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COMPACTNESS CRITERION IN WEIGHTED MUSIELAK–ORLICZ SEQUENCE SPACES AND APPLICATIONS

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Abstract. We study relatively compact sets in weighted Musielak–Orlicz sequence spaces. The characterization of such sets is given in the case of weighted Musielak–Orlicz sequence spaces. As an application, we establish a necessary and sufficient condition on weight functions for the compactness of a discrete Hardy operator on weighted Musielak–Orlicz sequence spaces. In particular, we get similar results for the dual operator of a discrete Hardy operator. The results are illustrated by a number of corollaries.

1. Introduction

Compactness results in ordinary Lebesgue spaces are often crucial in existence proofs for nonlinear partial differential equations. A necessary and sufficient condition for the compactness of a subset of ordinary Lebesgue spaces is given in the so-called Kolmogorov compactness theorem, or the Frechet-Kolmogorov theorem. Furthermore, we trace the historical roots of the Kolmogorov compactness theorem, which arose in [20].

We mention here some generalizations of the Riesz–Kolmogorov theorem. H. Rafeiro characterized precompact sets in variable exponent Lebesgue spaces on Euclidean spaces in [33]. Further characterizations of the relatively compact sets are studied in variable-exponent Lebesgue spaces on metric measure spaces. In [31], Kolmogorov's theorem for $p = 2$ is given in terms of the Fourier transform (see also [13] and [14]).

In the literature, many authors considered the following standard form of Hardy's discrete inequality in a discrete Lebesgue space with a constant exponent. Let $p > 1$, $p' = \frac{p}{p-1}$ and let $\{x_k\}_{k=1}^{\infty}$ be an arbitrary sequence of non-negative real numbers. Then

$$\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}x_k\right)^p\right)^{\frac{1}{p}} \le p'\left(\sum_{n=1}^{\infty}x_k^p\right)^{\frac{1}{p}}. \tag{1.1}$$

The constant p' in (1.1) is sharp. The first result concerning the weight characterization of (1.1) was proved by K. F. Andersen and H. P. Heinig in [1]. A full weight characterization of the discrete Hardy inequality was proved by G. Bennett in [6], etc. It is well known that an essential development for Hardy-type inequalities in the discrete case was given by C. A. Okpoti, L. -E. Persson, and A. Wedestig in [30]. A similar development for Hardy-type inequalities in the discrete case given by A. A. Kalybay, L. -E. Persson and A. M. Temirkhanova [18]. The history of Hardy type inequalities on the cones of monotone functions and sequences, as well as references to the related results, can be found in the papers of M. L. Gol'dman [12] and A. Gogatishvili and V. D. Stepanov [10, 11].

It is well known that the variable Lebesgue sequence space was first studied by W. Orlicz [31] in 1931. Hölder's inequality for a variable Lebesgue sequence space was proved in [31]. W. Orlicz also considered the variable Lebesgue space on the real line, and proved Hölder inequality in this setting. However, this paper is essentially the unique Orlicz contribution to the study of the variable Lebesgue spaces. The Musielak–Orlicz space was originally introduced by J. Musielak and W. Orlicz in [27]. The Musielak–Orlicz spaces are closely related to modular spaces. Based on the modular

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theory, Musielak and Orlicz developed in 1959 the theory of Musielak–Orlicz spaces [27]. These spaces have been studied for almost sixty years and are widely applied in various fields of analysis. Such spaces have also been generalized in many directions. For example, many authors have considered their generalizations to the vector-valued functions and spaces generated by families of Musielak–Orlicz modulars. Later, a more explicit version of these spaces, namely modular function spaces, was investigated by many mathematicians. The study of variable exponent spaces has been stimulated by the problems of elasticity, fluid dynamics, calculus of variations and differential equations with non-standard growth conditions (see, [8] and [19]). We firmly believe that these more general spaces will become increasingly important in the modeling of modern materials. Various characterizations of the mapping properties such as the boundedness and compactness of a continuous version of the Hardy operator in the variable Lebesgue spaces, were studied in [3–5, 7, 9, 17, 21–26, 29] etc. We especially note that the compactness characteristic of the continuous version of the Hardy operator on weighted Lebesgue spaces was proved in [9, 22] and [26]. A similar problem for the discrete Hardy operator on weighted variable Lebesgue sequence spaces was considered in [4].

In this paper, we obtain a necessary and sufficient condition of the precompact sets in the Musielak–Orlicz sequence spaces. We find sufficient conditions for the generalized Φ-functions that guarantee continuous embeddings between weighted Musielak–Orlicz sequence spaces. Depending on the sufficient conditions on the generalized Φ-functions, the weighted Musielak–Orlicz sequence spaces coincide. As applications, we establish the compactness criteria for the discrete Hardy operator and its dual operator defined on the weighted Musielak–Orlicz sequence spaces. The results are based on the paper [16].

The rest of the paper is structured as follows. Section 2 contains some preliminaries along with the standard ingredients used in the proofs. We also recall standard results from the theory of Musielak–Orlicz sequence spaces. In particular, in Section 2, we proved continuous embeddings between weighted Musielak–Orlicz sequence spaces. Our principal assertions are formulated and proved in Section 3. In the same Section 3, we obtained the necessary and sufficient conditions for the subsets of Musielak–Orlicz spaces to be relatively compact. In the next Section 4, we establish a necessary and sufficient condition on weight functions for the compactness of the discrete Hardy operator on weighted Musielak–Orlicz sequence spaces. A similar problem is also studied for the dual operator of the discrete Hardy operator.

2. Preliminaries

Let $\mathbb{N}$ be the set of natural numbers and let $p = \{p_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $1 \le \underline{p} \le p_n \le \overline{p} < \infty$, where $\underline{p} = \inf\limits_{n\ge1} p_n$ and $\overline{p} = \sup\limits_{n\ge1} p_n$. The conjugate exponent function of p_n is defined as $\frac{1}{p_n} + \frac{1}{p_n'} = 1$ for all $n \in \mathbb{N}$. Denote by χ_A the characteristic function of $A \subset \mathbb{N}$. Throughout this paper, $\underline{p}' = \frac{\underline{p}}{\underline{p}-1}$. Let $w = \{w_n\}_{n=1}^{\infty}$ be a sequence of non-negative numbers, i.e., w is a weight function defined on $\mathbb{N}$.

Next, we give the definition of the Musielak–Orlicz space.

