m|\le 2$, $m\in\{1,2,\ldots\}$, where $\Bbbk(\xi)=1+\sum_{m=1}^{\infty}\Bbbk_m\xi^m$.*

We know that $\Bbbk(\xi)\in\wp\Leftrightarrow\Bbbk(\frac{1}{\xi})\in\wp$, $\xi\in\mho$.

Define the functions $\Bbbk$ and ℓ in $\wp$ given by

$$\Bbbk(\xi)=\frac{1+u(\xi)}{1-u(\xi)}=1+\frac{\Bbbk_1}{\xi}+\frac{\Bbbk_2}{\xi^2}+\cdots,$$

and

$$\ell(\xi)=\frac{1+v(\xi)}{1-v(\xi)}=1+\frac{\ell_1}{\xi}+\frac{\ell_2}{\xi^2}+\cdots,$$

where $u(\xi)=\frac{\epsilon_1}{\xi}+\frac{\epsilon_2}{\xi^2}+\cdots+\frac{\epsilon_n}{\xi^n}+\cdots$, $|u(\xi)|<1$ and $v(\xi)=\frac{\varrho_1}{\xi}+\frac{\varrho_2}{\xi^2}+\cdots+\frac{\varrho_n}{\xi^n}+\cdots$, $|v(\xi)|<1$ $\xi\in\mho$ are the Schwarz functions (e.g., see [19]). It follows that

$$u(\xi)=\frac{\Bbbk(\xi)-1}{\Bbbk(\xi)+1}=\frac{\Bbbk_1}{2}\frac{1}{\xi}+\frac{1}{2}\left(\Bbbk_2-\frac{\Bbbk_1^2}{2}\right)\frac{1}{\xi^2}+\cdots,\tag{2.3}$$

and

$$v(\xi)=\frac{\ell(\xi)-1}{\ell(\xi)+1}=\frac{\ell_1}{2}\frac{1}{\xi}+\frac{1}{2}\left(\ell_2-\frac{\ell_1^2}{2}\right)\frac{1}{\xi^2}+\cdots.\tag{2.4}$$

In this article, we obtain the initial bounds of the class on meromorphic bi-univalent functions for $B\Upsilon(\ltimes,\chi;p,q)$, as well as for the classes $B\Upsilon(\Theta,\chi;p,q)$ and $B\Upsilon_\rho(\Theta,\chi;p,q)$ obtained by using subordination and quasi-subordination. We also give some results related to this class.

3. Main Theorems

Theorem 3.1. *Let s be presented by* (1.1), *is said to be in the class $B\Upsilon(\Theta,\chi;p,q)$ for $0<p<q\leq 1$, $\chi\geq 1$. Then*

$$|a_0|\leqslant \frac{B_1}{1-\chi}$$

and

$$|a_1|\leqslant \frac{B_1 pq}{pq-\chi(1+pq)}\,.$$

Proof. Let $s\in B\Upsilon(\Theta,\chi;p,q)$. Then there are analytic functions $u,v:\mho\to C$ with $u(\infty)=v(\infty)=0$, satisfying

$$(1-\chi)\frac{s(\xi)}{\xi}+\chi\partial_{p,q}s(\xi)=\Theta(u(\xi)),$$

and

$$(1-\chi)\frac{\vartheta(\varpi)}{\varpi}+\chi\partial_{p,q}\vartheta(\varpi)=\Theta(v(\varpi)),\quad (\vartheta=s^{-1}).$$

