

## UNVEILING BINARY SET FUNCTIONS: APPLICATIONS AND TOPOLOGICAL INSIGHTS

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**Abstract.** In this paper, we focus on the introduction, study and characterization of functions defined on binary sets in binary topological spaces. As a part of our applications, we demonstrate how a binary soft set can be viewed as a special case of the functions defined here. Furthermore, we propose a method for improving decision-making. Additionally, we present an algorithm for identifying binary topologies.

### 1. INTRODUCTION

In mathematics, two universal set structures are important, allowing us to explore relationships and interactions between distinct mathematical objects or systems. These structures find wide-ranging applications across various fields, including mathematics, physics, computer science, and engineering. Commonly encountered structures, associated with two universal sets, fuzzy sets, soft sets, and graphs [1–3, 10–12] provide a mechanism for information classifying, facilitating nuanced and adaptive reasoning and analysis.

In the study of structures involving two universal sets, the concept of binary topology, introduced in [8], plays an important role. It serves as a fundamental tool for establishing notions of proximity between such structures. The formulation of binary topology is a relatively recent development that has paved the way for further advancements in the field. Noteworthy contributions include the introduction of weaker notions of binary open sets and the subsequent derivation of various characterizations, as demonstrated in the studies of [4, 5, 7, 9, 13]. The significance of structures involving two universal sets cannot be understated, given their broad applicability and ability to elucidate intricate relationships. By employing binary topology, we gain insights into the proximity and interplay of these structures, contributing to a deeper understanding of their properties and behavior. The concept of classical soft sets extends into a binary context through the mathematical framework of binary soft sets. Initially proposed by Molodtsov in 1999 [12], soft sets provide a more flexible and uncertain representation of data, expanding upon the notion of classical sets. The introduction of binary soft sets further enhances the practicality and computational aspects of soft sets. Extensive research by Acikgoz and Tas [1] has focused on binary soft sets, including the introduction of binary operations and examination of their algebraic properties. Some applications of binary soft sets in decision-making were explored in [6].

In Section 2 of this paper, we compile the relevant preliminaries on binary topological spaces and duly cite them. These concepts will be utilized throughout the paper. Section 3 discusses the stability properties related to finite unions and intersections of various binary sets and presents a method for endowing finite sets with a binary topology. Moving on to Section 4, we introduce notions of continuity and open functions associated with binary sets, along with relevant characterizations and properties. Given that calculating open and closed sets, and consequently interiors and closures, can be a tedious and challenging task. Section 5 presents an algorithm that provides an efficient way to perform these calculations. Worth noting, all the examples presented in this paper were computed using this algorithm. Lastly, in Section 6, we demonstrate the relationship between soft sets and

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binary topological spaces and propose a method to find parameters with certain “similarities” that could eventually contribute to decision-making.

## 2. PRELIMINARIES

Throughout the entire work, we use the notation  $\mathcal{P}(X)$  to denote the power set of the set  $X$ , and similarly,  $\mathcal{P}(Y)$  represents the power set of set  $Y$ . If  $(A, B)$  and  $(C, D)$  are the elements of  $\mathcal{P}(X) \times \mathcal{P}(Y)$ , then we use the following notation:

- (1)  $(A, B) \subseteq (C, D)$  if  $A \subseteq C$  and  $B \subseteq D$ .
- (2)  $(A, B) = (C, D)$  if  $(A, B) \subseteq (C, D)$  and  $(C, D) \subseteq (A, B)$ .
- (3)  $(A, B) \subseteq (X, Y)$  or  $(A, B)$  is a subset of  $(X, Y)$  if  $(A, B) \in \mathcal{P}(X) \times \mathcal{P}(Y)$ .
- (4) The binary complement of  $(A, B)$  is  $(X \setminus A, Y \setminus B)$  and it is denoted as  $(X, Y) \setminus (A, B)$ .

**Definition 2.1** ([8]). Let  $X$  and  $Y$  be any two nonempty sets, a binary topology from  $X$  to  $Y$  is a binary structure  $\mathcal{M} \subset \mathcal{P}(X) \times \mathcal{P}(Y)$  that satisfies the following axioms:

- (1)  $(\emptyset, \emptyset) \in \mathcal{M}$  and  $(X, Y) \in \mathcal{M}$ .
- (2) If  $(A_1, B_1)$  and  $(A_2, B_2)$  are elements of  $\mathcal{M}$ , then

$$(A_1 \cap A_2, B_1 \cap B_2) \in \mathcal{M}.$$

- (3) If  $\{(A_i, B_i), i \in I\}$  is a family of elements of  $\mathcal{M}$ , then

$$(\cup_{i \in I} A_i, \cup_{i \in I} B_i) \in \mathcal{M}.$$

If  $\mathcal{M}$  is a binary topology from  $X$  to  $Y$ , then the triple  $(X, Y, \mathcal{M})$  is called a binary topological space and the members of  $\mathcal{M}$  are called the binary open sets of the binary topological space  $(X, Y, \mathcal{M})$ .  $(A, B)$  is called a binary closed set if  $(X \setminus A, Y \setminus B) \in \mathcal{M}$ . In [4] and [8], the examples of binary topological spaces are provided.