Definition 2.1 ([8, 27])**.** A convex, left-continuous function $\varphi : [0,\infty) \mapsto [0,\infty]$ with $\varphi(0) = 0$, $\lim\limits_{t\to+0} \varphi(t) = 0$ and $\lim\limits_{t\to\infty} \varphi(t) = \infty$ is called a Φ-function. It is called positive if $\varphi(t) > 0$ for all $t > 0$.

Definition 2.2 ([8, 27])**.** Let (A, Σ, μ) be a σ-finite, complete measure space. A real function $\varphi : A\times[0,\infty) \mapsto [0,\infty]$ is called a generalized Φ-function on (A, Σ, μ) if it satisfies the following conditions:

(a) $\varphi(x, \cdot)$ is a Φ-function for all $x \in A$.

(b) $\varphi(\cdot, t)$ is measurable for all $t \ge 0$.

If φ is a generalized Φ-function on (A, Σ, μ), we shortly write $\varphi \in \Phi(A, \mu)$.

We observe that any generalized Φ-function $\varphi(x, \cdot)$ is non-decreasing on $[0,\infty)$ for all $x \in A$.

Remark 2.1. Let $\varphi \in \Phi(A, \mu)$. Then, as a convex function, $\varphi(x, \cdot)$ is continuous if and only if $\varphi(x, \cdot)$ is finite on $[0,\infty)$ for all $x \in A$ (see, [8]).

Definition 2.3 ([8,27])**.** Let $\varphi \in \Phi(A,\mu)$ and let ρ_φ be given by

$$\rho_\varphi(f) := \int_A \varphi(x, |f(x)|)\, d\mu(x) \ \text{ for all } \ f \in L_0(\Omega).$$

We put $L_\varphi(A,\mu) = \{f \in L_0(A,\mu): \ \ \rho_\varphi(\lambda_0 f) < \infty \ \text{for some} \ \lambda_0 > 0\}$, and

$$\|f\|_{L_\varphi(A,\mu)} = \inf\left\{\lambda > 0: \quad \rho_\varphi\left(\frac{f}{\lambda}\right) \le 1\right\}.$$

The space $L_\varphi(A,\mu)$ is called the Musielak–Orlicz space.

Let μ be the counting measure on $\mathbb{N}$ and let $\ell_0(\mathbb{N})$ be the space of all sequences $\mathbf{x} = \{x_n\}_{n=1}^{\infty}$. Let $w = \{w_n\}_{n=1}^{\infty}$ be a weight function defined on $\mathbb{N}$. Then we have the definition of weighted Musielak–Orlicz sequence space $\ell_{\varphi,w}(\mathbb{N})$. We put

$$\ell_{\varphi,w}(\mathbb{N}) = \left\{\mathbf{x} \in \ell_0(\mathbb{N}): \ \sum_{n=1}^{\infty} \varphi\left(n, \frac{|x_n|\, w_n}{\lambda_0}\right) < \infty \ \text{for some} \ \lambda_0 > 0\right\}.$$

The functional

$$\|\mathbf{x}\|_{\ell_{\varphi,w}(\mathbb{N})} = \inf\left\{\lambda > 0: \ \sum_{n=1}^{\infty} \varphi\left(n, \frac{|x_n|\, w_n}{\lambda}\right) \le 1\right\}$$

is defined as the norm in $\ell_{\varphi,w}(\mathbb{N})$.

Suppose that $1 \le p_n < \infty$, $n \in \mathbb{N}$. In case $\varphi(n,t) = t^{p_n}$, $\ell_{\varphi,w}(\mathbb{N})$ is the variable exponent weighted Lebesgue sequence space $\ell_w^{p_n}(\mathbb{N})$. There are some examples on the weighted Musielak–Orlicz sequence space $\ell_{\varphi,w}(\mathbb{N})$. We give some examples on Φ-functions:

$\varphi_1(n,t) = t^{p_n} \log(1+t)$, $\varphi_2(n,t) = t^{p_n} + a_n\, t^{q_n}$, $1 \le q_n < \infty$ and $a_n \ge 0$, $\varphi_3(n,t) = e^t\, t^{p_n}$ for all $n \in \mathbb{N}$, and so on.

A ε-cover of a metric space is a cover of the space consisting of sets of diameter at most ε. A metric space is called totally bounded if it admits a finite ε-cover for all $\varepsilon > 0$. It is well known that a metric space is compact if and only if it is complete and totally bounded (or precompact) (see [16]). Since we are interested in the compactness results for subsets of Banach spaces, we focus our attention on a total boundedness.

Next, we need the following lemmas.

Lemma 2.1 ([16])**.** *Let (X, d_X) be a metric space. Suppose that for any $\varepsilon > 0$ there exists $\delta > 0$, a metric space (W, d_W), and a function $\Phi: X \mapsto W$ satisfying the following conditions:*

(a) *$\Phi[X]$ is totally bounded,*

(b) *if $x, y \in X$ and $d_W(\Phi(x), \Phi(y)) < \delta$, then $d_X(x,y) < \varepsilon$.*

Then X is totally bounded.

Lemma 2.2. *Let μ be the counting measure on $\mathbb{N}$ and let φ, $\psi \in \Phi(\mathbb{N},\mu)$. Suppose that $v = \{v_n\}_{n=1}^{\infty}$ is a weight function defined on $\mathbb{N}$, that $\psi(n,t)$ is finite on $[0,\infty)$ for all $n \in \mathbb{N}$, and that there exists $C > 0$ such that*

$$\varphi\left(n, \frac{t}{C}\right) \le \psi(n,t) \tag{2.1}$$

for all $n \in \mathbb{N}$ and $0 \le t \le \sup\limits_{n\in\mathbb{N}} \psi^{-1}(n,1) < \infty$.

Then $\ell_{\psi,v}(\mathbb{N}) \hookrightarrow \ell_{\varphi,v}(\mathbb{N})$ and the following inequality:

$$\|\mathbf{x}\|_{\ell_{\varphi,v}(\mathbb{N})} \le C\, \|\mathbf{x}\|_{\ell_{\psi,v}(\mathbb{N})}$$

holds.

