Transactions of A. Razmadze
Mathematical Institute
Vol. **180** (2026), issue 1, 25–34

SOME NEW $(H_p - L_p)$ TYPE INEQUALITIES FOR WEIGHTED MAXIMAL OPERATORS OF FEJÉR MEANS OF WALSH–FOURIER SERIES

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Abstract. In this paper, we introduce some new weighted maximal Fejér mean operators of the Walsh–Fourier series. We prove that for some "optimal" weights, $(H_p - L_p)$ type inequalities hold for these operators when $0 < p < 1/2$. We also prove the sharpness of this result. As a consequence, we obtain some new and well-known results.

1. Introduction

All symbols used in this introduction can be found in Section 2.

The weak $(1,1)$-type inequality for the maximal operator σ^* of Fejér means $\{\sigma_n\}$ with respect to the Walsh system

$$\sigma^* f := \sup_{n\in\mathbb{N}} |\sigma_n f|$$

can be found in the works of Schipp [19] and Pál, Simon [14] (see also [2] and [17]). Fujii [5] and Simon [21] proved that σ^* is bounded from H_1 to L_1. Weisz [30] generalized this result and proved the boundedness of σ^* from the martingale space H_p to the Lebesgue space L_p for $p > 1/2$. Simon [22] gave a counterexample showing that the boundedness does not hold for $0 < p < 1/2$. A counterexample for $p = 1/2$ was given by Goginava [7]. Moreover, in [9] (see also [23]) he proved that there exists a martingale $F \in H_p$ $(0 < p \le 1/2)$ such that

$$\sup_{n\in\mathbb{N}} \|\sigma_n F\|_p = +\infty.$$

Weisz [32] proved that the maximal operator σ^* of the Fejér means is bounded from the Hardy space $H_{1/2}$ to the space $weak - L_{1/2}$.

Concerning convergence and summability of Fejér means of Walsh-Fourier series we refer to [4,12, 13,15,26,27,30].

For $0 < p < 1/2$ in [25], the weighted maximal operator $\widetilde{\sigma}^{*,p}$ was studied, defined by

$$\widetilde{\sigma}^{*,p} F := \sup_{n\in\mathbb{N}} \frac{|\sigma_n F|}{(n+1)^{1/p-2}}, \tag{1.1}$$

and it was proved that the following inequality

$$\left\|\widetilde{\sigma}^{*} F\right\|_p \le c_p \|F\|_{H_p} \tag{1.2}$$

holds, where c_p is an absolute constant depending only on p. Moreover, it was proved that the rate of the sequence $(n+1)^{1/p-2}$, given in the denominator of (1.1), cannot be improved. In the case $p = 1/2$, similar results for the maximal operator $\widetilde{\sigma}^{*}$, defined by

$$\widetilde{\sigma}^{*} F := \sup_{n\in\mathbb{N}} \frac{|\sigma_n F|}{\log^2 (n+1)},$$

was proved in [24].

To study the convergence of subsequences of Fejér means and their bounded maximal operators on the martingale Hardy spaces H_p for $0 < p \le 1/2$, a central role is played by the fact that any natural

2020 *Mathematics Subject Classification.* 26015, 42C10, 42B30.

Key words and phrases. Inequalities; Walsh system; Fejér means; Martingale Hardy space; Weighted maximal operators.

number $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_j 2^j$, $n_j \in Z_2$ $(j \in \mathbb{N})$, where only a finite number of n_j differs from zero, and their important characters $[n]$, $|n|$, $\rho(n)$ and $V(n)$ are defined, respectively, as follows:

$$[n] := \min\{j \in \mathbb{N}, n_j \neq 0\}, \quad |n| := \max\{j \in \mathbb{N}, n_j \neq 0\}, \quad \rho(n) := |n| - [n],$$

$$V(n) := n_0 + \sum_{k=1}^{\infty} |n_k - n_{k-1}|, \quad \text{for all} \quad n \in \mathbb{N}.$$

Moreover, for any $\{n_{s_j}\}$, $j = 1, 2, \ldots, r$, satisfying the conditions

$$2^s \leq n_{s_1} \leq n_{s_2} \leq \cdots \leq n_{s_r} < 2^{s+1}, \quad s \in \mathbb{N},$$

we define the numbers

$$s_- := \min\{[n_{s_j}]\}, \; s_+ := \max\{|n_{s_j}|\} = s \text{ and } \rho_s(n_{s_j}) := s_+ - s_-. \tag{1.3}$$