Since

$$(1-\chi)\frac{s(\xi)}{\xi}+\chi\partial_{p,q}s(\xi)=1+(1-\chi)a_0\frac{1}{\xi}+\Big(1-\chi\Big(1+\frac{1}{pq}\Big)\Big)a_1\frac{1}{\xi^2},\tag{3.1}$$

and

$$(1-\chi)\frac{\vartheta(\varpi)}{\varpi}+\chi\partial_{p,q}\vartheta(\varpi)=1-(1-\chi)b_0\frac{1}{\varpi}-\Big(1-\chi\Big(1+\frac{1}{pq}\Big)\Big)b_1\frac{1}{\varpi^2}\tag{3.2}$$

and also,

$$\Theta(u(\xi))=1+\frac{B_1\mathbb{k}_1}{\xi}+\frac{B_1\mathbb{k}_2+B_2\mathbb{k}_1^2}{\xi^2}+\cdots,\tag{3.3}$$

and

$$\Theta(v(\varpi))=1+\frac{B_1\ell_1}{\varpi}+\frac{B_1\ell_2+B_2\varrho_1^2}{\ell^2}+\cdots,\tag{3.4}$$

then equating coefficients (3.1) with (3.3) and (3.2) with (3.4), we obtain the following equalities:

$$(1-\chi)a_0=B_1\mathbb{k}_1,\tag{3.5}$$

$$\Big[1-\chi\Big(1+\frac{1}{pq}\Big)\Big]a_1=B_1\mathbb{k}_2+B_2\mathbb{k}_1^2,\tag{3.6}$$

$$-(1-\chi)a_0=B_1\ell_1,\tag{3.7}$$

$$-\Big[1-\chi\Big(1+\frac{1}{pq}\Big)\Big]a_1=B_1\ell_2+B_2\ell_1^2.\tag{3.8}$$

Now, considering (3.5) and (3.7), we have

$$\mathbb{k}_1=-\ell_1.$$

Also, from (3.6) and (3.8), we find that

$$a_1=\frac{B_1(\mathbb{k}_2-\ell_2)}{2\left[1-\chi\left(1+\frac{1}{pq}\right)\right]},$$

which in view of the inequalities $|\mathbb{k}_m|\leq 1$ and $|\ell_m|\leq 1$ yield

$$|a_1|\leq\frac{B_1}{1-\chi\left(1+\frac{1}{pq}\right)}\,.$$

Since $B_1>0$, the last inequality leads to the required estimate given in Theorem 3.1.

From (3.5) and (3.6), using $|\mathbb{k}_1|\leq 1$ and $|\ell_1|\leq 1$, we have

$$|a_0|=\frac{B_1}{1-\chi}\,.$$

On the other hand, comparing the coefficient of (2.3) and (2.4), we get

$$(1-\chi)a_0 = B_1\frac{\Bbbk_1}{2}, \tag{3.9}$$

$$\Big[1-\chi\Big(1+\frac{1}{pq}\Big)\Big]a_1 = \frac{1}{2}B_1\left(\Bbbk_2 - \frac{\Bbbk_1^2}{2}\right) + \frac{1}{4}B_2\Bbbk_1^2, \tag{3.10}$$

$$-(1-\chi)a_0 = B_1\frac{\ell_1}{2}, \tag{3.11}$$

$$-\Big[1-\chi\Big(1+\frac{1}{pq}\Big)\Big]a_1 = \frac{1}{2}B_1\left(\ell_2 - \frac{\ell_1^2}{2}\right) + \frac{1}{4}B_2\ell_1^2. \tag{3.12}$$

From (3.9) and (3.11), we have $\Bbbk_1 = -\ell_1$. Considering the sums of (3.10) and (3.12) with $\Bbbk_1 = -\ell_1$, we have

$$2\Big[1-\chi\Big(1+\frac{1}{pq}\Big)\Big]a_1 = \frac{1}{2}B_1\left(\Bbbk_2 - \ell_2\right)$$

and hence,

$$|a_1| \le \frac{B_1}{1-\chi\left(1+\frac{1}{pq}\right)}.$$

□

Upon letting $p \to 1$ in Theorem 3.1, we obtain following

Corollary 3.1. *Let s of the form (1.1) be in the class $B\Upsilon(\Theta,\chi;q)$ for $0<q\le 1,\quad \chi\ge 1$. Then*

