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HARDY–LITTLEWOOD MAXIMAL OPERATOR ON ASSOCIATE SPACES

TENGIZ KOPALIANI AND ANI OZBETELASHVILI

Abstract. Let X be a Banach function space over a locally compact abelian group admitting a family of coverings. We show that if the Hardy–Littlewood maximal operator M is bounded on the space X, then its boundedness on the associate space X' is equivalent to a certain condition $\mathcal{A}_\infty$.

1. Intriduction

One of the central problems of Harmonic Analysis is the problem of the boundedness of the Hardy–Littlewood maximal operator M on the Banach function spaces. In 2005, Diening [4, Theorem 8.1] proved the following result: the Hardy–Littlewood maximal operator M (defined by the Euclidean balls or cubes) is bounded on the reflexive variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ if and only if it is bounded on its dual $L^{q(\cdot)}(\mathbb{R}^n)$ space. For the Euclidean setting, A. Lerner in [9] proved that if the Hardy–Littlewood maximal operator M is bounded on a Banach function space X, then its boundedness on the associate space X' is equivalent to a certain condition $\mathcal{A}_\infty$. Analogous result for BFS on a space of homogeneous type was obtained in [8]. The boundedness of the Hardy–Littlewood maximal operator M on the Banach function space was considered in [3] and [10]. In [5], the authors studied the weighted norm inequalities for maximal type operators such as the Hardy–Littlewood maximal operator associated with $\mathbb{E} := E_r(x)_{r\in\mathbb{I},x\in X}$, a family of open subsets of a topological space X endowed with a non-negative Borel measure μ satisfying certain basic conditions. However, there are many examples of important families of measurable sets arising in harmonic analysis and PDE that cannot be generated by a quasi-metric, and hence are not in the scope of spaces of homogeneous type (for examples, refer to [5]).

The aim of this paper is to present an extension of Lerner's above-mentioned theorem to the case of Locally Compact Abelian (LCA) groups G with covering families.

Suppose that G is a locally compact abelian group equipped with a nontrivial and inner regular measure μ such that $\mu(K) < \infty$ for all compact $K \subset G$. Note that μ is not necessarily a Haar measure, since we do not assume that μ is translation-invariant. The general assumption on the group will be that it admits a sequence of neighborhoods of 0 with certain properties that will be described in the next definition (see [6, Section 2.1]).

Definition 1.1. A family $\{U_i\}_{i\in\mathbb{Z}}$ is a covering family for G if:

(1) $\{U_i\}_{i\in\mathbb{Z}}$ is an increasing base of relatively compact neighborhoods of 0, $\cup_{k\in\mathbb{Z}}U_k = G$, and $\cap_{k\in\mathbb{Z}}U_k = \{0\}$,

(2) there exist a positive constant c_D and a mapping $\theta : \mathbb{Z} \to \mathbb{Z}$ such that for all $k \in \mathbb{Z}$ and all $x \in G$,

a) $k < \theta(k)$,
b) $U_k - U_k \subset U_{\theta(k)}$,
c) $\mu(x + U_{\theta(k)}) \leq c_D \mu(x + U_k)$.

We call condition c) the doubling property of measure μ with respect to θ, and call c_D the doubling constant. Note that $c_D \geq 1$ is required because $U_k \subset U_{\theta(k)}$. Any group G admitting a sequence $\{U_i\}_{i\in\mathbb{Z}}$ of neighbourhoods of 0 and satisfying the above postulates is said to have a family of coverings. This concept was introduced in [6] by *Edwards* and *Gaudry*.

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Some basic properties of a family of coverings follow directly from Definition 1.1. For instance: (1) for every $x \in G$ and $k \in \mathbb{Z}$, there is $\mu(x+U_k) > 0$; (2) the interiors U_k' of the base sets U_k cover G, i.e., $\cup_{k\in\mathbb{Z}} U_k' = G$. In particular, for every compact $K \subset G$, there is $k \in \mathbb{Z}$ such that $K \subset U_k$ (see [12, Proposition 1]).

Among the well known groups satisfying Definition 1.1, there are the groups $\mathbb{R}$, $\mathbb{Z}$, torus $\mathbb{T}$, p-adic group $\mathbb{Q}_p$ and finite products of these groups (for more details, see [6]). Moreover, if G is an LCA group equipped with a nontrivial Haar measure and with an increasing sequence $\{U_k\}_{k\in\mathbb{Z}}$ of compact open subgroups such that

$$\bigcup_{k\in\mathbb{Z}} U_k = G, \qquad \bigcap_{k\in\mathbb{Z}} U_k = \{0\},$$

then Definition 1.1 satisfies this property if and only if $\sup_{k\in\mathbb{Z}} |U_k/U_{k-1}| < \infty$ ($|U_k/U_{k-1}|$ is the order of factor groups U_k/U_{k-1}), where one can take $\theta(k) = k+1$.