Weisz [31] (see also [29]) proved that for any $F \in H_p$ $(p > 0)$, the maximal operator $\sup_{n \in \mathbb{N}} |\sigma_{2^n} F|$ is bounded from the Hardy space H_p to the Lebesgue space L_p. Persson and Tephnadze [16] generalized this result and proved that if $0 < p \leq 1/2$ and $\{n_k : k \geq 0\}$ is a sequence of positive integers such that

$$\sup_{k \in \mathbb{N}} \rho(n_k) \leq c < \infty, \tag{1.4}$$

then the maximal operator $\widetilde{\sigma}^{*,\nabla}$, defined by

$$\widetilde{\sigma}^{*,\nabla} F := \sup_{k \in \mathbb{N}} |\sigma_{n_k} F|, \tag{1.5}$$

is bounded from the space H_p to the space L_p. Moreover, if $0 < p < 1/2$ and $\{n_k : k \geq 0\}$ is a sequence of positive numbers such that $\sup_{k \in \mathbb{N}} \rho(n_k) = \infty$, then there exists a martingale $F \in H_p$ such that

$$\sup_{k \in \mathbb{N}} \|\sigma_{n_k} F\|_p = \infty.$$

A similar problem for $p = 1/2$ was studied in [1].

In [28] it was proved that if $F \in H_{1/2}$, then there exists an absolute constant c such that the following inequality:

$$\|\sigma_n F\|_{H_{1/2}} \leq c V^2(n) \|F\|_{H_{1/2}}$$

holds. Moreover, the rate of the sequence $V^2(n)$ cannot be improved.

In [28] it was also proved that if $0 < p < 1/2$ and $F \in H_p$, then there exists an absolute constant c_p, depending only on p, such that the inequality

$$\|\sigma_n F\|_{H_p} \leq c_p 2^{\rho(n)(1/p-2)} \|F\|_{H_p} \tag{1.6}$$

holds. Moreover, the rate of the sequence $2^{\rho(n)(1/p-2)}$ in inequality (1.6) is sharp.

In [3] it was proved that the weighted maximal operator $\widetilde{\sigma}^{*,*,p}$, defined by

$$\widetilde{\sigma}^{*,*,p} F := \sup_{n \in \mathbb{N}} \frac{|\sigma_n F|}{2^{\rho(n)(1/p-2)}},$$

is bounded from the Hardy space H_p to the space $weak - L_p$ and is not bounded from the space H_p to the Lebesgue space L_p for $0 < p < 1/2$.

One of the main goals of this paper is to generalize the estimate (1.2) for $f \in H_p$, $0 < p < 1/2$. Indeed, we investigate the much more general maximal operators $\widetilde{\sigma}^{*,\nabla}$, defined by

$$\widetilde{\sigma}^{*,\nabla} F := \sup_{s \in \mathbb{N}} \sup_{2^s \leq n_{s_i} < 2^{s+1}} \frac{|\sigma_{n_{s_i}} F|}{2^{\rho_s(n_{s_i})(1/p-2)}}. \tag{1.7}$$

In particular, we prove that (1.7) is bounded from the Hardy space H_p to the Lebesgue space L_p (see Theorem 3.1). Moreover, we also prove the sharpness of this result (see Theorem 3.2). As a consequence, we obtain some new and well-known results.

This paper is organized as follows: In order not to disturb our discussions later on, some preliminaries (definitions, notations and lemmas) are presented in Section 2. The main results and some of their consequences can be found in Section 3. Finally, the detailed proofs are given in Section 4.

2. Preliminaries

Let $\mathbb{N}_+$ denote the set of positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Denote by Z_2 the discrete cyclic group of order 2, that is $Z_2 := \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given so that the measure of a singleton is 1/2. Define the group G as the complete direct product of the group Z_2, with the product of the discrete topologies of Z_2. The elements of G are represented by sequences $x := (x_0, x_1, \ldots, x_j, \ldots)$, where $x_k = 0 \vee 1$.