**Definition 2.2** ([8]). Let  $(X, Y, \mathcal{M})$  be a binary topological space and let  $(A, B)$  be a subset of  $(X, Y)$ . Define the following:

- (1)  $(A, B)^{1^\circ} = \cup\{A_\alpha : (A_\alpha, B_\alpha) \in \mathcal{M} \text{ and } (A_\alpha, B_\alpha) \subseteq (A, B)\}$ .
- (2)  $(A, B)^{2^\circ} = \cup\{B_\alpha : (A_\alpha, B_\alpha) \in \mathcal{M} \text{ and } (A_\alpha, B_\alpha) \subseteq (A, B)\}$ .
- (3)  $(A, B)^{1^*} = \cap\{A_\alpha : (X \setminus A_\alpha, Y \setminus B_\alpha) \in \mathcal{M} \text{ and } (A, B) \subseteq (A_\alpha, B_\alpha)\}$ .
- (4)  $(A, B)^{2^*} = \cap\{B_\alpha : (X \setminus A_\alpha, Y \setminus B_\alpha) \in \mathcal{M} \text{ and } (A, B) \subseteq (A_\alpha, B_\alpha)\}$ .

The pair  $((A, B)^{1^\circ}, (A, B)^{2^\circ})$  is referred to as the binary interior of  $(A, B)$  and is denoted by  $\text{Int}((A, B))$  and the pair  $((A, B)^{1^*}, (A, B)^{2^*})$  is referred to as the binary closure of  $(A, B)$  and denoted by  $\text{Cl}(A, B)$ .

The properties of the binary interior and binary closure have already been studied in [4] and [8]. Below, we provide some of these properties.

**Theorem 2.1.** *Let  $(X, Y, \mathcal{M})$  be a binary topological space, and let  $(A, B)$  be a subset of  $(C, D)$  which is itself a subset of  $(X, Y)$ . Then the following statements hold:*

- (1)  $\text{Int}(A, B) \in \mathcal{M}$  and  $\text{Int}(A, B) \subseteq (A, B)$ .
- (2)  $(A, B) \in \mathcal{M}$  if and only if  $\text{Int}(A, B) = (A, B)$ .
- (3)  $\text{Cl}(A, B)$  is a binary closed set and  $(A, B) \subseteq \text{Cl}(A, B)$ .
- (4)  $(A, B)$  is a binary closed set if and only if  $\text{Cl}(A, B) = (A, B)$ .
- (5)  $\text{Int}(A, B) \subseteq \text{Int}(C, D)$  and  $\text{Cl}(A, B) \subseteq \text{Cl}(C, D)$  (monotony).
- (6)  $\text{Int}(\text{Int}(A, B)) = \text{Int}(A, B)$  and  $\text{Cl}(\text{Cl}(A, B)) = \text{Cl}(A, B)$ .
- (7)  $(X, Y) \setminus \text{Cl}(A, B) = \text{Int}(X \setminus A, Y \setminus B)$  and  $(X, Y) \setminus \text{Int}(A, B) = \text{Cl}(X \setminus A, Y \setminus B)$ .

As usual, weak notions of binary sets have been defined in terms of binary interior and binary closure, and the relationship between these weak notions of binary open sets was studied in [4].

**Definition 2.3.** Let  $(X, Y, \mathcal{M})$  be a binary topological space, and let  $(A, B)$  be a subset of  $(X, Y)$ . Then  $(A, B)$  is said to be:

- (1) Binary regular closed set [9], if  $(A, B) = \text{Cl}(\text{Int}(A, B))$ .
- (2) Binary semi-open set [7], if  $(A, B) \subseteq \text{Cl}(\text{Int}(A, B))$ .

- (3) Binary  $\alpha^*$ -set [4], if  $\text{Int}(A, B) = \text{Int}(\text{Cl}(\text{Int}(A, B)))$ .
- (4) Binary  $t$ -set [4], if  $\text{Int}(A, B) = \text{Int}(\text{Cl}(A, B))$ .
- (5) Binary  $s$ -set [4], if  $\text{Int}(A, B) = \text{Cl}(\text{Int}(A, B))$ .
- (6) Binary  $\beta^*$ -set [4], if  $\text{Int}(A, B) = \text{Cl}(\text{Int}(\text{Cl}(A, B)))$ .
- (7) Binary  $\mathcal{A}$ -set [4], if  $(A, B) = (U \cap C, V \cap D)$ , where  $(U, V) \in \mathcal{M}$  and  $(C, D)$  is a binary regular closed set.
- (8) Binary  $\mathcal{B}$ -set [4], if  $(A, B) = (U \cap C, V \cap D)$ , where  $(U, V) \in \mathcal{M}$  and  $(C, D)$  is a binary  $t$ -set.
- (9) Binary  $\mathcal{C}$ -set [4], if  $(A, B) = (U \cap C, V \cap D)$ , where  $(U, V) \in \mathcal{M}$  and  $(C, D)$  is a binary  $\alpha^*$ -set.
- (10) Binary locally closed set [4], if  $(A, B) = (U \cap C, V \cap D)$ , where  $(U, V) \in \mathcal{M}$  and  $(C, D)$  is a binary closed set.
- (11) Binary  $\beta$ -set [4], if  $(A, B) = (U \cap C, V \cap D)$ , where  $(U, V) \in \mathcal{M}$  and  $(C, D)$  is a binary  $\beta^*$ -set.
- (12) Binary  $\mathcal{S}$ -set [4], if  $(A, B) = (U \cap C, V \cap D)$ , where  $(U, V) \in \mathcal{M}$  and  $(C, D)$  is a binary  $s$ -set.

### 3. SOME PROPERTIES OF BINARY SETS

In this section, we present the stability properties related to the union and intersection of established binary set notions. Additionally, we introduce a method to generate a binary topology on finite sets from a given collection.