$$|a_0| \leqslant \frac{B_1}{1-\chi}$$

and

$$|a_1| \leqslant \frac{B_1 q}{q-\chi(1+q)}.$$

Upon letting $q \to 1$ in Corollary 3.1, we can get the following

Corollary 3.2. *Let s of the form (1.1) be in the class $B\Upsilon(\Theta,\chi)$ for $\chi\ge 1$. Then*

$$|a_0| \leqslant \frac{B_1}{1-\chi}$$

and

$$|a_1| \leqslant \frac{B_1}{1-2\chi}.$$

Theorem 3.2. *Let the function s given by the series (1.1) be in the class $B\Upsilon(\ltimes,\chi;p,q)$ for $0<p<q\le 1,\quad \chi\ge 1$. Then*

$$|a_0| \leqslant \frac{2\ltimes}{1-\chi}$$

and

$$|a_1| \leqslant \frac{2\ltimes}{1-\chi\left(1+\frac{1}{pq}\right)}.$$

Proof. Let the function s be a member of the class $B\Upsilon(\ltimes,\chi;p,q)$, $(0<\ltimes\le 1,\ \chi\ge 1)$. Then by Definition 2.2, we have the following:

$$(1-\chi)\frac{s(\xi)}{\xi} + \chi\partial_{p,q}s(\xi) = [\Bbbk(\xi)]^{\ltimes}, \quad (\xi\in\mho), \tag{3.13}$$

and

$$(1-\chi)\frac{\vartheta(\varpi)}{\varpi} + \chi\partial_{p,q}\vartheta(\varpi) = [\ell(\varpi)]^{\ltimes}, \quad (\varpi\in\mho), \tag{3.14}$$

where $\Bbbk(\xi)$ and $\ell(\varpi)$,

$$\Bbbk(\xi) = 1 + \frac{\Bbbk_1}{\xi} + \frac{\Bbbk_2}{\xi^2} + \cdots, \quad \text{and} \quad \ell(\xi) = 1 + \frac{\ell_1}{\xi} + \frac{\ell_2}{\xi^2} + \cdots$$

are the functions with a positive real part in $\mho$.

Now, equating the coefficients of (3.13) and (3.14), we have the following:

$$(1-\chi)a_0 = \ltimes \Bbbk_1, \tag{3.15}$$

$$\left[1 - \chi\left(1 + \frac{1}{pq}\right)\right]a_1 = \ltimes \Bbbk_2 + \frac{1}{2} \ltimes (\ltimes - 1)\Bbbk_1^2, \tag{3.16}$$

$$-(1-\chi)a_0 = \ltimes \ell_1, \tag{3.17}$$

$$-\left[1 - \chi\left(1 + \frac{1}{pq}\right)\right]a_1 = \ltimes \ell_2 + \frac{1}{2} \ltimes (\ltimes - 1)\ell_1^2. \tag{3.18}$$

Equations (3.15) and (3.17) yield the relation between $\Bbbk_1$ and ℓ_1 as $\Bbbk_1^2 = \ell_1^2$.

We shall first obtain a refined estimate for a_1 for our future use. For this purpose, we add (3.16) and (3.18), and also using $\Bbbk_1^2 = \ell_1^2$,

$$a_1 = \frac{\ltimes(\Bbbk_2 - \ell_2)}{2\left[1 - \chi\left(1 + \frac{1}{pq}\right)\right]},$$

and $|\Bbbk_2| \leq 2, |\ell_2| \leq 2$, we get the following:

$$|a_1| \leqslant \frac{2\ltimes}{1 - \chi\left(1 + \frac{1}{pq}\right)}.$$

From (3.15) and (3.17),

$$a_0^2 = \frac{\ltimes^2(\Bbbk_1^2 + \ell_1^2)}{2\left(1-\chi\right)^2},$$

and taking the modulus, we get the following:

$$|a_0| \leqslant \frac{2\ltimes}{1-\chi}. \qquad \square$$

Upon letting $p \to 1^-$ in Theorem 3.2, we obtain following

Corollary 3.3. *Let s of the form* (1.1) *be in the class $B\Upsilon(\ltimes, \chi; q)$. Then*