It is easy to give a base for the neighborhood of $x \in G$:

$$I_0(x) := G, \quad I_n(x) := \{y \in G : y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\} \ (n \in \mathbb{N}).$$

Denote $I_n := I_n(0)$, $\overline{I_n} := G \backslash I_n$ and

$$e_n := (0, \ldots, 0, x_n = 1, 0, \ldots) \in G, \quad \text{for} \quad n \in \mathbb{N}.$$

Then it is easy to prove that

$$\overline{I_M} = \left(\bigcup_{k=0}^{M-2} \bigcup_{l=k+1}^{M-1} I_{l+1}(e_k + e_l) \right) \bigcup \left(\bigcup_{k=0}^{M-1} I_M(e_k) \right). \tag{2.1}$$

The norms (or quasi-norms) of the spaces $L_p(G)$ and $weak - L_p(G)$, $(0 < p < \infty)$ are defined, respectively, by

$$\|f\|_p^p := \int_G |f|^p \, d\mu \quad \text{and} \quad \|f\|_{weak-L_p(G)}^p := \sup_{\lambda > 0} \lambda^p \mu(f > \lambda) < +\infty.$$

The k-th Rademacher function $r_k(x)$ is defined by

$$r_k(x) := (-1)^{x_k} \qquad (x \in G, \ k \in \mathbb{N}).$$

Now, define the Walsh system $w := (w_n : n \in \mathbb{N})$ on G as

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) = r_{|n|}(x) (-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \qquad (n \in \mathbb{N}).$$

The Walsh system is orthonormal and complete in $L_2(G)$ (see [20]).

If $f \in L_1$, we can define the Fourier coefficients, partial sums, Dirichlet kernels, Fejér means and Fejér kernels as follows:

$$\widehat{f}(n) := \int_G f w_n d\mu, \ (n \in \mathbb{N}),$$

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k) w_k, \quad D_n := \sum_{k=0}^{n-1} w_k, \quad (n \in \mathbb{N}_+, S_0 f := 0),$$

$$\sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f, \quad K_n := \frac{1}{n} \sum_{k=1}^{n} D_k, \quad (n \in \mathbb{N}_+).$$

Recall that (see [6, 10] and [20]) for any $t, n \in \mathbb{N}$,

$$D_{2^n}(x) = \begin{cases} 2^n & \text{if } x \in I_n, \\ 0 & \text{if } x \notin I_n \end{cases} \tag{2.2}$$

and

$$K_{2^n}(x) = \begin{cases} 2^{t-1}, & \text{if } x \in I_n(e_t), \ n > t, x \in I_t \backslash I_{t+1}, \\ (2^n + 1)/2, & \text{if } x \in I_n, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

Let $n=\sum_{i=1}^{r}2^{n^{i}},\ \ n^{1}>n^{2}>\cdots>n^{r}\geq 0$ and $n^{(k)}:=2^{n^{k+1}}+2^{n^{k+2}}+\cdots+2^{n^{r}}$. Then (see [10] and [20]), for any $n\in\mathbb{N}$,

$$nK_{n}=\sum_{A=1}^{r}\left(\prod_{j=1}^{A-1}w_{2^{n^{j}}}\right)\left(2^{n^{A}}K_{2^{n^{A}}}+n^{(A)}D_{2^{n^{A}}}\right). \tag{2.4}$$

The proof of the next lemma can be found in [16].

Lemma 2.1. *Let $n\in\mathbb{N}$, $[n]\neq|n|$ and $x\in I_{[n]+1}\left(e_{[n]-1}+e_{[n]}\right)$. Then the inequality*

$$|nK_{n}(x)|=\left|\left(n-2^{|n|}\right)K_{n-2^{|n|}}(x)\right|\geq\frac{2^{2[n]}}{4}$$

holds. Note that if $[n]=0$, we have the set $I_{2}\left(e_{0}\right)$.

We also need the following lemma (see [8]).

Lemma 2.2. *Let $n\geq 2^{M}$ and $x\in I_{M}^{k,l},\ k=0,\ldots,M-1$ and $l=k+1,\ldots,M$. Then the following inequality:*

$$\int_{I_{M}}\left|K_{n}\left(x+t\right)\right|d\mu\left(t\right)\leq\frac{c2^{k+l}}{2^{2M}}$$

holds.