**Theorem 3.1.** *Let  $(X, Y, \mathcal{M})$  be a binary topological space, and let  $(A, B)$  and  $(C, D)$  be the subsets of  $(X, Y)$ . Then the following hold:*

- (1) *If  $(A, B)$  and  $(C, D)$  are binary  $\alpha^*$ -sets, then  $(A \cap C, B \cap D)$  is a binary  $\alpha^*$ -set.*
- (2) *If  $(A, B)$  and  $(C, D)$  are binary  $t$ -sets, then  $(A \cap C, B \cap D)$  is a binary  $t$ -set.*
- (3) *If  $(A, B)$  and  $(C, D)$  are binary  $\mathcal{B}$ -sets, then  $(A \cap C, B \cap D)$  is a binary  $\mathcal{B}$ -set.*
- (4) *If  $(A, B)$  and  $(C, D)$  are binary  $\mathcal{C}$ -sets, then  $(A \cap C, B \cap D)$  is a binary  $\mathcal{C}$ -set.*
- (5) *If  $(A, B)$  and  $(C, D)$  are binary locally closed sets, then  $(A \cap C, B \cap D)$  is a binary locally closed set.*

*Proof.* We only demonstrate the first item, since the demonstrations of the other items are similar to this one. Let  $(A, B)$  and  $(C, D)$  be binary  $\alpha^*$ -sets, then  $\text{Int}(A, B) = \text{Int}(\text{Cl}(\text{Int}(A, B)))$  and  $\text{Int}(C, D) = \text{Int}(\text{Cl}(\text{Int}(C, D)))$ . For the monotonicity and by the hypothesis:

$$\text{Int}(\text{Cl}(\text{Int}(A \cap C, B \cap D))) \subseteq \text{Int}(\text{Cl}(\text{Int}(A, B))) = \text{Int}(A, B) \subseteq (A, B)$$

and

$$\text{Int}(\text{Cl}(\text{Int}(A \cap C, B \cap D))) \subseteq \text{Int}(\text{Cl}(\text{Int}(C, D))) = \text{Int}(C, D) \subseteq (C, D),$$

hence  $\text{Int}(\text{Cl}(\text{Int}(A \cap C, B \cap D))) \subseteq (A \cap C, B \cap D)$ . Therefore,  $\text{Int}(\text{Cl}(\text{Int}(A \cap C, B \cap D))) \subseteq \text{Int}(A \cap C, B \cap D)$ . Note that it is always true that  $\text{Int}(A \cap C, B \cap D) \subseteq \text{Int}(\text{Cl}(\text{Int}(A \cap C, B \cap D)))$ . So,  $(A \cap C, B \cap D)$  is a binary  $\alpha^*$ -set.  $\square$

Note that the previous result establishes that the collections of binary  $\alpha^*$ -sets, binary  $t$ -sets, binary  $\mathcal{B}$ -sets, binary  $\mathcal{C}$ -sets and binary locally closed sets satisfy condition (2) of Definition 2.1.

**Theorem 3.2.** *Let  $(X, Y, \mathcal{M})$  be a binary topological space, and let  $(A, B)$  and  $(C, D)$  be the subsets of  $(X, Y)$ . Then the following hold:*

- (1) *If  $(A, B)$  and  $(C, D)$  are binary semi-open sets, then  $(A \cup C, B \cup D)$  is a binary semi-open set.*
- (2) *If  $(A, B)$  and  $(C, D)$  are binary regular closed sets, then  $(A \cup C, B \cup D)$  is a binary regular closed set.*

*Proof.* 1. Let  $(A, B)$  and  $(C, D)$  be binary semi-open sets, then  $(A, B) \subseteq \text{Cl}(\text{Int}(A, B))$  and  $(C, D) \subseteq \text{Cl}(\text{Int}(C, D))$ . For the monotonicity and by the hypothesis:  $(A, B) \subseteq \text{Cl}(\text{Int}(A, B)) \subseteq \text{Cl}(\text{Int}(A \cup C, B \cup D))$  and  $(C, D) \subseteq \text{Cl}(\text{Int}(C, D)) \subseteq \text{Cl}(\text{Int}(A \cup C, B \cup D))$ , hence  $(A \cup C, B \cup D) \subseteq \text{Cl}(\text{Int}(A \cup C, B \cup D))$ . Therefore  $(A \cup C, B \cup D)$  is a binary semi-open set.

2. Let  $(A, B)$  and  $(C, D)$  be binary regular closed sets, then  $(A, B) = \text{Cl}(\text{Int}(A, B))$  and  $(C, D) = \text{Cl}(\text{Int}(C, D))$ , by the hypothesis:  $(A, B) = \text{Cl}(\text{Int}(A, B)) \subseteq \text{Cl}(\text{Int}(A \cup C, B \cup D))$  and  $(C, D) = \text{Cl}(\text{Int}(C, D)) \subseteq \text{Cl}(\text{Int}(A \cup C, B \cup D))$ , hence  $(A \cup C, B \cup D) \subseteq \text{Cl}(\text{Int}(A \cup C, B \cup D))$ . Now, it is always true that  $\text{Cl}(\text{Int}(A \cup C, B \cup D)) \subseteq \text{Cl}(A \cup C, B \cup D)$ . Note that  $(A, B)$  and  $(C, D)$  are binary closed

sets and therefore  $(A \cup C, B \cup D)$  is a binary closed set, as a result  $(A \cup C, B \cup D) = \text{Cl}(A \cup C, B \cup D)$ . The above allows us to conclude that  $\text{Cl}(\text{Int}(A \cup C, B \cup D)) \subseteq (A \cup C, B \cup D)$ . Based on this, we can conclude that  $(A \cup C, B \cup D)$  is a binary regular closed set.  $\square$