Let G be a LCA group with the Haar measure μ, and let H be a compact and open subgroup of G with $\mu(H) = 1$. Let A be an automorphism on G such that $H \subset AH$ and $\cap_{i<0} A^i H = \{0\}$. In addition, suppose that $G = \cup_{i\in\mathbb{Z}} A^i H$. Then $\{A^i H\}_{i\in\mathbb{Z}}$ satisfies the required properties to be a covering family (for details, see [11]). A structure of this type is considered in [1] for constructing wavelets on LCA groups with open and compact subgroups.

We call any set of the form $x+U_k$, where $x \in G$, $k \in \mathbb{Z}$, a base set and the collection of all base sets we denote by $\mathcal{B}$. For any natural number n, we denote $\theta^n(k) = \theta(\theta^{n-1}(k))$, $n > 1$, $\theta^1(k) = \theta(k)$, $k \in \mathbb{Z}$. For $V = x+U_k$, we denote $V^* = x + U_{\theta(k)}$ and $V^{**} = x + U_{\theta^2(k)}$. Note that without loss of generality we can assume that the base sets are symmetric. This is not a restriction at all, since we can always consider a new family of base sets formed by the difference sets $U_i - U_i$ that increases the doubling constant from c_D to c_D^2. We denote $2U_k := U_k - U_k = U_k + U_k$ and assume that the function θ is nondecreasing. Indeed, if we replace θ by

$$\widetilde{\theta}(k) = \min\{l \in \mathbb{Z} : \ l > k \text{ with } U_k - U_k \subset U_l\},$$

we find that the function $\widetilde{\theta}$ is nondecreasing, i.e., $\widetilde{\theta}(k) \le \theta(k)$, $k \in \mathbb{Z}$, and for all $x \in G$ and $k \in \mathbb{Z}$,

$$\mu(U_{\widetilde{\theta}(k)}) \le \mu(U_{\theta(k)}) \le c_D \mu(U_k).$$

In the Euclidean setting, the standard way to introduce maximal functions is to consider averages over cubes, balls, or more general families of convex sets. However, there are many LCA groups where we have a possibility to consider a family of base sets satisfying the fundamental property of the collection of cubes or balls: any point has a family of decreasing base sets shrinking to any point and, in addition, the entire space can be covered by an increasing union of such families. The notion of base sets allows us to define a direct analogue of the Hardy–Littlewood maximal function (non-centered):

$$Mf(x) = \sup_{x\in V\in\mathcal{B}} \frac{1}{\mu(V)} \int\limits_V |f| d\mu, \quad f \in L^1_{\mathrm{loc}}(G).$$

Analogously, we can define the centered maximal function for $f \in L^1_{\mathrm{loc}}(G)$,

$$\mathcal{M}f(x) = \sup_{k\in\mathbb{Z}} \frac{1}{\mu(x+U_k)} \int\limits_{x+U_k} |f| d\mu.$$

Note that the non-centered maximal operator is comparable to the centered maximal operator (see [12, Lemma 1]). More precisely, for $f \in L^1_{\mathrm{loc}}(G)$, we have

$$\mathcal{M}f \le Mf \le c_D^2 \mathcal{M}f.$$

Let us recall the definition of a Banach function space (BFS) (see, e.g., [2, Ch. 1, Definition 1.1]).

Definition 1.2. Let $L^0 := L^0(G,\mu)$ denote the set of all real-valued measurable functions on G and let $L^0_+ := L^0_+(G,\mu)$ be the set of all non-negative measurable functions on G. The characteristic function of a set $E \subset G$ is denoted by χ_E. A mapping $\rho : L^0_+ \to [0,\infty]$ is called a Banach function norm if for all functions $f, g, f_n \in L^0_+$ with $n \in \mathbb{N}$, for all constants $a \geq 0$, and for all measurable subsets E of G, the following properties hold:

$$(A1)\ \rho(f) = 0 \Leftrightarrow f = 0 \text{ a.e.},\ \rho(af) = a\rho(f),\ \rho(f+g) \leq \rho(f) + \rho(g),$$

$$(A2)\ 0 \leq g \leq f \text{ a.e.} \Rightarrow \rho(g) \leq \rho(f)\ \text{(the lattice property)},$$

$$(A3)\ 0 \leq f_n \uparrow f \text{ a.e} \Rightarrow \rho(f_n) \uparrow \rho(f)\ \text{(the Fatou property)},$$

$$(A4)\ \mu(E) < \infty \Rightarrow \rho(\chi_E) < \infty,$$

$$(A5)\ \int\limits_E f d\mu \leq C_E \rho(f)$$

with a constant $C_E \in (0, \infty)$ that depends on E and ρ, but is independent of f.

When the functions differing only on a set of measure zero are identifed, the set X of all functions $f \in L^0$ for which $\rho(|f|) < \infty$ is called a space of Banach functions. For each $f \in X$, the norm of f is defined by

$$\|f\|_X = \rho(|f|).$$

The set X under the natural linear space operations and under this norm becomes a space of Banach functions (see [2, Chapt. 1, Theorems 1.4 and 1.6]). If ρ is a norm of a Banach function, then its associate norm ρ' is defined on L^0_+ by

$$\rho'(g) = \sup\left\{ \int\limits_G fg d\mu :\ \ f \in L^0_+,\ \rho(f) \leq 1 \right\}.$$

It is known that the associated norm ρ' is itself a Banach function norm [2, Chapt. 1, Theorem 2.2]. The Banach function space X' determined by the Banach function norm ρ' is called the associate space of X. By the Lorentz–Luxemburg theorem, $\|f\|_X = \|f\|_{X''}$, where $X'' = (X')'$.

