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VARIATIONAL AND NUMERICAL ANALYSIS OF A DYNAMIC THERMO-VISCOELASTIC CONTACT PROBLEM WITH NORMAL COMPLIANCE FOR A LOCKING MATERIAL

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Abstract. This paper studies the dynamic problem of frictional contact between a perfectly locking thermo-viscoelastic material and a heat conductive foundation. The contact is described by the normal compliance condition, and the friction is modeled by the nonlocal Coulomb friction law. The mathematical model of the dynamic process is presented, and the variational formulation is derived. The existence and uniqueness of the weak solution are established. The continuous dependence of the solution on the friction coefficient, as well as on the initial displacement and temperature data, is studied. Finally, a fully discrete finite element scheme for the variational problem is proposed, and error estimates for the approximate solution are derived.

1. Introduction

The term 'locking material' refers to a physical phenomenon in which the deformation of a material is hindered or halted upon encountering an obstacle or reaching a specific threshold. In this case, the material can no longer be distorted independently of the force. Changes in stress occur that are not accompanied by corresponding changes in deformation.

Recent advances in the modeling of a differential variational-hemivariational inequality and a generalized nonlinear quasi-hemivariational inequality in the context of an optimal control problem can be found in [32, 33]. Applications of generalized quasi-variational inequalities, particularly those involving generalized sub-differentials in the Clarke sense, have been observed in various fields. Refer to [8, 31, 32] for further details.

The first studies of variational problems with locking materials, initiated by Prager, are cited in [22–24], and developed by F. Demengel and P. Suquet in [12, 13]. Bourichi et al. addressed a penalty method for an elastic-locking material in a unilateral contact problem with friction, as discussed in [6]. The mathematical model of an elastic locking material with memory was considered in [26]. Recently, in [16], the authors have presented a new model of a contact problem involving an electroelastic-locking material in friction with a conductive foundation.

Mathematical models describing the dynamic frictional contact between a thermo-viscoelastic body and a rigid foundation were studied in [10, 19], and more recently in [4]. Elliptic variational and hemivariational inequalities for the displacement field, arising in various types of contact problems with friction for elastic and viscoelastic materials, can be found in [16, 20]. References [1, 25] present the models for perfectly locking materials.

Several papers investigating numerical analysis schemes and their error estimates for the dynamic and quasi-static contact problem, with or without friction, were studied in [3, 5, 7, 9, 15].

In this paper, we study the mathematical and numerical analysis of a new dynamic contact problem for a thermo-viscoelastic perfectly locking material with a heat conductive foundation. The deformation is limited by a positive constant, beyond which the body becomes extremely rigid, causing blockage, while the stress and temperature continue to increase. Since temperature has an impact on

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displacement, the locking effect is also depends on it. This model includes the subdifferential of indicator functions, introducing a key novelty in considering the locking effect caused by the temperature intensity.

Furthermore, the temperature field will be constrained to belong to a certain convex set. In this context, the Fourier law of heat conduction is expressed in the following form:

$$q_{th}(t) \in -\mathscr{K}\nabla\theta(t) - \partial I_B(\nabla\theta(t)) \text{ in } \Omega \times (0,T),$$

where B is the closed convex set defined by the formula

$$B = \{\beta_B \in L^2(\Omega),\ \|\beta_B\| \leq M_B\}.$$

This new model leads to a complex system of a dynamic quasi-variational nonlinear inequality, and a second parabolic variational nonlinear inequality. The difficulties in solving this problem lie in the coupling of the viscoelastic and thermal aspects, as well as in the non-linearity of the boundary conditions. In addition, we get the non-linear terms that are hard to deal with in continuous and discrete cases.

The contact is modelled by the penalized normal compliance condition with the parameter of penalization ϵ. The friction is described by Coulomb's law. We derive the existence and uniqueness of the weak solution, the proof is based on the quasi-variational inequalities and the Banach fixed point theorem. We prove the dependence of the solution on the friction coefficient and its convergence result. Likewise, we present the discrete problem using the finite element method and the backward Euler finite difference method. We establish the convergence of its solution through a proof. The rest of the paper is structured as follows. Section 2 presents the method for our dynamic frictional contact problem, including the non-linear thermo-viscoelastic-locking constitutive low, the equilibrium equation, and the boundary conditions describing the material behavior. In Section 3, we provide the notations and assumptions regarding the problem data. Additionally, we derive its variational formulation. In Section 4, we present the proof of a weak solvability. In Section 5, we study a contact problem for thermo-elastic case with a small perturbation of the friction coefficient, and establish a convergence result. In Section 6, we analyze a fully discrete scheme and derive related error estimates. Finally, in Appendix, we present some results of the analysis.

2. A Mathematical Model of Contact

We consider a locking body that initially occupies an open bounded domain $\Omega \subset \mathbb{R}^d$ for $d = 1, 2, 3$. The boundary Γ of the domain Ω is assumed to be Lipschitz and is divided into three disjoint measurable parts: Γ_D, Γ_N and Γ_C such that $meas(\Gamma_D) > 0$. Let $T > 0$ be the time interval of interest, $[0,T]$, and let $\rho : \Omega \mapsto \mathbb{R}^+$ denote the mass density of the body.

The body is clamped on $\Gamma_D \times (0,T)$, causing a displacement field in that region to vanish. The body is subject to a volume force of density f_0 in $\Omega \times (0,T)$ and to a volume thermal force of density q_0. It is also subject to a surface traction of density f_1 on $\Gamma_N \times (0,T)$. On Γ_C, the body may come into contact with a heat conductive obstacle, the so-called foundation. We assume that its temperature is maintained at θ_F. The normalized gap between Γ_C and the conductive foundation is denoted by g.

We use $\mathbb{S}^d$ to denote the space of a second-order symmetric tensor on $\mathbb{R}^d$, and " $\cdot$ " and $\|\cdot\|$ represent the inner product and the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$, that is, for all $u,\ v \in \mathbb{R}^d$, and for all $\sigma,\ \tau \in \mathbb{S}^d$,

$$u \cdot v = u_i \cdot v_i,\ \ \|v\| = (v \cdot v)^{\frac{1}{2}},\ \text{ and }\ \sigma \cdot \tau = \sigma_{ij} \cdot \tau_{ij},\ \ \|\tau\| = (\tau \cdot \tau)^{\frac{1}{2}},$$

where the indices $i,\ j$ run between 1 and d, and the index following the comma indicates a partial derivative with respect to the corresponding component of the spatial variable, e.g., $u_{i,j} = \frac{\partial u_i}{\partial x_j}$.

We denote by $u : \Omega \times (0,T) \to \mathbb{R}^d$ the displacement field, by $\sigma = (\sigma_{ij}) : \Omega \times (0,T) \to \mathbb{S}^d$, the stress tensor, by $q_{th} = (q_{th_i}) : \Omega \times (0,T) \to \mathbb{R}$ the heat flux vector, and by $\varepsilon(u) = (\varepsilon_{ij}(u)) = \frac{1}{2}(u_{i.j} + u_{j.i})$ the linearized strain tensor.

Here and below, Div denotes the divergence operators for tensor and vector-valued functions, i.e., $\mathrm{Div}(\sigma) = (\sigma_{ij,j})$ and $\mathrm{div}(q_{th}) = (q_{th_{i,i}})$ are the vector-valued functions.

Moreover, $\nu = (\nu_i)$ denotes the outward unit normal at Γ, and $u_\nu = u \cdot \nu$, $u_\tau = u - u_\nu \nu$ are the normal and tangential components of u on Γ. Also, $\sigma_\nu = (\sigma\nu) \cdot \nu$, $\sigma_\tau = \sigma\nu - \sigma_\nu \nu$ are the normal and tangential stress on Γ, respectively.

Under the above assumptions, we are in a position to introduce the formulation of the contact problem.

• **Problem (P)**: Find a displacement field $u : \Omega\times]0,T[\to \mathbb{R}^d$ and a temperature field $\theta : \Omega\times]0,T[\to \mathbb{R}$ such that

$$\sigma(t) \in \mathscr{C}\varepsilon(\dot{u}(t)) + \mathscr{F}\varepsilon(u(t)) - \theta(t)\mathscr{M} + \partial I_A(\varepsilon(u(t))) \text{ in } \Omega \times (0,T), \tag{2.1}$$

$$q_{th}(t) \in -\mathscr{K}\nabla\theta(t) - \partial I_B(\nabla\theta(t)) \text{ in } \Omega \times (0,T), \tag{2.2}$$

$$\rho\ddot{u}(t) = \text{Div } \sigma(u(t)) + f_0(t) \text{ in } \Omega \times (0,T), \tag{2.3}$$

$$\dot{\theta}(t) + \text{div}(q_{th}(t)) = q_0(t) \text{ in } \Omega \times (0,T), \tag{2.4}$$

$$u(t) = 0 \text{ on } \Gamma_D \times (0,T), \tag{2.5}$$

$$\theta(t) = 0 \text{ on } (\Gamma_N \cup \Gamma_D) \times (0,T), \tag{2.6}$$

$$\sigma(t)\nu = f_1(t) \text{ on } \Gamma_N \times (0,T), \tag{2.7}$$

$$u(0,x) = u_0,\ \dot{u}(0,x) = \dot{u}_0,\ \theta(0,x) = \theta_0 \text{ in } \Omega, \tag{2.8}$$

$$\sigma_\nu(u_\nu(t) - g) = -\frac{1}{\epsilon}\left[u_\nu(t) - g\right]^+,\ \epsilon > 0 \text{ on } \Gamma_C \times (0,T), \tag{2.9}$$

$$\left.\begin{aligned} &\|\sigma_\tau(t)\| \le \mu(\|u_\tau(t)\|)|R\sigma_\nu(u(t))|, \\ &\dot{u}_\tau(t) \neq 0 \Rightarrow \sigma_\tau(t) = \mu(\|u_\tau(t)\|)|R\sigma_\nu(u(t))|\frac{\dot{u}_\tau(t)}{\|\dot{u}_\tau(t)\|} \end{aligned}\right\} \text{ on } \Gamma_C \times (0,T), \tag{2.10}$$

$$\frac{\partial q_{th}(t)}{\partial\nu} = k_c(u_\nu(t) - g)\phi_L(\theta(t) - \theta_F) \text{ on } \Gamma_C \times (0,T), \tag{2.11}$$

where $I_{P=A,B}$ is the indicator function defined by

$$I_P(\alpha) = \begin{cases} 0, & \text{if } \alpha \in P, \\ +\infty, & \text{otherwise,} \end{cases}$$

in the closed and convex sets A and B defined, respectively, by

$$A = \{\beta_A \in \mathbb{S}^d,\ \|\beta_A\| \le M_A\},$$
$$B = \{\beta_B \in L^2(\Omega),\ \|\beta_B\| \le M_B\}.$$

Recall that equations (2.1) and (2.2) represent the thermo-viscoelastic constitutive law for a locking material. In these equations, $\mathscr{F} = (f_{ijkl})$, $\mathscr{C} = (c_{ijkl})$, $\mathscr{M} = (m_{ij})$, and $\mathscr{K} = (k_{ij})$ correspond to the elasticity tensor, viscosity tensor, thermal expansion tensor, and heat conductivity tensor, respectively. Equations (2.3)–(2.4) are, respectively, the equations of motion and the Fourier law of heat conductivity. Conditions (2.5)–(2.7) represent the displacement, traction, and thermal boundary conditions. The initial conditions are specified by equation (2.8). Furthermore, equation (2.9) represents the normal compliance contact condition on Γ_C, where $\epsilon > 0$ denotes the penalty parameter. The Coulomb law of friction is considered in relation (2.10), where μ is the friction coefficient and R is a regularization operator. The function R can be selected as the convolution product with a regular function ω, which corresponds to the nonlocal friction law, as described in [11].

$$R\sigma_\nu(x) = \int_{\Gamma_C} \omega(\|x - z\|)\sigma_\nu(z)dz,$$

and

$$\omega(x) = \begin{cases} \varpi.\exp\left(\dfrac{\alpha^2}{\|x\|^2 - \alpha^2}\right) & \text{if } \|x\| \le \alpha, \\ 0 & \text{if } \|x\| > \alpha. \end{cases}$$

The normalization constant ϖ is determined such that $\int\limits_{-\infty}^{+\infty} w(x), dx = 1$, where $w(x)$ is the regular function. Additionally, α represents a length that characterizes the geometry of the material's asperities.