The σ-algebra generated by the intervals $\{I_{n}\left(x\right):x\in G\}$ is denoted by $\zeta_{n}\ \left(n\in\mathbb{N}\right)$. By $F=\left(F_{n},n\in\mathbb{N}\right)$ we denote a martingale with respect to $\zeta_{n}\ \left(n\in\mathbb{N}\right)$ (see, e.g., [18] and [29]). The maximal function F^{*} of a martingale F is defined by

$$F^{*}:=\sup_{n\in\mathbb{N}}\left|F_{n}\right|.$$

In the case $f\in L_{1}\left(G\right)$, the maximal function f^{*} is given by

$$f^{*}\left(x\right):=\sup_{n\in\mathbb{N}}\left(\frac{1}{\mu\left(I_{n}\left(x\right)\right)}\left|\int_{I_{n}(x)}f\left(u\right)d\mu\left(u\right)\right|\right).$$

For $0<p<\infty$, the Hardy martingale spaces $H_{p}\left(G\right)$ consists of all martingales for which

$$\left\|F\right\|_{H_{p}}:=\left\|F^{*}\right\|_{p}<\infty.$$

It is easy to check that for every martingale $F=\left(F_{n},n\in\mathbb{N}\right)$ and every $k\in\mathbb{N}$, the limit

$$\widehat{F}\left(k\right):=\lim_{n\rightarrow\infty}\int_{G}F_{n}\left(x\right)w_{k}\left(x\right)d\mu\left(x\right)$$

exists and it is called the k-th Walsh-Fourier coefficients of F.

If $F:=\left(S_{2^{n}}f:n\in\mathbb{N}\right)$ is a regular martingale, generated by $f\in L_{1}\left(G\right)$, then $\widehat{F}\left(k\right)=\widehat{f}\left(k\right)$, $k\in\mathbb{N}$.

A bounded measurable function a is called p-atom, if there exists a dyadic interval I such that

$$\operatorname{supp}\left(a\right)\subset I,\qquad\int_{I}ad\mu=0,\quad\left\|a\right\|_{\infty}\leq\mu\left(I\right)^{-1/p}.$$

The dyadic Hardy martingale spaces H_{p} for $0<p\leq 1$ have an atomic characterization. Namely, the following holds (see [18, 29, 31]).

Lemma 2.3. *A martingale $F=\left(F_{n},\ n\in\mathbb{N}\right)$ belongs to $H_{p}\left(0<p\leq 1\right)$ if and only if there exist a sequence $\left(a_{k},k\in\mathbb{N}\right)$ of p-atoms and a sequence $\left(\mu_{k},k\in\mathbb{N}\right)$ of real numbers such that for every $n\in\mathbb{N}$,*

$$\sum_{k=0}^{\infty}\mu_{k}S_{2^{n}}a_{k}=F_{n},\qquad\sum_{k=0}^{\infty}\left|\mu_{k}\right|^{p}<\infty. \tag{2.5}$$

Moreover, $\left\|F\right\|_{H_{p}}\backsim\inf\left(\sum_{k=0}^{\infty}\left|\mu_{k}\right|^{p}\right)^{1/p}$, where the infimum is taken over all decomposition of F of the form (2.5).

From Lemma 2.3 follows the following important lemma (see Weisz [29]).

Lemma 2.4. *Suppose that an operator T is σ-linear and*

$$\int_{\overline{I}} |Ta|^p \, d\mu \leq c_p < \infty, \quad (0 < p \leq 1)$$

for every p-atom a, where I denotes the support of the atom. If T is bounded from L_∞ to L_∞, then

$$\|TF\|_p \leq c_p \|F\|_{H_p}.$$

3. The Main Results

Our first main result reads as

Theorem 3.1. *Let $0 < p < 1/2$, $f \in H_p$ and $\{n_k : k \geq 0\}$ be a sequence of positive numbers and let $\{n_{s_i} : 1 \leq i \leq r\} \subset \{n_k : k \geq 0\}$ be numbers such that $2^s \leq n_{s_1} \leq n_{s_2} \leq \cdots \leq n_{s_r} \leq 2^{s+1}$, $s \in \mathbb{N}$. Then the weighted maximal operator $\widetilde{\sigma}^{*,\nabla}$, defined by (1.7), where $\rho_s(n_{s_i})$ are defined by (1.3), is bounded from the martingale Hardy space H_p to the Lebesgue space L_p.*

We also prove the sharpness of this result:

Theorem 3.2. *Let $0 < p < 1/2$, $f \in H_p(G)$, $\{n_k : k \geq 0\}$ be a sequence of positive numbers and let $\{n_{s_i} : 1 \leq i \leq r\} \subset \{n_k : k \geq 0\}$ be numbers such that*

$$2^s \leq n_{s_1} \leq n_{s_2} \leq \cdots \leq n_{s_r} \leq 2^{s+1}, \quad s \in \mathbb{N}.$$