The definition of $\|f\|_{X'}$ implies that

$$\int\limits_G |fg| d\mu \leq \|f\|_X \|g\|_{X'} \tag{1.1}$$

and

$$\|g\|_{X'} = \sup_{f \in X,\, \|f\|_X \leq 1} \int\limits_G |fg| d\mu. \tag{1.2}$$

Definition 1.3. We say that a collection $\mathcal{S} \subset \mathcal{B}$ is sparse if for every base set $V \in \mathcal{S}$, there is a measurable subset $E(V) \subset V$ such that $\mu(V) \leq 2\mu(E(V))$ and the sets $\{E(V)\}_{V \in \mathcal{S}}$ are pairwise disjoint.

Sparse domination is a recent technique allowing one to estimate many operators in harmonic analysis by using simple expressions of the form

$$\sum_{Q \in \mathcal{S}} f_{p,Q} \chi_Q,$$

where $f_{p,Q} = \left(\frac{1}{|Q|}\int_Q |f|^p\right)^{1/p}$ for $p \in (0, \infty)$, and $\mathcal{S}$ is a sparse family of cubes in $\mathbb{R}^n$.

Definition 1.4. (Condition $\mathcal{A}_\infty$). We say that a Banach function space X over an LCA group G satisfies the condition $\mathcal{A}_\infty$ if there exist the constants $C = C_{[\infty]}$ and $\gamma > 0$ such that for every finite sparse collection $\mathcal{S} \subset \mathcal{B}$, every family of non-negative numbers $\{\alpha_V\}_{V \in \mathcal{S}}$, and every family of pairwise disjoint measurable sets $\{G_V\}_{V \in \mathcal{S}}$ such that $G_V \subset V$, $V \in \mathcal{S}$, one has

$$\left\| \sum_{V \in \mathcal{S}} \alpha_V \chi_{G_V} \right\|_X \leq C_{[\infty]} \left(\max_{V \in \mathcal{S}} \frac{\mu(G_V)}{\mu(V)} \right)^\gamma \left\| \sum_{V \in \mathcal{S}} \alpha_V \chi_V \right\|_X. \tag{1.3}$$

The question about the conditions for a BFS X under which a maximal Hardy–Littlewood operator M is bounded on X (with respect to cubes or balls), leading to the boundedness of M on X', was solved by Lerner in terms of sparse families and $\mathcal{A}_\infty$-type conditions in the Euclidean setting. A similar problem for a BFS over a space of homogeneous type was studied by Karlovich in [8].

The aim of this paper is to prove the following

Theorem 1.5. *Let X be a BFS over an LCA group G with a covering family such that the corresponding Hardy–Littlewood maximal operator M is bounded on X. The following conditions are equivalent:*
(i) *M is bounded on X';*
(ii) *X satisfies the condition $\mathcal{A}_\infty$.*

The paper is organized as follows. In Section 2, we present some preliminary results. We investigate some properties of the weak reverse Hölder inequality defined by the family of covers. In Section 3, we prove the main theorem. Throughout the paper, we use C and c to denote an absolute positive constant, which may take different values in different cases.

2. Preliminaries

In the Theorem 44.18 [7], a version of the Lebesgue differentiation theorem with respect to the Haar measure is proved for LCA groups with a sequence D' (see [7, Definition 44.10]). Note that the result is still true with the obvious changes for measures that are not translation-invariant. Thus, since a covering family is, in particular, a sequence D', we have that the Lebesgue Differentiation Theorem holds in our context (for more details, see [11, Remark 2.6]).

For a given $V \in \mathcal{B}$, we denote by $j(V)$ the maximum integer such that $V = x + U_{j(V)}$ for some $x \in G$. Such a number exists (see for details [11, Remark 2.3]).

Let $V \in \mathcal{B}$ be a fixed base set and $k = j(V)$. The local base $\mathcal{B}_V$ is defined as

$$\mathcal{B}_V = \{y + U_j : y \in V, j \le k\}.$$

We also defined the enlarged set $\widehat{V}$ by the formula

$$\widehat{V} = \bigcup_{U \in \mathcal{B}_V} U.$$

We have the following geometric properties of $\widehat{V}$ (see [11]): For any $z \in V$, $\widehat{V} \subset z + U_{\theta^2(k)}$ and, as a consequence of this property, we have

$$\mu(\widehat{V}) \le \mu(z + U_{\theta^2(k)}) \le c_D^2 \mu(z + U_k). \tag{2.1}$$

Any covering family has the so-called engulfing property.

Lemma 2.1 ([11, Lemma 2.2])**.** *Let U, V be two base sets such that $U = x + U_i$ and $V = y + U_j$ with $i \le j$ and $x, y \in G$. If $U \cap V \ne \emptyset$, then $x + U_i \subset y + U_{\theta^2(j)}$.*

We need the following covering lemma. Note that Lemma 2.2 has been proved in [6] in the case of a translation-invariant measure μ. For a regular measure μ with doubling property with respect to θ, Lemma 2.2 is proved in [12].

