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ON THE SPACE OF GENERALIZED THETA-SERIES FOR CERTAIN QUADRATIC FORMS OF SIX VARIABLES

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Abstract. The vector spaces of generalized theta-series with spherical polynomials of order ν, corresponding to certain diagonal and non-diagonal quadratic forms in six variables, are considered. An upper bound on the dimension of these spaces is established.

1. Introduction

Let

$$Q(X)=Q(x_1,x_2,\dots,x_r)=\sum_{1\le i\le j\le r} b_{ij}x_ix_j$$

be an integer positive definite quadratic form of r variables and let $A=(a_{ij})$ be the symmetric $r\times r$ matrix of the quadratic form $Q(X)$, where $a_{ii}=2b_{ii}$ and $a_{ij}=a_{ji}=b_{ij}$, for $i<j$. If $X=(x_1\cdots x_r)^T$ denotes a column matrix and X^T its transpose, then $Q(X)=\frac{1}{2}X^TAX$. Let A_{ij} denote the cofactor to the element a_{ij} in A and let $A^{-1}=(a^*_{ij})^r_{ij=1}$ be the inverse matrix.

A homogeneous polynomial $P(X)=P(x_1,\dots,x_r)$ of degree ν with complex coefficients, satisfying the condition

$$\sum_{1\le i,j\le r} a^*_{ij}\Big(\frac{\partial^2 P}{\partial x_i\partial x_j}\Big)=0 \tag{1.1}$$

is called a spherical polynomial of order ν with respect to $Q(X)$ (see [5]).

Let $\mathcal{P}(\nu,Q)$ denote the vector space over $\mathbb{C}$ of spherical polynomials $P(X)$ of even order ν with respect to $Q(X)$.

Hecke [6] calculated the dimension of the space $\mathcal{P}(\nu,Q)$ and showed that

$$\dim\mathcal{P}(\nu,Q)=\binom{\nu+r-1}{r-1}-\binom{\nu+r-3}{r-1}.$$

He formed a basis for the space of spherical polynomials of order two ($\nu=2$) with respect to $Q(X)$.

Lomadze [7] constructed a basis of the space of spherical polynomials of order four ($\nu=4$) with respect to $Q(X)$. In the next section a basis of the space $\mathcal{P}(\nu,Q)$ is constructed in a simpler way.

Let

$$\vartheta(\tau,P,Q)=\sum_{n\in\mathbb{Z}^r} P(n)z^{Q(n)},\quad z=e^{2\pi i\tau},\quad \tau\in\mathbb{C},\quad \operatorname{Im}\tau>0$$

be the corresponding generalized r-fold theta-series. Schoeneberg [8] proved that the function $\vartheta(\tau,P,Q)$ is a modular form of weight $-(\frac{r}{2}+\nu)$ with respect to the congruent subgroup $\Gamma_0(N)$, where N is the least positive integer such that NA^{-1} is again an even integral symmetric matrix. The mapping that assigns to each P in $\mathcal{P}(\nu,Q)$ the modular form $\vartheta(\tau,P,Q)$ is a linear transformation.

Let $T(\nu,Q)$ denote the vector space over $\mathbb{C}$ of generalized multiple theta-series, i.e.,

$$T(\nu,Q)=\{\vartheta(\tau,P,Q)\,:\,P\in\mathcal{P}(\nu,Q)\}.$$

Gooding [4,5] calculated the dimension of the vector space $T(\nu,Q)$ for a reduced binary quadratic form Q and obtained an upper bound on the dimension of the space $T(\nu,Q)$ for some diagonal

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quadratic forms of r variables

$$\dim T(\nu,Q)\le\binom{\frac{\nu}{2}+r-2}{r-2}. \tag{1.2}$$

In [9–11], the upper bounds were established for the dimensions of the spaces $T(\nu,Q)$ for certain quadratic forms in three, four, five and r variables. In several cases, the dimensions were calculated and bases for these spaces were constructed.

Gaigalas [1–3] obtained upper bounds for the dimension of the spaces $T(4,Q)$ and $T(6,Q)$ for certain diagonal quadratic forms and presented upper bounds for the dimension of the spaces $T(\nu,Q)$ for some diagonal quadratic forms in six variables.