Then for any non-negative and non-decreasing function $\varphi : \mathbb{N}_+ \to \mathbb{R}$ satisfying the condition

$$\sup_{s \in \mathbb{N}} \sup_{2^s \leq n_{s_i} < 2^{s+1}} \frac{2^{\rho_s(n_{s_i})(1/p-2)}}{\varphi(n_{s_i})} = \infty, \tag{3.1}$$

there exist p-atoms f_s such that

$$\frac{\left\| \sup_{s\in\mathbb{N}} \sup_{2^s \leq n_{s_i} < 2^{s+1}} \frac{|\sigma_{n_{s_i}} f_s|}{\varphi(n_{s_i})} \right\|_p}{\|f_s\|_{H_p}} \to \infty, \quad as \quad s \to \infty.$$

From Theorems 3.1 and 3.2 follows immediately the following result (see [25]).

Corollary 3.1. a) *Let $0 < p < 1/2$ and $f \in H_p$. Then the weighted maximal operator $\widetilde{\sigma}^*$, defined by (1.1), is bounded from the martingale Hardy space H_p to the Lebesgue space L_p.*

b) *Let $\{\varphi_n\}$ be any nondecreasing sequence satisfying the condition*

$$\overline{\lim_{n\to\infty}} \frac{(n+1)^{1/p-2}}{\varphi_n} = \infty.$$

Then the maximal operator

$$\sup_{n\in\mathbb{N}} \frac{|\sigma_n f|}{\varphi_n}$$

is unbounded from the martingale Hardy space H_p to the Lebesgue space L_p, for any $0 < p < 1/2$.

We also have the following new consequences of Theorem 3.1 that all are sharp in a special sense (see Theorem 3.2).

Corollary 3.2. *Let $p > 0$ and $f \in H_p$. Then the restricted maximal operators $\widetilde{\sigma}_i^{*,\nabla}$, $i = 1, 2$, defined by*

$$\widetilde{\sigma}_1^{*,\nabla} F := \sup_{k\in\mathbb{N}} |\sigma_{2^k} F| \quad and \quad \widetilde{\sigma}_2^{*,\nabla} F := \sup_{k\in\mathbb{N}} |\sigma_{2^k + 2^{k-1}} F|,$$

are both bounded from the Hardy space H_p to the Lebesgue space L_p.

Corollary 3.3. *If $0<p<1/2$, $f\in H_p$ and $\{n_k : k\geq 0\}$ is any sequence of positive numbers, then the maximal operator, defined by* (1.5)*, is bounded from the Hardy space H_p to the Lebesgue space L_p if and only if condition* (1.4) *is fulfilled.*

Corollary 3.4. *Let $0<p<1/2$ and $f\in H_p$. Then the restricted maximal operators $\widetilde{\sigma}_i^{*,\nabla},\ i=3,4$, defined by*

$$\widetilde{\sigma}_3^{*,\nabla}F:=\sup_{k\in\mathbb{N}}\frac{|\sigma_{2^{2k}+2^k}F|}{2^{k(1/p-2)}} \qquad \textit{and} \qquad \widetilde{\sigma}_4^{*,\nabla}F:=\sup_{k\in\mathbb{N}}\frac{|\sigma_{2^{3k}+2^k}F|}{2^{2k(1/p-2)}},$$

are both bounded from the Hardy space H_p to the Lebesgue space L_p.

4. Proofs

Proof of Theorem 3.1. Since σ_n is bounded from L_∞ to L_∞, by Lemma 2.4, the proof of Theorem 3.1 will be complete if we prove that

$$\int\limits_{\overline{I_M}}\left(\sup_{s\in\mathbb{N}}\sup_{2^s\leq n_{s_i}<2^{s+1}}\left|\frac{\sigma_{n_{s_k}}a}{2^{\rho_s\left(n_{s_k}\right)(1/p-2)}}\right|\right)^p d\mu\leq c<\infty, \tag{4.1}$$