Lemma 2.2 (see [12, Lemma 2])**.** *Let E be a subset of G and $k : E \to \mathbb{Z}$ be a mapping bounded from above such that for every $k_0 \in \mathbb{Z}$, the set $\{x \in E : k(x) \ge k_0\}$ is relatively compact in G. Then there is a sequence (x_n) of elements of E, finite or infinite, such that*
(i) *the sequence $(k_n) := (k(x_n))$ is non-increasing,*
(ii) *the sets $x_n + U_{k_n}$ are pairwise disjoint, and*
(iii) *$E \subset \bigcup (x_n + 2U_{k_n})$.*

We now define the local maximal function as follows:

$$M_V f(x) = \sup_{x \in U \in \mathcal{B}_V} \frac{1}{\mu(U)} \int\limits_U |f(y)| d\mu$$

for any $x \in \widehat{V}$ and $M_V f(x) = 0$, otherwise.

A nonnegative, locally integrable function is called a weight. Below, we use the standard notation: for any measurable set E, we denote

$$w(E)=\int_E w\,d\mu, \text{ and } f_E=\frac{1}{\mu(E)}\int_E f\,d\mu.$$

Consider, for a fixed $V\in\mathcal{B}$, the level set for the local maximal function acting on a function w at level $\lambda>0$:

$$\Omega_\lambda=\{x\in\widehat{V}:\ M_V w(x)>\lambda\}.$$

The principal method for investigateting level sets is the following Calderón–Zygmund decomposition of the set Ω_λ.

Lemma 2.3 ([11, Lemma 2.5])**.** *Let $V\in\mathcal{B}$ be a fixed base set in G and w be a nonnegative and integrable function supported on $\widehat{V}$ and $\lambda>w_{\widehat{V}}$. If Ω_λ is nonempty, then there exist a finite or countable index set $Q\subset\mathbb{Z}$ and a family $\{y_i+U_{\alpha_i}\}_{i\in Q}$ of pairwise disjoint base sets from $\mathcal{B}_V$ such that:*

(a) *The sequence $\{\alpha_i\}_{i\in Q}$ is decreasing;*

(b) $\bigcup_{i\in Q} y_i+U_{\alpha_i}\subset\Omega_\lambda\subset\bigcup_{i\in Q} y_i+U_{\theta^2(\alpha_i)}$*;*

(c) *For any $i\in Q$, we have*

$$\lambda<\frac{1}{\mu(y_i+U_{\alpha_i})}\int_{y_i+U_{\alpha_i}} w\,d\mu; \tag{2.2}$$

(d) *Given $r>\alpha_i$ for some $i\in Q$, then*

$$\frac{1}{\mu(y_i+U_r)}\int_{y_i+U_r} w\,d\mu\le c_D^2\lambda. \tag{2.3}$$

From part (b) of Lemma 2.3 and the Lebesgue differentiation theorem we obtain

$$w(x)\le\lambda \text{ for a.e. } x\in\widehat{V}\setminus\bigcup_{i\in Q} y_i+U_{\theta^2(\alpha_i)}. \tag{2.4}$$

A weight w is an $A_1=A_1(G,d\mu)$ weight if

$$[w]_{A_1}:=\sup_{V\in\mathcal{B}}\left(\frac{1}{\mu(V)}\int_V w\,d\mu\right)\operatorname{ess\,sup}_V(w^{-1})<+\infty,$$

which is equivalent to w, having the property

$$Mw(x)\le[w]_{A_1}w(x)\ \ \mu-\text{a.e. } x\in G.$$

The Muckenhoupt A_1 weights can be characterized as those weight functions, such that the maximal operator is weakly bounded in the weighted function space $L_w^{(1)}(G)$ (see Theorem 4 in [12]).

We prove the following version of the reverse Hölder inequality.

Theorem 2.4 (Weak reverse Hölder inequality)**.** *Let $w\in A_1(G)$. Then there are positive constants c and δ such that for every base set V,*

$$\left(\frac{1}{\mu(\widehat{V})}\int_{\widehat{V}} w^{1+\delta}d\mu\right)^{\frac{1}{1+\delta}}\le c\frac{1}{\mu(\widehat{V})}\int_{\widehat{V}} w\,d\mu. \tag{2.5}$$

For the local maximal function M_V, we need the following kind of “reverse” weak $(1,1)$ inequality.