Proof. Let $\mathbf{x} \in \ell_{\psi,v}(\mathbb{N})$ and $\|\mathbf{x}\|_{\ell_{\psi,v}(\mathbb{N})} \le 1$. For a generalized Φ-function $\psi \in \Phi(\mathbb{N},\mu)$, we define $\psi^{-1}(n,1) = \sup\{t \ge 0: \ \ \psi(n,t) \le 1\}$. So, we have

$$\psi\left(n, \frac{|x_n|\, v_n}{\|\mathbf{x}\|_{\ell_{\psi,v}(\mathbb{N})}}\right) \le \sum_{k=1}^{\infty} \psi\left(k, \frac{|x_k|\, v_k}{\|\mathbf{x}\|_{\ell_{\psi,v}(\mathbb{N})}}\right) \le 1 \ \text{ for } \ \text{all } \ n \in \mathbb{N}.$$

Consequently, $\psi\Big(n, \frac{|x_n| v_n}{\|\mathbf{x}\|_{\ell_{\psi,v}(\mathbb{N})}}\Big)$ is a bounded function for all $n\in\mathbb{N}$. Next, one has $\frac{|x_n| v_n}{\|\mathbf{x}\|_{\ell_{\psi,v}(\mathbb{N})}} \le \psi^{-1}(n,1)$ and, therefore, $|x_n|\, v_n \le \sup\limits_{n\in\mathbb{N}} \psi^{-1}(n,1) < \infty$. Thus, by (2.1), we have

$$\sum_{k=1}^{\infty} \varphi\left(k, \frac{|x_k|\, v_k}{C\,\|\mathbf{x}\|_{\ell_{\psi,v}(\mathbb{N})}}\right) \le \sum_{k=1}^{\infty} \psi\left(k, \frac{|x_k|\, v_k}{\|\mathbf{x}\|_{\ell_{\psi,v}(\mathbb{N})}}\right) \le 1 \quad \text{for} \quad \text{all} \quad n\in\mathbb{N}.$$

This completes the proof of Lemma 2.2.

Lemma 2.3. *Let μ be the counting measure on $\mathbb{N}$ and $\varphi,\, \psi \in \Phi(\mathbb{N},\mu)$. Suppose that $v=\{v_n\}_{n=1}^{\infty}$ is a weight function defined on $\mathbb{N}$. Suppose that there exist $C>0$ and $\mathbf{h}=\{h_n\}_{n=1}^{\infty}\in \ell_1(\mathbb{N})$ with $\|\mathbf{h}\|_{\ell_1}\le 1$ such that*

$$\psi\left(n, \frac{t}{C}\right) \le \varphi(n,t) + h_n \tag{2.2}$$

for all $n\in\mathbb{N}$ and $t\ge 0$.

Then $\ell_{\varphi,v}(\mathbb{N}) \hookrightarrow \ell_{\psi,v}(\mathbb{N})$ and the following inequality

$$\|\mathbf{x}\|_{\ell_{\psi,v}(\mathbb{N})} \le 2C\,\|\mathbf{x}\|_{\ell_{\varphi,v}(\mathbb{N})}$$

holds.

Proof. Let $\mathbf{x}\in\ell_{\varphi,v}(\mathbb{N})$ and $\|\mathbf{x}\|_{\ell_{\varphi,v}(\mathbb{N})}\le 1$. By the unit ball property, we have $\sum\limits_{n=1}^{\infty} \varphi\left(n\,,\, |x_n|\, v_n\right) \le 1$. So, by (2.2), we have

$$\sum_{n=1}^{\infty} \psi\left(n, \frac{|x_n|\, v_n}{2C\,\|\mathbf{x}\|_{\ell_{\varphi,v}}}\right) \le \frac{1}{2}\sum_{n=1}^{\infty} \psi\left(n, \frac{|x_n|\, v_n}{C\,\|\mathbf{x}\|_{\ell_{\varphi,v}}}\right)$$
$$\le \frac{1}{2}\left(\sum_{n=1}^{\infty} \varphi\left(n, \frac{|x_n|\, v_n}{\|\mathbf{x}\|_{\ell_{\varphi,v}}}\right) + \sum_{n=1}^{\infty} |h_n|\right) \le 1.$$

Thus, by Definition 2.3, we have $\|\mathbf{x}\|_{\ell_{\psi,v}} \le 2C\,\|\mathbf{x}\|_{\ell_{\varphi,v}}$.

This completes the proof of Lemma 2.3.

From Lemma 2.2 and Lemma 2.3, we have the following

Theorem 2.1. *Let μ be the counting measure on $\mathbb{N}$ and $\varphi,\, \psi\in\Phi(\mathbb{N},\mu)$. Suppose that $v=\{v_n\}_{n=1}^{\infty}$ is a weight function defined on $\mathbb{N}$. Let $\psi(n,t)$ be finite on $[0,\infty)$ for all $n\in\mathbb{N}$. Suppose that there exist constants $C_1, C_2>0$ such that conditions (2.1) and (2.2) of Lemma 2.2 and Lemma 2.3 are satisfied, respectively.*

Then $\ell_{\varphi,v}(\mathbb{N}) = \ell_{\psi,v}(\mathbb{N})$ and the following inequality

$$\frac{1}{C_1}\,\|\mathbf{x}\|_{\ell_{\varphi,v}(\mathbb{N})} \le \|\mathbf{x}\|_{\ell_{\psi,v}(\mathbb{N})} \le 2C_2\,\|\mathbf{x}\|_{\ell_{\varphi,v}(\mathbb{N})}.$$

holds.

Let $1\le p<\infty$ and $\varphi\in\Phi(\mathbb{N},\mu)$. Suppose that there exists $C_1>0$ such that

$$\varphi\left(n, \frac{t}{C_1}\right) \le t^p \tag{2.3}$$

for all $n\in\mathbb{N}$ and $0\le t\le 1$. Also, suppose that there exist $C_2>0$ and $\mathbf{h}=\{h_n\}_{n=1}^{\infty}\in\ell_1(\mathbb{N})$ with $\|\mathbf{h}\|_{\ell_1}\le 1$ such that

$$\left(\frac{t}{C_2}\right)^p \le \varphi(n,t) + h_n \tag{2.4}$$

for all $n\in\mathbb{N}$ and $t\ge 0$.

Corollary 2.1. *Let $1\le p<\infty$, and let $\psi(n,t)=t^p$ and $\varphi\in\Phi(\mathbb{N},\mu)$. Suppose that $v=\{v_n\}_{n=1}^{\infty}$ is a weight function defined on $\mathbb{N}$. Assume that there exist the constants $C_1, C_2>0$ such that conditions (2.3) and (2.4) are satisfied, respectively.*

Then $\ell_{\varphi,v}(\mathbb{N}) = \ell_v^p(\mathbb{N})$ *and the following inequality*

$$\frac{1}{C_1}\,\|\mathbf{x}\|_{\ell_{\varphi,v}(\mathbb{N})} \le \|\mathbf{x}\|_{\ell_v^p(\mathbb{N})} \le 2C_2\,\|\mathbf{x}\|_{\ell_{\varphi,v}(\mathbb{N})}$$

holds.

