

## CHARACTERIZATIONS FOR THE G-FRACTIONAL INTEGRAL OPERATORS IN GENERALIZED G-MORREY SPACES ON $\mathbb{R}_+$

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**Abstract.** In this paper, we study the boundedness of the G-fractional integral operators  $J_G^\alpha$  on  $\mathbb{R}_+ = [0, \infty)$  in the generalized Morrey G-spaces  $\mathcal{M}_{p,\nu,\lambda}(\mathbb{R}_+)$ . We characterize the strong and weak types of boundedness of  $J_G^\alpha$  on the generalized G-Morrey spaces, respectively.

### 1. INTRODUCTION

As is known, the maximal functions, singular integrals, and potentials generated by different differential operators are important in applications and various questions in harmonic analysis, making their study highly topical. Non-accidentally, there is a vast literature devoted to different properties of the aforementioned object of harmonic analysis. We cite only the works that are relevant to the question discussed in this paper. More detailed information the reader can find in the above-mentioned paper [4].

Based on our investigation, Gegenbauer differential operator G was introduced in [2]  $G \equiv G_\lambda = (x^2 - 1)\frac{d^2}{dx^2} + (2\lambda + 1)x\frac{d}{dx}$ ,  $x \in [1, \infty)$ ,  $\lambda \in (0, \frac{1}{2})$ . The shift operator  $A_{cht}^\lambda$  generated by G is given as follows [10]:

$$A_{cht}^\lambda f(chx) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda)\Gamma(\frac{1}{2})} \int_0^\pi f(chxcht - shxsht \cos \varphi) (\sin \varphi)^{2\lambda-1} d\varphi,$$

where  $F(x) = \int_0^\infty e^{-tx} t^{-1} dt$ ,  $x > 0$  is a Gamma function.

Let  $H_r = (0, r)$ ,  $r \in (0, \infty)$ . For any  $E \subset \mathbb{R}_+ = [0, \infty)$ ,  $278E = |E|_\lambda = \int_E sh^{2\lambda} t dt$ . The Gegenbauer maximal operator  $M_G$  and the Gegenbauer potential  $I_G^\alpha$  are introduced in [11] as follows:

$$M_G f(chx) = \sup_{r>0} \frac{1}{|H_r|_\lambda} \int_{H_r} A_{cht}^\lambda |f(chx)| sh^{2\lambda} t dt,$$

$$I_G^\alpha f(chx) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty \left( \int_0^\infty r^{\frac{\alpha}{2}-1} h_r(cht) dr \right) A_{cht}^\lambda f(chx) sh^{2\lambda} t dt,$$

where

$$h_r(cht) = \int_1^\infty e^{-\nu(\nu+2\lambda)r} P_\nu^\lambda(cht) (\nu^2 - 1)^{\lambda-\frac{1}{2}} d\nu, \quad 0 < \alpha < 2\lambda + 1,$$

and

$$P_\nu^\lambda(cht) = \frac{\Gamma(\nu + 2\lambda) \cos \pi \lambda}{\Gamma(\lambda)\Gamma(\lambda + 1)} (2cht)^{-\nu-2\lambda} {}_2F_1\left(\frac{\nu}{2} + \lambda, \frac{\nu}{2} + \lambda + \frac{1}{2}; \nu + \lambda + 1; (cht)^{-2}\right)$$

is the eigenfunction of the operator G, and  ${}_2F_1(\alpha, \beta; \gamma; x)$  is the Gauss hypergeometric function.

We also consider the Gegenbauer fractional maximal operator  $M_G^\alpha$  and Gegenbauer fractional integral  $J_G^\alpha$  introduced in [16] as follows:

$$M_G^\alpha f(chx) = \sup_{r>0} \frac{1}{|H_r|_\lambda^{1-\frac{\alpha}{\gamma}}} \int_{H_r} A_{cht}^\lambda |f(chx)| sh^{2\lambda} t dt, \quad 0 \leq \alpha < \gamma \leq 2\lambda + 1,$$

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and

$$J_G^\alpha f(chx) = \int_0^\infty |H_y|_\lambda^{\frac{\alpha}{\gamma}-1} A_{chy}^\lambda f(chx) sh^{2\lambda} y dy, \quad 0 < \alpha < \gamma \leq 2\lambda + 1.$$

Note that  $M_G^0 \equiv M_G$ .

The operators  $M_G^\alpha$ ,  $I_G^\alpha$  and  $J_G^\alpha$  play an important role in harmonic analysis and applications.

In the present work, we study the Spanne-Guliyev type boundedness of the operator  $J_G^\alpha$  from  $\mathcal{M}_{p,\omega_1,\gamma}(\mathbb{R}_+)$  to  $\mathcal{M}_{q,\omega_2,\gamma}(\mathbb{R}_+)$ ,  $1 < p < q < \infty$ , and from  $\mathcal{M}_{1,\omega_1,\gamma}(\mathbb{R}_+)$  to the weak  $W\mathcal{M}_{q,\omega_2,\gamma}(\mathbb{R}_+)$ ,  $1 < q < \infty$ . We also study the Adams-Guliyev type boundedness of the operator  $J_G^\alpha$  from  $\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}(\mathbb{R}_+)$  to  $\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}(\mathbb{R}_+)$ ,  $1 < p < q < \infty$  and from the space  $\mathcal{M}_{1,\omega_1,\gamma}(\mathbb{R}_+)$  to the weak space  $W\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}(\mathbb{R}_+)$ ,  $1 < q < \infty$ .

We characterize the Spanne-Guliyev and Adams-Guliyev boundedness type of the operator  $J_G^\alpha$  on the generalized G-Morrey space, including weak versions. These results are analogues to those presented in [4]. Here and further,  $\omega, \omega_1$  and  $\omega_2$  are the positive measurable functions on  $\mathbb{R}_+$ .

Next,  $A \lesssim B$  denotes that there exists some positive constant  $C$  depending on some nonessential parameters such that  $A \lesssim CB$ . If  $A \lesssim B$  and  $B \lesssim A$ , then we write  $A \approx B$  and say that  $A$  and  $B$  are equivalent.

## 2. NOTATIONS AND PRELEMINARY RESULTS

We denote by  $L_{p,\lambda}(\mathbb{R}_+)$ ,  $1 \leq p \leq \infty$  the space of functions  $\mu_\lambda(x) = sh^{2\lambda}x-$ , measurable on  $\mathbb{R}_+$  with the finite norm

$$\|f\|_{L_{p,\lambda}(\mathbb{R}_+)} = \left( \int_{\mathbb{R}_+} |f(cht)|^p d\mu_\lambda(t) \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \quad d\mu_\lambda(t) = sh^{2\lambda} t dt,$$

$$\|f\|_{L_{\infty,\lambda}} = \|f\|_{L_\infty} = \operatorname{ess\,sup}_{t \in \mathbb{R}_+} |f(cht)|, \quad p = \infty.$$

Let  $f \in L_{p,\lambda}(\mathbb{R}_+)$ ,  $1 \leq p < \infty$ . Then for any  $y \in \mathbb{R}_+$ , the following inequality (see [10, Lemma 2])

$$\|A_{chy}^\lambda f\|_{L_{p,\lambda}(\mathbb{R}_+)} \leq \|f\|_{L_{p,\lambda}(\mathbb{R}_+)} \quad (2.1)$$

holds.

We also denote by  $WL_{p,\lambda}(\mathbb{R}_+)$  the weak space defined as the set of locally integrable functions on  $\mathbb{R}_+$  with the finite norm

$$\|f\|_{WL_{p,\lambda}(\mathbb{R}_+)} = \sup_{t>0} t(|\{x \in \mathbb{R}_+ : |f(chx)| > t\}|_\lambda)^{\frac{1}{p}}.$$

The main objective of this paper is to obtain the results similar to [4].

In the study of local properties of solutions of partial differential equations, together with weighted Lebesgue spaces, Morrey spaces play an important role. They were introduced by C.Morrey in 1938 [18].

Consider the Riesz potential (see [1])

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 < \alpha < n.$$

In [1], Adams studied the boundedness of the Riesz potential on the Morrey space and proved the following

**Theorem A** ([1]). *Let  $0 < \alpha < n$  and  $0 \leq \lambda < n$ ,  $1 \leq p < (n-\lambda)/\alpha$ .*

(1) *If  $1 < p < (n-\lambda)/\alpha$ , then the condition*

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$$

*is necessary and sufficient for the boundedness of  $I_\alpha$  from  $M_{p,\lambda}(\mathbb{R}^n)$  to  $M_{q,\lambda}(\mathbb{R}^n)$ .*

(2) If  $p = 1$ , then the condition

$$1 - \frac{1}{q} = \frac{\alpha}{n - \lambda}$$

is necessary and sufficient for the boundedness of  $I_\alpha$  from  $M_{1,\lambda}(\mathbb{R}^n)$  to  $WM_{q,\lambda}(\mathbb{R}^n)$ .