Lemma 2.5. *Let $w\in L^1(G)$ be a nonnegative function, V be a base set, and $\lambda>w_{\widehat{V}}$. Then there exists a constant $C>0$ such that*

$$\int_{\{x\in\widehat{V}:\,w(x)>\lambda\}} w\,d\mu\le C\lambda\mu(\{x\in\widehat{V}:\ M_V w>\lambda\}). \tag{2.6}$$

Proof. Let $\{y_i+U_{\alpha_i}\}_{i\in Q}$ be the Calderón–Zygmund decomposition of the set Ω_λ according to Lemma 2.3 . Using (2.2), (2.3) and (2.4), we obtain

$$\begin{aligned}\int\limits_{\{x\in\widehat{V}:\,w(x)>\lambda\}} wd\mu &\le c_D^2\lambda\sum\mu(y_i+U_{\theta^2(\alpha_i)})\\ &\le c_D^4\lambda\sum\mu(y_i+U_{\alpha_i})\le C\lambda\mu(\{x\in\widehat{V}:\,M_Vw>\lambda\}).\end{aligned}$$ ☐

Proof of Theorem 2.4. We use inequality (2.6) and the fact that $w\in A_1$ to write

$$\begin{aligned}w(\{x\in\widehat{V}:\,w(x)>\lambda\})&\le C\lambda\mu(\{x\in\widehat{V}:\,M_Vw(x)>\lambda\})\\ &\le C\lambda\mu(\{x\in\widehat{V}:\,w(x)>[w]_{A_1}^{-1}\lambda\})\end{aligned}$$

for $\lambda>w_{\widehat{V}}$.

Using (2.6), we obtain

$$\begin{aligned}\int\limits_{\widehat{V}} w^r d\mu &= (r-1)\int\limits_0^\infty \lambda^{r-2}w(\{x\in\widehat{V};\,w(x)>\lambda\})d\lambda\\ &=(r-1)\int\limits_0^{w_{\widehat{V}}}\lambda^{r-2}w(\{x\in\widehat{V};\,w(x)>\lambda\})d\lambda\\ &+(r-1)\int\limits_{w_{\widehat{V}}}^\infty\lambda^{r-2}w(\{x\in\widehat{V};\,w(x)>\lambda\})d\lambda\\ &\le w_{\widehat{V}}^{r-1}w(\widehat{V})+C(r-1)\int\limits_{w_{\widehat{V}}}^\infty\lambda^{r-1}\mu(\{x\in\widehat{V};\,w(x)>[w]_{A_1}^{-1}\lambda\})d\lambda\\ &\le\frac{w(\widehat{V})^r}{\mu(\widehat{V})^{r-1}}+C\frac{r-1}{r}[w]_{A_1}^r\int\limits_{\widehat{V}}w^r d\mu.\end{aligned}$$

Choosing $r=1+\delta$ close enough to 1, the last term can be absorbed by the left-hand side, and hence (2.5) follows. ☐

Corollary 2.6. *Let $w\in A_1(G)$. Then there are positive constants c and δ such that for every base set V,*

$$\left(\frac{1}{\mu(V)}\int\limits_V w^{1+\delta}d\mu\right)^{\frac{1}{1+\delta}}\le c\frac{1}{\mu(\widehat{V})}\int\limits_{\widehat{V}}wd\mu. \tag{2.7}$$

Proof. After using (2.1), then from (2.5), we obtain (2.7). ☐

Corollary 2.7. *Let $w\in A_1(G)$. Then there are positive constants c and γ such that for every base set V and any measurable subset $E\subset V$,*

$$\frac{w(E)}{w(V)}\le c\left(\frac{\mu(E)}{\mu(V)}\right)^\gamma. \tag{2.8}$$

Proof. Using the condition $w\in A_1$ and (2.1), we obtain

$$\begin{aligned}w(\widehat{V})&\le w(V^{**})\le c\mu(V^{**})\text{ess inf}_{x\in V^{**}}w(x)\\ &\le c\mu(V)\text{ ess inf}_{x\in V}w(x)\le cw(V).\end{aligned}$$

Combining Hölder's inequalities (1.1) and (2.7), for any base set V and measurable subset $E \subset V$, we get

$$\begin{aligned} w(E) &= \int\limits_E w d\mu \le \Big(\int\limits_V w^{1+\delta} d\mu\Big)^{1/(1+\delta)} \mu(E)^{\delta/(1+\delta)} \\ &\le c \left(\frac{\mu(E)}{\mu(V)}\right)^{\delta/(1+\delta)} w(\widehat{V}) \le c \left(\frac{\mu(E)}{\mu(V)}\right)^{\delta/(1+\delta)} w(V). \end{aligned}$$

□

3. Proof of Theorem 1.5

First, let us prove (i) ⇒ (ii). Let $g \in L^0_+$ with $\|g\|_{X'} \le 1$. Using an idea of Rubio de Francia, we put

$$\mathcal{R}g(x) = \sum_{k=0}^{\infty} \frac{M^k g(x)}{(2\|M\|_{X'})^k}, \quad x \in G,$$

where M^k denotes the k-th iteration of M and $M^0 g = g$, and $|M\|_{X'}$ stands for the norm of the operator M on X'. Then $g \le \mathcal{R}g$ and $\|\mathcal{R}g\|_{X'} \le 2$. Since M is sublinear, we have

$$\begin{aligned} M\mathcal{R}g(x) &\le \sum_{k=0}^{\infty} \frac{M^{k+1} g(x)}{(2\|M\|_{X'})^k} \le 2\|M\|_{X'} \sum_{k=0}^{\infty} \frac{M^{k+1} g(x)}{(2\|M\|_{X'})^{k+1}} \\ &\le 2\|M\|_{X'} \sum_{k=0}^{\infty} \frac{M^{k} g(x)}{(2\|M\|_{X'})^{k}} \le 2\|M\|_{X'} \mathcal{R}g(x), \end{aligned}$$

whence $\mathcal{R}g \in A_1$ with $[\mathcal{R}g]_{A_1} \le 2\|M\|_{X'}$.