Just as in the general topology, we can obtain a binary topology from a smaller collection that satisfies a certain condition. Below, we present both the condition that this collection must satisfy and the method for constructing the binary topology. Let  $\Delta \subseteq \mathcal{P}(X) \times \mathcal{P}(Y)$  be a collection such that for all  $(x, y) \in X \times Y$  there exist  $(A, B), (C, D) \in \Delta$  such that  $(\{x\}, \{y_1\}) \subseteq (A, B)$  and  $(\{x_1\}, \{y\}) \subseteq (C, D)$  for some  $x_1 \in X$  and some  $y_1 \in Y$ ; consider the collection

$$\mathcal{B} = \left\{ (C, D) : (C, D) = \left( \bigcap_{i=1}^n A_i, \bigcap_{i=1}^n B_i \right), \text{ where } (A_i, B_i) \in \Delta \right\}.$$

Then the collection

$$\mathcal{M} = \left\{ (U, V) : (U, V) = \left( \bigcup_{i=1}^n C_i, \bigcup_{i=1}^n D_i \right), \text{ where } (C_i, D_i) \in \mathcal{B} \right\} \cup \{(\emptyset, \emptyset)\},$$

is a binary topology from  $X$  to  $Y$  for  $X, Y$  finite sets. We refer to  $\Delta$  as a binary subbase and to  $\mathcal{B}$  as a binary base for  $\mathcal{M}$ , respectively.

#### 4. SOME TYPES OF BINARY CONTINUOUS FUNCTIONS

In this section, we introduced the notion of binary continuous functions by using any class  $\mathcal{W}$  of binary sets. Let  $X, Y$ , and  $Z$  be nonempty sets,  $f : Z \rightarrow \mathcal{P}(X) \times \mathcal{P}(Y)$  be a function, and  $(A, B)$  be a subset of  $(X, Y)$ . Then  $f^{-1}(A, B) = \{z \in Z : f(z) = (C, D) \subseteq (A, B)\}$ , and  $f^{-1}(A, B)$  is called a preimage of  $(A, B)$ .

**Definition 4.1.** Let  $(X, Y, \mathcal{M})$  be a binary topological space,  $(Z, \tau)$  be a topological space, and  $\mathcal{W}$  be a class of subsets of  $(X, Y)$ . A function  $f : Z \rightarrow \mathcal{P}(X) \times \mathcal{P}(Y)$  is called binary  $\mathcal{W}$ -continuous if  $f^{-1}(A, B) \in \tau$  for all  $(A, B) \in \mathcal{W}$ .

**Remark 4.1.** Depending on the above-considered class  $\mathcal{W}$ , we obtain specific kind of continuous function, for instance, if  $\mathcal{W}$  is the collection of all binary open sets, then  $f$  is said to be binary continuous.

Now, we are going to prove a pointwise characterization of binary  $\mathcal{W}$ -continuous functions.

**Theorem 4.1.** Let  $(X, Y, \mathcal{M})$  be a binary topological space, and  $(Z, \tau)$  be a topological space. A function  $f : Z \rightarrow \mathcal{P}(X) \times \mathcal{P}(Y)$  is a binary  $\mathcal{W}$ -continuous function if and only if for every  $z \in Z$  and for every  $(A, B) \in \mathcal{W}$  with  $f(z) \subseteq (A, B)$ , there exists an open set  $U \subseteq Z$  such that  $f(U) \subseteq (A, B)$ .

*Proof.* Let  $f : Z \rightarrow \mathcal{P}(X) \times \mathcal{P}(Y)$  be a binary  $\mathcal{W}$ -continuous function,  $(A, B) \in \mathcal{W}$  and  $z \in Z$  such that  $f(z) = (C, D) \subseteq (A, B)$ , then  $U = f^{-1}(A, B)$  is open in  $Z$ ,  $z \in U$  and  $f(U) \subseteq (A, B)$ . Now, let  $(A, B) \in \mathcal{W}$  and  $z \in f^{-1}(A, B)$ , then  $f(z) = (C, D) \subseteq (A, B)$ . By the hypotheses, there exists an open set  $U_z \subseteq Z$  such that  $f(U_z) \subseteq (A, B)$  and therefore  $U_z \subseteq f^{-1}(A, B)$ . So,  $f^{-1}(A, B) = \bigcup_{z \in f^{-1}(A, B)} U_z$ . Hence  $f^{-1}(A, B)$  is open in  $Z$ , since it is a union of open sets in  $Z$ .  $\square$

**Theorem 4.2.** Let  $(X, Y, \mathcal{M})$  be a binary topological space,  $Z$  be a non empty set and  $f : Z \rightarrow \mathcal{P}(X) \times \mathcal{P}(Y)$  be a function, then the collection  $\mu = \{f^{-1}(A, B) : (A, B) \in \mathcal{M}\}$  is stable under finite intersections. Moreover, if  $\mathcal{W}$  is a class of binary sets that satisfies condition (2) of Definition 2.1, then the collection  $\beta = \{f^{-1}(A, B) : (A, B) \in \mathcal{W}\}$  is stable under finite intersections.