Corollary 2.2. *Let* $1 \le p_n \le q_n \le \overline{q} < \infty$ *and* $\Omega_1 = \{n \in \mathbb{N} : p_n < q_n\}$ *and* $\Omega_2 = \{n \in \mathbb{N} : p_n = q_n\}$. *Assume that* $\frac{1}{r_n} = \frac{1}{p_n} - \frac{1}{q_n}$ *and let* $\|1\|_{\ell^{r_n}(\mathbb{N})} < \infty$. *Let* $\psi(n,t) = t^{p_n}$ *and* $\varphi(n,t) = t^{q_n}$. *Suppose that* $v = \{v_n\}_{n=1}^{\infty}$ *is a weight function defined on* $\mathbb{N}$.

Then $\ell_v^{p_n}(\mathbb{N}) = \ell_v^{q_n}(\mathbb{N})$ *and the following inequality*

$$\|\mathbf{x}\|_{\ell_v^{p_n}(\mathbb{N})} \le \|\mathbf{x}\|_{\ell_v^{q_n}(\mathbb{N})} \le \Big(A + B + \|\chi_{\Omega_2}\|_{\ell^\infty(\mathbb{N})}\Big)^{\frac{1}{p}}\ \|1\|_{\ell^{r_n}(\mathbb{N})}\,\|\mathbf{x}\|_{\ell_v^{p_n}(\mathbb{N})}$$

holds, where $A = \sup\limits_{n\in\Omega_1} \frac{p_n}{q_n}$ *and* $B = \sup\limits_{n\in\Omega_1} \frac{q_n - p_n}{q_n}$.

Remark 2.2. Note that the embeddings between the variable Lebesgue sequence spaces were proved by A. Nekvinda in [28]. The next step in the development of the embeddings between variable Lebesgue spaces with measure was studied in [2] and in monograph [8].

We need the following

Lemma 2.4. *Let* $1 \le q < \infty$, $B_n = \sum\limits_{k=n}^{\infty} \left(\frac{\omega_k}{k}\right)^q$, *and let* $\omega_n > 0$ *for all* $n \in \mathbb{N}$. *Then the following statements*

(a) *If* $0 < \gamma < 1$, *then*

$$\gamma\, B_n^{\gamma-1}\left(\frac{\omega_n}{n}\right)^q \le B_n^\gamma - B_{n+1}^\gamma \le \gamma\, B_{n+1}^{\gamma-1}\left(\frac{\omega_n}{n}\right)^q \quad \textit{for all } n \in \mathbb{N};$$

(b) *If* $\gamma < 0$ *or* $\gamma \ge 1$, *then*

$$\gamma\, B_{n+1}^{\gamma-1}\left(\frac{\omega_n}{n}\right)^q \le B_n^\gamma - B_{n+1}^\gamma \le \gamma\, B_n^{\gamma-1}\left(\frac{\omega_n}{n}\right)^q \quad \textit{for all } n \in \mathbb{N}.$$

hold.

Proof. Let $f(x) = x^\gamma$ and $0 < \gamma < 1$. Then by the mean value theorem, we have

$$\gamma\, x^{\gamma-1}(x-y) \le x^\gamma - y^\gamma \le \gamma\, y^{\gamma-1}(x-y) \ \text{ for } \ 0 < y < x. \tag{2.5}$$

By inequality (2.5), we get

$$\gamma\, B_n^{\gamma-1}\,(B_n - B_{n+1}) \le B_n^\gamma - B_{n+1}^\gamma \le \gamma\, B_{n+1}^{\gamma-1}\,(B_n - B_{n+1})\,.$$

So, we have

$$B_n^{\gamma-1}\left(\frac{\omega_n}{n}\right)^q \le \frac{1}{\gamma}\left(B_n^\gamma - B_{n+1}^\gamma\right) \le B_{n+1}^{\gamma-1}\left(\frac{\omega_n}{n}\right)^q.$$

It is obvious that

$$\sum_{m=n}^{\infty} B_m^{\gamma-1}\left(\frac{\omega_m}{m}\right)^q \le \frac{1}{\gamma}\sum_{m=n}^{\infty}\left(B_m^\gamma - B_{m+1}^\gamma\right) = \frac{1}{\gamma}\,B_n^\gamma. \tag{2.6}$$

In a similar way, we can prove statements (b). Therefore we omit the proof.

3. Precompactness in $\ell_{\varphi,v}(\mathbb{N})$

In this section, we give a characterization of relatively compact sets in $\ell_{\varphi,v}(\mathbb{N})$.

Theorem 3.1. *Suppose that* $v = \{v_n\}_{n=1}^{\infty}$ *is a weight function defined on* $\mathbb{N}$. *Let* $\mathcal{F} \subset \ell_{\varphi,v}(\mathbb{N})$ *and let* $\varphi(n,t) < \infty$ *for all* $n \in \mathbb{N}$ *and* $t > 0$. *Then the set* $\mathcal{F} = \left\{a^i\right\}_{i\in I}$ *is precompact in* $\ell_{\varphi,v}(\mathbb{N})$ *if and only if the following conditions are satisfied:*

(i) $\mathcal{F}$ *is bounded, i.e., there exists* $M > 0$ *such that for every* $a^i \in \mathcal{F}$

$$\left\|a^i\right\|_{\ell_{\varphi,v}(\mathbb{N})} \le M,$$

(ii) *for every* $\varepsilon > 0$, *there exists some* $n \in \mathbb{N}$ *such that for every* $a^i \in \mathcal{F}$

$$\left\| a_k^i \right\|_{\ell_{\varphi,v}(k>n)} < \varepsilon.$$

Proof. Let $\mathcal{F} \subset \ell_{\varphi,v}(\mathbb{N})$ and let conditions (i)–(ii) be satisfied. Let us fix $\varepsilon > 0$. We choose $n \in \mathbb{N}$ from condition (ii) for $\frac{\varepsilon}{3}$ and define a mapping $\Phi : \mathcal{F} \mapsto \mathbb{R}^n$ by

$$\Phi\left(a^i\right) = \left(a_1^i, \ldots, a_n^i\right).$$

We notice that the boundedness of $\mathcal{F}$ implies the boundedness $\Phi[\mathcal{F}]$. Moreover, $\Phi[\mathcal{F}]$ is totally bounded, since $\Phi[\mathcal{F}] \subset \mathbb{R}^n$.