$$|a_0| \leqslant \frac{2\ltimes}{1-\chi}$$

and

$$|a_1| \leqslant \frac{2\ltimes}{1 - \chi\left(1 + \frac{1}{q}\right)}.$$

Upon letting $q \to 1^-$ in Corollary 3.3, we obtain the following

Corollary 3.4. *Let s of the form* (1.1) *be in the class $B\Upsilon(\ltimes, \chi)$. Then*

$$|a_0| \leqslant \frac{2\ltimes}{1-\chi}$$

and

$$|a_1| \leqslant \frac{2\ltimes}{1-2\chi}.$$

Throughout this paper, it is assumed that Θ is analytic in $\mho$ with $\Theta(0) = 1$ and let

$$\Theta(\xi) = 1 + \frac{B_1}{\xi} + \frac{B_2}{\xi^2} + \frac{B_3}{\xi^3} + \cdots, \qquad B_1 > 0. \tag{3.19}$$

Also, let

$$h(\xi) = D_0 + \frac{D_1}{\xi} + \frac{D_2}{\xi^2} + \frac{D_3}{\xi^3} + \cdots, \qquad h(\xi) \leq 1. \tag{3.20}$$

Theorem 3.3. *Let the function s given by the series* (1.1) *be in the class $B\Upsilon_\rho(\Theta, \chi; p, q)$ for $0 < p < q \leq 1$, $\chi \geq 1$. Then*

$$|a_0| \leqslant \frac{D_0 B_1}{1-\chi}$$

and

$$|a_1| \leqslant \frac{pq\,(D_0 B_2 + D_1 B_1)}{pq - \chi\,(1 + pq)}.$$

Proof. If $s \in B\Upsilon_\rho(\Theta, \chi; p, q)$, then there are two analytic functions $u, v : \mho \to \mho$ with $u(0) = v(0) = 0$, $|u(\xi)| < 1$, $|v(\varpi)| < 1$, and a function h given by (3.20) such that

$$\left[(1-\chi)\frac{s(\xi)}{\xi} + \chi \partial_{p,q} s(\xi) - 1\right] = h(\xi)\left(\Theta(u(\xi)) - 1\right) \tag{3.21}$$

and

$$\left[(1-\chi)\frac{\vartheta(\varpi)}{\varpi} + \chi \partial_{p,q} \vartheta(\varpi) - 1\right] = h(\varpi)\left(\Theta(v(\varpi)) - 1\right). \tag{3.22}$$

Determine the functions $\Bbbk(\xi)$ and $\ell(\xi)$ given by

$$\Bbbk(\xi) = \frac{1 + u(\xi)}{1 - u(\xi)} = 1 + \frac{\Bbbk_1}{\xi} + \frac{\Bbbk_2}{\xi} + \cdots$$

and

$$\ell(\varpi) = \frac{1 + v(\varpi)}{1 - v(\varpi)} = 1 + \frac{\ell_1}{\varpi} + \frac{\ell_2}{\varpi} + \cdots.$$

Thus

$$u(\xi) = \frac{\Bbbk(\xi) - 1}{\Bbbk(\xi) + 1} = \frac{\Bbbk_1}{2}\frac{1}{\xi} + \frac{1}{2}\left(\Bbbk_2 - \frac{\Bbbk_1^2}{2}\right)\frac{1}{\xi^2} + \cdots \tag{3.23}$$

and

$$v(\varpi) = \frac{\ell(\varpi) - 1}{\ell(\varpi) + 1} = \frac{\ell_1}{2}\frac{1}{\varpi} + \frac{1}{2}\left(\ell_2 - \frac{\ell_1^2}{2}\right)\frac{1}{\varpi^2} + \cdots. \tag{3.24}$$

In fact, $\Bbbk$ and ℓ are analytic in $\mho$ with $\Bbbk(0) = \ell(0) = 1$.