In [7], there was introduced Gegenbauer–Morrey (G–Morrey) space associated with the Gegenbauer differential operator  $G$  as the set of locally integrable functions  $f(chx)$ ,  $x \in \mathbb{R}_+$  with the finite norm

$$\|f\|_{L_{p,\nu,\lambda}(\mathbb{R}_+)} = \sup_{x,r \in \mathbb{R}_+} \left( r^{-\nu} \int_0^r A_{cht}^\lambda |f(chx)|^p d\mu_\lambda(t) \right)^{\frac{1}{p}},$$

$1 \leq p < \infty$ ,  $0 < \lambda < 1/2$  and  $0 \leq \nu \leq 2\lambda + 1$ , and also the weak space  $WL_{p,\nu,\lambda}(\mathbb{R}_+)$  with the finite norm

$$\|f\|_{WL_{p,\nu,\lambda}(\mathbb{R}_+)} = \sup_{r>0} r \sup_{\substack{x \in \mathbb{R}_+ \\ t>0}} (t^{-\nu} |\{y \in [0, t] : A_{chy}^\lambda |f(chx)| > r\}|_\lambda)^{\frac{1}{p}}.$$

The Hardy–Littlewood–Sobolev theorem for the Gegenbauer potential  $I_G^\alpha$  in the G–Morrey space, which is analogous to Theorem A, is proved in [7].

**Theorem B** ([7, Theorem 2.1]). *Let  $0 < \alpha < 2\lambda + 1$ ,  $0 < \nu < 2\lambda + 1 - \alpha$  and  $1 \leq p < (2\lambda + 1 - \nu)/\alpha$ .*

(i) *If  $1 < p < (2\lambda + 1 - \nu)/\alpha$ , then the condition*

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda + 1 - \nu}$$

*is necessary and sufficient for the boundedness of  $I_G^\alpha$  from  $L_{p,\nu,\lambda}(\mathbb{R}_+)$  to  $L_{q,\nu,\lambda}(\mathbb{R}_+)$ .*

(ii) *If  $p = 1 < (2\lambda + 1 - \nu)/\alpha$ , then the condition*

$$1 - \frac{1}{q} = \frac{\alpha}{2\lambda + 1 - \nu}$$

*is necessary and sufficient for the boundedness of  $I_G^\alpha$  from  $L_{1,\nu,\lambda}(\mathbb{R}_+)$  to  $L_{q,\nu,\lambda}(\mathbb{R}_+)$ .*

If we take  $\nu = 0$  in Theorem B, then we obtain the following result.

**Theorem C** ([8, Theorem 3]). *Suppose that  $0 < \lambda < 1/2$ ,  $0 < \alpha < 2\lambda + 1$  and  $1 \leq p < (2\lambda + 1)/\alpha$ .*

(a) *If  $1 < p < (2\lambda + 1)/\alpha$ , then the condition*

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda + 1}$$

*is necessary and sufficient for the boundedness of  $I_G^\alpha$  from  $L_{p,\lambda}(\mathbb{R}_+)$  to  $L_{q,\lambda}(\mathbb{R}_+)$ .*

(b) *If  $p = 1$ , then the condition*

$$1 - \frac{1}{q} = \frac{\alpha}{2\lambda + 1}$$

*is necessary and sufficient for the boundedness of  $I_G^\alpha$  from  $L_{1,\lambda}(\mathbb{R}_+)$  to  $WL_{q,\lambda}(\mathbb{R}_+)$ .*

We denote  $H_r = (0, r) \subset \mathbb{R}_+$ . Further, we need the following relation (see [8]):

$$|H_r|_\lambda = \int_{H_r} sh^{2\lambda} t dt \approx \left( sh \frac{r}{2} \right)^\gamma, \quad (2.2)$$

where  $0 < \lambda < 1/2$  and

$$\gamma = \gamma_\lambda(r) = \begin{cases} 2\lambda + 1, & \text{if } 0 < r < 2, \\ 4\lambda, & \text{if } 2 \leq r < \infty. \end{cases}$$

According to (2.2), in [16], the following concept was introduced.

**Definition 2.1.** Let  $1 \leq p < \infty$ ,  $0 < \lambda < 1/2$  and  $0 \leq \nu \leq \gamma$ . We denote by  $L_{p,\nu,\lambda}(\mathbb{R}_+)$  the G-Morrey space associated with the Gegenbauer differential operator  $G$  as the set of locally integrable functions  $f(chx)$ ,  $x \in \mathbb{R}_+$  with the finite norm

$$\begin{aligned} \|f\|_{L_{p,\nu,\lambda}(\mathbb{R}_+)} &= \sup_{\substack{x \in \mathbb{R}_+ \\ r > 0}} \left( |H_r|_\lambda^{-\frac{\nu}{\gamma}} \int_0^r A_{chy}^\lambda |f(chx)|^p d\mu_\lambda(y) \right)^{\frac{1}{p}} \\ &= \sup_{\substack{x \in \mathbb{R}_+ \\ r \in (0,2)}} \left( |H_r|_\lambda^{-\frac{\nu}{2\lambda+1}} \int_0^r A_{chy}^\lambda |f(chx)|^p d\mu_\lambda(y) \right)^{\frac{1}{p}} \\ &\quad + \sup_{\substack{x \in \mathbb{R}_+ \\ r \in [2,\infty)}} \left( |H_r|_\lambda^{-\frac{\nu}{4\lambda}} \int_0^r A_{chy}^\lambda |f(chx)|^p d\mu_\lambda(y) \right)^{\frac{1}{p}}. \end{aligned}$$

Thus,  $L_{p,\nu,\lambda}(\mathbb{R}_+) = \{f \in L_{p,\lambda}^{\text{loc}}(\mathbb{R}_+) : \|f\|_{L_{p,\nu,\lambda}(\mathbb{R}_+)} < \infty\}$ .

In [17], it has been proven that if  $L_{p,0,\lambda}(\mathbb{R}_+) = L_{p,\lambda}(\mathbb{R}_+)$ ,  $L_{p,\gamma,\lambda}(\mathbb{R}_+) = L_\infty(\mathbb{R}_+)$  for  $\nu < 0$  or  $\nu > \gamma$ , then  $L_{p,\nu,\lambda}(\mathbb{R}_+) = \Theta(\mathbb{R}_+)$ , where  $\Theta$  is the set of functions equivalent to zero on  $\mathbb{R}_+$ .

**Definition 2.2.** Let  $1 \leq p < \infty$  and  $0 \leq \nu \leq \gamma$ . We denote by  $WL_{p,\nu,\lambda}(\mathbb{R}_+)$  the weak space  $L_{p,\nu,\lambda}(\mathbb{R}_+)$  defined as the set of locally integrable functions  $f(chx)$ ,  $x \in \mathbb{R}_+$  with the finite norm

$$\begin{aligned} \|f\|_{WL_{p,\nu,\lambda}(\mathbb{R}_+)} &= \sup_{r>0} r \sup_{t>0, x \in \mathbb{R}_+} \left( \left( sh \frac{t}{2} \right)^{-\nu} \left| \{y \in [0, t] : A_{chy}^\lambda |f(chx)| > r\} \right|_\lambda \right)^{\frac{1}{p}} \\ &= \sup_{r>0} r \sup_{t>0, x \in \mathbb{R}_+} \left( \left( sh \frac{t}{2} \right)^{-\nu} \int_{\{y \in [0, t] : A_{chy}^\lambda |f(chx)| > r\}} sh^{2\lambda} y dy \right)^{\frac{1}{p}}. \end{aligned}$$

**Remark 2.3.** In [11, Corollary 3.1], it has been proven that

$$|J_G^\alpha f(chx)| \lesssim \int_0^\infty \frac{|A_{chy}^\lambda f(chx)|}{(shy)^{2\lambda+1-\alpha}} d\mu_\lambda(y), \quad 0 < \alpha < 2\lambda + 1.$$

According to (2.2), we can write

$$\begin{aligned} |J_G^\alpha f(chx)| &\lesssim \int_0^\infty \frac{|A_{chy}^\lambda f(chx)|}{(shy)^{2\lambda+1-\alpha}} d\mu_\lambda(y) \lesssim \int_0^2 \frac{|A_{chy}^\lambda f(chx)|}{(sh\frac{y}{2})^{2\lambda+1-\alpha}} d\mu_\lambda(y) \\ &\quad + \int_2^\infty \frac{|A_{chy}^\lambda f(chx)|}{(sh\frac{y}{2})^{4\lambda-\alpha}} d\mu_\lambda(y) \lesssim \int_0^\infty \frac{A_{chy}^\lambda |f(chx)|}{|H_y|_\lambda^{1-\frac{\alpha}{\gamma}}} d\mu_\lambda(y) = J_G^\alpha(|f|)(chx). \end{aligned} \quad (2.3)$$

From the inequality (see [16, Proposition 2.4])  $t \leq sht \leq e^A t$ , where  $t \in [0, A]$  and  $A > 0$ , it follows that Definition 2.1 and Definition 2.2 are equivalent to the G-Morrey space introduced in [7].

This approach seemed natural to us, since it is an absolutely continuous measure of the intervals  $(0,2)$  and  $[2,\infty)$ , respectively. Therefore, the relation (2.2) was used in the formulation of the theorem, as well as in its proof.