for every p-atom a. We may assume that a is an arbitrary p-atom with a support $I,\ \mu(I)=2^{-M}$ and $I=I_M$. It is easy to see that $\sigma_{n_{s_k}}a=0$, when $n_{s_k}<2^M$. Therefore, we can suppose that $n_{s_k}\geq 2^M$. Let $2^s\leq n_{s_i}<2^{s+1}$, for some $s\geq M$. Since $\|a\|_\infty\leq 2^{M/p}$, we find that

$$\begin{aligned}\frac{\left|\sigma_{n_{s_i}}a\right|}{2^{\rho_s\left(n_{s_i}\right)(1/p-2)}}&\leq\frac{1}{2^{\rho_s\left(n_{s_i}\right)(1/p-2)}}\|a\|_\infty\int\limits_{I_M}\left|K_{n_{s_i}}(x+t)\right|d\mu(t)\\&\leq\frac{1}{2^{\rho_s\left(n_{s_i}\right)(1/p-2)}}2^{M/p}\int\limits_{I_M}\left|K_{n_{s_i}}(x+t)\right|d\mu(t).\end{aligned}$$

Let $x\in I_{l+1}(e_k+e_l)$, $0\leq k<l\leq[n_{s_i}]\leq M$. Then $x+t\in I_{l+1}(e_k+e_l)$, $0\leq k<l\leq[n_{s_i}]\leq M$ and by applying (2.2), (2.3) and (2.4), we get

$$K_{n_{s_i}}(x+t)=0, \quad \text{for} \quad t\in I_M$$

and $\left|\sigma_{n_{s_i}}a\right|=0$, for any $2^s\leq n_{s_i}<2^{s+1}$. Since $[n_{s_i}]\geq s_-$, we obtain

$$\sup_{2^s\leq n_{s_i}<2^{s+1}}\frac{\left|\sigma_{n_{s_i}}a\right|}{2^{\rho_s\left(n_{s_i}\right)(1/p-2)}}=0,\ \text{for } x\in I_{l+1}(e_k+e_l),\ 0\leq k<l<s_-. \tag{4.2}$$

Next, we suppose that $x\in I_{l+1}(e_k+e_l)$, where either $[n_{s_i}]\leq k<l\leq M$ or $k\leq[n_{s_i}]<l\leq M$. Since $n_{s_i}\geq 2^M$ and $|n_{s_i}|=s$, applying Lemma 2.2 we find that

$$\frac{\left|\sigma_{n_{s_i}}a\right|}{2^{\rho_s\left(n_{s_i}\right)(1/p-2)}}\quad\leq\quad\frac{2^{M(1/p-2)+k+l}}{2^{\rho_s\left(n_{s_i}\right)(1/p-2)}}\leq\frac{2^{M(1/p-2)}}{2^{s(1/p-2)}}2^{k+l+s_-(1/p-2)}. \tag{4.3}$$

Applying (4.3) for any $x\in I_{l+1}(e_k+e_l)$, $s_-\leq k<l\leq M$ or $k\leq s_-<l\leq M$, we obtain

$$\sup_{2^s\leq n_{s_i}<2^{s+1}}\frac{\left|\sigma_{n_{s_i}}a\right|}{2^{\rho_s\left(n_{s_i}\right)(1/p-2)}}\leq\frac{2^{M(1/p-2)}}{2^{s(1/p-2)}}2^{k+l+s_-(1/p-2)}. \tag{4.4}$$

If we now define $\widetilde{\sigma}_s^*$ by

$$\widetilde{\sigma}_s^*f:=\sup_{2^s\leq n_{s_i}<2^{s+1}}\frac{\left|\sigma_{n_{s_i}}f\right|}{2^{\rho_s\left(n_{s_i}\right)(1/p-2)}},$$

then we can conclude that

$$\left(\sup_{s\in\mathbb{N}}\sup_{k\in\mathbb{N}}\left|\frac{\sigma_{n_{s_k}}a}{2^{\rho_s\left(n_{s_k}\right)(1/p-2)}}\right|\right)^p\leq\sum_{s=M}^{\infty}\left|\widetilde{\sigma}_s^*a\right|^p.$$