Finally, equation (2.11) represents the regularized thermal condition, as described in [14], such that

$$\phi_L(s) = \begin{cases} -L & \text{if } s < -L, \\ s & \text{if } -L \leq s \leq L, \\ L & \text{if } s > L, \end{cases} \quad \text{and} \quad \begin{cases} k_c(r) = 0 & \text{if } r < 0, \\ k_c(r) > 0 & \text{if } r \geq 0, \end{cases}$$

where L is a large positive constant.

3. Weak Formulation

In order to present the variational formulation of **Problem (P)**, we require some additional notation and preliminary concepts. Let X be a Banach space, we use the classical notations for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, where $1 \leq p \leq \infty$ and $k = 1, \ldots,$ endowed with the norm

$$\|u\|^p_{L^p(0,T;X)} = \int\limits_0^T \|u\|^p_X dt.$$

We introduce the following functional spaces:

$$H = L^2(\Omega)^d = \left\{u = (u_i)\ ;\ u_i \in L^2(\Omega)\right\},\ H_1 = H^1(\Omega)^d,$$
$$\mathscr{H} = \left\{\tau = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega)\right\},\ \mathscr{H}_1 = \{\sigma \in \mathscr{H} \mid \text{Div } \sigma \in H\}.$$

These spaces are the real Hilbert spaces equipped with inner products corresponding to the associated norms defined as follows:

$$(u,v)_H = \int\limits_\Omega \rho u_i v_i dx,\ (u,v)_{H_1} = (u,v)_H + (\varepsilon(u), \varepsilon(v))_{\mathscr{H}},$$

$$(\sigma,\tau)_{\mathscr{H}} = \int\limits_\Omega \sigma_{ij}\tau_{ij} dx,\ (u,v)_{\mathscr{H}_1} = (\sigma,\tau)_{\mathscr{H}} + (\text{Div } \sigma, \text{Div } \tau)_{\mathscr{H}},$$

and the associated norms $\|\cdot\|_H$, $\|\cdot\|_{H_1}$, $\|\cdot\|_{\mathscr{H}}$ and $\|\cdot\|_{\mathscr{H}_1}$, respectively.

Let $H_\Gamma = H^{\frac{1}{2}}(\Gamma)^d$ and $\gamma : H \to H_\Gamma$ be the trace map. For every element $v \in H$, we also use the notation v to denote the trace γv of v on Γ.

Let $H_\Gamma^{'}$ be the dual of H_Γ and $\langle \cdot, \cdot \rangle$ denote the duality pairing between $H_\Gamma^{'}$ and H_Γ. For every $\sigma \in \mathscr{H}$, $\sigma\nu$ can be defined as the element in $H_\Gamma^{'}$ satisfying

$$\langle \sigma\nu, \gamma v\rangle = (\sigma, \varepsilon(v))_{\mathscr{H}} + (\text{Div } \sigma, v)_H,\ \ \forall v \in H_1. \tag{3.1}$$

Moreover, if σ is continuously differentiable on $\overline{\Omega}$, then

$$\langle \sigma\nu, \gamma v\rangle = \int\limits_\Gamma \sigma\nu \cdot v da,\ \ \forall v \in H_1, \tag{3.2}$$

where da is the surface measure element.

Combining equations (3.1) through (3.2), we obtain the following Green's formula in elasticity theory:

$$(\sigma, \varepsilon(v))_{\mathscr{H}} + (\text{Div } \sigma, v)_H = \int\limits_\Gamma \sigma\nu \cdot v da. \tag{3.3}$$

Notice that if $q \in H^1(\Omega)$ is a sufficiently regular function, the following Green's type formula

$$(q, \nabla\eta)_H + (\text{div } q, \eta)_{L^2(\Omega)} = \int\limits_\Gamma q \cdot \nu\eta da,\ \ \forall \eta \in H^1(\Omega) \tag{3.4}$$

holds. Taking into account the boundary conditions (2.5)–(2.9), we introduce the closed subspace of H_1 as defined below,

$$V = \{v \in H_1 \ : \ v = 0 \ \text{ on } \ \Gamma_D\},$$
$$Q = \left\{\eta \in H^1(\Omega) \ : \ \eta = 0 \ \text{ on } \ \Gamma_D \cup \Gamma_N\right\},$$
$$V_1 = \{v \in V \ : \ v_\nu - g \leq 0 \ \text{ on } \ \Gamma_C\},$$

which are endowed with the inner products and norms defined as follows:

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathscr{H}}, \quad \|v\|_V = (v, v)_V^{\frac{1}{2}},$$
$$(\theta, \eta)_Q = (\nabla\theta, \nabla\eta)_H, \qquad \|\eta\|_Q = (\eta, \eta)_Q^{\frac{1}{2}}.$$

Since $meas(\Gamma_D) > 0$, the following Korn inequality (see [21])

$$\|\varepsilon(v)\|_{\mathscr{H}} \geq c_t \|v\|_{H_1}, \ \text{ for all } \ v \in V,$$

holds, where c_t is a non-negative constant depending only on Ω and Γ_D.

The following Frierichs–Poincaré inequality

$$\|\nabla\eta\|_H \geq C_F \|\eta\|_Q, \ \text{ for all } \ \eta \in Q,$$

holds on Q. Furthermore, according to the Sobolev trace theorem, there exist the constants $c_0 > 0$ and $c_1 > 0$, depending on Ω, Γ_C, and Γ_D, such that

$$\|v\|_{[L^2(\Gamma_C)]^d} \leq c_0 \|v\|_V, \quad \|\eta\|_{L^2(\Gamma_C)} \leq c_1 \|\eta\|_Q, \ \text{ for all } \ (v, \eta) \in V \times Q. \tag{3.5}$$

Next, utilizing the Riesz representation theorem, we define the elements $f(t) \in V$ and $q(t) \in Q$ for all $v \in V$ and $\eta \in Q$ as follows:

$$(f(t), v)_V = \int_\Omega f_0(t) \cdot v dx + \int_{\Gamma_N} f_1(t) \cdot v da, \tag{3.6}$$

$$(q(t), \eta)_Q = \int_\Omega q_0(t) \cdot \eta dx. \tag{3.7}$$

Also, we define $j : V \times V \to \mathbb{R}$, $\chi : V \times Q \times Q \to \mathbb{R}$ and $\Phi : V \times V \to \mathbb{R}$ by

$$j(u(t), v) = \int_{\Gamma_C} \mu(\|u_\tau(t)\|)|R\sigma_\nu(u(t))|\|v_\tau\| da, \tag{3.8}$$

$$\chi(u(t), \theta(t), \eta) = \int_{\Gamma_C} k_c(u_\nu(t) - g)\phi_L(\theta(t) - \theta_F)\eta da, \tag{3.9}$$

and

$$\Phi(u(t), v) = \frac{1}{\epsilon} \int_{\Gamma_C} [u_\nu(t) - g]^+ v_\nu da = \frac{1}{\epsilon} \left\langle [u_\nu(t) - g]^+, v_\nu \right\rangle_{\Gamma_C}. \tag{3.10}$$

It should be noted that the integrals (3.8) and (3.9) are well-defined due to the conditions (HP_7) and (HP_8) below, respectively. To proceed with the investigation of the mechanical **Problem (P)**, we require the following assumptions:

(HP_1) The elasticity tensor $\mathscr{F} = (f_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies the usual properties of symmetry, boundedness, ellipticity and continuity:
- i) $f_{ijkl} = f_{jikl} = f_{lkij} \in L^\infty(\Omega)$, for all i, j, k, l;
- ii) $\mathscr{F}(x, \xi).\sigma = \xi.\mathscr{F}(x, \sigma)$, for all $\xi, \sigma \in \mathbb{S}^d$, a.e. $x \in \Omega$;
- iii) $f_{ijkl}(x)\xi_{ij}\xi_{kl} \geq m_{\mathscr{F}} \|\xi\|^2$, with $m_{\mathscr{F}} > 0$, for all $\xi \in \mathbb{S}^d$, a.e. $x \in \Omega$;
- iv) $\|(\mathscr{F}\varepsilon(u), \varepsilon(v))\|_{\mathscr{H}} \leq M_{\mathscr{F}} \|u\|_V \|v\|_V$.

(HP_2) The viscosity tensor $\mathscr{C} = (c_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies the following conditions:
- i) $c_{ijkl} = c_{jikl} = c_{lkij} \in L^\infty(\Omega)$;

ii) $\mathscr{C}(x,.)$ is monotone on $\mathbb{S}^d$, i.e.,

$$(\mathscr{C}(x,\xi_1)-\mathscr{C}(x,\xi_2)).(\xi_1-\xi_2)\geq 0, \text{ for all } \xi_1,\xi_2\in\mathbb{S}^d, \text{ a.e. } x\in\Omega;$$

iii) $\mathscr{C}(x,.)$ is coercive on $\mathbb{S}^d$, i.e.,

$$c_{ijkl}(x)\xi_{ij}\xi_{kl}\geq m_{\mathscr{C}}\|\xi\|^2, \text{ with } m_{\mathscr{C}}>0, \text{ for all } \xi\in\mathbb{S}^d, \text{ a.e. } \in\Omega;$$

iv) There exist $r\in L^\infty(\Omega)$ and $s\in L^2(\Omega)$ such that

$$|\mathscr{C}(x,\xi)|\leq r(x)|\xi|+s(x), \ \forall\xi\in\mathbb{S}^d, \text{ a.e. } x\in\Omega;$$

v) $\mathscr{C}(x,.)$ is continuous on $\mathbb{S}^d$, a.e. $x\in\Omega$

$$\|(\mathscr{C}\varepsilon(u),\varepsilon(v))\|_{\mathscr{H}}\leq M_{\mathscr{C}}\|u\|_V\|v\|_V;$$

vi) $\mathscr{C}(.,\xi)$ is Lebesgue measurable on Ω for all $\xi\in\mathbb{S}^d$.

(HP_3) The thermal conductivity tensor $\mathscr{K}=(k_{ij}):\Omega\times\mathbb{R}^d\to\mathbb{R}^d$ satisfies
i) $k_{ij}=k_{ji}\in L^\infty(\Omega)$;
ii) $k_{ij}(x)\xi_i\xi_j\geq m_{\mathscr{K}}\|\xi\|^2$, with $m_{\mathscr{K}}>0$, for all $\xi\in\mathbb{R}^d$, $x\in\Omega$;
iii) $\|(\mathscr{K}\nabla\theta,\nabla\eta)\|_H\leq M_{\mathscr{K}}\|\theta\|_Q\|\eta\|_Q$.

(HP_4) The thermal expansion tensor $\mathscr{M}=(m_{ij}):\Omega\times\mathbb{R}\to\mathbb{R}$ satisfies
i) $m_{ij}=m_{ji}\in L^\infty(\Omega)$;
ii) $\|(\mathscr{M}\theta,\varepsilon(v))\|_{\mathscr{H}}\leq M_{\mathscr{M}}\|\theta\|_Q\|v\|_V$.