*Proof.* Let  $\{(A_i, B_i)\}_{i=1}^n$  be a finite collection of elements of  $\mathcal{M}$ , then

$$f^{-1} \left( \left( \bigcap_{i=1}^n A_i, \bigcap_{i=1}^n B_i \right) \right) = \bigcap_{i=1}^n f^{-1}(A_i, B_i).$$

The result follows owing to  $(\bigcap_{i=1}^n A_i, \bigcap_{i=1}^n B_i) \in \mathcal{M}$ .  $\square$

It would be natural to think that the previously defined collection is stable for arbitrary unions; however, this is generally not true, as shown in the following example.

**Example.** Take  $X = \{1, 2\}$ ,  $Y = \{5, 6\}$ ,

$$\mathcal{M} = \{(\emptyset, \emptyset), (X, Y), (\{1\}, \{6\}), (\{2\}, Y), (\emptyset, \{6\})\},$$

and  $Z = \{p, q, r, s\}$ . Define  $f : Z \rightarrow \mathcal{P}(X) \times \mathcal{P}(Y)$  as follows:  $f(p) = (\{1\}, \{6\})$ ,  $f(q) = (\{2\}, \{5\})$ ,  $f(r) = (\{2\}, \{6\})$ ,  $f(s) = (\{1\}, \{5\})$ . Note that:  $f^{-1}(\{1\}, \{6\}) \cup f^{-1}(\{2\}, Y) = \{p, q, r\}$  and  $f^{-1}(\{1\} \cup \{2\}, \{6\} \cup Y) = \{p, q, r, s\}$ . Therefore  $f^{-1}(\{1\}, \{6\}) \cup f^{-1}(\{2\}, Y) \neq f^{-1}(\{1\} \cup \{2\}, \{6\} \cup Y)$ .

However, as a direct consequence of the previous result, it follows that if  $(X, Y, \mathcal{M})$  is a binary topological space,  $Z$  is a nonempty set and  $f : Z \rightarrow \mathcal{P}(X) \times \mathcal{P}(Y)$  is a function, then the collection  $\beta = \{f^{-1}(A, B) : (A, B) \in \mathcal{M}\}$  is a basis for a topology on  $Z$ . If on  $Z$  we consider the topology generated by the previously described base  $\beta$ , then  $f$  becomes a binary continuous function.

Below, the open functions in binary spaces are defined, and a characterization of these functions is provided.

**Definition 4.2.** Let  $(X, Y, \mathcal{M})$  be a binary topological space,  $(Z, \tau)$  be a topological space, and  $\mathcal{W}$  be a class of subsets of  $(X, Y)$ . A function  $f : Z \rightarrow \mathcal{P}(X) \times \mathcal{P}(Y)$  is called binary  $\mathcal{W}$ -open if  $f(U) \in \mathcal{W}$  for all  $U \in \tau$ .

**Theorem 4.3.** Let  $(X, Y, \mathcal{M})$  be a binary topological space,  $(Z, \tau)$  be a topological space. A function  $f : Z \rightarrow \mathcal{P}(X) \times \mathcal{P}(Y)$  is binary  $\mathcal{M}$ -open if and only if  $f(\text{Int}(G)) \subseteq \text{Int}(f(G))$  for all  $G \subseteq Z$ .

*Proof.* Let  $f : Z \rightarrow \mathcal{P}(X) \times \mathcal{P}(Y)$  be a binary  $\mathcal{W}$ -open function and let  $G \subseteq Z$ . Since  $\text{Int}(G) \subseteq G$ , by the monotonicity of a direct image, it follows that  $f(\text{Int}(G)) \subseteq f(G)$  and therefore

$$f(\text{Int}(G)) = \text{Int}(f(\text{Int}(G))) \subseteq \text{Int}(f(G)).$$

Reciprocally, consider  $G \in \tau$ , then  $f(G) = f(\text{Int}(G)) \subseteq \text{Int}(f(G))$ , this implies that  $f(G) \in \mathcal{M}$ .  $\square$

The following result states that the composition of a continuous function and a binary  $\mathcal{W}$ -open function is a  $\mathcal{W}$ -open function.

**Theorem 4.4.** Let  $(X, Y, \mathcal{M})$  be a binary topological space,  $(Z, \tau)$  and  $(W, \Gamma)$  be topological spaces. Consider the functions  $f : Z \rightarrow \mathcal{P}(X) \times \mathcal{P}(Y)$  and  $g : W \rightarrow Z$ . If  $f \circ g$  is a binary  $\mathcal{W}$ -open function and  $g$  is a continuous and surjective function, then  $f$  is also a binary  $\mathcal{W}$ -open function.

*Proof.* Let  $G$  be an open set in  $Z$ , since  $g : W \rightarrow Z$  is a continuous function  $g^{-1}(G)$  is open in  $W$ . Due to  $f \circ g$  is a binary  $\mathcal{W}$ -open function, we obtain  $(f \circ g)(g^{-1}(G)) = f(G) \in \mathcal{W}$   $\square$

## 5. ALGORITHM

When building finite topologies, manual calculations are characterized by being tedious, long and complicated. This is something that a computer can help with, and we take advantage of it using the simple algorithms presented in this section.

For any given finite  $\Delta$  set, one can apply Algorithm 1 and then Algorithm 2 to create a binary topology. These algorithms follow the procedure of building a topology with a given subbase.