Hence, combining (2.1), (4.2) and (4.4), we obtain

$$\begin{aligned}
&\int_{I_M}\left(\sup_{s\in\mathbb{N}}\sup_{k\in\mathbb{N}}\left|\frac{\sigma_{n_{s_k}}a}{2^{\rho_s\left(n_{s_k}\right)(1/p-2)}}\right|\right)^p d\mu \le \sum_{s=M}^{\infty}\int_{I_M}\left|\widetilde{\sigma}_s^* a\right|^p d\mu \\
&= \sum_{s=M}^{\infty}\left(\sum_{k=0}^{s_--2}\sum_{l=k+1}^{s_--1}+\sum_{k=0}^{s_--1}\sum_{l=s_-}^{M-1}+\sum_{k=s_-}^{M-2}\sum_{l=k+1}^{M-1}\right)\int_{I_{l+1}(e_k+e_l)}\left|\widetilde{\sigma}_s^* a\right|^p d\mu \\
&+\sum_{s=M}^{\infty}\sum_{k=0}^{M-1}\int_{I_M(e_k)}\left|\widetilde{\sigma}_s^* a\right|^p d\mu \\
&\le c_p\sum_{s=M}^{\infty}\left(\frac{2^{M(1/p-2)}}{2^{s(1/p-2)}}\right)^p\sum_{k=0}^{s_--1}\sum_{l=s_-+1}^{M-1}\frac{1}{2^l}2^{s_-(1-2p)}2^{p(k+l)} \\
&+c_p\sum_{s=M}^{\infty}\left(\frac{2^{M(1/p-2)}}{2^{s(1/p-2)}}\right)^p\sum_{k=s_-}^{M-2}\sum_{l=k+1}^{M-1}\frac{1}{2^l}2^{s_-(1-2p)}2^{p(k+l)} \\
&+c_p\sum_{s=M}^{\infty}\left(\frac{2^{M(1/p-2)}}{2^{s(1/p-2)}}\right)^p\frac{2^{s_-(1-2p)}}{2^M}\sum_{k=0}^{s_-}2^{p(k+M)} \\
&:= I+II+III. \qquad (4.5)
\end{aligned}$$

Since $\sum_{s=M}^{\infty}\left(\frac{2^{M(1/p-2)}}{2^{s(1/p-2)}}\right)^p < c < \infty$, we find that

$$I \le c_p\sum_{k=0}^{s_--1}\sum_{l=s_-+1}^{M-1}\frac{1}{2^l}2^{s_-(1-2p)}2^{p(k+l)} < c_p < \infty, \qquad (4.6)$$

$$II \le c_p\sum_{k=s_-}^{M-2}\sum_{l=k+1}^{M-1}\frac{1}{2^l}2^{s_-(1-2p)}2^{p(k+l)} < c_p < \infty, \qquad (4.7)$$

and

$$III \le \frac{c_p 2^{s_-(1-2p)}}{2^M}\sum_{k=0}^{s_-}2^{p(k+M)} < c_p < \infty. \qquad (4.8)$$

Combining (4.5)–(4.8), we can conclude that (4.1) holds. Thus the proof is complete. □

Proof of Theorem 3.2. Let $n_{s_k} := q_{n_k}^s \in \mathbb{N}$ be such that $2^{n_k} \le q_{n_k}^s \le 2^{n_k+1}$, where $0 \le s < 2^{n_k}$. If s_0 is such that $q_{n_k}^{s_0} = s_-$, then we get $\rho_{n_k}(q_{n_k^s}) = n_k - s_- = \rho(q_{n_k}^{s_0})$. In view of (3.1), we have

$$\frac{2^{\rho(q_{n_k}^{s_0})(1/p-2)}}{\varphi(q_{n_k}^{s_0})} \to \infty, \quad \text{as} \quad k \to \infty. \qquad (4.9)$$

Set

$$f_{n_k} = D_{2^{n_k+1}} - D_{2^{n_k}}, \qquad n_k \ge 3.$$

It is evident that

$$\widehat{f_{n_k}}(i) = \begin{cases} 1, & \text{if } i = 2^{n_k},\ldots,2^{n_k+1}-1, \\ 0, & \text{otherwise}. \end{cases}$$

Then we can easily derive that

$$S_i f_{n_k} = \begin{cases} D_i - D_{2^{n_k}}, & \text{if } i = 2^{n_k},\ldots,2^{n_k+1}-1, \\ f_{n_k}, & \text{if } i \ge 2^{n_k+1}, \\ 0, & \text{otherwise}. \end{cases} \qquad (4.10)$$