(HP_5) i) The forces, the traction, and the thermal flux satisfy

$$f_0\in W^{1,1}(0,T;H), \quad f_1\in W^{1,1}\left(0,T;L^2(\Gamma_N)^d\right), \quad q_0\in L^2\left(0,T;L^2(\Omega)\right);$$

ii) The thermal potential and the gap function satisfy

$$\theta_F\in L^2(0,T;L^2(\Gamma_C)), \quad g\in L^2(\Gamma_C), \ g\geq 0.$$

iii) The mapping $R:H'_{\Gamma_C}\to L^\infty(\Gamma_C)$ is linear compact and continuous with

$$c_R=\|R\|_{H'}.$$

iv) The mass density satisfies $\rho\in L^\infty(\Omega)$, and there exists $\rho^*>0$ such that

$$\rho\in L^\infty(\Omega) \text{ and } \rho(x)\geq\rho^* \text{ a.e. } x\in\mathbb{R}.$$

(HP_6) i) The initial data u_0, $\dot{u}_0$ and θ_0 of **Problem (P)** satisfy

$$u_0\in V, \ \dot{u}_0\in D(\partial j), \ \theta_0\in Q,$$

where ∂j denotes the sub-differential of j and $D(\partial j)$ represents its domain.

ii) There exists $h\in L^2(\Omega)^d$ for all $v\in V$ such that

$$c(\dot{u}_0,v-\dot{u}_0)+a(u_0,\dot{u}_0)+j(u_0,v)-j(u_0,\dot{u}_0)\geq(h,v-\dot{u}_0).$$

iii) The functional j is proper, convex, and lower semi-continuous on $V\times V$.

(HP_7) The coefficient of friction $\mu:\Gamma_C\times\mathbb{R}^+\to\mathbb{R}^+$ satisfies
i) There exists a positive constant $L_\mu>0$, for all $x,y\in\mathbb{R}^+$,

$$|\mu(.,x)-\mu(.,y)|\leq L_\mu|x-y| \text{ a.e. on } \Gamma_C;$$

ii) There exists a positive constant $M_\mu>0$ such that

$$|\mu(x,u)|\leq M_\mu, \text{ for all } u\in\mathbb{R}^+ \text{ and } x\in\Gamma_C;$$

iii) The mapping $x\to\mu(x,u)$ is measurable on Γ_C for all $u\in\mathbb{R}^+$.

(HP_8) The coefficient of heat exchange $k_c:\Gamma_C\times\mathbb{R}\to\mathbb{R}^+$ satisfies ies
i) There exists $M_{k_c}>0$ such that $|k_c(x,u)|<M_{k_c}$ for all $u\in\mathbb{R}, x\in\Gamma_C$ $x\mapsto k_c(x,u)$ is measurable on Γ_C for all $x\in\mathbb{R}$, $k_c(x,u)=0$ for all $x\in\Gamma_C$ and $u\leq 0$;
ii) There exists $L_{k_c}>0$ such that $|k_c(x,u_1)-k_c(x,u_2)|\leq L_{k_c}|u_1-u_2|$, for all $u_1,u_2\in\mathbb{R}$.

Using $(HP_6)(iv)$, the inclusion mapping of $(V, \|\cdot\|_V)$ into $(H, \|\cdot\|_H)$ is continuous and dense. Then we identify H and H' and write $V \subset H \equiv H' \subset V'$.

Since Q is dense in $L^2(\Omega)$, we write $Q \subset L^2(\Omega) \subset Q'$.

We denote by $a : V \times V \to \mathbb{R}$, $c : V \times V \to \mathbb{R}$, $d : Q \times Q \to \mathbb{R}$ and $m : Q \times V \to \mathbb{R}$ are the bilinear operators given by

$$a(u,v) = (\mathscr{F}\varepsilon(u), \varepsilon(v))_{\mathscr{H}}, \quad c(u,v) = (\mathscr{C}\varepsilon(u), \varepsilon(v))_{\mathscr{H}},$$
$$d(\theta,\eta) = (\mathscr{K}\nabla\theta, \nabla\eta)_H, \quad m(\theta,v) = (\mathscr{M}\theta, \varepsilon(v))_{\mathscr{H}}.$$

From the constitutive laws (2.1)–(2.2), we have

$$\sigma(t) = \mathscr{C}\varepsilon(\dot{u}(t)) + \mathscr{F}\varepsilon(u(t)) - \theta(t)\mathscr{M} + \mathscr{A}\,(u(t)),$$
$$q_{th}(t) = -\mathscr{K}\nabla\theta(t) - \mathscr{B}\,(\theta(t)),$$

where

$$\mathscr{A}u(t) \in \partial I_A(\varepsilon(u(t))) \text{ and } \mathscr{B}\theta(t) \in \partial I_B(\nabla\theta(t)) \text{ in } \Omega.$$

For almost all $t \in]0,T[$, let us define the following subsets:

$$V_2 = \{u(t) \in V \;;\; \sup\{\|\varepsilon(u(t))\|_{\mathscr{H}}, \|\varepsilon(\dot{u}(t))\|_{\mathscr{H}}\} \le M_A \text{ a.e. in } \Omega\},$$

and

$$Q_1 = \{\theta(t) \in Q \;;\; \|\nabla\theta(t)\|_H \le M_B \text{ a.e. in } \Omega\}.$$

For all $u, v \in V_2$ and $\theta, \eta \in Q_1$, we obtain

$$\langle \mathscr{A}u(t), \varepsilon(v) - \varepsilon(\dot{u}(t))\rangle \le I_A(\varepsilon(v)) - I_A(\varepsilon(\dot{u}(t))) \le 0,$$
$$\langle \mathscr{B}\theta(t), \nabla\eta - \nabla\theta(t)\rangle \le I_B(\nabla\eta) - I_B(\nabla\theta(t)) \le 0.$$

Then,

$$(\sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t)))_{\mathscr{H}} \le (\mathscr{C}\varepsilon(\dot{u}(t)) + \mathscr{F}\varepsilon(u(t)) - \theta(t)\mathscr{M}, \varepsilon(v) - \varepsilon(\dot{u}(t)))_{\mathscr{H}}, \tag{3.11}$$
$$(-q_{th}(t), \nabla\eta - \nabla\theta(t))_H \le (\mathscr{K}\nabla\theta(t), \nabla\eta - \nabla\theta(t))_H. \tag{3.12}$$

Next, we assume that $\{u, \sigma, \theta, q_{th}\}$ are regular functions satisfying (2.1)–(2.11) and let $v \in V_1 \cap V_2$, $\eta \in Q_1$. Using (3.3)–(3.4), we have

$$-\,(\mathrm{Div}\;\sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t)))_{\mathscr{H}} + \int_{\Gamma} (\sigma\nu.\varepsilon(v) - \varepsilon(\dot{u}(t)))\,da$$
$$\le (\mathscr{C}\varepsilon(\dot{u}(t)) + \mathscr{F}\varepsilon(u(t)) - \theta(t)\mathscr{M}, \varepsilon(v) - \varepsilon(\dot{u}(t)))_{\mathscr{H}},$$

$$(\mathrm{div}\;q_{th}(t), \nabla\eta - \nabla\theta(t))_H - \int_{\Gamma} (q_{th}\nu.\nabla\eta - \nabla\theta(t))\,da \le (\mathscr{K}\nabla\theta(t), \nabla\eta - \nabla\theta(t))_H.$$

By conditions (2.5)–(2.7) and (3.6)–(3.7), we deduce

$$(-\rho\ddot{u}(t) - \mathscr{C}\varepsilon(\dot{u}(t)) - \mathscr{F}\varepsilon(u(t)) + \theta(t)\mathscr{M}, \varepsilon(v) - \varepsilon(\dot{u}(t)))_{\mathscr{H}}$$
$$+ \int_{\Gamma_C} (\sigma\nu.\varepsilon(v) - \varepsilon(\dot{u}(t)))\,da \le -\,(f(t), \varepsilon(v) - \varepsilon(\dot{u}(t)))_V,$$

$$\Big(-\dot{\theta}(t) - \mathscr{K}\nabla\theta(t), \nabla\eta - \nabla\theta(t)\Big)_H - \int_{\Gamma_C} (q_{th}.\nu, \nabla\eta - \nabla\theta(t))\,da \le -\,(q(t), \nabla\eta - \nabla\theta(t))_H.$$

Taking into consideration equations (3.11)–(3.12), (2.9)–(2.11), (3.8)–(3.10), as well as the inner product in H, we obtain the following weak formulation.

• **Problem (PV)**: Find a displacement field $u \in (V_1 \cap V_2) \times]0;T[$ and a temperature field $\theta \in Q_1 \times]0,T[$, for all $v \in V_1 \cap V_2$ and $\eta \in Q_1$ such that:

$$\begin{aligned} &(\ddot{u}(t), v-\dot{u}(t))_H + c(\dot{u}(t), v-\dot{u}(t)) + a(u(t), v-\dot{u}(t)) - m(\theta(t), v-\dot{u}(t)) \\ &+\Phi(u(t), v-\dot{u}(t)) + j(u(t),v) - j(u(t),\dot{u}(t)) \geq (f(t), v-\dot{u}(t))_V, \\ &\left(\dot{\theta}(t), \eta-\theta(t)\right)_{Q'\times Q} + d(\theta(t), \eta-\theta(t)) + \chi(u(t),\theta(t),\eta-\theta(t)) \geq (q(t),\eta-\theta(t))_Q, \\ &u(0)=u_0, \quad \dot{u}(0)=\dot{u}_0, \quad \theta(0)=\theta_0. \end{aligned} \tag{3.13}$$

4. Existence and Uniqueness of the Weak Solution

The existence and uniqueness of the weak solution to **Problem (PV)** are established as follows:

Theorem 4.1. *Assuming that (HP_1)–(HP_8) hold, there exists a unique solution to **Problem (PV)** that satisfies*

$$u \in W^{1,\infty}(0,T;V) \cap W^{2,\infty}(0,T;H), \quad \theta \in W^{1,2}(0,T;Q) \cap L^2(0,T;Q).$$

The proof of Theorem 4.1 is carried out in several steps and based on the arguments involving quasi-variational inequalities and the Banach fixed point theorem.

At the first step, let $\lambda \in W^{1,1}(0,T;H)$ and $\gamma \in L^2(0,T;L^2(\Omega))$ be given, and consider the following intermediate variational problems.

• **Problem** (PV_λ): Find a displacement field $u_\lambda \in (V_1 \cap V_2)\times]0,T[$ for all $v \in V_1 \cap V_2$ such that:

$$(\ddot{u}_\lambda(t), v-\dot{u}_\lambda(t))_H + c(\dot{u}_\lambda(t), v-\dot{u}_\lambda(t)) + a(u_\lambda(t), v-\dot{u}_\lambda(t))$$

$$+(\lambda(t), v-\dot{u}_\lambda(t))_V + j(u_\lambda(t),v) - j(u_\lambda(t),\dot{u}_\lambda(t)) \geq (f(t), v-\dot{u}_\lambda(t))_V, \tag{4.1}$$

$$u_\lambda(0)=u_0, \quad \dot{u}_\lambda(0)=\dot{u}_0. \tag{4.2}$$

• **Problem** (PV_γ): Find a temperature field $\theta_\gamma \in Q_1 \times]0,T[$ for all $\eta \in Q_1$ such that:

$$\left(\dot{\theta}_\gamma(t), \eta-\theta_\gamma(t)\right)_{Q'\times Q} + d(\theta_\gamma(t), \eta-\theta_\gamma(t)) + (\gamma(t), \eta-\theta_\gamma(t))_Q \geq (q(t), \eta-\theta_\gamma(t))_Q, \tag{4.3}$$

$$\theta_\gamma(0)=\theta_0. \tag{4.4}$$

At the second step, we present and prove the solvability result for the intermediate problems.

Lemma 4.1. *There exists a unique solution to **Problem** (PV_λ) that satisfies*

$$u_\lambda \in W^{1,\infty}(0,T;V) \cap W^{2,\infty}(0,T;H).$$

Proof. Using the Riesz representation theorem, we can define the operator

$$(f_\lambda(t), v)_V = (f(t),v)_V - (\lambda(t),v)_V.$$

Then **Problem** (PV_λ) can be written as follows:

$$\begin{aligned} &(\ddot{u}_\lambda(t), v-\dot{u}_\lambda(t))_H + c(\dot{u}_\lambda(t), v-\dot{u}_\lambda(t)) + a(u_\lambda(t), v-\dot{u}_\lambda(t)) \\ &+j(u_\lambda(t),v) - j(u_\lambda(t),\dot{u}_\lambda(t)) \geq (f_\lambda(t), v-\dot{u}_\lambda(t))_V, \\ &u_\lambda(0)=u_0, \quad \dot{u}_\lambda(0)=\dot{u}_0. \end{aligned}$$

Keeping in mind (HP_5)(i), (3.6), we have $f \in W^{1,1}(0,T,H)$, and from the regularity of $\lambda \in W^{1,1}(0,T;V)$, we can conclude that $f_\lambda \in W^{1,1}(0,T;H)$.