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**Algorithm 1:** Finite Intersections

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**Input:** Set  $\Delta \subseteq \mathcal{P}(X) \times \mathcal{P}(Y)$ **Output:** Set  $\mathcal{B}$ , the binary base

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1  $\mathcal{B} \leftarrow$  empty list;
2 foreach possible combination of elements of  $\Delta$  do
3   | Intersect that finite collection of sets;
4   | if the result is not in  $\mathcal{B}$  then
5   |   | Append the result to  $\mathcal{B}$ ;
6   | end
7 end
8 return  $\mathcal{B}$ ;
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**Algorithm 2:** Finite unions

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**Input:** Set  $\mathcal{B}$ , the binary base**Output:** Set  $\mathcal{M}$ , a binary topology

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1  $\mathcal{M} \leftarrow$  empty list;
2 foreach possible combination of elements of  $\mathcal{B}$  do
3   | Join that finite collection of sets;
4   | if the result is not in  $\mathcal{M}$  then
5   |   | Append the result to  $\mathcal{M}$ ;
6   | end
7 end
8 return  $\mathcal{M}$ ;
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One can also compute the binary interior and binary closure of a given set using Algorithms 3 and 4 respectively. These procedures follow the usual definitions of binary interior and binary closure, applied in the finite space.

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**Algorithm 3:** Get binary Interior

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**Input** : Set  $(A, B) \subset \mathcal{P}(X) \times \mathcal{P}(Y)$  and binary topology  $\mathcal{M}$ **Output:** Interior of  $(A, B)$ 

```

1  $l \leftarrow$  empty list;
2 foreach binary open set in  $\mathcal{M}$  do
3   | if binary open set is a subset of  $(A, B)$  then
4   |   | Append binary open set to list  $l$ ;
5   | end
6 end
7 interior  $\leftarrow$  union of list  $l$ ;
8 return interior;
```

---

**Algorithm 4:** Get binary Closure**Input** : Set  $(A, B) \subset \mathcal{P}(X) \times \mathcal{P}(Y)$  and binary topology  $\mathcal{M}$ **Output:** Closure of  $(A, B)$ 


---

```

1  $l \leftarrow$  empty list;
2 foreach binary closed set in  $\mathcal{M}$  do
3   | if  $(A, B)$  is a subset of the binary closed set then
4   |   | Append binary closed set to list  $l$ ;
5   | end
6 end
7 closure  $\leftarrow$  intersection of list  $l$ ;
8 return closure;
```

---

Following the established in section 4, we also synthesize the calculation of the preimage of a function in Algorithm 5.

**Algorithm 5:** Get Preimage**Input** : Set  $(A, B) \subset \mathcal{P}(X) \times \mathcal{P}(Y)$  and a function  $f : Z \rightarrow \mathcal{P}(X) \times \mathcal{P}(Y)$ **Output:** Preimage of  $(A, B)$ 


---

```

1 preimage  $\leftarrow$  empty list;
2 foreach  $z \in Z$  do
3   | if  $f(z)$  is a subset of  $(A, B)$  then
4   |   | Append  $z$  to preimage;
5   | end
6 end
7 return preimage;
```

---

By utilizing the algorithms defined earlier, we can demonstrate multiple examples, including the one presented in the following section.

## 6. APPLICATIONS TO BINARY SOFT SET

We begin this section with recalling the concept of binary soft sets over finite sets  $X$  and  $Y$ , defined in [1]. Let  $E$  be a set of parameters, and consider a subset  $A$  of  $E$ . We define a function  $F : A \rightarrow \mathcal{P}(X) \times \mathcal{P}(Y)$ , where each element  $e \in A$  is mapped to a pair  $(U, V)$ , where  $U$  is a subset of  $X$  and  $V$  is a subset of  $Y$ . We call the pair  $(F, A)$  as a binary soft set over  $X$  and  $Y$ . This notion allows us to encompass and explore the concepts presented in the previous section within the framework of binary soft sets. In this way, we not only extend the scope of binary set theory, but also open up new possibilities for its application in various fields. The flexibility and generality of binary soft sets enable us to establish connections with other mathematical theories and utilize them in innovative ways. Through the study of binary soft sets, we can establish novel concepts, techniques, and applications that go beyond traditional set theory. The interplay between binary soft sets and other mathematical frameworks gives rise to exciting opportunities for research and practical implementations in fields such as decision-making and information processing. It is evident that the concepts and topics discussed in the previous section can be encompassed within the framework of binary soft sets, leading to the emergence of novel concepts and applications within this theory. In the following discussion, we will demonstrate these new concepts and explore their practical implications. Consider  $X = \{c_1, c_2, c_3, c_4\}$ ,  $Y = \{p_1, p_2, p_3, p_4\}$ ,  $E = \{e_1, e_2, e_3, e_4, e_5\}$  and the soft set  $(F, E)$  given by  $F(e_1) = (\{c_4\}, \{p_4\})$ ;  $F(e_2) = (\{c_1, c_2, c_3\}, \{p_1, p_2, p_3\})$ ;  $F(e_3) = (\{c_1, c_2\}, \{p_1, p_3, p_4\})$ ;  $F(e_4) = (\{c_3, c_4\}, \{p_1\})$  and  $F(e_5) = (\{c_1, c_2\}, \{p_2, p_3, p_4\})$ . Considering the attributes assigned to