Since

$$D_{j+2^{n_k}} - D_{2^{n_k}} = w_{2^{n_k}} D_j, \qquad j = 1, 2, .., 2^{n_k}, \tag{4.11}$$

it follows from (2.2) that

$$\|f_{n_k}\|_{H_p} \quad = \quad \left\|\sup_{i\in\mathbb{N}} |S_{2^i} f_{n_k}|\right\|_p = \|D_{2^{n_k+1}} - D_{2^{n_k}}\|_p \le 2^{n_k(1-1/p)}. \tag{4.12}$$

Combining (4.10) and (4.11), we can conclude that

$$\begin{aligned} \left|\sigma_{q_{n_k}^s} f_{n_k}\right| &= \frac{1}{q_{n_k}^s}\left|\sum_{j=2^{n_k}}^{q_{n_k}^s-1} S_j f_{n_k}\right| = \frac{1}{q_{n_k}^s}\left|\sum_{j=2^{n_k}}^{q_{n_k}^s-1} (D_j - D_{2^{n_k}})\right| \\ &= \frac{1}{q_{n_k}^s}\left|\sum_{j=0}^{q_{n_k}^s-2^{n_k}-1} (D_{j+2^{n_k}} - D_{2^{n_k}})\right| \\ &= \frac{1}{q_{n_k}^s}\left|\sum_{j=0}^{q_{n_k}^s-2^{n_k}-1} D_j\right| = \frac{q_{n_k}^s - 2^{n_k} - 1}{q_{n_k}^s}\left|K_{q_{n_k}^s-2^{n_k}-1}\right|. \end{aligned}$$

Let $q_{n_k}^{s_0}$ be n_{s_j} so that $[q_{n_k}^{s_0}] = s_-$, that is, $s_0 = s_-$ and $x \in I_{s_0+1}(e_{s_0-1} + e_{s_0})$. Using Lemma 2.1, we have

$$\left|\sigma_{q_{n_k}^{s_0}} f_{n_k}\right| \ge \frac{c2^{2s_0}}{2^{n_k}} \quad \text{and} \quad \frac{\left|\sigma_{q_{n_k}^{s_0}} f_{n_k}\right|}{\varphi\left(q_{n_k}^{s_0}\right)} \ge \frac{c2^{2s_0}}{2^{n_k}\varphi\left(q_{n_k}^{s_0}\right)}.$$

Hence

$$\begin{aligned} \int_G \left(\sup_{k\in\mathbb{N}} \frac{\left|\sigma_{q_{n_k}^s} f_{n_k}\right|}{\varphi\left(q_{n_k}^s\right)}\right)^p d\mu &\ge \int_{I_{s_0+1}\left(e_{s_0-1}+e_{s_0}\right)} \left(\frac{\left|\sigma_{q_{n_k}^{s_0}} f_{n_k}\right|}{\varphi\left(q_{n_k}^{s_0}\right)}\right)^p d\mu \\ &\ge c_p \frac{1}{2^{s_0}} \frac{2^{2ps_0}}{2^{pn_k}\varphi^p\left(q_{n_k}^{s_0}\right)} \ge \frac{C_p 2^{(2p-1)s_0}}{2^{pn_k}\varphi^p\left(q_{n_k}^{s_0}\right)}. \end{aligned}$$

Finally, combining (4.9) and (4.12), we find that

$$\begin{aligned} &\frac{\left(\int_G \left(\sup_{k\in\mathbb{N}} \sup_{0\le s<n_k} \frac{\left|\sigma_{q_{n_k}^s} f_{n_k}\right|}{\varphi\left(q_{n_k}^s\right)}\right)^p d\mu\right)^{1/p}}{\|f_{n_k}\|_{H_p}} \\ \ge &\frac{C_p 2^{(2-1/p)s_0}}{2^{n_k}\varphi(q_{n_k}^{s_0})} \frac{1}{2^{n_k(1-1/p)}} \ge \frac{c_p 2^{\rho(q_{n_k}^{s_0})(1/p-2)}}{\varphi(q_{n_k}^{s_0})} \to \infty, \quad \text{as} \quad k \to \infty. \end{aligned}$$

Thus the proof is complete. □

Final remark. The research presented in this paper is partly inspired by the research published in [11]. The authors plan to continue the study of maximal operators discussed in this paper and to explore their ability in finding specific geometric constructions of more general means (than power means) and their related inequalities.

Acknowledgement

The research was supported by The University of Georgia grant "Mathematical Analysis and PDE on Lie Groups".

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(Received: 09.05.2024; Accepted: 17.08.2024; Published online: 10.02.2026)

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