By the assumption (HP_1), the operator a is continuous, symmetric, bilinear, and satisfies the following coerciveness condition

$$\langle cu, u\rangle_{V'\times V} + \alpha\|u\|^2 \geq w\|u\|^2, \;\; \forall u \in V,$$

with $\alpha = 0$ and $w = m_{\mathscr{C}}$.

Next, we define the set-valued operator $M : V \to V'$ by $M = c + \partial j$.

From (HP_2), we have

$$(\mathscr{C}(\varepsilon(u_1)) - \mathscr{C}(\varepsilon(u_2)), \varepsilon(u_1)-\varepsilon(u_2))_{\mathscr{H}} \geq 0, \quad \forall u_1,u_2 \in V,$$

then the operator c is monotone.

Since c is continuous, in accordance with (HP_2)(iv), we can deduce that c is bounded.

Using (HP_6)(iii), the operator j is proper, convex and lower semi-continuous on $V \times V$, which implies that ∂j is maximal monotone.

Consequently, since c is monotone, bounded and continuous in $V \times V$, we conclude [2, P. 39] that $M = c + \partial j$ is maximal monotone. Moreover, by (HP_6)(i)–(ii), the initial data u_0 and $\dot{u}_0$ satisfy the condition

$$\{au_0 + c\dot{u}_0\} \cap H \neq \varnothing.$$

It follows now from Theorem 6.2 that there exists a unique function satisfying $u_\lambda \in W^{1,\infty}(0,T;V) \cap W^{2,\infty}(0,T;H)$. □

Lemma 4.2. *There exists a unique solution to* ***Problem*** (PV_γ) *that satisfies the given condition*

$$\theta_\gamma \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;Q).$$

Proof. By the Riesz representation theorem, there exists an operator $q_\gamma(t) \in Q$ such that

$$(q_\gamma(t), \eta)_Q = (q(t), \eta)_Q - (\gamma(t), \eta)_Q\,, \quad \forall \eta \in Q_1. \tag{4.5}$$

Then problem (4.3)–(4.4) can be written as follows:

$$\left(\dot{\theta}_\gamma(t), \eta - \theta_\gamma(t)\right)_{Q' \times Q} + d\left(\theta_\gamma(t), \eta - \theta_\gamma(t)\right) \geq \left(q_\gamma(t), \eta - \theta_\gamma(t)\right)_Q,$$

for all $\eta \in Q_1$. From (3.7), (4.5) and the regularity of q_0, we can observe that $q_\gamma(t) \in L^2(0,T;L^2(\Omega))$.

The hypothesis (HP_3) assures that the operator d is a continuous, symmetric bilinear and satisfies the following condition:

$$d(\theta,\theta) + c_0\|\theta\|^2_{L^2(\Omega)} \geq \alpha\|\theta\|^2_Q, \quad \forall \theta \in Q,$$

with $c_0 = 0$ and $\alpha = m_{\mathscr{K}}$. Now, utilizing Theorem 6.3, we can conclude the result. □

From now on, the constant denoted by C may differ line to line.

At the last step, for $(\lambda, \gamma) \in W^{1,1}(0,T;H) \times L^2(0,T;L^2(\Omega))$, we define the mapping

$$\Lambda(\lambda,\gamma)(t) = (\Lambda_1(\lambda,\gamma)(t), \Lambda_2(\lambda,\gamma)(t)) \in V \times Q, \tag{4.6}$$

given by

$$(\Lambda_1(\lambda,\gamma)(t), v) = \Phi\left(u_\lambda(t), v\right) - m\left(\theta_\gamma(t), v\right), \tag{4.7}$$

$$(\Lambda_2(\lambda,\gamma)(t), \eta) = \chi\left(u_\lambda(t), \theta_\gamma(t), \eta\right), \tag{4.8}$$

for all $v \in V_1 \cap V_2$ and $\eta \in Q_1$. We have the following

Lemma 4.3. *For* $(\lambda, \gamma) \in W^{1,1}(0,T;H) \times L^2(0,T;L^2(\Omega))$, *the operator* $\Lambda : [0,T] \to V \times Q$ *is continuous. Moreover, there exists a unique* $(\lambda^*, \gamma^*) \in (\lambda,\gamma) \in W^{1,1}(0,T;H) \times L^2(0,T;L^2(\Omega))$ *such that* $\Lambda(\lambda^*, \gamma^*) = (\lambda^*, \gamma^*)$.

Proof. Let $(\lambda, \gamma) \in W^{1,1}(0,T;H) \times L^2(0,T;L^2(\Omega))$ and $t_1, t_2 \in]0,T[$.

Using (3.10), (4.7), (HP_4) and the inequality $|[x]^+ - [y]^+| \leq |x - y|$, we obtain

$$\begin{aligned} &\|\Lambda_1(\lambda,\gamma)(t_1) - \Lambda_1(\lambda,\gamma)(t_2)\|_{V\times Q} \\ &\leq \sup\left(M_{\mathscr{M}}, \frac{c_0^2}{\epsilon}\right)\left(\|\theta_\gamma(t_1) - \theta_\gamma(t_2)\|_Q + \|u_\lambda(t_1) - u_\lambda(t_2)\|_V\right). \end{aligned} \tag{4.9}$$

By the regularities of u_λ and θ_γ, we conclude that $\Lambda_1(\lambda,\gamma) \in C([0,T];V)$.

On the other hand, by (3.9), (4.8), and (HP_8), it follows that

$$\|\Lambda_2(\lambda,\gamma)(t_1) - \Lambda_2(\lambda,\gamma)(t_2)\|_{V\times Q} \leq M_{k_c} c_1^2 L\,\|\theta_\gamma(t_1) - \theta_\gamma(t_2)\|_Q. \tag{4.10}$$

Then $\Lambda_2(\lambda,\gamma) \in C([0,T];Q)$. Consequently, $\Lambda(\lambda,\gamma) \in C([0,T];V \times Q)$.

Let now $(\lambda_1,\gamma_1), (\lambda_2,\gamma_2) \in W^{1,1}(0,T;H) \times L^2(0,T;L^2(\Omega))$ and $t \in [0,T]$.

Using the arguments similar to (4.9) and (4.10), we can deduce the existence of a positive constant C such that

$$\|\Lambda(\lambda_1,\gamma_1)(t) - \Lambda(\lambda_2,\gamma_2)(t)\|_{V\times Q} \leq C\left(\|\theta_{\gamma_1}(t) - \theta_{\gamma_2}(t)\|_Q + \|u_{\lambda_1}(t) - u_{\lambda_2}(t)\|_V\right). \tag{4.11}$$

Note that $u_{\lambda_i}(t) = \int\limits_0^t \dot{u}_{\lambda_i}(s)ds + u_{\lambda_i}(0)$, for $i = 1, 2$, implies

$$\|u_{\lambda_1}(t) - u_{\lambda_2}(t)\|_V^2 \leq C \int\limits_0^t \|\dot{u}_{\lambda_1}(s) - \dot{u}_{\lambda_2}(s)\|_V^2 \, ds. \tag{4.12}$$

From (4.1), we write

$$\begin{aligned} &(\ddot{u}_{\lambda_1}(t) - \ddot{u}_{\lambda_2}(t), \dot{u}_{\lambda_1}(t) - \dot{u}_{\lambda_2}(t)) + c\,(\dot{u}_{\lambda_1}(t) - \dot{u}_{\lambda_2}(t), \dot{u}_{\lambda_1}(t) - \dot{u}_{\lambda_2}(t)) \\ &+a\,(u_{\lambda_1}(t) - u_{\lambda_2}(t), \dot{u}_{\lambda_1}(t) - \dot{u}_{\lambda_2}(t)) + (\lambda_1(t) - \lambda_2(t), \dot{u}_{\lambda_1}(t) - \dot{u}_{\lambda_2}(t)) \\ &+j\,(u_{\lambda_1}(t), \dot{u}_{\lambda_1}(t)) - j\,(u_{\lambda_1}(t), \dot{u}_{\lambda_2}(t)) - j\,(u_{\lambda_2}(t), \dot{u}_{\lambda_1}(t)) + j\,(u_{\lambda_2}(t), \dot{u}_{\lambda_2}(t)) \leq 0. \end{aligned} \tag{4.13}$$

We first bound the terms of j,

$$\begin{aligned} &|j\,(u_{\lambda_1}(t), \dot{u}_{\lambda_1}(t)) - j\,(u_{\lambda_1}(t), \dot{u}_{\lambda_2}(t)) - j\,(u_{\lambda_2}(t), \dot{u}_{\lambda_1}(t)) + j\,(u_{\lambda_2}(t), \dot{u}_{\lambda_2}(t))| \\ &\leq c_0^2 L_\mu c_R \|u_{\lambda_1}(t) - u_{\lambda_2}(t)\|_V \|\dot{u}_{\lambda_1}(t) - \dot{u}_{\lambda_2}(t)\|_V. \end{aligned} \tag{4.14}$$

Integrating (4.13) over $[0, t]$ and combining it with (4.14), (HP_1), and (HP_2), we can deduce

$$\begin{aligned} m_{\mathscr{C}} \int\limits_0^t \|\dot{u}_{\lambda_1}(s) - \dot{u}_{\lambda_2}(s)\|_V^2 \, ds \leq &\left(M_{\mathscr{F}} + c_0^2 L_\mu c_B\right) \int\limits_0^t \|u_{\lambda_1}(s) - u_{\lambda_2}(s)\|_V \|\dot{u}_{\lambda_1}(s) - \dot{u}_{\lambda_2}(s)\|_V \, ds \\ &+\frac{1}{2} \|\dot{u}_{\lambda_1}(t) - \dot{u}_{\lambda_2}(t)\|_V - \int\limits_0^t (\lambda_1(s) - \lambda_2(s), \dot{u}_{\lambda_1}(s) - \dot{u}_{\lambda_2}(s))\, ds. \end{aligned}$$

Using the inequality $xy \leq \alpha x^2 + \frac{1}{4\alpha} y^2$, $(\alpha > 0)$, (4.12) and Gronwall's inequality, it follows that

$$\|u_{\lambda_1}(t) - u_{\lambda_2}(t)\|_V^2 \leq C \int\limits_0^t \|\lambda_1(s) - \lambda_2(s)\|_V^2 \, ds. \tag{4.15}$$

Moreover, by (4.3), we conclude that

$$\begin{aligned} &\left(\dot{\theta}_{\gamma_1}(t) - \dot{\theta}_{\gamma_2}(t), \theta_{\gamma_1}(t) - \theta_{\gamma_2}(t)\right)_{Q' \times Q} + d\,(\theta_{\gamma_1}(t) - \theta_{\gamma_2}(t), \theta_{\gamma_1}(t) - \theta_{\gamma_2}(t)) \\ &+ (\gamma_1(t) - \gamma_2(t), \theta_{\gamma_1}(t) - \theta_{\gamma_2}(t))_Q \leq 0. \end{aligned}$$

Similarly, from (HP_3), we obtain

$$\|\theta_{\gamma_1}(t) - \theta_{\gamma_2}(t)\|_Q^2 \leq C \int\limits_0^t \|\gamma_1(s) - \gamma_2(s)\|_Q^2 \, ds.$$

Combining (4.11) through (4.15) and (4.16), we get

$$\|\Lambda(\lambda_1, \gamma_1)(t) - \Lambda(\lambda_2, \gamma_2)(t)\|_{V \times Q}^2 \leq C \int\limits_0^t \|(\lambda_1, \gamma_1)(s) - (\lambda_2, \gamma_2)(s)\|_{V \times Q}^2. \tag{4.16}$$

Finally, using the result presented in [27, P. 41–45], we deduce that the operator Λ possesses a unique fixed point. □

We are now ready to prove Theorem 4.1.