**Theorem 2.4** ([16]). *Let  $0 < \alpha < \gamma_\lambda(r)$ ,  $0 < \nu < \gamma_\lambda(r) - \alpha$  and  $1 \leq p < (\gamma_\lambda(r) - \nu)/\alpha$ .*

(i) *If  $1 < p < (\gamma_\lambda(r) - \nu)/\alpha$ , then the condition*

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma_\lambda(r) - \nu}$$

*is necessary and sufficient for the boundedness of  $J_G^\alpha$  from  $L_{p,\nu,\lambda}(\mathbb{R}_+)$  to  $L_{q,\nu,\lambda}(\mathbb{R}_+)$ .*

(ii) If  $p = 1 < (\gamma_\lambda(r) - \nu)/\alpha$ , then the condition

$$1 - \frac{1}{q} = \frac{\alpha}{\gamma_\lambda(r) - \nu}$$

is necessary and sufficient for the boundedness of  $J_G^\alpha$  from  $L_{1,\nu,\lambda}(\mathbb{R}_+)$  to  $WL_{q,\nu,\lambda}(\mathbb{R}_+)$ .

Note that from (2.3), under the conditions of Theorem 2.4, the boundedness of the Gegenbauer potential  $I_G^\alpha$  follows from the boundedness of G-fractional Gegenbauer integral  $J_G^\alpha$ .

### 3. GENERALIZED MORREY G-SPACES

In this section, we give the following generalization of the G-Morrey spaces.

**Definition 3.1.** Let  $1 \leq p < \infty$  and  $\omega(x, r)$  be a positive measurable function on  $\mathbb{R}_+ \times (0, \infty)$ . We have defined the generalized G-Morrey space  $\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}_+)$  associated with the Gegenbauer differential operator  $G$  for all functions  $f \in L_{p,\lambda}^{\text{loc}}(\mathbb{R}_+)$  by the finite norm

$$\begin{aligned} \|f\|_{\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}_+)} &= \sup_{\substack{x \in \mathbb{R}_+ \\ r > 0}} \omega(x, r)^{-1} \left( sh \frac{r}{2} \right)^{-\frac{\gamma}{p}} \|f\|_{L_{p,\lambda}(H_r)} \\ &= \sup_{\substack{x \in \mathbb{R}_+ \\ r \in (0,2)}} \omega(x, r)^{-1} \left( sh \frac{r}{2} \right)^{-\frac{2\lambda+1}{p}} \|f\|_{L_{p,\lambda}(H_r)} \\ &\quad + \sup_{\substack{x \in \mathbb{R}_+ \\ r \in [2,\infty)}} \omega(x, r)^{-1} \left( sh \frac{r}{2} \right)^{-\frac{4\lambda}{p}} \|f\|_{L_{p,\lambda}(H_r)}. \end{aligned}$$

We also defined the weak generalized the G-Morrey space  $W\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}_+)$  of locally integrable functions  $f(chx)$ ,  $x \in \mathbb{R}_+$  with the finite norm

$$\begin{aligned} \|f\|_{W\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}_+)} &= \sup_{\substack{x \in \mathbb{R}_+ \\ r > 0}} \omega(x, r)^{-1} \left( sh \frac{r}{2} \right)^{-\frac{\gamma}{p}} \|f\|_{WL_{p,\lambda}(H_r)} \\ &= \sup_{\substack{x \in \mathbb{R}_+ \\ r \in (0,2)}} \omega(x, r)^{-1} \left( sh \frac{r}{2} \right)^{-\frac{2\lambda+1}{p}} \|f\|_{WL_{p,\lambda}(H_r)} \\ &\quad + \sup_{\substack{x \in \mathbb{R}_+ \\ r \in [2,\infty)}} \omega(x, r)^{-1} \left( sh \frac{r}{2} \right)^{-\frac{4\lambda}{p}} \|f\|_{WL_{p,\lambda}(H_r)}. \end{aligned}$$

By analogy with [19], in [14] was introduced the following notation.

**Definition 3.2.** Let  $0 \leq p < \infty$  and  $\varphi(r) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a Lebesgue measurable function. The generalized Gegenbauer–Morrey (G-Morrey) space  $M_{p,\varphi,\lambda}(\mathbb{R}_+)$  associated with the Gegenbauer differential operator  $G$  are the set of locally integrable functions  $f(chx)$ ,  $x \in \mathbb{R}_+$  with the finite norm

$$\|f\|_{M_{p,\varphi,\lambda}(\mathbb{R}_+)} \equiv \|f\|_{M_{p,\varphi,\lambda}} := \sup_{\substack{x \in \mathbb{R}_+ \\ r > 0}} \left( \frac{1}{\varphi(r)} \int_{H_r} A_{cht}^\lambda |f(chx)|^p d\mu_\lambda(t) \right)^{\frac{1}{p}},$$

and the weak G-Morrey space  $WM_{p,\varphi,\lambda}(\mathbb{R}_+)$  are the set of locally integrable functions  $f(chx)$ ,  $x \in \mathbb{R}_+$  with the finite norm

$$\begin{aligned} \|f\|_{WM_{p,\varphi,\lambda}(\mathbb{R}_+)} &\equiv \|f\|_{WM_{p,\varphi,\lambda}} \\ &= \sup_{r > 0} r \sup_{\substack{x \in \mathbb{R}_+ \\ t > 0}} \left( \frac{1}{\varphi(t)} |\{y \in H_t : A_{cht}^\lambda |f(chx)| > r\}| \right)^{\frac{1}{p}} \end{aligned}$$

$$= \sup_{r>0} r \sup_{\substack{x \in \mathbb{R}_+ \\ r>0}} \left( \frac{1}{\varphi(t)} \int_{\{y \in H_t : A_{cht}^\lambda |f(chx) > r\}} d\mu_\lambda(y) \right)^{\frac{1}{p}}.$$

Let  $0 < \delta \leq 1$ . Assume that for any  $r > 0$ ,  $\varphi(r)$  satisfies the conditions

$$r \leq t \leq 2r \Rightarrow \varphi(t) \approx \varphi(r), \quad (\text{a})$$

$$\int_r^\infty \frac{\varphi(t)}{t^{\nu\delta+1}} dt \lesssim \begin{cases} r^{-(2\lambda+1)\delta} \varphi(r), & \nu = 2\lambda + 1, 0 < r < 2, \\ r^{-4\lambda\delta} \varphi(r), & \nu = 4\lambda, 2 \leq r < \infty. \end{cases} \quad (\text{b})$$

**Theorem D** ([14]). *Let  $0 < \lambda < 1/2$ ,  $0 < \alpha < 2\lambda + 1$ ,  $1 \leq p < \alpha/(2\lambda + 1)$  and  $1/p - 1/q = \alpha/(2\lambda + 1)$ . Assume that  $\varphi$  satisfies conditions (a) and (b). Then:*

(i) *If  $p > 1$  and  $f \in M_{p,\varphi,\lambda}(\mathbb{R}_+)$ , then*

$$\|J_G^\alpha f\|_{M_{q,\varphi^{p/q},\lambda}} \lesssim \|f\|_{M_{p,\varphi,\lambda}};$$

(ii) *If  $p = 1$  and  $f \in M_{1,\varphi,\lambda}(\mathbb{R}_+)$ , then*

$$\|J_G^\alpha f\|_{WM_{q,\varphi,\lambda}(\mathbb{R}_+)} \lesssim \|f\|_{M_{1,\varphi,\lambda}(\mathbb{R}_+)}.$$

**Lemma 3.3** ([12]). *Let  $\omega(x, r)$  be a positive measurable function on  $(\mathbb{R}_+) \times (0, \infty)$ .*

(i) *If*

$$\sup_{t < r < \infty} \left( sh \frac{r}{2} \right)^{-\frac{\gamma}{p}} \omega(x, r)^{-1} = \infty$$

*is true for some  $t > 0$  and any  $x \in \mathbb{R}_+$ , then  $\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}_+) \Theta(\mathbb{R}_+)$ ;*

(ii) *If*

$$\sup_{0 < r < t} \omega(x, r)^{-1} = \infty$$

*is true for some  $t > 0$  and any  $x \in \mathbb{R}_+$ , then  $\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}_+) = \Theta(\mathbb{R}_+)$ .*

**Remark 3.4.** We denote by  $\Omega_p^\gamma$  the sets of all positive measurable functions  $\omega$  on  $\mathbb{R}_+ \times (0, \infty)$  such that for all  $t \in (0, \infty)$ ,  $\sup_{x \in \mathbb{R}_+} \left\| \frac{(sh \frac{x}{2})^{-\frac{\gamma}{p}}}{\omega(x, r)} \right\|_{L_\infty(t, \infty)} < \infty$  and  $\sup_{x \in \mathbb{R}_+} \|\omega(x, r)^{-1}\|_{L_\infty(0, t)} < \infty$ , respectively. In the following, keeping in mind Lemma 3.3, we assume that  $\omega \in \Omega_p^\gamma$ .

A function  $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is said to be almost increasing (resp., almost decreasing) if  $\omega(r) \lesssim \omega(s)$  for  $r \leq s$ . Let  $1 \leq p < \infty$ . Denote by  $\Phi_p^\gamma$  the set of all almost decreasing functions  $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $t \rightarrow (sh \frac{t}{2})^{\frac{\gamma}{p}} \omega(t)$  is almost increasing.