For every sparse collection $\mathcal{S} \subset \mathcal{B}$, every family of pairwise disjoint measurable subsets $G_V \subset V$, $V \in \mathcal{S}$ and every family of non-negative numbers α_V, $V \in \mathcal{S}$ and every $g \in L^0_+$, using the properties of $\mathcal{R}g$ along with (2.8), one has

$$\begin{aligned} &\int\limits_G \Big(\sum_{V \in \mathcal{S}} \alpha_V \chi_{G_V}\Big) g d\mu \le \sum_{V \in \mathcal{S}} \alpha_V \int\limits_{G_V} g d\mu \\ &\le \sum_{V \in \mathcal{S}} \alpha_V \int\limits_{G_V} \mathcal{R}g d\mu \le C \sum_{V \in \mathcal{S}} \alpha_V \Big(\frac{\mu(G_V)}{\mu(V)}\Big)^{\gamma} \int\limits_V \mathcal{R}g d\mu \\ &\le C \Big(\max_{V \in \mathcal{S}} \frac{\mu(G_V)}{\mu(V)}\Big)^{\gamma} \int\limits_G \Big(\sum_{V \in \mathcal{S}} \alpha_V \chi_V\Big) \mathcal{R}g d\mu \\ &\le C \Big(\max_{V \in \mathcal{S}} \frac{\mu(G_V)}{\mu(V)}\Big)^{\gamma} \Big\| \sum_{V \in \mathcal{S}} \alpha_V \chi_V \Big\|_X \|\mathcal{R}g\|_{X'} \\ &\le C \Big(\max_{V \in \mathcal{S}} \frac{\mu(G_V)}{\mu(V)}\Big)^{\gamma} \Big\| \sum_{V \in \mathcal{S}} \alpha_V \chi_V \Big\|_X . \end{aligned}$$

It remains to take here the supremum over all $g \in L^0_+$ with $\|g\|_{X'} \le 1$ and use (1.2).

Turn to proof of (ii) ⇒ (i). Let us show that there exists $C > 0$ such that for every $f \in L^1(G) \cap X'$, $f \ge 0$

$$\|Mf\|_{X'} \le C \|f\|_{X'}. \tag{3.1}$$

Note that (3.1) implies the boundedness of M on X'. Indeed, having (3.1) established, for an arbitrary nonnegative $f \in X'$, we apply $f_n = f\chi_{U_n}$ (clearly, $f_n \in L^1(G) \cap X'$). Letting then $n \to \infty$ and using the Fatou property $(A3)$ of Definition 1.2, we find that (3.1) holds for any $f \in X'$.

We consider only the case $\mu(G) = \infty$, the case $\mu(G) < \infty$ needs minor modification. Suppose $f \in L^1(G) \cap X'$, $f \ge 0$ and $||f||_{X'} \le 1$. It suffices to prove that (see (1.2)) there exists a positive constant C (independent of f) such that for any non-negative function $g \in X$, with $\|g\|_X \leq 1$,

$$\int\limits_G Mf(x)g(x)d\mu \leq C. \tag{3.2}$$

Let A be any number not less than $c_D^2 + 4$. For each integer k, we set

$$\Omega_k = \{x \in G \ : \ Mf(x) > A^k\}.$$

Denote $D_k = \Omega_k \backslash \Omega_{k+1}$. Let F_k be an arbitrary compact subset of D_k. We prove that

$$\int\limits_{\cup F_k} Mf(x)g(x)d\mu \leq C. \tag{3.3}$$

Recall that the group G is σ-compact, since $G = \cup_{k\in\mathbb{Z}}\overline{U}_k$. Using the fact that μ is inner regular measure, by a simple limiting argument from (3.3), we obtain (3.2).

Without loss of generality, we may assume that $\mu(F_k) > 0$ for all $k \in \mathbb{Z}$. For $x \in F_k$ there exists a base set $V\alpha_x = y(x) + U_{\alpha_x}$ such that $|f|_{V_{\alpha_x}} > A^k$. Note that $\bigcup V\alpha_x \subset \Omega_k$ and $\alpha_x : F_k \to \mathbb{Z}$ bounded from above mapping (see [12, Theorem 3]). We may select from collection $\{V_{\alpha_x}, x \in F_k\}$ a sequence finite or infinite of pairwise disjoint base sets V_{k_j} such that $F_k \subset \bigcup_j V_{k_j}^{**}$ (see Lemma 2.2).

Define the sets

$$E_1^k = V_{k_1}^{**} \cap F_k, \qquad E_j^k = \Big(V_{k_j}^{**} \backslash \bigcup_{s<j} V_{k_s}^{**}\Big) \cap F_k,\ j > 1.$$

Note that the sets E_j^k are pairwise disjoint and $\cup_j E_j^k = F_k$.