• **Existence:** Let $(\lambda^*, \gamma^*) \in W^{1,1}(0, T; H) \times L^2(0, T; L^2(\Omega))$ be the fixed point of Λ, and denote by u_λ^*, θ_γ^* the solutions of problems (4.1)–(4.2) and (4.3)–(4.4), respectively. For $(\lambda, \gamma) = (\lambda^*, \gamma^*)$, by definition (4.6) of Λ, we deduce that the pair $(u_\lambda^*, \theta_\gamma^*)$ is a solution of **Problem (PV)**.

• **Uniqueness:** The uniqueness of the solution follows from the uniqueness of the fixed point of the operator Λ.

5. Continuous Dependence Result

In this section, we consider $\{\mu_\delta\}_{\delta\geq 1}$ as a family of perturbations of the friction coefficient μ. We aim to study the dependence of the solution of **Problem (PV)** on the perturbation of μ, with respect to the initial condition $(u_0, \dot{u}_0, \theta_0)$.

We introduce the following variational problem involving a perturbed friction coefficient μ_δ and denote its solution as $(u_\delta, \theta_\delta)$ with respect to the initial data $(u_{\delta 0}, \theta_{\delta 0})$.

• **Problem** (PV_δ): Find a displacement field $u_\delta \in (V_1 \cap V_2)\times]0;T[$ and a temperature field $\theta_\delta \in Q_1\times]0,T[$, for all $v \in V_1 \cap V_2$ and $\eta \in Q_1$ such that:

$$\begin{aligned}(\ddot{u}_\delta(t), v-\dot{u}_\delta(t))_H + c\left(\dot{u}_\delta(t), v-\dot{u}_\delta(t)\right) + a\left(u_\delta(t), v-\dot{u}_\delta(t)\right) - m\left(\theta_\delta(t), v-\dot{u}_\delta(t)\right)\\ +\Phi\left(u_\delta(t), v-\dot{u}_\delta(t)\right) + j(u_\delta(t), v) - j(u_\delta(t), \dot{u}_\delta(t)) \geq (f(t), v-\dot{u}_\delta(t))_V\,,\end{aligned} \tag{5.1}$$

$$\left(\dot{\theta}_\delta(t), \eta-\theta_\delta(t)\right)_{Q'\times Q} + d\left(\theta_\delta(t), \eta-\theta_\delta(t)\right) + \chi\left(u_\delta(t), \theta_\delta(t), \eta-\theta_\delta(t)\right) \geq (q(t), \eta-\theta_\delta(t))_Q\,, \tag{5.2}$$

$$u_\delta(0) = u_{\delta 0},\ \dot{u}_\delta(0) = \dot{u}_{\delta 0},\ \theta_\delta(0) = \theta_{\delta 0}.$$

Problem (PV_δ) has a unique solution, and the arguments of the proof are similar to that used in Section 4.

Now, we present the following convergence result.

Theorem 5.1. *Assume that the conditions*

$$\begin{aligned}\|\mu\left(\|u_\tau\|\right) - \mu_\delta\left(\|u_{\delta\tau}\|\right)\|_{L^2(0,T,L^2(\Gamma_C))} \to 0, \quad as\ \ \delta \to +\infty,\\ \left\{\|u_0 - u_{\delta 0}\|_V + \|\dot{u}_0 - \dot{u}_{\delta 0}\|_H + \|\theta_0 - \theta_{\delta 0}\|_Q\right\} \to 0,\ \ as\ \delta \to +\infty\end{aligned} \tag{5.3}$$

hold. Then the solution of ***Problem*** (PV_δ) *converges to the solution of* ***Problem (PV)****, in accordance with the following estimate:*

$$\begin{aligned}\Big\{\|u(t) - u_\delta(t)\|_V^2 + \|\dot{u}(t) - \dot{u}_\delta(t)\|_H^2 + \|\theta(t) - \theta_\delta(t)\|_Q^2\\ + \|\theta(t) - \theta_\delta(t)\|_{L^2(0,T,Q)}^2\Big\} \to 0, \quad as\ \ \delta \to +\infty.\end{aligned}$$

Proof. Substituting $v = \dot{u}_\delta(t)$ into (3.13) and $v = \dot{u}(t)$ into (5.1), we deduce

$$\begin{aligned}(\ddot{u}(t) - \ddot{u}_\delta(t), \dot{u}(t) - \dot{u}_\delta(t))_H + c\left(\dot{u}(t) - \dot{u}_\delta(t), \dot{u}(t) - \dot{u}_\delta(t)\right) + a\left(u(t) - u_\delta(t), \dot{u}(t) - \dot{u}_\delta(t)\right)\\ -m\left(\theta(t) - \theta_\delta(t), \dot{u}(t) - \dot{u}_\delta(t)\right) + \Phi\left(u(t) - u_\delta(t), \dot{u}(t) - \dot{u}_\delta(t)\right)\\ \leq j\left(u(t), \dot{u}_\delta(t)\right) - j\left(u(t), \dot{u}(t)\right) - j\left(u_\delta(t), \dot{u}(t)\right) + j\left(u_\delta(t), \dot{u}_\delta(t)\right).\end{aligned}$$

In other words, referring to (3.5), (HP_5)(iii) and (HP_7), there exists a positive constant C depending on c_R, M_μ, and c_0 such that

$$\begin{aligned}|j\left(u(t), \dot{u}_\delta(t)\right) - j\left(u(t), \dot{u}(t)\right) - j\left(u_\delta(t), \dot{u}(t)\right) + j\left(u_\delta(t), \dot{u}_\delta(t)\right)|\\ \leq C\left(\|u(t) - u_\delta(t)\|_V^2 + \|\dot{u}(t) - \dot{u}_\delta(t)\|_V^2 + \|\mu\left(\|u_\tau\|\right) - \mu_\delta\left(\|u_{\delta\tau}\|\right)\|_{L^2(0,T,L^2(\Gamma_C))}^2\right).\end{aligned} \tag{5.4}$$

Employing hypotheses (HP_1)–(HP_3) along with the Young inequality in conjunction with (5.4), we can conclude that

$$\begin{aligned}\|\dot{u}(t) - \dot{u}_\delta(t)\|_V^2 + \frac{1}{2}\frac{d}{dt}\|\dot{u}(t) - \dot{u}_\delta(t)\|_H^2 \leq C\Big\{\|u(t) - u_\delta(t)\|_V^2 + \|\dot{u}(t) - \dot{u}_\delta(t)\|_V^2\\ + \|\theta(t) - \theta_\delta(t)\|_Q^2 + \|\mu\left(\|u_\tau\|\right) - \mu_\delta\left(\|u_{\delta\tau}\|\right)\|_{L^2(0,T,L^2(\Gamma_C))}\Big\}.\end{aligned} \tag{5.5}$$

Similarly to (4.12), we observe that

$$\|u(t) - u_\delta(t)\|_V^2 \leq C\left\{\int_0^t \|\dot{u}(s) - \dot{u}_\delta(s)\|_V^2\, ds + \|u_0 - u_{\delta 0}\|_V^2\right\}.$$

Integrating equality (5.5) over the interval $[0, t]$ and applying Grownwall's inequality, we arrive at

$$\|u(t) - u_\delta(t)\|_V^2 + \|\dot{u}(t) - \dot{u}_\delta(t)\|_H^2 \leq C \Big\{ \|\theta(t) - \theta_\delta(t)\|_Q^2 + \|u_0 - u_{\delta 0}\|_V^2 + \|\dot{u}_0 - \dot{u}_{\delta 0}\|_H^2 + \|\mu(\|u_\tau\|) - \mu_\delta(\|u_{\delta\tau}\|)\|_{L^2(0,T,L^2(\Gamma_C))} \Big\}. \tag{5.6}$$

Take $\eta = \theta_\delta(t)$ in (3.13) and $\eta = \theta(t)$ in (5.2), we have

$$\Big(\dot{\theta}(t) - \dot{\theta}_\delta(t), \theta(t) - \theta_\delta(t)\Big)_{Q' \times Q} + d\left(\theta(t) - \theta_\delta(t), \theta(t) - \theta_\delta(t)\right) \leq \chi\left(u(t), \theta(t), \theta_\delta(t) - \theta(t)\right) - \chi\left(u_\delta(t), \theta_\delta(t), \theta_\delta(t) - \theta(t)\right). \tag{5.7}$$

We deduce from (3.9) and (HP_8) that

$$|\chi\left(u(t), \theta(t), \theta_\delta(t) - \theta(t)\right) - \chi\left(u_\delta(t), \theta_\delta(t), \theta_\delta(t) - \theta(t)\right)| \leq M_{k_c} L c_0 c_1 \|u(t) - u_\delta(t)\|_V \|\theta(t) - \theta_\delta(t)\|_Q. \tag{5.8}$$

Using the property of the operator d combined with (5.7)–(5.8), we get

$$\|\theta(t) - \theta_\delta(t)\|_Q^2 + \frac{1}{2}\frac{d}{dt}\|\theta(t) - \theta_\delta(t)\|_Q^2 \leq C\|u(t) - u_\delta(t)\|_V^2. \tag{5.9}$$

We integrate the two sides of (5.9) and, using the Gronwall inequality, there exists a constant $C > 0$ such that

$$\|\theta(t) - \theta_\delta(t)\|_Q^2 + \|\theta(t) - \theta_\delta(t)\|_{L^2(0,T;Q)}^2 \leq C\Big\{ \int_0^t \|u(s) - u_\delta(s)\|_V^2 + \|\theta_0 - \theta_{\delta 0}\|_Q^2 \Big\}. \tag{5.10}$$

Finally, combining (5.6) with (5.10) and performing some algebraic manipulations, we can deduce the existence of a positive constant C such that

$$\|u(t) - u_\delta(t)\|_V^2 + \|\dot{u}(t) - \dot{u}_\delta(t)\|_H^2 + \|\theta(t) - \theta_\delta(t)\|_Q^2 + \|\theta(t) - \theta_\delta(t)\|_{L^2(0,T;Q)}^2 \leq C\Big\{ \|u_0 - u_{\delta 0}\|_V^2 + \|\dot{u}_0 - \dot{u}_{\delta 0}\|_H^2 + \|\theta_0 - \theta_{\delta 0}\|_Q^2 + \|\mu(\|u_\tau\|) - \mu_\delta(\|u_{\delta\tau}\|)\|_{L^2(0,T,L^2(\Gamma_C))} \Big\}.$$

Hence, by the hypothesis stated in relation (5.1), we can derive the result presented in Theorem 5.1. □

6. Fully Discrete Approximation

In this section, we introduce a fully discrete numerical scheme for the solution of **Problem (PV)** and derive the error estimate. We use the finite element spaces V^h and Q^h and introduce a partition of the time interval $[0; T]$: $0 = t_0 < t_1 < \cdots < t_N = T$. Let $k_n > 0$ denote the time step size defined as $k_n = t_n - t_{n-1}$ for $n = 1, \ldots, N$. We allow a non-uniform partitioning of the time interval and denote $k = \max_n k_n$ as the maximum step size.

For a continuous function $u(t)$, taking values in a function space, we denote $u_n = u(t_n)$ for $0 \leq n \leq N$. We represent the difference as $\delta u_n = \frac{1}{k}(u_n - u_{n-1})$, where k is the step size.