**Lemma 3.5** ([12, Lemma 2.4]). *Let  $\omega \in \Phi_p^\gamma$ ,  $1 \leq p < \infty$ ,  $H_0 = (0, r_0)$  and  $\chi_{H_0}$  be the characteristic function of the interval  $H_0$ . Then  $\chi_{H_0} \in \mathcal{M}_{p,\omega,\gamma}(\mathbb{R}_+)$ .*

Moreover,

$$\frac{1}{\omega(r_0)} \leq \|\chi_{H_0}\|_{WM_{p,\omega,\gamma}(\mathbb{R}_+)} \leq \|\chi_{H_0}\|_{\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}_+)} \lesssim \frac{1}{\omega(r_0)}.$$

#### 4. G-FRACTIONAL INTEGRAL OPERATOR IN THE SPACES $\mathcal{M}_{p,\omega,\lambda}(\mathbb{R}_+)$

**4.1. Spanne–Guliyev type result.** Denote  $E_\gamma = \begin{cases} (0, 2) & \text{if } \gamma = 2\lambda + 1 \\ [2, \infty) & \text{if } \gamma = 4\lambda \end{cases}$  and let

$$\begin{aligned} & \left( sh \frac{t}{2} \right)^{\frac{\gamma}{q}} \int_t^\infty \left( sh \frac{s}{2} \right)^{-\frac{\gamma}{q}-1} \|f\|_{L_{p,\lambda}(H_s)} \left( ch \frac{s}{2} \right) ds, \quad t \in E_\gamma \\ &= \begin{cases} \left( sh \frac{t}{2} \right)^{\frac{2\lambda+1}{q}} \int_t^\infty \left( sh \frac{s}{2} \right)^{-\frac{2\lambda+1}{q}-1} \|f\|_{L_{p,\lambda}(H_s)} \left( ch \frac{s}{2} \right) ds, & t \in (0, 2), \\ \left( sh \frac{t}{2} \right)^{\frac{4\lambda}{q}} \int_t^\infty \left( sh \frac{s}{2} \right)^{-\frac{4\lambda}{q}-1} \|f\|_{L_{p,\lambda}(H_s)} \left( ch \frac{s}{2} \right) ds, & t \in [2, \infty). \end{cases} \end{aligned}$$

The following result is an analogue of Theorem 4.2 from [4].

**Theorem 4.1.** Let  $1 \leq p < \infty$ ,  $0 < \alpha < \gamma_\lambda(r)$ ,  $f \in L_{p,\lambda}^{\text{loc}}(\mathbb{R}_+)$  and

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma_\lambda(r)}. \quad (4.1)$$

Then for  $p > 1$ ,

$$\|J_G^\alpha f\|_{L_{q,\lambda}(H_t)} \lesssim \left(sh \frac{t}{2}\right)^{\frac{\gamma}{q}} \int_t^\infty \left(sh \frac{s}{2}\right)^{-\frac{\gamma}{q}-1} \|f\|_{L_{p,\lambda}(H_s)} \left(ch \frac{s}{2}\right) ds, \quad t \in E_\gamma \quad (4.2)$$

and for  $p = 1$ ,

$$\|J_G^\alpha f\|_{WL_{q,\lambda}(H_t)} \lesssim \left(sh \frac{t}{2}\right)^{\frac{\gamma}{q}} \int_t^\infty \left(sh \frac{s}{2}\right)^{-\frac{\gamma}{q}-1} \|f\|_{L_{1,\lambda}(H_s)} \left(ch \frac{s}{2}\right) ds, \quad t \in E_\gamma. \quad (4.3)$$

*Proof.* For a given interval  $H_t = (0, t) \subset \mathbb{R}_+$ , we split the function  $f$  as  $f = f_1 + f_2$ , where  $f_1 = f\chi_{H_t}$ ,  $f_2 = f\chi_{(H_t)^c} = f\chi_{(t,\infty)}$ . Then

$$J_G^\alpha f(chx) = J_G^\alpha f_1(chx) + J_G^\alpha f_2(chx).$$

Let  $1 < p < \infty$ ,  $0 < \alpha < \gamma/p$  and (4.1) hold. Since  $f_1 \in L_{p,\lambda}(\mathbb{R}_+)$ , by the boundedness of the operator  $J_G^\alpha$  from  $L_{p,\lambda}(\mathbb{R}_+)$  to  $L_{q,\lambda}(\mathbb{R}_+)$  (see Theorem 2.4 by  $\nu = 0$ ), it follows that

$$\begin{aligned} \|J_G^\alpha f_1\|_{L_{q,\lambda}(\mathbb{R}_+)} &\lesssim \|f_1\|_{L_{p,\lambda}(\mathbb{R}_+)} = \|f\|_{L_{p,\lambda}(H_t)} \\ &\lesssim \left(sh \frac{t}{2}\right)^{\frac{\gamma}{q}} \int_t^\infty \left(sh \frac{s}{2}\right)^{-\frac{\gamma}{q}-1} \|f\|_{L_{p,\lambda}(H_s)} \left(ch \frac{s}{2}\right) ds, \quad t \in E_\gamma. \end{aligned} \quad (4.4)$$

Estimate the integral  $J_G^\alpha f_2$  as

$$\begin{aligned} J_G^\alpha f_2(chx) &= \int_t^\infty A_{chy}^\lambda |f(chx)| \left(sh \frac{y}{2}\right)^{\alpha-\gamma} d\mu_\lambda(y) \\ &\lesssim \int_t^\infty A_{chy}^\lambda |f(chx)| \left(\int_y^\infty \left(sh \frac{s}{2}\right)^{\alpha-\gamma-1} \left(ch \frac{s}{2}\right) ds\right) d\mu_\lambda(y) \\ &= \int_t^\infty \left(\int_{t < y < s} A_{chy}^\lambda |f(chx)| d\mu_\lambda(y)\right) \left(sh \frac{s}{2}\right)^{\alpha-\gamma-1} \left(ch \frac{s}{2}\right) ds \\ &\lesssim \int_t^\infty \left(\int_{H_s} A_{chy}^\lambda |f(chx)| d\mu_\lambda(y)\right) \left(sh \frac{s}{2}\right)^{\alpha-\gamma-1} \left(ch \frac{s}{2}\right) ds. \end{aligned}$$

Using Hölder's inequalities (2.1) and (2.2), we have

$$\begin{aligned} \int_{H_s} A_{chy}^\lambda |f(chx)| d\mu_\lambda(y) &\lesssim \left(\int_{H_s} A_{chy}^\lambda |f(chx)|^p d\mu_\lambda(y)\right)^{\frac{1}{p}} \left(\int_{H_s} d\mu_\lambda(y)\right)^{\frac{1}{p'}} \\ &\lesssim \left(sh \frac{s}{2}\right)^{\frac{\gamma}{p'}} \|A_{chy}^\lambda f\|_{L_{p,\lambda}(H_s)} \\ &\lesssim \left(sh \frac{s}{2}\right)^{\gamma(1-\frac{1}{p})} \|f\|_{L_{p,\lambda}(H_s)}, \end{aligned}$$

where  $p + p' = pp'$ .

Then we obtain

$$\begin{aligned} J_G^\alpha f_2(chx) &\lesssim \int_t^\infty \left(sh\frac{s}{2}\right)^{\alpha-\frac{\gamma}{p}-1} \|f\|_{L_{p,\lambda}(H_s)} \left(ch\frac{s}{2}\right) ds \\ &= \int_t^\infty \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{q}-1} \|f\|_{L_{p,\lambda}(H_s)} \left(ch\frac{s}{2}\right) ds. \end{aligned} \quad (4.5)$$

From (4.5) and (2.2), we get

$$\begin{aligned} \|J_G^\alpha f_2\|_{L_{q,\lambda}(H_t)} &\lesssim \|\chi_{H_t}\|_{L_{q,\lambda}(\mathbb{R}_+)} \int_t^\infty \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{q}-1} \|f\|_{L_{p,\lambda}(H_s)} \left(ch\frac{s}{2}\right) ds \\ &= \|\chi_{H_t}\|_{L_{q,\lambda}(H_t)} \int_t^\infty \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{q}-1} \|f\|_{L_{p,\lambda}(H_s)} \left(ch\frac{s}{2}\right) ds \\ &= |H_t|^\frac{1}{q} \int_t^\infty \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{q}-1} \|f\|_{L_{p,\lambda}(H_s)} \left(sh\frac{s}{2}\right) ds \\ &\approx \left(sh\frac{t}{2}\right)^\frac{\gamma}{q} \int_t^\infty \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{q}-1} \|f\|_{L_{p,\lambda}(H_s)} \left(sh\frac{s}{2}\right) ds, \quad t \in E_\gamma. \end{aligned} \quad (4.6)$$

Combining (4.4) and (4.6), we obtain (4.2).

Finally, in case  $p = 1$ , by the boundedness of the operator  $J_G^\alpha$  from  $L_{1,\lambda}(\mathbb{R}_+)$  to  $WL_{q,\lambda}(\mathbb{R}_+)$  (see Theorem 2.4 by  $\nu = 0$ ), it follows that

$$\begin{aligned} \|J_G^\alpha f_1\|_{WL_{q,\lambda}(H_t)} &\lesssim \|f_1\|_{L_{1,\lambda}(H_t)} = \|f\|_{L_{1,\lambda}(H_t)} \\ &\lesssim \left(sh\frac{t}{2}\right)^\frac{\gamma}{q} \int_t^\infty \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{q}-1} \|f\|_{L_{1,\lambda}(H_s)} \left(sh\frac{s}{2}\right) ds, \quad t \in E_\gamma. \end{aligned} \quad (4.7)$$

Note that inequality (4.6) is also true in case  $p = 1$ . Then by (4.6) and (4.7), we get inequality (4.3).  $\square$

We use the following statement on the boundedness of the weighted Hardy operator

$$H_w g(t) = \int_t^\infty g(s)w(s)ds, \quad 0 < t < \infty.$$

The following theorem is true.