TABLE 1. Binary topology  $\mathcal{M}$ 

$(\{c_2\}, \{p_1\})$	$(\{c_3, c_4\}, \{p_1\})$	$(\{c_1, c_2\}, \{p_2, p_3, p_4\})$
$(\{c_4\}, \{p_1\})$	$(\{c_2\}, \{p_2, p_4\})$	$(\{c_1, c_2, c_3\}, \{p_1, p_3, p_4\})$
$(\{c_3\}, \{p_1\})$	$(\{c_4\}, \{p_1, p_4\})$	$(\{c_2, c_3, c_4\}, \{p_1, p_2, p_4\})$
$(\{c_2\}, \{p_4\})$	$(\{c_3\}, \{p_1, p_4\})$	$(\{c_1, c_2, c_4\}, \{p_2, p_3, p_4\})$
$(\{c_2\}, \{p_2\})$	$(\{c_2, c_4\}, \{p_2, p_4\})$	$(\{c_2, c_3, c_4\}, \{p_1, p_2\})$
$(\{c_4\}, \{p_4\})$	$(\{c_2, c_3\}, \{p_1, p_2\})$	$(\{c_1, c_2\}, \{p_1, p_3, p_4\})$
$(\{c_2\}, \emptyset)$	$(\{c_1, c_2\}, \{p_3, p_4\})$	$(\{c_1, c_2, c_4\}, \{p_3, p_4\})$
$(\{c_4\}, \emptyset)$	$(\{c_3, c_4\}, \{p_1, p_4\})$	$(\{c_1, c_2, c_3, c_4\}, \{p_1, p_3, p_4\})$
$(\emptyset, \{p_1\})$	$(\{c_2, c_3\}, \{p_1, p_4\})$	$(\{c_2, c_3, c_4\}, \{p_1\})$
$(\emptyset, \{p_4\})$	$(\{c_2, c_4\}, \{p_1, p_4\})$	$(\{c_2\}, \{p_1, p_2, p_4\})$
$(\emptyset, \{p_1, p_4\})$	$(\{c_2, c_4\}, \{p_2\})$	$(\{c_2, c_3, c_4\}, \{p_1, p_4\})$
$(\{c_2, c_4\}, \emptyset)$	$(\{c_1, c_2\}, \{p_1, p_2, p_3, p_4\})$	$(\{c_2\}, \{p_1, p_4\})$
$(\{c_2, c_4\}, \{p_1\})$	$(\{c_2, c_3\}, \{p_1, p_2, p_3, p_4\})$	$(\{c_1, c_2, c_4\}, \{p_1, p_3, p_4\})$
$(\{c_2, c_3\}, \{p_1\})$	$(\{c_2, c_4\}, \{p_1, p_2, p_4\})$	$(\{c_1, c_2, c_4\}, \{p_1, p_2, p_3, p_4\})$
$(\{c_2\}, \{p_1, p_2\})$	$(\{c_2, c_3\}, \{p_1, p_2, p_4\})$	$(X, Y), (\emptyset, \emptyset)$

each parameter, we can create the following list:

$$\Delta = \{(\{c_4\}, \{p_4\}), (\{c_2, c_3\}, \{p_1, p_2\}), (\{c_1, c_2\}, \{p_1, p_3, p_4\}), (\{c_3, c_4\}, \{p_1\}), (\{c_1, c_2\}, \{p_2, p_3, p_4\})\}.$$

Using the algorithm described in the previous section, we obtain the binary topology  $\mathcal{M}$ , as presented in Table 1.

Then we calculate the preimage of each element in  $\mathcal{M}$  to obtain a basis  $\beta$  and, consequently, a topology on  $E$ .

$$\beta = \{\emptyset, \{e_1\}, \{e_3\}, \{e_4\}, \{e_5\}, \{e_1, e_4\}, \{e_1, e_3\}, \{e_1, e_5\}, \{e_2, e_3, e_5\}, \{e_1, e_3, e_4\}, \{e_3, e_5\}, \{e_1, e_2, e_3, e_4, e_5\}, \{e_1, e_3, e_5\}\}$$

This procedure turns  $F$  into a binary continuous function. The idea is to consider the closure of the parameters (elements of the domain) and calculate the image of this closure. This allows us to determine which other parameters are “similar” to the originally assigned parameters and thus expand the range of assignment for decision-making. For example, for the parameter  $e_3$ , we obtain  $\text{Cl}(\{e_3\}) = \{e_2, e_3\}$ , and therefore  $F(\text{Cl}(\{e_3\})) = \{(\{c_1, c_2, c_3\}, \{p_1, p_2, p_3\}), (\{c_1, c_2\}, \{p_1, p_3, p_4\})\}$ . This allows for the interpretation that it is possible to expand the parameter assignment and consider the set  $(\{c_1, c_2, c_3\}, \{p_1, p_2, p_3, p_4\})$  as parameters associated with  $e_3$ , thereby expanding the possibilities for

decision-making. Observe that we obtain new attributes associated with the selected parameter when the parameter in question is not a closed set. In the case where the parameter is a closed set, the assignment remains unchanged. This is the reason why the parameter  $e_3$  was chosen to illustrate for this example.