Since $u, v : \mho \to \mho$, the functions $\Bbbk, \ell$ have a positive real part in $\mho$, and the relations $|\Bbbk_m| < 2$ and $|\ell_m| < 2$ are true. Using (3.23) and (3.24) together with (3.19) and (3.20), in the right side of correlations (3.21) and (3.22), we have

$$h(\xi)\left(\Theta(u(\xi)) - 1\right) = \frac{1}{2}D_0 B_1 \Bbbk_1 \frac{1}{\xi} + \left(\frac{1}{2}D_1 B_1 \Bbbk_1 + \frac{1}{2}D_0 B_1\left(\Bbbk_2 - \frac{\Bbbk_1^2}{2}\right) + \frac{1}{4}D_0 B_2 \Bbbk_1^2\right)\frac{1}{\xi^2} + \cdots$$

and

$$h(\varpi)\left(\Theta(v(\varpi)) - 1\right) = \frac{1}{2}D_0 B_1 \ell_1 \frac{1}{\varpi} + \left(\frac{1}{2}D_1 B_1 \ell_1 + \frac{1}{2}D_0 B_1\left(\ell_2 - \frac{\ell_1^2}{2}\right) + \frac{1}{4}D_0 B_2 \ell_1^2\right)\frac{1}{\varpi^2} + \cdots.$$

In view of (3.21) and (3.22), we have

$$(1-\chi)a_0 = \frac{D_0 B_1 \Bbbk_1}{2}, \tag{3.25}$$

$$\Big[1-\chi\Big(1+\frac{1}{pq}\Big)\Big]a_1 = \frac{D_1 B_1 \Bbbk_1}{2} + \frac{D_0 B_1}{2}\left(\Bbbk_2 - \frac{\Bbbk_1^2}{2}\right) + \frac{D_0 B_2 \Bbbk_1^2}{4}, \tag{3.26}$$

$$-(1-\chi)a_0 = \frac{D_0 B_1 \ell_1}{2}, \tag{3.27}$$

$$\Big[1-\chi\Big(1+\frac{1}{pq}\Big)\Big]a_1 = \frac{D_1 B_1 \ell_1}{2} + \frac{D_0 B_1}{2}\left(\ell_2 - \frac{\ell_1^2}{2}\right) + \frac{D_0 B_2 \ell_1^2}{4}. \tag{3.28}$$

Now, due to (3.25) and (3.27), we have

$$\Bbbk_1 = -\ell_1 \tag{3.29}$$

and

$$2(1-\chi)a_0 = \frac{D_0 B_1 (\Bbbk_1 - \ell_1)}{2}. \tag{3.30}$$

Thus we obtain

$$|a_0| \leqslant \frac{D_0 B_1}{1-\chi}.$$

Adding (3.26) and (3.28), we get

$$2\left[1-\chi\Big(1+\frac{1}{pq}\Big)\right]a_1 = \frac{D_1 B_1 (\Bbbk_1 + \ell_1)}{2} + \frac{D_0 B_1 (\Bbbk_2 + \ell_2)}{2} + \frac{D_0 (B_2 - B_1)\left(\Bbbk_1^2 + \ell_1^2\right)}{4}, \tag{3.31}$$

Using (3.29), (3.30) and $|\Bbbk_m| \le 2, \quad |\ell_m| \le 2$ in (3.31), we obtain

$$|a_1| \leqslant \frac{D_0 B_2 + D_1 B_1}{1-\chi\left(1+\frac{1}{pq}\right)}.$$

This completes the proof of Theorem 3.3. □

Upon letting $p \to 1^-$ in Theorem 3.3, we obtain the following

Corollary 3.5. *Let the function s given by the series* (1.1) *be in the class $B\Upsilon_\rho(\Theta,\chi;q)$. Then*