**Theorem 4.2** ([4]). *Let  $v_1, v_2$  and  $w$  be the weights on  $\mathbb{R}_+$  and let  $v_1(t)$  be bounded outside a neighborhood of the origin. The inequality*

$$\operatorname{ess\,sup}_{t>0} v_2(t)H_w g(t) \leq C \operatorname{ess\,sup}_{t>0} v_1(t)g(t) \quad (4.8)$$

*holds for some  $C > 0$  for all nonnegative and nondecreasing  $g$  on  $\mathbb{R}_+$  if and only if*

$$B := \operatorname{ess\,sup}_{t>0} \int_t^\infty \frac{w(s)ds}{\operatorname{ess\,sup}_{s<r<\infty} v_1(r)} < \infty.$$

Moreover, the value  $C=B$  is the best constant for (4.8).

**Theorem 4.3.** *Let  $1 \leq p < \infty$ ,  $0 < \alpha < \gamma/p$ , condition (4.1) hold. Also, let the pair  $(\omega_1, \omega_2)$ , where  $\omega_1 \in \Omega_p^\gamma$ ,  $\omega_2 \in \Omega_q^\gamma$ , satisfy the condition*

$$\int_t^\infty \frac{\operatorname{ess\,inf}_{r<s<\infty} \omega_1(x,s) \left(sh\frac{s}{2}\right)^\frac{\gamma}{p}}{\left(sh\frac{t}{2}\right)^\frac{\gamma}{q}+1} \left(ch\frac{r}{2}\right) dr \lesssim \omega_2(x,t), \quad t \in E_\gamma. \quad (4.9)$$

Then for  $p > 1$ , the operator  $J_G^\alpha$  is bounded from  $\mathcal{M}_{p,\omega_1,\gamma}(\mathbb{R}_+)$  to  $\mathcal{M}_{q,\omega_2,\gamma}(\mathbb{R}_+)$  and for  $p = 1$ , the operator  $J_G^\alpha$  is bounded from  $\mathcal{M}_{1,\omega_1,\gamma}(\mathbb{R}_+)$  to  $W\mathcal{M}_{q,\omega_2,\gamma}(\mathbb{R}_+)$ .

*Proof.* By Theorems 4.1 and 4.2 with  $v_2(r) = \omega_2(x,r)^{-1}$ ,  $v_1(r) = (sh\frac{r}{2})^{-\frac{\gamma}{p}}\omega_1(x,r)^{-1}$ ,  $w(r) = (sh\frac{r}{2})^{-\frac{\gamma}{q}-1}ch\frac{r}{2}$ ,  $g(r) = \|f\|_{L_{p,\lambda}(H_r)}$ , we have

$$\begin{aligned} \|J_G^\alpha f\|_{\mathcal{M}_{q,\omega_2,\gamma}} &= \sup_{\substack{x \in \mathbb{R}_+ \\ t \in E_\gamma}} \omega_2(x,t)^{-1} \left( sh\frac{t}{2} \right)^{-\frac{\gamma}{q}} \|J_G^\alpha f\|_{L_{q,\lambda}(H_t)} \\ &\lesssim \sup_{\substack{x \in \mathbb{R}_+ \\ t \in E_\gamma}} \omega_2(x,t)^{-1} \int_t^\infty \left( sh\frac{r}{2} \right)^{-\frac{\gamma}{q}-1} \left( ch\frac{r}{2} \right) \|f\|_{L_{p,\lambda}(H_r)} dr \\ &\lesssim \sup_{\substack{x \in \mathbb{R}_+ \\ t \in E_\gamma}} \omega_1(x,t)^{-1} \left( sh\frac{t}{2} \right)^{-\frac{\gamma}{p}} \|f\|_{L_{p,\lambda}(H_t)} = \|f\|_{\mathcal{M}_{p,\omega_1,\gamma}}, \end{aligned}$$

and for  $p > 1$ , we have

$$\begin{aligned} \|J_G^\alpha\|_{W\mathcal{M}_{q,\omega_2,\gamma}} &\lesssim \sup_{\substack{x \in \mathbb{R}_+ \\ t \in E_\gamma}} \omega_2(x,t)^{-1} \int_t^\infty \left( sh\frac{r}{2} \right)^{-\frac{\gamma}{q}-1} \left( ch\frac{r}{2} \right) \|f\|_{L_{1,\lambda}(H_r)} dr \\ &\lesssim \sup_{\substack{x \in \mathbb{R}_+ \\ t \in E_\gamma}} \omega_1(x,t)^{-1} \left( sh\frac{t}{2} \right)^{-\gamma} \|f\|_{L_{1,\lambda}(H_t)} = \|f\|_{\mathcal{M}_{1,\omega_1,\gamma}}, \end{aligned}$$

for  $p = 1$ . □

**Remark 4.4.** Note that the analogue of Theorems 4.1 and 4.3 is proved by another method for the Gegenbauer potential  $I_G^\alpha$  in [15].

In order to prove our main result, we need the following estimate.

**Lemma 4.5.** *Let  $H_0 = (0, r_0)$ , then*

$$\left( sh\frac{r}{2} \right)^\alpha \lesssim J_G^\alpha \chi_{H_0}(chx)$$

for every  $x \in H_0$ .

*Proof.* By (2.2), we have

$$\begin{aligned} J_G^\alpha \chi_{H_0}(chx) &= \int_0^\infty \frac{A_{cht}^\lambda \chi_{H_0}(chx)}{\left( sh\frac{t}{2} \right)^{\gamma-\alpha}} d\mu_\lambda(t) = \int_0^{r_0} \left( sh\frac{t}{2} \right)^{\alpha-\gamma} sh^{2\lambda} t dt \\ &\gtrsim \left( sh\frac{r_0}{2} \right)^{\alpha-\gamma} \left( sh\frac{r_0}{2} \right)^\gamma \gtrsim \left( sh\frac{r_0}{2} \right)^\alpha. \end{aligned} \quad \square$$

The following theorem is one of our main results.

**Theorem 4.6.** *Let  $0 < \alpha < \gamma$ ,  $p, q \in [1, \infty)$ ,  $\omega_1 \in \Omega_p^\gamma$ ,  $\omega_2 \in \Omega_q^\gamma$ ,  $1 \leq p < \frac{\gamma}{\alpha}$  and  $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma}$ .*

*If  $\omega_1 \in \Phi_p^\gamma$  satisfies the condition*

$$\int_t^\infty \left( sh\frac{r}{2} \right)^{\alpha-1} \left( ch\frac{r}{2} \right) \omega_1(r) dr \lesssim \left( sh\frac{t}{2} \right)^\alpha \omega_1(t), \quad t \in E_\gamma \quad (4.10)$$

then the condition

$$\left( sh\frac{t}{2} \right)^\alpha \omega_1(t) \lesssim \omega_2(t), \quad t \in E_\gamma \quad (4.11)$$

is necessary and sufficient for the boundedness of  $J_G^\alpha$  from  $\mathcal{M}_{1,\omega_1,\gamma}(\mathbb{R}_+)$  to  $W\mathcal{M}_{q,\omega_2,\gamma}(\mathbb{R}_+)$  by  $p = 1$ .

Moreover, if  $1 < p < \gamma/\alpha$ , then condition (4.11) is necessary and sufficient for the boundedness of  $J_G^\alpha$  from  $\mathcal{M}_{p,\omega_1,\gamma}(\mathbb{R}_+)$  to  $\mathcal{M}_{q,\omega_2,\gamma}(\mathbb{R}_+)$ .

*Proof. Sufficiency.* Let (4.11) hold, then (4.10) implies (4.9) and by Theorem 4.3,  $J_G^\alpha$  is bounded from  $\mathcal{M}_{p,\omega_1,\gamma}(\mathbb{R}_+)$  to  $\mathcal{M}_{q,\omega_2,\gamma}(\mathbb{R}_+)$ .

*Necessity.* Now, let  $J_G^\alpha$  be bounded from  $\mathcal{M}_{p,\omega_1,\gamma}(\mathbb{R}_+)$  to  $\mathcal{M}_{q,\omega_2,\gamma}(\mathbb{R}_+)$ . Let  $H_0 = (0, r_0)$  and  $x \in H_0$ , then by Lemmas 3.5 and 4.5, we have

$$\begin{aligned} \left( sh \frac{t_0}{2} \right)^\alpha &\lesssim |H_0|_\lambda^{-\frac{1}{q}} \|J_G^\alpha \chi_{H_0}\|_{L_{q,\lambda}(H_0)} \lesssim \omega_2(t_0) \|J_G^\alpha \chi_{H_0}\|_{\mathcal{M}_{q,\omega_2,\gamma}} \\ &\lesssim \omega_2(t_0) \|\chi_{H_0}\|_{\mathcal{M}_{p,\omega_1,\gamma}} \lesssim \frac{\omega_2(t_0)}{\omega_1(t_0)}, \end{aligned}$$

or

$$\left( sh \frac{t_0}{2} \right)^\alpha \lesssim \frac{\omega_2(t_0)}{\omega_1(t_0)} \Leftrightarrow \left( sh \frac{t_0}{2} \right)^\alpha \omega_1(t_0) \lesssim \omega_2(t_0),$$

for all  $t_0 > 0$ .