**6.1. Additional examples.** Now, in order to complement the theoretical concepts presented in Section 3, we provide some examples obtained through the algorithm proposed in Section 6. Consider  $X = \{c_1, c_2, c_3, c_4\}$ ,  $Y = \{p_1, p_2, p_3, p_4\}$  and  $\mathcal{M}$  defined as in Table 1, then:

- (1)  $(A, B) = (\{c_3, c_4\}, \{p_4\})$  and  $(C, D) = (\{c_1\}, \{p_4\})$  are binary  $t$ -sets, but  $(A \cup C, B \cup D) = (\{c_1, c_3, c_4\}, \{p_4\})$  isn't a binary  $t$ -set.
- (2)  $(A, B) = (\{c_1, c_2, c_4\}, \{p_2\})$  and  $(C, D) = (\{c_3\}, \{p_1, p_4\})$  are binary  $\alpha^*$ -sets but  $(A \cup C, B \cup D) = (\{c_1, c_2, c_3, c_4\}, \{p_1, p_2, p_4\})$  isn't a binary  $\alpha^*$ -set.
- (3)  $(A, B) = (\{c_1\}, \{p_4\})$  and  $(C, D) = (\{c_1, c_2\}, \{p_3\})$  are binary semi-open sets, but  $(A \cap C, B \cap D) = (\{c_1\}, \emptyset)$  isn't a binary semi-open set.

**6.2. Benchmarking with other works.** It is important to note that the functions we have been working with throughout this article, and their properties, differ subtly from the functions introduced in [8]. For example, in [8], the functions  $f : Z \rightarrow X \times Y$  are considered, whereas we consider the functions  $f : Z \rightarrow \mathcal{P}(X) \times \mathcal{P}(Y)$ . The work [8], defines  $f^{-1}(A, B) = \{z \in Z : f(z) = (x, y) \in (A, B)\}$ , and under these conditions proves that

$$Z \setminus f^{-1}(A, B) = f^{-1}(X \setminus A, B) \cup f^{-1}(A, Y \setminus B) \cup f^{-1}(X \setminus A, Y \setminus B).$$

However, in our context, this is generally not true, as shown in the following example.

**Example.** Consider  $X = \{c_1, c_2, c_3, c_4\}$ ,  $Y = \{p_1, p_2, p_3, p_4\}$ ,  $E = \{e_1, e_2, e_3, e_4, e_5\}$  and the function  $f : Z \rightarrow \mathcal{P}(X) \times \mathcal{P}(Y)$  as follows:  $f(e_1) = (\{c_1\}, \{p_1\})$ ;  $f(e_2) = (\{c_2, c_3, c_4\}, \{p_1\})$ ;  $f(e_3) = (\{c_1\}, \{p_2, p_3, p_4\})$ ;  $f(e_4) = (\emptyset, \{p_2, p_3, p_4\})$  and  $f(e_5) = (\{c_2, c_3, c_4\}, \emptyset)$ . Then for  $(A, B) = (\emptyset, \{p_2, p_3, p_4\})$ , we have:  $f^{-1}(A, B) = \{e_4\}$ ;  $f^{-1}(X \setminus A, B) = \{e_3, e_4, e_5\}$ ;  $f^{-1}(A, Y \setminus B) = \emptyset$  and  $f^{-1}(X \setminus A, Y \setminus B) = \{e_1, e_2, e_5\}$ .

$$\text{So, } f^{-1}(X \setminus A, B) \cup f^{-1}(A, Y \setminus B) \cup f^{-1}(X \setminus A, Y \setminus B) = Z \neq Z \setminus f^{-1}(A, B) = Z \setminus \{e_4\}.$$

This small, but subtle and significant change in the range of the function is what allows us to establish a connection between the binary sets and the soft sets.

## REFERENCES

1. A. Açıkgöz, N. Taş, Binary soft set theory. *Eur. J. Pure Appl. Math.* **9** (2016), no. 4, 452–463.
2. A. Anıyan, S. Naduvath, Subspace graph topological space of graphs. *Proyecciones* **42** (2023), no. 2, 521–532.
3. N. Çağman, S. Enginoğlu, F. Citak, Fuzzy soft set theory and its applications. *Iran. J. Fuzzy Syst.* **8** (2011), no. 3, 137–147, 170.
4. C. Carpintero, E. Rosas, M. Salas-Brown, L. Vásquez, On sets in binary topological spaces. *Trans. A. Razmadze Math. Inst.* **178** (2024), no. 2, 213–223.
5. C. Granados, On binary semi-open sets and binary semi- $\omega$ -open sets in binary topological spaces. *Trans. A. Razmadze Math. Inst.* **175** (2021), no. 3, 365–373.
6. S. Hussain, Binary soft connected spaces and an application of binary soft sets in decision making problem. *Fuzzy Information and Engineering*, **11** (2019), no. 4, 506–521.
7. S. N. Jothi, Binary Semi open sets in Binary topological Spaces. *International Journal of Mathematical Archive*, **7** (2016), no. 9, 73–76.
8. S. N. Jothi, P. Thangavelu, Topology between two sets. *J. Math. Sci. Comput. Appl.* **1** (2011), no. 3, 95–107.
9. S. N. Jothi, P. Thangavelu, Generalized binary regular closed sets. *IRA-International Journal of Applied Sciences* **4** (2016), no. 2, 259–263.
10. G. Lin, L. Xie, J. Li, J. Chen, Y. Kou, Local double quantitative fuzzy rough sets over two universes. *Applied Soft Computing* **145** (2023), 110556.
11. P. K. Maji, R. Biswas, A. R. Roy, Soft set theory. *Comput. Math. Appl.* **45** (2003), no. 4-5, 555–562.
12. D. Molodtsov, Soft set theory—first results. Global optimization, control, and games, III. *Comput. Math. Appl.* **37** (1999), no. 4-5, 19–31.

13. P. G. Patil, N. N. Bhat, New separation axioms in binary soft topological spaces. *Italian Journal of Pure and Applied Mathematics*, **44** (2020), 775–783.

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