Let— $\{\mathscr{T}^h\}$ be a regular family of triangular finite element partition of $\overline{\Omega}$, compatible with the boundary decomposition $\Gamma = \overline{\Gamma}_C \cup \overline{\Gamma}_D \cup \overline{\Gamma}_N$. We then define a finite element space $V^h \subset V$ and $Q^h \subset Q$ for the approximations of the displacement field u and the temperature θ defined by

$$V^h = \Big\{ v^h \in \left[C(\overline{\Omega})\right]^d;\ v^h_{|Tr} \in [\mathbb{P}_1(Tr)]^d;\ \forall Tr \in \mathscr{T}^h;\ v^h = 0 \text{ on } \overline{\Gamma}_D \Big\},$$

$$Q^h = \Big\{ \eta^h \in \left[C(\overline{\Omega})\right];\ \eta^h_{|Tr} \in [\mathbb{P}_1(Tr)];\ \forall Tr \in \mathscr{T}^h;\ \eta^h = 0 \text{ on } \overline{\Gamma}_D \cup \overline{\Gamma}_N \Big\}.$$

We also have $V_1^h = V_1 \cap V^h$, $V_2^h = V_2 \cap V^h$ and $Q_1^h = Q_1 \cap Q^h$.
To simplify again the notations, we introduce the following notations for $n = 1, \ldots, N$:

$$w_n^{hk} = \delta u_n^{hk},\ u_n^{hk} = \sum_{j=1}^n k_j w_j^{hk} + u_0^h \text{ and } \theta_n^{hk} = \sum_{j=1}^n k_j \delta\theta_j^{hk} + \theta_0^h.$$

The fully discrete approximation method is based on a backward Euler scheme, it has the following form.

• **Problem** (PV_{hk}): Find a displacement field $\left\{u_n^{hk}\right\} \subset V_1^h \cap V_2^h$ and a temperature $\left\{\theta_n^{hk}\right\} \subset Q_1^h$, for all $v^h \in V_1^h \cap V_2^h$, $\eta^h \in Q_1^h$ and $n = 1, \ldots, N$ such that

$$\begin{aligned}\left(\delta w_n^{hk}, v^h - w_n^{hk}\right)_H + c\left(w_n^{hk}, v^h - w_n^{hk}\right) + a\left(u_n^{hk}, v^h - w_n^{hk}\right) - m\left(\theta_{n-1}^{hk}, v^h - w_n^{hk}\right)\\ +\Phi\left(u_{n-1}^{hk}, v^h - w_n^{hk}\right) + j\left(u_n^{hk}, v^h\right) - j\left(u_n^{hk}, w_n^{hk}\right) \geq \left(f_n, v^h - w_n^{hk}\right)_V,\end{aligned} \tag{6.1}$$

$$\left(\delta\theta_n^{hk}, \eta^h - \theta_n^{hk}\right)_{L^2(\Omega)} + d\left(\theta_n^{hk}, \eta^h - \theta_n^{hk}\right) + \chi\left(u_{n-1}^{hk}, \theta_{n-1}^{hk}, \eta^h - \theta_n^{hk}\right) \geq \left(q_n, \eta^h - \theta_n^{hk}\right)_Q, \tag{6.2}$$

$$u_0^{hk} = u_0^h,\ w_0^{hk} = w_0^h,\ \theta_n^{hk} = \theta_0^h,$$

with $u_0^h \in V^h$, $w_0^h \in V^h$ and $\theta_0^h \in Q^h$, are the approximations of u_0, w_0 and $\theta_{0,}$ respectively.

Remark 6.1. The choice of u_{n-1}^{hk} and θ_{n-1}^{hk} in (6.2) is motivated by the fixed point method employed in the previous section.

Applying a discrete analogue of Theorem 4.1, we observe that given $u_{n-1}^{hk} \in V_1^h \cap V_2^h$ and $\theta_{n-1}^{hk} \in Q_1^h$, **Problem** (PV_{hk}) possesses a unique solution $u_n^{hk} \in V_1^h \cap V_2^h$ and $\theta_n^{hk} \in Q_1^h$.

Now, we derive the following convergence result in the fully discrete solution.

Theorem 6.1. *Assume the initial values $u_0^h \in V_1^h \cap V_2^h$, $\theta_0^h \in Q_1^h$ and the assumptions $(HP_1)-(HP_8)$. Then if*

$$\left\|u_0 - u_0^h\right\|_V \to 0, \quad \left\|w_0 - w_0^h\right\|_V \to 0, \quad \left\|\theta_0 - \theta_0^h\right\|_Q \to 0, \quad as\ \ h \to 0,$$

*the fully discrete solution of **Problem** (PV_{hk}) converges:*

$$\max_{1\leq n\leq N}\left\{\left\|w_n - w_n^{hk}\right\|_V^2 + \left\|u_n - u_n^{hk}\right\|_V^2 + \left\|\theta_n - \theta_n^{hk}\right\|_Q^2\right\} \to 0, \quad as\ \ h, k \to 0.$$

Proof. We rewrite (6.1) in the form

$$\begin{aligned}\left(\delta w_n^{hk}, w_n - w_n^{hk}\right)_H + c\left(w_n^{hk}, w_n - w_n^{hk}\right) + a\left(u_n^{hk}, w_n - w_n^{hk}\right) - \left(\theta_{n-1}^{hk}, w_n - w_n^{hk}\right)\\ +\Phi\left(u_{n-1}^{hk}, w_n - w_n^{hk}\right) + j\left(u_n^{hk}, v_n^h\right) - j\left(u_n^{hk}, w_n^{hk}\right) \geq \left(\delta w_n^{hk}, w_n - v_n^h\right)_H + c\left(w_n^{hk}, w_n - v_n^h\right)\\ +a\left(u_n^{hk}, w_n - v_n^h\right) - \left(\theta_{n-1}^{hk}, w_n - v_n^h\right) + \Phi\left(u_{n-1}^{hk}, w_n - v_n^h\right) + \left(f_n, v_n^h - w_n^{hk}\right)_V.\end{aligned} \tag{6.3}$$

Substituting v with $w_n^{hk} \in V^h$ at time $t = t_n$ in equation (3.13), we obtain

$$\begin{aligned}\left(\dot{w}_n, w_n^{hk} - w_n\right)_H + c\left(w_n, w_n^{hk} - w_n\right) + a\left(u_n, w_n^{hk} - w_n\right) - m\left(\theta_n, w_n^{hk} - w_n\right)\\ +\Phi\left(u_n, w_n^{hk} - w_n\right) + j\left(u_n, w_n^{hk}\right) - j\left(u_n, w_n\right) \geq \left(f_n, w_n^{hk} - w_n\right)_V.\end{aligned} \tag{6.4}$$

Combining relations (6.3) and (6.4), we have

$$\begin{aligned}\left(\dot{w}_n - \delta w_n^{hk}, w_n - w_n^{hk}\right)_H + c\left(w_n - w_n^{hk}, w_n - w_n^{hk}\right) + a\left(u_n - u_n^{hk}, w_n - w_n^{hk}\right)\\ -m\left(\theta_n - \theta_{n-1}^{hk}, w_n - w_n^{hk}\right) + \Phi\left(u_n - u_{n-1}^{hk}, w_n - w_n^{hk}\right)\\ -j\left(u_n, w_n^{hk}\right) + j\left(u_n, w_n\right) - j\left(u_n^{hk}, v_n^h\right) + j\left(u_n^{hk}, w_n^{hk}\right)\\ \leq \left(\delta w_n^{hk}, v_n^h - w_n\right)_H + c\left(w_n^{hk}, v_n^h - w_n\right) + a\left(u_n^{hk}, v_n^h - w_n\right)\\ -m\left(\theta_{n-1}^{hk}, v_n^h - w_n\right) + \Phi\left(u_{n-1}^{hk}, v_n^h - w_n\right) + \left(f_n, w_n - v_n^h\right)_V.\end{aligned}$$

We write

$$\begin{aligned}\left(\dot{w}_n - \delta w_n^{hk}, w_n - w_n^{hk}\right)_H + \left(\delta w_n^{hk}, w_n - v_n^h\right)_H = \left(\delta w_n - \delta w_n^{hk}, w_n - w_n^{hk}\right)_H\\ + \left(\delta w_n - \delta w_n^{hk}, v_n^h - w_n\right)_H + \left(\dot{w}_n - \delta w_n, v_n^h - w_n^{hk}\right)_H + \left(\dot{w}_n, w_n - v_n^h\right)_H.\end{aligned}$$

After performing some algebraic manipulations, we deduce

$$\begin{aligned}\left(\delta w_n - \delta w_n^{hk}, w_n - w_n^{hk}\right)_H + c\left(w_n - w_n^{hk}, w_n - w_n^{hk}\right) \leq \left(\delta w_n - \delta w_n^{hk}, w_n - v_n^h\right)_H\\ +c\left(w_n^{hk} - w_n, v_n^h - w_n\right) + a\left(u_n^{hk} - u_n, v_n^h - w_n\right) - m\left(\theta_{n-1}^{hk} - \theta_n, v_n^h - w_n\right)\\ +\Phi\left(u_{n-1}^{hk} - u_n, v_n^h - w_n^{hk}\right) + \left(\delta w_n - \dot{w}_n, v_n^h - w_n^{hk}\right) + R\left(w_n, v_n^h\right) + R_j,\end{aligned} \tag{6.5}$$

where

$$R\left(w_n, v_n^h\right)=\left(\dot{w}_n, v_n^h-w_n\right)_H+c\left(w_n, v_n^h-w_n\right)+a\left(u_n, v_n^h-w_n\right)-m\left(\theta_n, v_n^h-w_n\right)$$
$$+\Phi\left(u_n, v_n^h-w_n\right)+j\left(u_n, v_n^h\right)-j\left(u_n, w_n\right)-\left(f_n, w_n-v_n^h\right)_V,$$

and

$$R_j=j\left(u_n, w_n^{hk}\right)-j\left(u_n, v_n^h\right)+j\left(u_n^{hk}, v_n^h\right)-j\left(u_n^{hk}, w_n^{hk}\right). \tag{6.6}$$

From the relation

$$\delta w_n-\delta w_n^{hk}=\frac{1}{k}\left(w_n-w_{n-1}\right)-\frac{1}{k}\left(w_n^{hk}-w_{n-1}^{hk}\right),$$

and after some calculation, it follows that

$$\left\|w_n-w_n^{hk}\right\|_H^2-\left\|w_{n-1}-w_{n-1}^{hk}\right\|_H^2 \leq 2k\left(\delta w_n-\delta w_n^{hk}, w_n-w_n^{hk}\right)_H. \tag{6.7}$$

We conclude from (HP_1)–(HP_3) that

$$c\left(w_n-w_n^{hk}, w_n-w_n^{hk}\right) \geq m_{\mathscr{C}}\left\|w_n-w_n^{hk}\right\|_V^2, \tag{6.8}$$

$$\left|c\left(w_n-w_n^{hk}, w_n-v_n^h\right)\right| \leq M_{\mathscr{C}}\left\|w_n-w_n^{hk}\right\|_V\left\|w_n-v_n^h\right\|_V, \tag{6.9}$$

$$\left|a\left(u_n-u_{n-1}^{hk}, w_n^{hk}-v_n^h\right)\right| \leq M_{\mathscr{F}}\left\|u_n-u_{n-1}^{hk}\right\|_V\left\|w_n^{hk}-v_n^h\right\|_V, \tag{6.10}$$

and

$$\left|m\left(\theta_n-\theta_{n-1}^{hk}, w_n^{hk}-v_n^h\right)\right| \leq M_{\mathscr{K}}\left\|\theta_n-\theta_{n-1}^{hk}\right\|_Q\left\|w_n^{hk}-v_n^h\right\|_V. \tag{6.11}$$

Similar to (4.9), we observe that

$$\left\|\Phi\left(u_n-u_{n-1}^{hk}, w_n^{hk}-v_n^h\right)\right\| \leq \frac{c_0^2}{\epsilon}\left\|u_n-u_{n-1}^{hk}\right\|_V\left\|w_n^{hk}-v_n^h\right\|_V. \tag{6.12}$$

Applying (3.5), (3.8), (6.6), (HP_5)(iii) and (HP_7), we have

$$\left|R_2\right| \leq M_\mu c_R c_0^2\left\|u_n-u_n^{hk}\right\|_V\left\|w_n^{hk}-v_n^h\right\|_V. \tag{6.13}$$