Since this is true for every  $t_0 > 0$ , we are done.  $\square$

**Remark 4.7.** If we take  $\omega_1(t) = (sh \frac{t}{2})^{\frac{\nu-\gamma}{p}}$  and  $\omega_2(t) = (sh \frac{t}{2})^{\frac{\mu-\gamma}{q}}$  in Theorem 4.6, then conditions (4.11) and (4.10) are equivalent to  $0 < \nu < \gamma - \alpha p$  and  $\frac{\nu}{p} = \frac{\mu}{q}$ , respectively.

**Corollary 4.8** ([17]). *Let  $0 < \alpha < \gamma$ ,  $1 \leq p < \gamma/\alpha$ ,  $0 < \nu < \gamma - \alpha p$ , and (4.1) hold. Then the operator  $J_G^\alpha$  is bounded from  $L_{1,\nu,\lambda}(\mathbb{R}_+)$  to  $WL_{q,\mu,\lambda}(\mathbb{R}_+)$  if and only if  $\nu = \mu/q$ .*

*Moreover, if  $1 < p < \gamma/\alpha$ , then the operator  $J_G^\alpha$  is bounded from  $L_{p,\nu,\lambda}(\mathbb{R}_+)$  to  $L_{q,\mu,\lambda}(\mathbb{R}_+)$  if and only if  $\nu/p = \mu/q$ .*

**4.2. Adams–Guliyev type results.** The following pointwise estimate plays a key role in proving our main results.

**Theorem 4.9.** *Let  $1 \leq p < \infty$ ,  $0 < \alpha < \gamma$  and  $f \in L_{p,\lambda}^{\text{loc}}(\mathbb{R}_+)$ . Then*

$$\begin{aligned} J_G^\alpha f(chx) &\lesssim \left( sh \frac{t}{2} \right)^\alpha M_G f(chx) \\ &+ \int_t^\infty \left( sh \frac{r}{2} \right)^{\alpha - \frac{\gamma}{p} - 1} \left( ch \frac{r}{2} \right) \|f\|_{L_{p,\lambda}(H_r)} dr, \quad t \in E_\gamma. \end{aligned} \quad (4.12)$$

*Proof.* Write  $f = f_1 + f_2$ , where  $f_1 = f\chi_{(H_t)}$  and  $f_2 = f\chi_{(H_t)^c}$ . Then

$$J_G^\alpha f(chx) = J_G^\alpha f_1(chx) + J_G^\alpha f_2(chx).$$

First, we show that

$$|J_G^\alpha f_1(chx)| \lesssim \left( sh \frac{t}{2} \right)^\alpha M_G f(chx), \quad (4.13)$$

for all  $x \in \mathbb{R}_+$

In fact,

$$\begin{aligned} J_G^\alpha f_1(chx) &= \int_0^t \frac{A_{cht}^\lambda |f(chx)|}{(sh \frac{y}{2})^{\gamma-\alpha}} d\mu_\lambda(y) \\ &\lesssim \sum_{k=0}^\infty \int_{2^{-k-1}t}^{2^{-k}t} \frac{A_{cht}^\lambda |f(chx)|}{(sh \frac{y}{2})^{\gamma-\alpha}} d\mu_\lambda(y) \\ &\leq \sum_{k=0}^\infty \frac{sh^\alpha \left( \frac{t}{2^{k+1}} \right)}{sh^\gamma \left( \frac{t}{2^{k+1}} \right)} \int_0^{\frac{t}{2^k}} A_{chy}^\lambda |f(chx)| d\mu_\lambda(y) \end{aligned}$$

$$\lesssim \left( sh \frac{t}{2} \right)^\alpha M_G f(chx) \sum_{k=0}^{\infty} 2^{-k\alpha} \lesssim \left( sh \frac{t}{2} \right)^\alpha M_G f(chx).$$

For  $J_G^\alpha f_2(chx)$  with  $x \in \mathbb{R}_+$ , from (4.5), we have

$$\begin{aligned} J_G^\alpha f_2(chx) &\lesssim \int_t^\infty \left( sh \frac{r}{2} \right)^{-\frac{\gamma}{q}-1} \left( ch \frac{r}{2} \right) \|f\|_{L_{p,\lambda}(H_r)} dr \\ &= \int_t^\infty \left( sh \frac{r}{2} \right)^{\alpha-\frac{\gamma}{p}-1} \left( ch \frac{r}{2} \right) \|f\|_{L_{p,\lambda}(H_r)} dr, \quad t \in E_\gamma. \end{aligned}$$

From this and (4.13), we obtain (4.12).  $\square$

**Theorem E** ([12]). *Suppose  $1 \leq p < \infty$ ,  $\omega_1 \in \Omega_p^\gamma$ ,  $\omega_2 \in \Omega_q^\gamma$ . Moreover, let condition (4.1) hold and also the pair  $(\omega_1, \omega_2)$  satisfy the condition*

$$\sup_{s>r} \left( sh \frac{s}{2} \right)^\alpha \omega_1(x, s) \lesssim \omega_2(x, r).$$

*Then  $M_G^\alpha$  is bounded from  $\mathcal{M}_{p,\omega_1,\gamma}(\mathbb{R}_+)$  to  $\mathcal{M}_{q,\omega_2,\gamma}(\mathbb{R}_+)$  for  $p > 1$ , and  $M_G^\alpha$  is bounded from  $\mathcal{M}_{1,\omega,\gamma}(\mathbb{R}_+)$  to  $W\mathcal{M}_{q,\omega_2,\gamma}(\mathbb{R}_+)$ .*

*Using Theorem E for  $\alpha = 0$  (then  $q = p$ ) and  $\omega_1(x, r) = \omega_2(x, r) = \omega(x, r)$ , we get the following*

**Corollary 4.10.** *Let  $1 \leq p < \infty$  and  $\omega$  satisfy the condition*

$$\sup_{s>r>0} \omega(x, s) \lesssim \omega(x, r). \quad (4.14)$$

*Then  $M_G$  is bounded from  $\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}_+)$  to  $\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}_+)$  for  $p > 1$  and  $M_G$  is bounded from  $\mathcal{M}_{1,\omega,\gamma}(\mathbb{R}_+)$  to  $W\mathcal{M}_{1,\omega,\gamma}(\mathbb{R}_+)$  for  $p = 1$ .*

**Theorem 4.11** (Adams-Guliyev type results). *Let  $1 \leq p < q < \infty$ ,  $0 < \alpha < \gamma/p$  and let  $\omega \in \Omega_p^\gamma$  satisfy condition (4.14) and*

$$\int_r^\infty \left( sh \frac{t}{2} \right)^{\alpha-1} \left( ch \frac{t}{2} \right) \omega(x, t)^{\frac{1}{p}} dt \lesssim \left( sh \frac{r}{2} \right)^{-\frac{\alpha p}{q-p}}, \quad r > 0. \quad (4.15)$$

*Then for  $p > 1$ , the operator  $J_G^\alpha$  is bounded from  $\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}(\mathbb{R}_+)$  to  $\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}(\mathbb{R}_+)$  and for  $p = 1$ , the operator  $J_G^\alpha$  is bounded from  $\mathcal{M}_{1,\omega,\gamma}(\mathbb{R}_+)$  to  $W\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}(\mathbb{R}_+)$ .*

*Proof.* Let  $1 \leq p < \infty$  and  $f \in \mathcal{M}_{p,\omega,\gamma}(\mathbb{R}_+)$ . By Theorem 4.9, inequality (4.12) is valid. Then from condition (4.15) and inequality (4.12), we get

$$\begin{aligned} J_G^\alpha f(chx) &\lesssim \left( sh \frac{t}{2} \right)^\alpha M_G f(chx) + \int_t^\infty \left( sh \frac{r}{2} \right)^{\alpha-\frac{\gamma}{p}-1} \left( ch \frac{r}{2} \right) \|f\|_{L_{p,\lambda}(H_r)} dr \\ &\lesssim \left( sh \frac{t}{2} \right)^\alpha M_G f(chx) + \int_t^\infty \frac{\left( sh \frac{r}{2} \right)^{\alpha-1} \omega(x, r)^{\frac{1}{p}} \|f\|_{L_{p,\lambda}(H_r)}}{\left( sh \frac{r}{2} \right)^{\frac{\gamma}{p}} \omega(x, r)^{\frac{1}{p}}} \left( ch \frac{r}{2} \right) dr \\ &\lesssim \left( sh \frac{t}{2} \right)^\alpha M_G f(chx) + \|f\|_{\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}(\mathbb{R}_+)} \int_t^\infty \frac{\omega(x, r)^{\frac{1}{p}} \left( ch \frac{r}{2} \right)}{\left( sh \frac{r}{2} \right)^{1-\alpha}} dr \\ &\lesssim \left( sh \frac{t}{2} \right)^\alpha M_G f(chx) + \left( sh \frac{t}{2} \right)^{-\frac{\alpha p}{q-p}} \|f\|_{\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}(\mathbb{R}_+)}, \quad t \in E_\gamma. \end{aligned}$$