Let $a^i = \left\{a_k^i\right\}_{k=1}^{\infty}$, $b^i = \left\{b_k^i\right\}_{k=1}^{\infty} \in \mathcal{F}$ with

$$\left|\Phi\left(a^i\right) - \Phi\left(b^i\right)\right|_{\phi,v} = \left\| a_k^i - b_k^i \right\|_{\ell_{\varphi,v}(k\le n)} < \frac{\varepsilon}{3}.$$

So, we have

$$\left\| a^i - b^i \right\|_{\ell_{\varphi,v}(\mathbb{N})} \le \left\| a_k^i - b_k^i \right\|_{\ell_{\varphi,v}(k\le n)} + \left\| a_k^i - b_k^i \right\|_{\ell_{\varphi,v}(k>n)}$$
$$\le \left|\Phi\left(a^i\right) - \Phi\left(b^i\right)\right|_{\phi,v} + \left\| a_k^i \right\|_{\ell_{\varphi,v}(k>n)} + \left\| b_k^i \right\|_{\ell_{\varphi,v}(k>n)} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Thus, by Lemma 2.1, $\mathcal{F}$ is totally bounded.

Conversely, suppose that the family F is a totally bounded subset of $\ell_{\varphi,v}(\mathbb{N})$. We show that conditions (i) and (ii) are satisfied.

(i) The existence of a finite ε-cover for $\mathcal{F}$, for any $\varepsilon > 0$, implies clearly the boundedness of $\mathcal{F}$.

(ii) We fix $\varepsilon > 0$ and choose $\left\{a^1, a^2, \ldots, a^m\right\} \subset \mathcal{F}$, an $\frac{\varepsilon}{2}$-net of $\mathcal{F}$. For any $j = 1, \ldots, m$, we choose $n_j \in \mathbb{N}$ satisfying the inequality

$$\left\| a_k^j \right\|_{\ell_{\varphi,v}(k>n_j+1)} \le \frac{\varepsilon}{2}.$$

Let $n = \max\left\{n_j : \ j = 1, \ldots, m\right\}$. For any fixed $a^0 \in \mathcal{F}$, there exists a^j such that

$$\left\| a_k^0 - a_k^j \right\|_{\ell_{\varphi,v}(\mathbb{N})} \le \frac{\varepsilon}{2}.$$

Thus we have

$$\left\| a_k^0 \right\|_{\ell_{\varphi,v}(k>n)} \le \left\| a_k^0 - a_k^j \right\|_{\ell_{\varphi,v}(k\le n)} + \left\| a_k^j \right\|_{\ell_{\varphi,v}(k>n)} < \varepsilon.$$

This completes the proof of Theorem 3.1.

Corollary 3.1 ([3]). *Suppose that* $v = \{v_n\}_{n=1}^{\infty}$ *is a weight function defined on* $\mathbb{N}$. *Let* $1 \le p_n \le \overline{p} < \infty$ *and* $\varphi(n,t) = t^{p_n}$ *for all* $n \in \mathbb{N}$ *and* $t \ge 0$. *Then the set* $\mathcal{F} = \left\{a^i\right\}_{i\in I} \subset \ell_v^{p_n}(\mathbb{N})$ *is precompact in* $\ell_v^{p_n}(\mathbb{N})$ *if and only if the following conditions are satisfied:*

(i) $\mathcal{F}$ *is bounded, i.e., there exists* $M > 0$ *such that for every* $a^i \in \mathcal{F}$

$$\sum_{n=1}^{\infty} \left(\left|a_n^i\right| v_n\right)^{p_n} \le M;$$

(ii) *for every* $\varepsilon > 0$, *there exists some* $k \in \mathbb{N}$ *such that for every* $a^i \in \mathcal{F}$

$$\sum_{n=k+1}^{\infty} \left(\left|a_n^i\right| v_n\right)^{p_n} < \varepsilon.$$

Remark 3.1. Note that Corollary 3.1 in the case $v = 1$ was proved in [15]. Also, for $p_n = p = \text{const}$, $n \in \mathbb{N}$ and $v = 1$, Corollary 3.1 was proved in [16].

4. Applications

Let $\{x_n\}_{n=1}^{\infty} \in \ell_{v_n}^{p_n}(\mathbb{N})$ be an arbitrary sequence of real numbers. Suppose that $H_n = \frac{1}{n}\sum\limits_{k=1}^{n} x_k$ and $H_n^{\star} = \sum\limits_{k=n}^{\infty} \frac{x_k}{k}$.

In this section we give a compactness characterization of the discrete Hardy operator from $\ell_{\phi,v}(\mathbb{N})$ into $\ell_{\psi,w}(\mathbb{N})$.

First,the we give the boundedness characterization of the discrete Hardy operator on the weighted Musielak–Orlicz sequence spaces.

Theorem 4.1 ([5])**.** *Let $1 < p < \infty$ and $0 < \alpha < 1$. Let μ be a counting measure on $\mathbb{N}$ and let $\psi\left(n, t^{\frac{1}{p}}\right) \in \Phi(\mathbb{N},\mu)$. Let $\varphi \in \Phi(\mathbb{N},\mu)$ be a function satisfying conditions* (2.3) *and* (2.4)*, respectively. Suppose that $v = \{v_n\}_{n=1}^{\infty}$ and $\omega = \{\omega_n\}_{n=1}^{\infty}$ are the sequences of positive numbers. Then the inequality*

$$\|H_n\|_{\ell_{\psi,\omega}(\mathbb{N})} \le C\|\mathbf{x}\|_{\ell_{\varphi,v}(\mathbb{N})} \tag{4.1}$$

holds if and only if

$$R(\alpha) = \sup_{k\ge 1}\left(\sum_{n=1}^{k} v_n^{-p'}\right)^{\frac{\alpha}{p'}} \left\| \frac{\left(\sum\limits_{m=1}^{n} v_m^{-p'}\right)^{\frac{1-\alpha}{p'}}}{n} \right\|_{\ell_{\psi,\omega}(n\ge k)} < \infty. \tag{4.2}$$

Moreover, if $C > 0$ is the best possible constant in (4.1)*, then*

$$\frac{1}{C_1}\sup_{0<\alpha<1}\left(\frac{\alpha(p-1)}{\alpha(p-1)+1}\right)^{\frac{1}{p}} R(\alpha) \le C \le 2\,C_2 \inf_{0<\alpha<1}\frac{R(\alpha)}{(1-\alpha)^{\frac{1}{p'}}}.$$