We present the following estimate:

$$\mu(V_{k_j} \cap D_{k+l}) \leq \frac{\widetilde{C}}{A^{l-1}}\mu(V_{k_j}) \quad l \geq 3. \tag{3.4}$$

Let $x \in V_{k_j}$ and V be an arbitrary base set such that $x \in V$. Observe that either $V \subset V_{k_j}^{**}$ or $V_{k_j} \subset V^{**}$ (see Lemma 2.1). If the second inclusion holds, then $V^{**} \cap D_k \neq \emptyset$, and hence

$$|f|_V \leq c_D^2 |f|_{V^{**}} \leq c_D^2 \cdot A^{k+1} \leq A^{k+l} \ (l \geq 3).$$

Therefore, if $|f|_V > A^{k+l}$, $(l \geq 3)$, then $V \subset V_{k_j}^{**}$. From this and from a weak type property of M (see [12, Theorem 3]), we get (3.4). Indeed,

$$\mu(V_{k_j} \cap D_{k+l}) \leq \mu\{x \in V_{k_j} \ : \ M(f\chi_{V_{k_j}^{**}})(x) > A^{k+l}\}$$

$$\leq \frac{C}{A^{k+l}} \int\limits_{V_{k_j}^{**}} |f|d\mu \leq C\frac{\mu(V_{k_j})}{A^{k+l}}|f|_{V_{k_j}^{**}} \leq C\frac{A^{k+1}}{A^{k+l}}\mu(V_{k_j}) = \frac{\widetilde{C}}{A^{l-1}}\mu(V_{k_j}).$$

Using the fact that $\Omega_{k+l} = \bigcup_{i\geq k+l} D_i$, from (3.4), we deduce that there exists the constant $\widetilde{C}$ such that

$$\mu(V_{k_j} \cap \Omega_{k+l}) \leq \frac{\widetilde{C}}{A^{l-1}}\mu(V_{k_j}), \quad l \geq 3. \tag{3.5}$$

From (3.5), we deduce that if we fix $A = \widetilde{A} > c_D^2 + 4$ such that $\widetilde{C}/\widetilde{A} < 1/2$, we obtain

$$\mu(V_{k_j} \backslash \Omega_{k+l}) \geq \frac{1}{2}\mu(V_{k_j}) \ \ (l \geq 3)$$

for any pair of indices (k, j).

Define

$$Tg(x) = \sum_{k\in\mathbb{Z}}\sum_j \Bigg(\frac{1}{\mu(V_{k_j})}\int\limits_{E_j^k} g d\mu\Bigg)\chi_{V_{k_j}}(x).$$

Using the above definition, we get

$$\int\limits_{\cup_k F_k} Mf(x)g(x)d\mu \leq A^{k+1}\sum_{k\in\mathbb{Z}}\sum_j \int\limits_{E_j^k} gd\mu \leq A\sum_{k\in\mathbb{Z}}\sum_j f_{V_{k_j}}\int\limits_{E_j^k} gd\mu$$
$$= A\int\limits_G fTg \leq 2A\|f\|_{X'}\|Tg\|_X,$$

and, consequently, for proving (3.3), it suffices to show that $\|Tg\|_X \leq C$.

Let $\mathcal{K}$ be a finite index set of pairs (k,j) of integer numbers from the definition of function Tg. Fix a natural number $\nu > 2$. Let $\mathbb{Z}_i$, $i = 0,1,\dots,\nu-1$ be the equivalent classes of $\mathbb{Z}/\nu\mathbb{Z}$. Define the index sets $\mathcal{K}_i = \{(k,j):\ (k,j)\in\mathcal{K},\ k\in Z_i\}$ $(i = 0,1,\dots,\nu-1)$. From (3.5), we deduce that if the collection $\{V_{k_j} : (k,j)\in\mathcal{K}_i\}$ $(i = 0,1,\dots,\nu-1)$ is non-empty, then it is a sparse set.

Define

$$T_{\mathcal{K}}g(x) = \sum_{(k,j)\in\mathcal{K}}\left(\frac{1}{\mu(V_{k_j})}\int\limits_{E_j^k} gd\mu\right)\chi_{V_{k_j}}(x).$$

Using the Fatou property for obtaining the estimate $\|Tg\|_X \leq C$, it suffices to prove the validity of the estimate $\|T_{\mathcal{K}}g\|_X \leq C$ for any finite subfamily $\mathcal{K}$ of indices.

Take $\nu\in\mathbb{N}, \nu > 2$ such that

$$\nu C_{[\infty]}\sum_{l=\nu}^{\infty}\frac{\widetilde{C}^{\gamma}}{\widetilde{A}^{(l-1)\gamma}} \leq \frac{1}{2}, \tag{3.6}$$

where the constants $C_{[\infty]}$ and γ are from (1.3).