$$|a_0| \leqslant \frac{D_0 B_1}{1-\chi}$$

and

$$|a_1| \leqslant \frac{q\,(D_0 B_2 + D_1 B_1)}{q-\chi\,(1+q)}.$$

Upon letting $q \to 1^-$ in the above Corollary 3.5, we obtain the following

Corollary 3.6. *Let the function s given by the series* (1.1) *be in the class $B\Upsilon_\rho(\Theta,\chi)$. Then*

$$|a_0| \leqslant \frac{D_0 B_1}{1-\chi}$$

and

$$|a_1| \leqslant \frac{D_0 B_2 + D_1 B_1}{1-2\chi}.$$

4. Conclusion

We introduced a subclass of meromorphic bi-quasi-subordinate functions using the (p,q)-Jackson derivative. We tried to find the coefficient bounds for these functions and explore several special consequences of our results. We extended our understanding of meromorphic function classes and their behavior under generalized derivatives, offering new insights and potential applications in complex analysis. We believe that the readers will be able to find useful results if they use new notions from the fractional calculus and combining them with the appropriate concepts from this area.

Acknowledgement

The first authors was supported by Dicle University. The second author was supported by Istanbul Technical University. The authors express their gratitude to dear unknown referees for their helpful suggestions which improved the final version of this paper.