Theorem 4.2. *Let $1 < p < \infty$ and $0 < \alpha < 1$. Let μ be the counting measure on $\mathbb{N}$ and let $\psi\left(n, t^{\frac{1}{p}}\right) \in \Phi(\mathbb{N},\mu)$. Let $\varphi \in \Phi(\mathbb{N},\mu)$ be the function satisfying conditions* (2.3) *and* (2.4)*, respectively. Suppose that $v = \{v_n\}_{n=1}^{\infty}$ and $\omega = \{\omega_n\}_{n=1}^{\infty}$ are the sequences of positive numbers.*

Then H_n is compact from $\ell_{\varphi,v}(\mathbb{N})$ into $\ell_{\psi,\omega}(\mathbb{N})$ if and only if

$$\lim_{k\to\infty}\left(\sum_{n=1}^{k} v_n^{-p'}\right)^{\frac{\alpha}{p'}} \left\| \frac{\left(\sum\limits_{m=1}^{n} v_m^{-p'}\right)^{\frac{1-\alpha}{p'}}}{n} \right\|_{\ell_{\psi,\omega}(n\ge k)} = 0. \tag{4.3}$$

Proof. Sufficiency. Let condition (4.3) be provided. Then condition (4.2) of Theorem 4.1 is valid. Therefore, by Theorem 4.1, the operator H_n is bounded from $\ell_{\varphi,v}(\mathbb{N})$ into $\ell_{\psi,\omega}(\mathbb{N})$. Assume that $\mathcal{A} = \{f^m\}_{m\in I} \subset \ell_{\varphi,v}(\mathbb{N})$ and $M = \sup\limits_{m\ge 1}\|f_m\|_{\ell_{\varphi,v}(\mathbb{N})}$. Let us show that the set $\{H_n f_m\}$ is precompact in $\ell_{\psi,\omega}(\mathbb{N})$. By Theorem 4.1, we have

$$\|H_n f_m\|_{\ell_{\psi,\omega}(\mathbb{N})} \le C\ \|f_m\|_{\ell_{\varphi,v}(\mathbb{N})} \le C\,M.$$

So, the set $\{H_n f_m\}$ is uniformly bounded in $\ell_{\psi,\omega}(\mathbb{N})$.

For $s > 1$, we set $\overline{\omega}_s = \{\overline{\omega}_{s,i}\}_{i=1}^{\infty}$, $\overline{\omega}_{s,i} = \begin{cases} 0, & \text{if} \quad 1 \le i \le s-1, \\ \omega_i, & \text{if} \quad i \ge s. \end{cases}$

By Theorem 4.1, the inequality

$$\|H_n\|_{\ell_{\psi,\overline{\omega}_s}(\mathbb{N})} = \|H_n\|_{\ell_{\psi,\omega}(n\ge s)} \le C\|f\|_{\ell_{\varphi,v}(\mathbb{N})} \tag{4.4}$$

holds if and only if $R_s(\alpha)<\infty$, where

$$R_s(\alpha)=\sup_{k\ge s}\left(\sum_{n=1}^{k} v_n^{-p'}\right)^{\frac{\alpha}{p'}}\left\|\frac{\left(\sum\limits_{m=1}^{n} v_m^{-p'}\right)^{\frac{1-\alpha}{p'}}}{n}\right\|_{\ell_{\psi,\omega}(n\ge k)}=\sup_{k\ge s}F_k(\alpha).$$

Obviously, if $C>0$ is the best possible constant in (4.4), then $C\le 2\,C_2\,\frac{R_s(\alpha)}{(1-\alpha)^{\frac{1}{p'}}}$ for all $0<\alpha<1$. Next, by (4.4), we have

$$\sup_{\|f_m\|_{\ell_{\phi,v}(\mathbb{N})}\le M}\|H_nf_m\|_{\ell_{\psi,\overline{\omega}_s}(n\ge s)}=\sup_{\|f_m\|_{\ell_{\phi,v}(\mathbb{N})}\le M}\|\overline{\omega}_{s,n}\,H_nf_m\|_{\ell_{\psi}(\mathbb{N})}$$
$$\le C\sup_{\|f_m\|_{\ell_{\phi,v}(\mathbb{N})\le M}}\|f_m\|_{\ell_{\phi,v}(\mathbb{N})}\le 2\,M\,C_2\,\frac{R_s(\alpha)}{(1-\alpha)^{\frac{1}{p'}}}\quad\text{for all }0<\alpha<1. \tag{4.5}$$

So, by (4.5), we get

$$\lim_{s\to\infty}\left(\sup_{\|f_m\|_{\ell_{\phi,v}(\mathbb{N})}\le M}\|\omega_{s,n}\,H_nf_m\|_{\ell_{\psi}(n\ge s)}\right)$$
$$\le 2\,M\,C_2\,\frac{1}{(1-\alpha)^{\frac{1}{p'}}}\lim_{s\to\infty}R_s(\alpha)$$
$$=2\,M\,C_2\,\frac{1}{(1-\alpha)^{\frac{1}{p'}}}\lim_{s\to\infty}\sup_{k\ge s}F_k(\alpha)$$
$$=2\,M\,C_2\,\frac{1}{(1-\alpha)^{\frac{1}{p'}}}\overline{\lim_{s\to\infty}}F_s(\alpha)$$
$$=2\,M\,C_2\,\frac{1}{(1-\alpha)^{\frac{1}{p'}}}\lim_{s\to\infty}F_s(\alpha)=0.$$

Necessity. Let the operator H_n be compact from $\ell_{\phi,v}(\mathbb{N})$ into $\ell_{\psi,\omega}(\mathbb{N})$. For $s\ge 1$, we introduce the following sequence $f_s=\{f_{s,j}\}_{j=1}^{\infty}$,

$$f_{s,j}=\begin{cases} v_j^{-p'}, & \text{if}\quad 1\le j\le s,\\ 0, & \text{if}\quad j>s.\end{cases}$$