The right-hand side attains its minimum at

$$sh \frac{t}{2} = \left( \frac{p}{q-p} \frac{\|f\|_{\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}(\mathbb{R}_+)}}}{M_G f(chx)} \right)^{\frac{q-p}{\alpha q}}$$

for all  $x \in \mathbb{R}_+$ , we have

$$J_G^\alpha f(chx) \lesssim (M_G f(chx))^{\frac{p}{q}} \|f\|_{\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}(\mathbb{R}_+)}^{1-\frac{p}{q}}.$$

Hence, the statement of the theorem follows from the boundedness of the G-maximal operator  $M_G$  in  $\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}_+)$ , provided by Corollary 4.10, by virtue of condition (4.14),

$$\begin{aligned} \|J_G^\alpha f\|_{\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}} &\lesssim \|f\|_{\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}}^{1-\frac{p}{q}} \sup_{\substack{x \in \mathbb{R}_+ \\ r > 0}} \omega(x,r)^{-\frac{1}{q}} \left( sh \frac{r}{2} \right)^{-\frac{\gamma}{q}} \|M_G f\|_{L_{p,\lambda}(H_r)}^{\frac{p}{q}} \\ &= \|f\|_{\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}}^{1-\frac{p}{q}} \left( \sup_{\substack{x \in \mathbb{R}_+ \\ r > 0}} \omega(x,r)^{-\frac{1}{p}} \left( sh \frac{r}{2} \right)^{-\frac{\gamma}{p}} \|M_G f\|_{L_{p,\lambda}(H_r)} \right)^{\frac{p}{q}} \\ &= \|f\|_{\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}} \|M_G f\|_{\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}}^{\frac{p}{q}} \lesssim \|f\|_{\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}}, \end{aligned}$$

for  $1 < p < q < \infty$ , and

$$\begin{aligned} \|J_G^\alpha f\|_{W\mathcal{M}_{q,\omega^{\frac{1}{q}}}} &\lesssim \|f\|_{\mathcal{M}_{1,\omega,\gamma}}^{1-\frac{1}{q}} \sup_{\substack{x \in \mathbb{R}_+ \\ r > 0}} \omega(x,r)^{-\frac{1}{q}} \left( sh \frac{r}{2} \right)^{-\frac{\gamma}{q}} \|M_G f\|_{WL_{1,\lambda}(H_r)}^{\frac{1}{q}} \\ &\lesssim \|f\|_{\mathcal{M}_{1,\omega,\gamma}}^{1-\frac{1}{q}} \|f\|_{\mathcal{M}_{1,\omega,\gamma}}^{\frac{1}{q}} = \|f\|_{\mathcal{M}_{1,\omega,\gamma}}, \end{aligned}$$

for  $p = 1 < q < \infty$ . □

The following theorem is one of our main results.

**Theorem 4.12.** *Let  $0 < \alpha < \gamma$ ,  $1 \leq p < q < \infty$  and (4.14) hold.*

*If  $\omega \in \Phi_p^\gamma$  satisfies the condition*

$$\int_r^\infty \left( sh \frac{t}{2} \right)^{\alpha-1} \left( ch \frac{t}{2} \right) \omega(t)^{\frac{1}{p}} dt \lesssim \omega(r)^{\frac{1}{p}} \left( sh \frac{r}{2} \right)^\alpha, \quad (4.16)$$

*then the condition*

$$\omega(r)^{\frac{1}{p}} \left( sh \frac{r}{2} \right)^\alpha \lesssim \left( sh \frac{r}{2} \right)^{-\frac{\alpha p}{q-p}} \quad (4.17)$$

*is necessary and sufficient for the boundedness of  $J_G^\alpha$  from  $\mathcal{M}_{1,\omega,\gamma}(\mathbb{R}_+)$  to  $W\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}(\mathbb{R}_+)$  if  $p = 1$ , and from  $\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}(\mathbb{R}_+)$  to  $\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}(\mathbb{R}_+)$  if  $1 < p < \infty$ .*

*Proof. Sufficiency.* Let (4.17) hold, then condition (4.16) implies (4.15) and by Theorem 4.11, it follows that  $J_G^\alpha$  is bounded from  $\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}(\mathbb{R}_+)$  to  $\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}(\mathbb{R}_+)$  if  $1 < p < \infty$ , and from  $\mathcal{M}_{1,\omega,\gamma}(\mathbb{R}_+)$  to  $W\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}(\mathbb{R}_+)$  if  $p = 1$ .

*Necessity.* Let  $J_G^\alpha$  be bounded from  $\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}(\mathbb{R}_+)$  to  $\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}(\mathbb{R}_+)$  if  $1 < p < q < \infty$  and from  $\mathcal{M}_{1,\omega,\gamma}(\mathbb{R}_+)$  to  $\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}(\mathbb{R}_+)$  if  $p = 1$ . Let  $H_0 = (0, t_0)$  and  $x \in H_0$ . By Lemma 4.5, we have  $(sh \frac{t_0}{2})^\alpha \lesssim J_G^\alpha \chi_{H_0}(chx)$ . Therefore, by Lemma 3.5 and Lemma 4.5, we get

$$\begin{aligned} \left( sh \frac{t_0}{2} \right)^\alpha &\lesssim |H_0|^{-\frac{1}{q}} \|J_G^\alpha \chi_{H_0}\|_{L_{q,\lambda}(H_0)} \lesssim \omega(t_0)^{\frac{1}{q}} \|J_G^\alpha \chi_{H_0}\|_{\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}} \\ &\lesssim \omega(t_0)^{\frac{1}{q}} \|\chi_{H_0}\|_{\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}} \lesssim \omega(t_0)^{\frac{1}{q} - \frac{1}{p}}, \end{aligned}$$

or

$$\left( sh \frac{t_0}{2} \right)^\alpha \omega(t_0)^{\frac{1}{p}-\frac{1}{q}} \lesssim 1 \iff \left( sh \frac{t_0}{2} \right)^\alpha \omega(t_0)^{\frac{1}{p}} \lesssim \left( sh \frac{t_0}{2} \right)^{-\frac{\alpha p}{q-p}},$$

for all  $t_0 > 0$ . Since this is true for every  $x \in \mathbb{R}_+$  and  $t_0 > 0$ , we are done.  $\square$

**Theorem 4.13** (Adams–Gunawan type result). *Let  $0 < \alpha < \gamma$ ,  $1 \leq p < q < \infty$  and  $\omega \in \Omega_p^\gamma$  satisfy condition (4.14) and*

$$\left( sh \frac{t}{2} \right)^\alpha \omega(x, t) + \int_t^\infty \left( sh \frac{r}{2} \right)^{\alpha-1} \left( ch \frac{r}{2} \right) \omega(x, r) dr \lesssim \omega(x, t)^{\frac{p}{q}}, \quad t \in E_\gamma. \quad (4.18)$$

Then for  $p > 1$ , the operator  $J_G^\alpha$  is bounded from  $\mathcal{M}_{p, \omega^{\frac{1}{p}}, \gamma}(\mathbb{R}_+)$  to  $\mathcal{M}_{q, \omega^{\frac{1}{q}}, \gamma}(\mathbb{R}_+)$  and for  $p = 1$ , the operator  $J_G^\alpha$  is bounded from  $\mathcal{M}_{1, \omega, \gamma}(\mathbb{R}_+)$  to  $W\mathcal{M}_{q, \omega^{\frac{1}{q}}, \gamma}(\mathbb{R}_+)$ .

*Proof.* Let  $1 \leq p < \infty$  and  $f \in \mathcal{M}_{p, \omega, \gamma}(\mathbb{R}_+)$ . By Theorem 4.9, inequality (4.12) is valid. Then from condition (4.15) and inequality (4.12), we get

$$\begin{aligned} |J_G^\alpha f(chx)| &\lesssim \left( sh \frac{t}{2} \right)^\alpha M_G f(chx) + \int_t^\infty \left( sh \frac{r}{2} \right)^{\alpha-\frac{\gamma}{p}-1} \left( ch \frac{r}{2} \right) \|f\|_{L_{p, \lambda}(H_r)} dr \\ &\lesssim \left( sh \frac{t}{2} \right)^\alpha M_G f(chx) + \|f\|_{\mathcal{M}_{p, \omega, \gamma}} \int_t^\infty \left( sh \frac{r}{2} \right)^{\alpha-1} \left( ch \frac{r}{2} \right) \omega(x, r) dr, \quad t \in E_\gamma. \end{aligned}$$