References

1. A. G. Alamoush, M. Darus, Faber polynomial coefficient estimates for a new subclass of meromorphic bi-univalent functions. *Adv. Inequal. Appl.* **2016** (2016), no. 3, 8 pp.
2. T. Al-Hawary, F. Yousef, B. A. Frasin, Subclasses of analytic functions of complex order involving Jackson's (p, q)-derivative. In: *Proceedings of International Conference on Fractional Differentiation and its Applications (ICFDA)*, 2018.
3. A. Alsoboh, M. Darus, Faber polynomial coefficient bounds of the meromorphic bi-univalent functions associated with Jackson's (p, q)-derivative. In: *The 6th International Arab Conference on Mathematics and Computations (IACMC)*, 2019.
4. Ş. Altınkaya, S. Yalçın, Certain classes of bi-univalent functions of complex order associated with quasi-subordination involving (p, q)-derivative operator. *Kragujevac J. Math.* **44** (2020), no. 4, 639–649.
5. S. Araci, U. Duran, M. Acikgoz, H. M. Srivastava, A certain (p, q)-derivative operator and associated divided differences. *J. Inequal. Appl.* **2016**, paper no. 301, 8 pp.
6. A. Aral, V. Gupta, R. P. Agarwal, *Applications of q-calculus in Operator Theory.* Springer, New York, 2013.
7. A. Boutiara, J. Alzabut, M. Ghaderi, S. Rezapour, On a coupled system of fractional (p, q)-differential equation with Lipschitzian matrix in generalized metric space. *AIMS Math.* **8** (2022), no. 1, 1566–1591.
8. R. Chakrabarti, R. A. Jagannathan, A (p, q)-oscillator realization of two-parameter quantum algebras. *J. Phys. A* **24** (1991), no. 13, L711–L718.
9. P. L. Duren, *Univalent functions.* Grundlehren der M athematischen Wissenschaften, 259. Springer-Verlag, New York, 1983.
10. S. Etemad, A. Hussain, J. Alzabut, S. Rezapour, A. G. M. Selvam, On a fractional cantilever beam model in the q-difference inclusion settings via special multi-valued operators. *J. Inequal. Appl.* **2021**, paper no. 174, 20 pp.
11. S. Etemad, M. A. Ragusa, S. Rezapour, A. Zada, Existence property of solutions for multi-order q-difference FBVPs based on condensing operators and end-point technique. *Fixed Point Theory* **25** (2024), no. 1, 115–141.
12. M. Ghaderi, S. Rezapour, A study on a fractional q-integro-differential inclusion by quantum calculus with numerical and graphical simulations. *Sahand Commun. Math. Anal.* **21**, no. 1 (2024), 189–206.
13. S. A. Halim, S. G. Hamidi, V. Ravichandran, Coefficient estimates for meromorphic bi-univalent functions, (preprint) 2011. `https://arxiv.org/pdf/1108.4089`.
14. S. Hamidi, S. Halim, J. M. Jahangiri, Coefficient estimates for a class of meromorphic bi-univalent functions. *C. R. Math. Acad. Sci. Paris* **351** (2013), no. 9-10, 349–352.
15. F. H. Jackson, XI.–On q-functions and a certain difference operator. *Earth and Environmental Science Transactions of the Royal Society of Edinburgh* **46** (1909), no. 2, 253–281.
16. F. H. Jackson, q-Difference Equations. *Amer. J. Math.* **32** (1910), no. 4, 305–314.
17. S. Kanas, D. Răducanu, Some class of analytic functions related to conic domains. *Math. Slovaca* **64** (2014), no. 5, 1183–1196.
18. T. H. MacGregor, Majorization by univalent functions. *Duke Math. J.* **34** (1967), 95–102.
19. C. Pommerenke, *Univalent Functions.* With a chapter on quadratic differentials by Gerd Jensen. Studia Mathematica/Mathematische Lehrbücher, Band XXV. Vandenhoeck & Ruprecht, Göttingen, 1975.
20. S. D. Purohit, R. K. Raina, Fractional q-calculus and certain subclasses of univalent analytic functions. *Mathematica* **55(78)** (2013), no. 1, 62–74.
21. F. Y. Ren, S. Owa, S. Fukui, Some inequalities on quasi-subordinate functions. *Bull. Austral. Math. Soc.* **43** (1991), no. 2, 317–324.
22. S. Rezapour, C. Thaiprayoon, S. Etemad, W. Sudsutad, C. T. Deressa, A. Zada, An existence study on the fractional coupled nonlinear q-difference systems via quantum operators along with Ulam-Hyers and Ulam-Hyers-Rassias stability. *J. Funct. Spaces* **2022**, Art. ID 4483348, 17 pp.
23. M. S. Robertson, Quasi-subordination and coefficient conjectures. *Bull. Amer. Math. Soc.* **76** (1970), 1–9.
24. F. M. Sakar, Estimating coefficients for certain subclasses of meromorphic and bi-univalent functions. *J. Inequal. Appl.* **2018**, paper no. 283, 8 pp.
25. F. M. Sakar, On sense of Yamakawa family of meromorphic bi-univalent and bi-subordinate functions. *Turkish J. Math.* **45** (2021), no. 4, 1887–1898.
26. F. M. Sakar, M. Aydoğan, Z. Karahüseyin, Quasi subordinations for bi-univalent functions with symmetric conjugate points. *Sigma J. Eng. Nat. Sci.* (2024) **42**, no. 1, 57-62.
27. F. M. Sakar, S. G. A. Shah, S. Hussain, A. Rasheed, N. Muhammad, Q-meromorphic close-to-convex functions related with Janowski function. *Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat.* **71** (2022), no. 1, 25–38.
28. G. Schober, Coefficients of inverses of meromorphic univalent functions. *Proc. Amer. Math. Soc.* **67** (1977), no. 1, 111–116.

29. M. Shabibi, M. E. Samei, M. Ghaderi, S. Rezapour, Some analytical and numerical results for a fractional q-differential inclusion problem with double integral boundary conditions. *Adv. Difference Equ.* **2021**, paper no. 466, 17 pp.
30. G. Springer, The coefficient problem for schlicht mappings of the exterior of the unit circle. *Trans. Amer. Math. Soc.* **70** (1951), 421–450.
31. A. Toto, L.I. Cotîrlă, Preserving classes of meromorphic functions through integral operators. *Symmetry* **14** (2022), no. 8, 1545.

(Received: 27.08.2024; Accepted: 09.12.2024; Published online: 10.02.2026)

[1]Department of Management, Dicle University, Diyarbakir, Turkey

[2]Department of Mathematics, Istanbul Technical University, Istanbul, Turkey
Email address: mugesakar@hotmail.com
Email address: aydogansm@itu.edu.tr