By combining (6.5), (6.7)–(6.13), and utilizing the given inequality, we derive

$$\left\|w_n^{hk}-v_n^h\right\|_V \leq\left\|w_n^{hk}-w_n\right\|_V+\left\|w_n-v_n^h\right\|_V,$$

there exists a positive constant c such that

$$\begin{aligned}
&\left\|w_n-w_n^{hk}\right\|_H^2-\left\|w_{n-1}-w_{n-1}^{hk}\right\|_H^2+ck\left\|w_n-w_n^{hk}\right\|_V^2 \\
&\leq ck\left\{\left\|\dot{w}_n-\delta w_n\right\|_H^2+\left\|u_n-u_n^{hk}\right\|_V^2+\left\|u_n-u_{n-1}^{hk}\right\|_V^2\right. \\
&\left.+\left\|\theta_n-\theta_{n-1}^{hk}\right\|_Q^2+\left\|w_n-v_n^h\right\|_V^2+\left\|w_n-v_n^h\right\|_H^2+R\left(w_n, v_n^h\right)\right\} \\
&+2k\left(\delta w_n-\delta w_n^{hk}, w_n-v_n^h\right)_H.
\end{aligned} \tag{6.14}$$

Substituting $\eta=\theta_n^{hk} \in Q^h$ at time $t=t_n$ into (3.13), we have

$$\left(\dot{\theta}_n, \theta_n^{hk}-\theta_n\right)_{L^2(\Omega)}+d\left(\theta_n, \theta_n^{hk}-\theta_n\right)+\chi\left(u_n, \theta_n, \theta_n^{hk}-\theta_n\right) \geq\left(q_n, \theta_n^{hk}-\theta_n\right)_Q. \tag{6.15}$$

We rewrite (6.2) in the following form:

$$\begin{aligned}
&\left(\delta \theta_n^{hk}, \theta_n-\theta_n^{hk}\right)_{L^2(\Omega)}+d\left(\theta_n^{hk}, \theta_n-\theta_n^{hk}\right)+\chi\left(u_{n-1}^{hk}, \theta_{n-1}^{hk}, \eta_n^h-\theta_n^{hk}\right) \\
&\geq\left(\delta \theta_n^{hk}, \theta_n-\eta_n^h\right)_{L^2(\Omega)}+d\left(\theta_n^{hk}, \theta_n-\eta_n^h\right)+\left(q_n, \eta_n^h-\theta_n^{hk}\right)_Q.
\end{aligned} \tag{6.16}$$

Adding (6.15) and (6.16), we have

$$\begin{aligned}
&\left(\dot{\theta}_n-\delta \theta_n^{hk}, \theta_n-\theta_n^{hk}\right)_{L^2(\Omega)}+\left(\delta \theta_n^{hk}, \theta_n-\eta_n^h\right)_{L^2(\Omega)}+d\left(\theta_n-\theta_n^{hk}, \theta_n-\theta_n^{hk}\right) \\
&\leq \chi\left(u_n, \theta_n, \theta_n^{hk}-\theta_n\right)+\chi\left(u_{n-1}^{hk} n \theta_{n-1}^{hk}, \eta_n^h-\theta_n^{hk}\right)+d\left(\theta_n^{hk}, \eta_n^h-\theta_n\right)+\left(q_n, \theta_n-\eta_n^h\right)_Q.
\end{aligned}$$

Using the inequality

$$\left(\dot{\theta}_n-\delta\theta_n^{hk},\theta_n-\theta_n^{hk}\right)_{L^2(\Omega)}+\left(\delta\theta_n^{hk},\theta_n-\eta_n^h\right)_{L^2(\Omega)}=\left(\delta\theta_n-\delta\theta_n^{hk},\theta_n-\theta_n^{hk}\right)_{L^2(\Omega)}$$
$$+\left(\delta\theta_n-\delta\theta_n^{hk},\eta_n^h-\theta_n\right)_{L^2(\Omega)}+\left(\dot{\theta}_n-\delta\theta_n,\eta_n^h-\theta_n^{hk}\right)_{L^2(\Omega)}+\left(\dot{\theta}_n,\theta_n-\eta_n^h\right)_{L^2(\Omega)},$$

we deduce

$$\left(\delta\theta_n-\delta\theta_n^{hk},\theta_n-\theta_n^{hk}\right)_{L^2(\Omega)}+d\left(\theta_n-\theta_n^{hk},\theta_n-\theta_n^{hk}\right)\le d\left(\theta_n^{hk}-\theta_n,\eta_n^h-\theta_n\right)$$
$$+R_\chi+R\left(\theta_n,\eta_n^h\right)+\left(\delta\theta_n-\delta\theta_n^{hk},\theta_n-\eta_n^h\right)_{L^2(\Omega)}+\left(\delta\theta_n-\dot{\theta}_n,\eta_n^h-\theta_n^{hk}\right)_{L^2(\Omega)},\tag{6.17}$$

where

$$R\left(\theta_n,\eta_n^h\right)=\left(\dot{\theta}_n,\eta_n^h-\theta_n\right)_{L^2(\Omega)}+d\left(\theta_n,\eta_n^h-\theta_n\right)+\chi\left(u_n,\theta_n,\eta_n^h-\theta_n\right)+\left(q_n,\theta_n-\eta_n^h\right)_Q,$$

and

$$R_\chi=\chi\left(u_n,\theta_n,\theta_n^{hk}-\theta_n\right)+\chi\left(u_{n-1}^{hk},\theta_{n-1}^{hk},\eta_n^h-\theta_n^{hk}\right)-\chi\left(u_n,\theta_n,\eta_n^h-\theta_n\right).$$

Since

$$\delta\theta_n-\delta\theta_n^{hk}=\frac{1}{k}\left(\theta_n-\theta_n^{hk}\right)-\frac{1}{k}\left(\theta_{n-1}-\theta_{n-1}^{hk}\right),$$

similar to (6.7), we have

$$\left\|\theta_n-\theta_n^{hk}\right\|_{L^2(\Omega)}^2-\left\|\theta_{n-1}-\theta_{n-1}^{hk}\right\|_{L^2(\Omega)}^2\le 2k\left(\delta\theta_n-\delta\theta_n^{hk},\theta_n-\theta_n^{hk}\right)_{L^2(\Omega)}.\tag{6.18}$$

Using (3.5), (3.9) and (HP_8), we obtain

$$|R_\chi|\le c\left\{\left\|\theta_n-\theta_{n-1}^{hk}\right\|_Q^2+\left\|u_n-u_{n-1}^{hk}\right\|_V^2+\left\|\eta_n^h-\theta_n\right\|_Q^2\right\}.\tag{6.19}$$

Now, we combine (6.17), (6.18)–(6.19) and (6.14), using the properties of the operator d and this inequality

$$\left\|\theta_n^{hk}-\eta_n^h\right\|_Q\le\left\|\theta_n^{hk}-\theta_n\right\|_Q+\left\|\theta_n-\eta_n^h\right\|_Q,$$

there exists a positive constant c such that

$$\left\|w_n-w_n^{hk}\right\|_H^2+\left\|\theta_n-\theta_n^{hk}\right\|_{L^2(\Omega)}^2-\left\|w_{n-1}-w_{n-1}^{hk}\right\|_H^2+\left\|\theta_{n-1}-\theta_{n-1}^{hk}\right\|_{L^2(\Omega)}^2$$
$$+ck\left\{\left\|w_n-w_n^{hk}\right\|_V^2+\left\|\theta_n-\theta_n^{hk}fz\right\|_Q^2\right\}$$
$$\le ck\left\{\left\|\dot{w}_n-\delta w_n\right\|_H^2+\left\|u_n-u_n^{hk}\right\|_V^2+\left\|u_n-u_{n-1}^{hk}\right\|_V^2+\left\|w_n-v_n^h\right\|_V^2\right.$$
$$\left.+\left\|\dot{\theta}_n-\delta\theta_n\right\|_{L^2(\Omega)}^2+\left\|\theta_n-\theta_{n-1}^{hk}\right\|_Q^2+\left\|\theta_n-\eta_n^h\right\|_Q^2+R\left(w_n,v_n^h\right)+R\left(\theta_n,\eta_n^h\right)\right\}$$
$$+2k\left\{\left(\delta w_n-\delta w_n^{hk},w_n-v_n^h\right)_H+\left(\delta\theta_n-\delta\theta_n^{hk},\theta_n-\eta_n^h\right)_{L^2(\Omega)}\right\}.$$

Replacing n by j and summing over j from 1 to n, we obtain

$$\left\|w_n-w_n^{hk}\right\|_H^2+\left\|\theta_n-\theta_n^{hk}\right\|_{L^2(\Omega)}^2 ck\sum_{j=1}^n\left\|w_j-w_j^{hk}\right\|_V^2+ck\sum_{j=1}^n\left\|\theta_j-\theta_j^{hk}\right\|_Q^2$$
$$\le\left\|w_0-w_0^{hk}\right\|_H^2+\left\|\theta_0-\theta_0^{hk}\right\|_{L^2(\Omega)}^2$$
$$+ck\sum_{j=1}^n\left\{\left\|\dot{w}_j-\delta w_j\right\|_H^2+\left\|u_j-u_j^{hk}\right\|_V^2+\left\|u_j-u_{j-1}^{hk}\right\|_V^2+\left\|w_j-v_j^h\right\|_H^2\right.$$
$$\left.+\left\|\dot{\theta}_j-\delta\theta_j\right\|_{L^2(\Omega)}^2+\left\|\theta_j-\theta_{j-1}^{hk}\right\|_Q^2+\left\|\theta_j-\eta_j^h\right\|_Q^2+R\left(w_j,v_j^h\right)+R\left(\theta_j,\eta_j^h\right)\right\}$$
$$+2k\sum_{j=1}^n\left\{\left(\delta w_j-\delta w_j^{hk},w_j-v_j^h\right)_H+\left(\delta\theta_j-\delta\theta_j^{hk},\theta_j-\eta_j^h\right)_{L^2(\Omega)}\right\}.$$

Estimate the term $\sum_{j=1}^{n} k\left(\delta w_j - \delta w_j^{hk}, w_j - v_j^h\right)_H$ as follows:

$$\begin{aligned}
k\sum_{j=1}^{n}\left(\delta w_j - \delta w_j^{hk}, w_j - v_j^h\right)_H &= \sum_{j=1}^{n}\left(\left(w_j - w_j^{hk}\right) - \left(w_{j-1} - w_{j-1}^{hk}\right), w_j - v_j^h\right)_H \\
&+ \left(w_n - w_n^{hk}, w_n - v_j^h\right)_H - \left(w_0 - w_0^{hk}, w_1 - v_1^h\right)_H \\
&+ \sum_{j=1}^{n-1}\left(w_j - w_j^{hk}, w_j - v_j^h - \left(w_{j+1} - v_{j+1}^h\right)\right)_H \\
&\le c\left\{\left\|w_n - w_n^{hk}\right\|_H^2 + \left\|w_n - v_n^h\right\|_H^2 + \left\|w_0 - w_0^h\right\|_H^2 + \left\|w_1 - v_1^h\right\|_H^2\right\} \\
&+4\sum_{j=1}^{n-1} k\left\|w_j - w_j^{hk}\right\|_H^2 + \sum_{j=1}^{n-1}\frac{1}{k}\left\|w_j - v_j^h - \left(w_{j+1} - v_{j+1}^h\right)\right\|_H^2 .
\end{aligned}$$

Using the same approach, we can conclude that

$$\begin{aligned}
&k\sum_{j=1}^{n}\left(\delta\theta_j - \delta\theta_j^{hk}, \theta_j - \eta_j^h\right)_{L^2(\Omega)} \\
&+c\left\{\left\|\theta_n - \theta_n^{hk}\right\|_{L^2(\Omega)}^2 + \left\|\theta_n - \eta_n^h\right\|_{L^2(\Omega)}^2 + \left\|\theta_0 - \theta_0^h\right\|_{L^2(\Omega)}^2 + \left\|\theta_1 - \eta_1^h\right\|_{L^2(\Omega)}^2\right\} \\
&+\sum_{j=1}^{n-1}\left\|\theta_j - \theta_j^{hk}\right\|_{L^2(\Omega)}\left\|\left(\theta_j - \eta_j^h\right) - \left(\theta_{j+1} - \eta_{j+1}^h\right)\right\|_{L^2(\Omega)} \\
&\le c\left\{\left\|\theta_n - \theta_n^{hk}\right\|_{L^2(\Omega)}^2 + \left\|\theta_n - \eta_n^h\right\|_{L^2(\Omega)}^2 + \left\|\theta_0 - \theta_0^h\right\|_{L^2(\Omega)}^2 + \left\|\theta_1 - \eta_1^h\right\|_{L^2(\Omega)}^2\right\} \\
&+k\sum_{j=1}^{n-1}\left\|\theta_j - \theta_j^{hk}\right\|_{L^2(\Omega)}^2 + \frac{1}{k}\sum_{j=1}^{n-1}\left\|\left(\theta_j - \eta_j^h\right) - \left(\theta_{j+1} - \eta_{j+1}^h\right)\right\|_{L^2(\Omega)}^2 .
\end{aligned}$$