Let $g_s=\left\{\frac{f_{s,j}}{\|f_s\|_{\ell_{\phi,v}(\mathbb{N})}}\right\}_{j=1}^{\infty}$. It is obvious that $\|g_s\|_{\ell_{\phi,v}(\mathbb{N})}=1$ for all $1\le j\le s$. Since the operator H_n is compact from $\ell_{\phi,v}(\mathbb{N})$ into $\ell_{\psi,\omega}(\mathbb{N})$, this implies that the set $\left\{H_n\varphi,\ \|\varphi\|_{\ell_{\phi,v}(\mathbb{N})=1}\right\}$ is precompact in $\ell_{\psi,\omega}(\mathbb{N})$. Thus, by the criteria on the precompactness of the sets in $\ell_{\psi,\omega}(\mathbb{N})$ (Theorem 3.1), we have

$$\lim_{s\to\infty}\left(\sup_{\|\varphi\|_{\ell_{\phi,v}(\mathbb{N})}=1}\|\omega_n\,H_n\varphi\|_{\ell_{\psi}(n\ge s)}\right)=0. \tag{4.6}$$

By Corollary 2.1, we have

$$\sup_{\|\varphi\|_{\ell_{\phi,v}(\mathbb{N})}=1}\|\omega_n\,H_n\varphi\|_{\ell_{\psi}(n\ge s)}\ge\|\omega_n\,H_ng_s\|_{\ell_{\psi}(n\ge s)}=\left\|\frac{\omega_n}{n}\,\frac{\sum\limits_{j=1}^{n}f_{s,j}}{\|f_s\|_{\ell_{\phi,v}(\mathbb{N})}}\right\|_{\ell_{\psi}(n\ge s)}$$

$$\geq \frac{1}{C_1}\left\|\frac{\omega_n}{n}\frac{\sum_{j=1}^{n} f_{s,j}}{\|f_s\|_{\ell^p_v(\mathbb{N})}}\right\|_{\ell_\psi(n\geq s)} = \frac{1}{C_1}\left\|\frac{\omega_n}{n}\frac{\sum_{j=1}^{n} v_j^{-p'}}{\left(\sum_{j=1}^{s} v_j^{-p'}\right)^{\frac{1}{p}}}\right\|_{\ell_\psi(n\geq s)}$$

$$= \frac{1}{C_1}\left\|\frac{\omega_n}{n}\left(\frac{\sum_{j=1}^{n} v_j^{-p'}}{\left(\sum_{j=1}^{s} v_j^{-p'}\right)^{\frac{1}{p}}}\right)^{\alpha}\left(\frac{\sum_{j=1}^{n} v_j^{-p'}}{\left(\sum_{j=1}^{s} v_j^{-p'}\right)^{\frac{1}{p}}}\right)^{1-\alpha}\right\|_{\ell_\psi(n\geq s)}$$

$$\geq \frac{1}{C_1}\left(\frac{\left(\sum_{j=1}^{s} v_j^{-p'}\right)}{\left(\sum_{j=1}^{s} v_j^{-p'}\right)^{\frac{1}{p}}}\right)^{\alpha}\left\|\frac{\omega_n}{n}\left(\frac{\left(\sum_{j=1}^{n} v_j^{-p'}\right)}{\left(\sum_{j=1}^{n} v_j^{-p'}\right)^{\frac{1}{p}}}\right)^{1-\alpha}\right\|_{\ell_\psi(n\geq s)}$$

$$= \frac{1}{C_1}\left(\sum_{j=1}^{s} v_j^{-p'}\right)^{\frac{\alpha}{p'}}\left\|\frac{\left(\sum_{j=1}^{n} v_j^{-p'}\right)^{\frac{1-\alpha}{p'}}}{n}\right\|_{\ell_{\psi,\omega}(n\geq s)}. \tag{4.7}$$

So, by (4.6) and (4.7), we have condition (4.3). □

A similar theorem holds for the dual operator of the discrete Hardy operator.

Theorem 4.3. *Let $1 < p < \infty$ and $0 < \beta < 1$. Let μ be the counting measure on $\mathbb{N}$ and let $\psi\left(n, t^{\frac{1}{p}}\right) \in \Phi(\mathbb{N},\mu)$. Let $\varphi \in \Phi(\mathbb{N},\mu)$ be the function satisfying conditions* (2.3) *and* (2.4), *respectively. Suppose that $v = \{v_n\}_{n=1}^{\infty}$ and $\omega = \{\omega_n\}_{n=1}^{\infty}$ are the sequences of positive numbers.*

Then $H_n^{\star}$ is compact from $\ell_{\varphi,v}(\mathbb{N})$ into $\ell_{\psi,\omega}(\mathbb{N})$ if and only if

$$\lim_{k\to\infty}\left(\sum_{n=k}^{\infty}\frac{v_n^{-p'}}{n^{p'}}\right)^{\frac{\beta}{p'}}\left\|\left(\sum_{m=n}^{\infty}\frac{v_m^{-p'}}{m^{p'}}\right)^{\frac{1-\beta}{p'}}\right\|_{\ell_{\psi,\omega}(n\leq k)} = 0. \tag{4.8}$$

In particular, we have the following corollaries.

Corollary 4.1 ([4])**.** *Let $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ be the sequences of real numbers such that $1 < \underline{p} \leq q_n \leq \overline{q} < \infty$, and let $\varphi(n,t) = t^{p_n}$ and $\psi(n,t) = t^{q_n}$ for all $n \in \mathbb{N}$ and $t \geq 0$. Suppose that $\frac{1}{r_n} = \frac{1}{\underline{p}} - \frac{1}{p_n}$ for all $n \in \mathbb{N}$ and let $\|1\|_{\ell^{r_n}(\mathbb{N})} < \infty$. Let $\omega = \{\omega_n\}_{n=1}^{\infty}$ and $v = \{v_n\}_{n=1}^{\infty}$ be the sequences of positive numbers.*

Then H_n is compact from $\ell_v^{p_n}(\mathbb{N})$ into $\ell_\omega^{q_n}(\mathbb{N})$ if and only if

$$\lim_{k\to\infty}\left(\sum_{n=1}^{k} v_n^{-\underline{p}'}\right)^{\frac{1}{\underline{q}\,\underline{p}'}}\left\|\frac{\left(\sum_{m=1}^{n} v_m^{-\underline{p}'}\right)^{\frac{1}{\underline{p}'\,\underline{q}'}}}{n}\right\|_{\ell_\omega^{q_n}(n\geq k)} = 0.$$

Corollary 4.2 ([4])**.** *Let $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ be the sequences of real numbers such that $1 < \underline{p} \leq q_n \leq \overline{q} < \infty$ and let $\varphi(n,t) = t^{p_n}$ and $\psi(n,t) = t^{q_n}$ for all $n \in \mathbb{N}$ and $t \geq 0$. Suppose that $\frac{1}{r_n} = \frac{1}{\underline{p}} - \frac{1}{p_n}$*

for all $n \in \mathbb{N}$ and let $\|1\|_{\ell^{r_n}(\mathbb{N})} < \infty$. Let $\omega = \{\omega_n\}_{n=1}^{\infty}$ and $v = \{v_n\}_{n=1}^{\infty}$ be the sequences of positive numbers.