Denote $\alpha_{k,j} = \frac{1}{\mu(V_{k_j})}\int_{E_j^k} fd\mu$. Then for all $x\in G$,

$$T_{\mathcal{K}}g(x) = \sum_{(k,j)\in\mathcal{K}}\left(\frac{1}{\mu(V_{k_j})}\int\limits_{E_j^k} gd\mu\right)\chi_{V_{k_j}\setminus\Omega_{k+\nu}}(x)$$
$$+\sum_{(k,j)\in\mathcal{K}}\left(\frac{1}{\mu(V_{k_j})}\int\limits_{E_j^k} gd\mu\right)\chi_{V_{k_j}\cap\Omega_{k+\nu}}(x) = T^1_{\mathcal{K}}g + T^2_{\mathcal{K}}g,$$

we have

$$\Omega_k\setminus\Omega_{k+\nu} = \bigcup_{i=0}^{\nu-1}\Omega_{k+i}\setminus\Omega_{k+i+1}. \tag{3.7}$$

It is easy to see that if $x\in V_{k_j}$, then

$$\alpha_{j,k} \leq \frac{1}{\mu(V_{k_j})}\int\limits_{V_{k_j}} gd\mu \leq Mg(x), \tag{3.8}$$

and from (3.7), (3.8), for $x\in G$, we get

$$T^1_{\mathcal{K}}g(x) = \sum_{j,k}\alpha_{j,k}\chi_{V_{k_j}\setminus\Omega_{k+\nu}}(x) \leq Mg(x)\sum_k\chi_{\Omega_k\setminus\Omega_{k+\nu}}(x)$$
$$\leq Mg(x)\sum_{i=0}^{\nu-1}\sum_{k:\,(k,j)\in\mathcal{K}}\chi_{\Omega_{k+i}\setminus\Omega_{k+i+1}} \leq \nu Mg(x).$$

Consequently, we have

$$\|T^1_{\mathcal{K}}g\|_X \leq \nu\|Mg\|_X \leq \nu\|M\|_X\|g\|_X. \tag{3.9}$$

We also have

$$T^2_{\mathcal{K}}g(x) = \sum_{i=0}^{\nu-1}\sum_{(k,j)\in\mathcal{K}_i}\alpha_{j,k}\chi_{V_{k_j}\cap\Omega_{k+\nu}} = \sum_{i=0}^{\nu-1}T^2_{\mathcal{K}_i}g(x). \tag{3.10}$$

For $x \in G$, it follows that

$$T^2_{\mathcal{K}_i} g(x) = \sum_{(k,j)\in\mathcal{K}_i} \alpha_{j,k}\chi_{V_{k_j}\cap\Omega_{k+\nu}} = \sum_{(k,j)\in\mathcal{K}_i} \alpha_{j,k} \sum_{l=\nu}^{\infty} \chi_{V_{k_j}\cap(\Omega_{k+l}\setminus\Omega_{k+l+1})}(x)$$
$$= \sum_{l=\nu}^{\infty} \sum_{(k,j)\in\mathcal{K}_i} \alpha_{j,k}\chi_{V_{k_j}\cap(\Omega_{k+l}\setminus\Omega_{k+l+1})}(x). \tag{3.11}$$

Applying (3.4) and the fact that $\{V_{k_j} : (k,j) \in \mathcal{K}_i\}$ is a sparse collection, for all $l \geq \nu$, we obtain

$$\Big\| \sum_{(j,k)\in\mathcal{K}_i} \alpha_{j,k}\chi_{V_{k_j}\cap(\Omega_{k+l}\setminus\Omega_{k+l+1})} \Big\|_X$$
$$\leq C_{[\infty]} \left(\max_{(k,j)\in\mathcal{K}_i} \frac{\mu(V_{k_j}\cap(\Omega_{k+l}\setminus\Omega_{k+l+1}))}{\mu(V_{k_j})} \right)^{\gamma} \|T_{\mathcal{K}_i} g\|_X$$
$$\leq C_{[\infty]} \frac{\widetilde{C}^{\gamma}}{\widetilde{A}^{(l-1)\gamma}} \|T_{\mathcal{K}_i} g\|_X. \tag{3.12}$$

Combining the above inequalities with (3.11), (3.12) and (3.6), we get

$$\|T^2_{\mathcal{K}_i} g\|_X \leq \frac{1}{2\nu}\|T_{\mathcal{K}_i} g\|_X \leq \frac{1}{2\nu}\|T_{\mathcal{K}} g\|_X \quad (i = 0, 1, \ldots, \nu - 1). \tag{3.13}$$

Combining (3.13), (3.9) and (3.10), we have

$$\|T_{\mathcal{K}} g\|_X \leq \nu\|\mathcal{M}\|_X\|g\|_X + \frac{1}{2}\|T_{\mathcal{K}} g\|_X.$$

Since $\mathcal{K}$ is finite, we obtain $\|T_{\mathcal{K}} g\|_X < \infty$. Hence,

$$\|T_{\mathcal{K}} g\|_X \leq 2\nu\|\mathcal{M}\|_X\|f\|_X$$

and this completes the proof of the implication (ii) ⇒ (i).

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FACULTY OF EXACT AND NATURAL SCIENCES, I. JAVAKHISHVILI TBILISI STATE UNIVERSITY, 13 UNIVERSITY STR., TBILISI 0143, GEORGIA
Email address: `tengiz.kopaliani@tsu.ge`
Email address: `aniozbetelashvili29@gmail.com`