We would like to recall the following classical inequality:

$$\left\|u_j - u_j^{hk}\right\|_V \le \left\|u_0 - u_0^h\right\|_V + \sum_{l=1}^{j} k\left\|w_l - w_l^{hk}\right\|_V + I_1,$$

where

$$I_1 = \left\|\int_0^{t_j} w(s)ds - \sum_{l=1}^{j} kw_l\right\|_V \le k\|u\|_{H^2(0,T;V)}.$$

Then

$$\left\|u_j - u_j^{hk}\right\|_V^2 \le c\left\{\left\|u_0 - u_0^h\right\|_V^2 + j\sum_{l=1}^{j} k^2\left\|w_l - w_l^{hk}\right\|_V^2 + k^2\|u\|_{H^2(0,T;V)}\right\}.$$

Employing the inequality for $j \le n \le N$ and $Nk = T$, we can deduce

$$\sum_{j=1}^{n} k\left\|u_j - u_j^{hk}\right\|_V^2 \le cT\left(\left\|u_0 - u_0^h\right\|_V^2 + k^2\|u\|_{H^2(0,T;V)}\right) + T\sum_{j=1}^{n} k\sum_{l=1}^{j}\left\|w_l - w_l^{hk}\right\|_V^2 .$$

Similarly, we have

$$\sum_{j=1}^{n} k\left\|\theta_j - \theta_{j-1}^{hk}\right\|_Q^2 \le cT\left(\left\|\theta_0 - \theta_0^h\right\|_Q^2 + k^2\|\theta\|_{H^1(0,T;Q)}\right) + T\sum_{j=1}^{n} k\sum_{l=1}^{j}\left\|\delta\theta_l - \delta\theta_l^{hk}\right\|_Q^2 , \tag{6.20}$$

$$\sum_{j=1}^{n} k\left\|u_j - u_{j-1}^{hk}\right\|_V^2 \le cT\left(\left\|u_0 - u_0^h\right\|_V^2 + k^2\|u\|_{H^2(0,T;V)}\right) + T\sum_{j=1}^{n-1} k\sum_{l=1}^{j}\left\|w_l - w_l^{hk}\right\|_V^2 , \tag{6.21}$$

and

$$\sum_{j=1}^{n} k \left\|\theta_j - \theta_{j-1}^{hk}\right\|_Q^2 \leq cT \left(\left\|\theta_0 - \theta_0^h\right\|_Q^2 + k^2 \|\theta\|_{H^1(0,T;Q)} \right) + T \sum_{j=1}^{n-1} k \sum_{l=1}^{j} \left\|\delta\theta_l - \delta\theta_l^{hk}\right\|_Q^2 . \tag{6.22}$$

Let us denote

$$e_n = \left\|w_n - w_n^{hk}\right\|_H^2 + \left\|\theta_n - \theta_n^{hk}\right\|_{L^2(\Omega)}^2 + ck \sum_{j=1}^{n} \left(\left\|w_j - w_j^{hk}\right\|_V^2 + \left\|\theta_j - \theta_j^{hk}\right\|_Q^2 \right) \tag{6.23}$$

and

$$\begin{aligned} g_n = {} & \left\|w_0 - w_0^{hk}\right\|_H^2 + \left\|u_0 - u_0^{hk}\right\|_V^2 + \left\|w_1 - v_1^h\right\|_H^2 + \left\|w_n - v_n^h\right\|_H^2 \\ & + \left\|\theta_0 - \theta_0^{hk}\right\|_{L^2(\Omega)}^2 + \left\|\theta_0 - \theta_0^{hk}\right\|_Q^2 + \left\|\theta_1 - \eta_1^h\right\|_{L^2(\Omega)}^2 + \left\|\theta_n - \eta_n^h\right\|_{L^2(\Omega)}^2 \\ & + k \sum_{j=1}^{n} \left(\|\dot{w}_j - \delta w_j\|_H^2 + \left\|w_j - v_j^h\right\|_V^2 + R\left(w_j, v_j^h\right) \right) \\ & + \sum_{j=1}^{n} \left(\left\|\dot{\theta}_j - \delta\theta_j\right\|_{L^2(\Omega)}^2 + \left\|\theta_j - \eta_j^h\right\|_Q^2 + R\left(\theta_j, \eta_j^h\right) \right) \\ & + \frac{1}{k} \sum_{j=1}^{n-1} \left(\left\|\left(w_j - v_j^n\right) - \left(w_{j+1} - v_{j+1}^h\right)\right\|_H^2 + \left\|\left(\theta_j - \eta_j^n\right) - \left(\theta_{j+1} - \eta_{j+1}^h\right)\right\|_{L^2(\Omega)}^2 \right). \end{aligned} \tag{6.24}$$

Now, we keep the assumptions stated in Theorem 4.1 under the regularity conditions

$$\begin{gathered} u \in C^1\left(0, T; H^2\left(\Omega; \mathbb{R}^d\right)\right) \cap H^3(0, T; H), \\ \dot{u}_{|\Gamma_C} \in C\left(0, T; H^2\left(\Gamma_C, \mathbb{R}^d\right)\right), \\ \theta \in C\left(0, T; H^2(\Omega)\right) \cap H^2\left(0, T; L^2(\Omega)\right), \quad \dot{\theta} \in L^2\left(0, T; H^1(\Omega)\right). \end{gathered}$$

Let $v_j^h \in V^h$ and $\eta_j^h \in Q^h$ be the finite element interpolates of u_j and θ_j, respectively. It is important to mention [17,18] the following approximation properties:

$$\begin{aligned} \max_{1 \leq n \leq N} \left\|w_n - v_n^h\right\|_V \leq ch \|w\|_{C(0,T;H^2(\Omega)^d)}, \\ \max_{1 \leq n \leq N} \left\|\theta_n - \eta_n^h\right\|_Q \leq ch \|\theta\|_{C(0,T;H^2(\Omega))}, \end{aligned} \tag{6.25}$$

$$\begin{aligned} \left\|w_0 - w_0^h\right\|_V \leq ch \|w_0\|_{H^2(\Omega,\mathbb{R}^d)}, \\ \left\|u_0 - u_0^h\right\|_H \leq ch \|u_0\|_{H^1(\Omega,\mathbb{R}^d)}, \\ \left\|\theta_0 - \theta_0^h\right\|_{L^2(\Omega)} \leq ch \|\theta_0\|_{L^2(\Omega)}, \end{aligned} \tag{6.26}$$

and

$$\begin{aligned} & k \sum_{j=1}^{n} \left(\|\dot{w}_j - \delta w_j\|_H + \left\|\dot{\theta}_j - \delta\theta_j\right\|_{L^2(\Omega)} \right) \leq ck^2 \|u\|_{H^2(0,T;L^2(\Omega))} + ck^2 \|\theta\|_{H^2(0,T;L^2(\Omega)}^2, \\ & \frac{1}{k} \sum_{j=1}^{n-1} \left(\left\|\left(w_j - v_j^n\right) - \left(w_{j+1} - v_{j+1}^h\right)\right\|_H^2 + \left\|\left(\theta_j - \eta_j^n\right) - \left(\theta_{j+1} - \eta_{j+1}^h\right)\right\|_{L^2(\Omega)}^2 \right) \\ & \qquad \leq ch^2 \|u\|_{H^2(0,T;V)}^2 + ch^2 \|\theta\|_{H^2(0,T;Q)}. \end{aligned} \tag{6.27}$$

Next, using a similar proof technique as presented in [18,30], we can obtain

$$\left|R\left(w_j, v_j^h\right)\right| \leq c \left\|w_n - v_n^h\right\|_{L^2(\Gamma_C)^d} \leq ch^2 \|w_n\|_{C(0,T;H^2(\Omega)^d)}. \tag{6.28}$$

and

$$\left|R\left(\theta_j, \eta_j^h\right)\right| \leq ch^2 \|\theta_n\|_{C(0,T;H^2(\Omega))}. \tag{6.29}$$

Finally, combining relations (6.20)–(6.24), (6.25)–(6.29) and applying the discrete Gronwall inequality [28], we can conclude that

$$\max_{1\leq n\leq N}\left\{\left\|w_n-w_n^{hk}\right\|_H+\left\|u_n-u_n^{hk}\right\|_V+\left\|\theta_n-\theta_n^{hk}\right\|_{L^2(\Omega)}\right\}\leq c(h+k). \qquad \square$$

The Theorem is proved.

Appendix

In this section, we recall some existence and uniqueness results concerning evolution problems, which can be found in references [2, 28].

Definition 6.1. Let X be a topological vector space and $f : X \to \overline{\mathbb{R}}$ be a function.

The subdifferential of f at the point x on the space X is defined as follows:

$$\partial f(x)=\left\{z\in X' \mid \langle z,y-x\rangle\leq f(y)-f(x),\ \ \forall y\in X\right\},$$

where X' is the topological space of X.

Theorem 6.2. *Let V and H be two real Hilbert spaces such that $V \subset H$ and let the inclusion mapping of V into H be continuous and densely defined. We suppose that V is endowed with the norm $\|\cdot\|$ induced by the inner product $(\cdot,\cdot)$ and H is endowed with the norm $|\cdot|$. We denote by V' the dual space of V, by $\langle\cdot,\cdot\rangle_{V',V}$ the duality pairing between an element of V and an element of V', and H is identified with its own dual H'. We assume that M is a maximal monotone set in $V \times V'$ and A is a linear, continuous and symmetric operator from V to V' satisfying the following coerciveness condition:*

$$\langle Au,u\rangle_{V'\times V}+\alpha\|u\|^2\geq\omega\|u\|^2,\ \ \forall u\in V,$$

where $\alpha\in\mathbb{R}$ and $\omega>0$. Let g be in $W^{1,1}(0,T;H)$ and u_0, v_0 be given with

$$u_0\in V,\quad v_0\in D(M),\quad \{Au_0+Mv_0\}\cap H\neq\phi.$$

Then there exists a unique solution u to the problem

$$\begin{cases}\dfrac{d^2u}{dt^2}+Au+M\Big(\dfrac{du}{dt}\Big)\ni g(t) & a.e.\ \ on\ (0,T),\\ u(0)=u_0, & \dfrac{du}{dt}(0)=v_0.\end{cases}$$

which satisfies $u\in W^{1,\infty}(0,T;V)\cap W^{2,\infty}(0,T;H)$.

Theorem 6.3. *Let $V \subset H \subset V'$ be a Gelfand triple. Let K be a nonempty, closed, and convex set of V. Assume that $a(.,.) : V\times V\to\mathbb{R}$ is a continuous and symmetric bilinear form such that for some constants $\alpha>0$ and c_0,*

$$a(v,v)+c_0\|v\|_H^2\geq\alpha\|v\|_V^2,\ \ \forall v\in V.$$

Then for every $u_0 \in K$ and $f \in L^2(0,T;H)$, there exists a unique function $u \in H^1(0,T;H)\cap L^2(0,T;V)$ such that $u(0)=u_0$, $u(t)\in K$ for all $t\in[0,T]$, and for almost all $t\in(0,T)$,

$$\langle\dot u(t),v-u(t)\rangle_{V'\times V}+a\left(u(t),v-u(t)\right)\geq\left(f(t),v-u(t)\right)_H,\ \ \forall v\in K.$$

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