From (4.18), we have

$$\omega(x, t) \left( sh \frac{t}{2} \right)^\alpha \lesssim \omega(x, t)^{p/q} \Rightarrow \left( sh \frac{t}{2} \right)^\alpha \lesssim \omega(x, t)^{p/q-1}, \quad (4.19)$$

and

$$\begin{aligned} \int_t^\infty \left( sh \frac{r}{2} \right)^{\alpha-1} \left( ch \frac{r}{2} \right) \omega(x, r) &\lesssim \omega(x, t)^{p/q} \\ \Rightarrow \|f\|_{\mathcal{M}_{p, \omega, \gamma}} \int_t^\infty \left( sh \frac{r}{2} \right)^{\alpha-1} \left( ch \frac{r}{2} \right) \omega(x, r) dr &\lesssim \|f\|_{\mathcal{M}_{p, \omega, \gamma}} \omega(x, t)^{p/q}. \end{aligned} \quad (4.20)$$

Summing (4.19) and (4.20), we get

$$|J_G^\alpha f(chx)| \lesssim \omega(x, t)^{\frac{p}{q}-1} M_G f(chx) + \omega(x, t)^{\frac{p}{q}} \|f\|_{\mathcal{M}_{p, \omega, \gamma}}.$$

If  $\omega(x, t)^{\frac{p}{q}-1} M_G f(chx) \leq \omega(x, t)^{\frac{p}{q}} \|f\|_{\mathcal{M}_{p, \omega, \gamma}}$ , then  $\omega(x, t) \geq M_G f(chx) \|f\|_{\mathcal{M}_{p, \omega, \gamma}}^{-1}$ , and we have

$$\begin{aligned} |J_G^\alpha f(chx)| &\lesssim \omega(x, t)^{\frac{p}{q}-1} M_G f(chx) \lesssim M_G f(chx) \left( \frac{\|f\|_{\mathcal{M}_{p, \omega, \gamma}}}{M_G f(chx)} \right)^{1-\frac{p}{q}} \\ &= (M_G f(chx))^{\frac{p}{q}} \|f\|_{\mathcal{M}_{p, \omega, \gamma}}^{1-\frac{p}{q}}. \end{aligned}$$

Since  $\omega(x, t)^{\frac{p}{q}-1} M_G f(chx) \geq \omega(x, t)^{\frac{p}{q}} \|f\|_{\mathcal{M}_{p, \omega, \gamma}}$ , therefore  $\omega(x, t) \leq M_G f(chx) \|f\|_{\mathcal{M}_{p, \omega, \gamma}}^{-1}$ , and we get

$$\begin{aligned} |J_G^\alpha f(chx)| &\lesssim \omega(x, t)^{\frac{p}{q}} \|f\|_{\mathcal{M}_{p, \omega, \gamma}} \leq \left( \frac{M_G f(chx)}{\|f\|_{\mathcal{M}_{p, \omega, \gamma}}} \right)^{\frac{p}{q}} \|f\|_{\mathcal{M}_{p, \omega, \gamma}} \\ &= (M_G f(chx))^{\frac{p}{q}} \|f\|_{\mathcal{M}_{p, \omega, \gamma}}^{1-\frac{p}{q}}. \end{aligned}$$

Thus,

$$|J_G^\alpha f(chx)| \lesssim (M_G f(chx))^{\frac{p}{q}} \|f\|_{\mathcal{M}_{p, \omega, \gamma}}^{1-\frac{p}{q}}. \quad (4.21)$$

From Corollary 4.10 and (4.21), we obtain (see proof of Theorem 4.11)

$$\|J_G^\alpha f\|_{\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}} \lesssim \|f\|_{\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}}^{1-\frac{p}{q}} \|M_G f\|_{\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}}^{\frac{p}{q}} \lesssim \|f\|_{\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}},$$

for  $1 < p < q < \infty$  and

$$\|J_G^\alpha f\|_{W\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}} \lesssim \|f\|_{\mathcal{M}_{1,\omega,\gamma}}^{1-\frac{1}{q}} \|M_G f\|_{\mathcal{M}_{1,\omega,\gamma}}^{\frac{1}{q}} \lesssim \|f\|_{\mathcal{M}_{1,\omega,\gamma}},$$

for  $p = 1 < q < \infty$ .  $\square$

The following theorem is one of our main results.

**Theorem 4.14.** *Let  $0 < \alpha < \gamma$ ,  $1 \leq p < q < \infty$  and  $\omega \in \Omega_p^\gamma$ .*

*If  $\omega \in \Phi_p^\gamma$  satisfies condition (4.18), then the condition*

$$\left(sh\frac{r}{2}\right)^\alpha \omega(r)^{\frac{1}{p}} \lesssim \omega(r)^{\frac{1}{q}} \quad (4.22)$$

*is necessary and sufficient for the boundedness of  $J_G^\alpha$  from  $\mathcal{M}_{1,\omega,\gamma}(\mathbb{R}_+)$  to  $W\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}(\mathbb{R}_+)$  if  $p = 1$  and from  $\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}(\mathbb{R}_+)$  to  $\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}(\mathbb{R}_+)$  if  $1 < p < q < \infty$ .*

*Proof.* Let (4.22) hold. Note that if  $\omega \in \Phi_p^\gamma$ , then

$$\begin{aligned} \int_t^\infty \left(sh\frac{r}{2}\right)^{\alpha-1} \left(ch\frac{r}{2}\right) \omega(r) dr &= \int_t^\infty \left(sh\frac{r}{2}\right)^{\frac{\gamma}{p}-\frac{\gamma}{q}-1} \left(ch\frac{r}{2}\right) \omega(r) dr \\ &\geq \left(sh\frac{t}{2}\right)^{\frac{\gamma}{p}} \omega(t) \int_t^\infty \left(sh\frac{r}{2}\right)^{-\frac{\gamma}{q}-1} \left(ch\frac{r}{2}\right) dr \\ &= q \left(sh\frac{t}{2}\right)^\alpha \omega(t), \quad i.e. \end{aligned}$$

$$\left(sh\frac{t}{2}\right)^\alpha \omega(t) \leq \int_t^\infty \left(sh\frac{r}{2}\right)^{\alpha-1} \left(ch\frac{r}{2}\right) \omega(r) dr \lesssim \omega(t)^{p/q}, \quad t \in E_\gamma.$$

Suppose that  $\omega(r) = \varphi(r)^{\frac{1}{p}}$ . Then condition (4.18) is equivalent to the condition

$$\int_t^\infty \left(sh\frac{r}{2}\right)^\alpha \left(ch\frac{r}{2}\right) \omega(r)^{\frac{1}{p}} dr \lesssim \omega(t)^{\frac{1}{q}}.$$

Then by Theorem 4.13, the statement of Theorem 4.14 is proved.

*Necessity.* Now, let  $J_G^\alpha$  be bounded from  $\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}(\mathbb{R}_+)$  to  $\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}(\mathbb{R}_+)$ . Let  $H_0 = (0, t_0)$  and  $x \in H_0$ . By Lemma 4.5, we get  $(sh\frac{t_0}{2})^\alpha \lesssim J_G^\alpha \chi_{H_0}(chx)$ . Therefore, by Lemma 3.5 and Lemma 4.5, we have

$$\begin{aligned} \left(sh\frac{t_0}{2}\right)^\alpha &\lesssim |H_0|^{-\frac{1}{q}} \|J_G^\alpha \chi_{H_0}\|_{L_{q,\lambda}(H_0)} \\ &\lesssim \omega(t_0)^{\frac{1}{q}} \|J_G^\alpha \chi_{H_0}\|_{\mathcal{M}_{q,\omega^{\frac{1}{q}},\gamma}} \lesssim \omega(t_0)^{\frac{1}{q}} \|\chi_{H_0}\|_{\mathcal{M}_{p,\omega^{\frac{1}{p}},\gamma}} \lesssim \omega(t_0)^{\frac{1}{q}-\frac{1}{p}}, \end{aligned}$$

or

$$\left(sh\frac{t_0}{2}\right)^\alpha \omega(t_0)^{\frac{1}{p}-\frac{1}{q}} \lesssim 1 \iff \left(sh\frac{t_0}{2}\right)^\alpha \omega(t_0)^{\frac{1}{p}} \lesssim \omega(t_0)^{\frac{1}{q}},$$

for all  $t_0 > 0$ .

Since this is true for every  $x \in \mathbb{R}_+$  and  $t_0 > 0$ , we are done.  $\square$

**Remark 4.15.** If we take  $\omega(t) = (sh\frac{t}{2})^{\nu-\gamma}$  in Theorem 4.12, then condition (4.16) is equivalent to  $0 < \nu < \gamma - \alpha p$  and condition (4.17) is equivalent to  $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma-\nu}$ . Thus we get the following

**Corollary 4.16** (Theorem 2.4). *Let  $0 < \alpha < \gamma$ ,  $1 \leq p < q < \infty$  and  $0 < \nu < \gamma - \alpha p$ . Then the operator  $J_G^\alpha$  is bounded from  $L_{1,\nu,\lambda}(\mathbb{R}_+)$  to  $WL_{q,\nu,\lambda}(\mathbb{R}_+)$  if and only if  $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma - \nu}$ .*

*Moreover, if  $1 < p < q < \infty$ , then the operator  $J_G^\alpha$  is bounded from  $L_{p,\nu,\lambda}(\mathbb{R}_+)$  to  $L_{q,\nu,\lambda}(\mathbb{R}_+)$  if and only if  $1 - \frac{1}{q} = \frac{\alpha}{\gamma - \nu}$ .*

**Corollary 4.17.** *Taking  $\gamma = 2\lambda + 1$  in Corollary 4.16, we obtain Theorem B.*

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