Then $H_n^\star$ is compact from $\ell_v^{p_n}(\mathbb{N})$ into $\ell_\omega^{q_n}(\mathbb{N})$ if and only if

$$\lim_{k\to\infty}\left(\sum_{n=k}^{\infty}\frac{v_n^{-p'}}{n^{p'}}\right)^{\frac{1}{q\,p'}}\left\|\left(\sum_{m=n}^{\infty}\frac{v_m^{-p'}}{m^{p'}}\right)^{\frac{1}{p'\,q'}}\right\|_{\ell_\omega^{q_n}(n\le k)}=0.$$

In the case of classical weighted Lebesgue sequence spaces, we have the following lemma.

Lemma 4.1. *Let $1 < p \le q < \infty$, $\varphi(n,t) = t^p$ and let $\psi(n,t) = t^q$ for all $n \in \mathbb{N}$. Suppose $\{\omega_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are sequences of nonnegative numbers. Let $\{x_n\}_{n=1}^{\infty} \in \ell_v^p(\mathbb{N})$ be an arbitrary sequence of real numbers. Then condition (4.3) is equivalent to the condition*

$$\lim_{k\to\infty}\left(\sum_{n=1}^{k}v_n^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{n=k}^{\infty}\left(\frac{\omega_n}{n}\right)^q\right)^{\frac{1}{q}}=0. \tag{4.9}$$

Proof. Assume that condition (4.3) holds. Let $M_k = \left(\sum\limits_{n=1}^{k}v_n^{-p'}\right)^{\frac{1}{p'}}\left(\sum\limits_{n=k}^{\infty}\left(\frac{\omega_n}{n}\right)^q\right)^{\frac{1}{q}}$. It is obvious that

$$F_k(\alpha)=\left(\sum_{n=1}^{k}v_n^{-p'}\right)^{\frac{\alpha}{p'}}\left\|\frac{\left(\sum\limits_{m=1}^{n}v_m^{-p'}\right)^{\frac{1-\alpha}{p'}}}{n}\right\|_{\ell_\omega^q(n\ge k)}\ge\left(\sum_{n=1}^{k}v_n^{-p'}\right)^{\frac{\alpha}{p'}}\left(\sum_{m=1}^{k}v_m^{-p'}\right)^{\frac{1-\alpha}{p'}}\left\|\frac{1}{n}\right\|_{\ell_\omega^q(n\ge k)}$$

$$=\left(\sum_{n=1}^{k}v_n^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{n=k}^{\infty}\left(\frac{\omega_n}{n}\right)^q\right)^{\frac{1}{q}}=M_k.$$

Conversely, suppose that condition (4.9) is satisfied. So, we have $M = \sup\limits_{k\ge1} M_k < \infty$. Therefore we get

$$\left(\sum_{n=1}^{k}v_n^{-p'}\right)^{\frac{1}{p'}}\le M\left(\sum_{n=k}^{\infty}\left(\frac{\omega_n}{n}\right)^q\right)^{-\frac{1}{q}}\quad\text{for all}\quad k\in\mathbb{N}.$$

Next, by inequality (2.6), we have

$$\left\|\frac{\left(\sum\limits_{m=1}^{n}v_m^{-p'}\right)^{\frac{1-\alpha}{p'}}}{n}\right\|_{\ell_\omega^q(n\ge k)}\le M^{1-\alpha}\left\|\frac{\left(\sum\limits_{m=n}^{\infty}\left(\frac{\omega_m}{m}\right)^q\right)^{-\frac{1-\alpha}{q}}}{n}\right\|_{\ell_\omega^q(n\ge k)}$$

$$=M^{1-\alpha}\left(\sum_{n=k}^{\infty}\left(\sum_{m=n}^{\infty}\left(\frac{\omega_m}{m}\right)^q\right)^{\alpha-1}\left(\frac{\omega_n}{n}\right)^q\right)^{\frac{1}{q}}\le\frac{1}{\alpha^{1/q}}M^{1-\alpha}\left(\sum_{n=k}^{\infty}\left(\frac{\omega_n}{n}\right)^q\right)^{\frac{\alpha}{q}} \tag{4.10}$$

for all $k \in \mathbb{N}$. Thus, by (4.10), we have

$$\left(\sum_{n=1}^{k}v_n^{-p'}\right)^{\frac{\alpha}{p'}}\left\|\frac{\left(\sum\limits_{m=1}^{n}v_m^{-p'}\right)^{\frac{1-\alpha}{p'}}}{n}\right\|_{\ell_\omega^q(n\ge k)}\le\frac{1}{\alpha^{1/q}}\,M^{1-\alpha}\left(\sum_{n=1}^{k}v_n^{-p'}\right)^{\frac{\alpha}{p'}}\left(\sum_{n=k}^{\infty}\left(\frac{\omega_n}{n}\right)^q\right)^{\frac{\alpha}{q}}$$

$$=\frac{1}{\alpha^{1/q}}\,M^{1-\alpha}\,M_k^{\alpha}$$

for all $k \in \mathbb{N}$. Finally, we have

$$M_k \leq F_k(\alpha) \leq \frac{1}{\alpha^{1/q}} M^{1-\alpha} M_k^{\alpha} \text{ for all } k \geq 1 \text{ and } 0 < \alpha < 1.$$

This completes the proof of Lemma 4.1.

Similarly, we can prove the following

Lemma 4.2. *Let $1 < p \leq q < \infty$, $\varphi(n,t) = t^p$ and let $\psi(n,t) = t^q$ for all $n \in \mathbb{N}$. Suppose $\{\omega_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are the sequences of nonnegative numbers. Let $\{x_n\}_{n=1}^{\infty} \in \ell_v^p(\mathbb{N})$ be an arbitrary sequence of real numbers. Then condition (4.8) is equivalent to the condition*

$$\lim_{k\to\infty} \left(\sum_{n=k}^{\infty} \frac{v_n^{-p'}}{n^{p'}} \right)^{\frac{1}{p'}} \left(\sum_{n=1}^{k} \omega_n \right)^{\frac{1}{q}} = 0.$$

Remark 4.1. In the case of classical weighted Lebesgue sequence spaces, the problem of the compactness of a class of matrix operators was considered in [29], etc. In particular, a class of matrix operators includes the discrete Hardy operator.

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