In this paper, upper bounds are obtained for the dimension of the spaces $T(\nu,Q)$ for certain diagonal and non-diagonal quadratic forms in six variables. The dimension of the space $T(2,Q)$ is calculated, and a basis for this space is constructed.

In the sequel, we use the following definition and results.

An integral $r\times r$ matrix U is called an integral automorphism of the quadratic form $Q(X)$ in r variables if $U^TAU=A$.

Lemma 1.1 ([5, p. 37])**.** *Let $Q(X)=Q(x_1,\dots,x_r)$ be a positive definite quadratic form in r variables and $P(X)=P(x_1,\dots,x_r)\in P(\nu,Q)$. Let G be the set of all integral automorphisms of Q. Suppose*

$$\sum_{i=1}^{t}P(U_iX)=0\quad \text{for some}\quad U_1,\dots,U_t\subseteq G,$$

then $\vartheta(\tau,P,Q)=0$.

Lemma 1.2 ([11, p. 92])**.** *Let $Q_1(X)$ be the non-diagonal quadratic form of r variables, given by $Q_1(X)=b_{11}x_1^2+b_{22}x_2^2+b_{33}x_3^2+\dots+b_{rr}x_r^2+b_{12}x_1x_2$, then*

$$\dim T(\nu,Q_1)\le\frac{1}{r-2}\binom{\frac{\nu}{2}+r-3}{r-3}(\nu+r-2). \tag{1.3}$$

2. Basis of the Space $\mathcal{P}(\nu,Q)$

Let

$$P(X)=P(x_1,x_2,x_3,x_4,x_5,x_6)=\sum_{k=0}^{\nu}\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{l=0}^{j}\sum_{s=0}^{l}a_{kijls}x_1^{\nu-k}x_2^{k-i}x_3^{i-j}x_4^{j-l}x_5^{l-s}x_6^{s}$$

be a spherical function of order ν with respect to the positive quadratic form $Q(x_1,x_2,x_3,x_4,x_5,x_6)$ of six variables and let

$$L=\big(a_{00000},a_{10000},a_{11000},a_{11100},a_{11110},a_{11111},a_{20000},a_{21000},\dots,a_{\nu,\nu,\nu,\nu,\nu}\big)^T$$

be the column vector, where a_{kijls} ($\nu\ge k\ge i\ge j\ge l\ge s\ge 0$) are the coefficients of polynomial $P(X)$.

According to (1.1), the condition

$$\frac{1}{|A|}\sum_{1\le i,j\le 6}A_{ij}\Big(\frac{\partial^2P}{\partial x_i\partial x_j}\Big)=0$$

is satisfied. Considering

$$\frac{\partial^2P}{\partial x_1^2}=\sum_{k=0}^{\nu}\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{l=0}^{j}\sum_{s=0}^{l}(\nu-k)(\nu-k-1)a_{kijls}x_1^{\nu-k-2}x_2^{k-i}x_3^{i-j}x_4^{j-l}x_5^{l-s}x_6^{s}$$

$$=\sum_{k=1}^{\nu-1}\sum_{i=0}^{k-1}\sum_{j=0}^{i}\sum_{l=0}^{j}\sum_{s=0}^{l}(\nu-k+1)(\nu-k)a_{k-1ijls}x_1^{\nu-k-1}x_2^{k-i-1}x_3^{i-j}x_4^{j-l}x_5^{l-s}x_6^{s}$$

and also to obtain similar formulas for other second partial derivatives, condition (1.1) takes the form

$$\frac{1}{|A|}\sum_{k=1}^{\nu-1}\sum_{i=0}^{k-1}\sum_{j=0}^{i}\sum_{l=0}^{j}\sum_{s=0}^{l}\big(A_{11}(\nu-k+1)(\nu-k)a_{k-1ijls}+2A_{12}(\nu-k)(k-i)a_{kijls}$$
$$+2A_{13}(\nu-k)(i-j+1)a_{ki+1jls}+2A_{14}(\nu-k)(j-l+1)a_{ki+1j+1ls}$$
$$+2A_{15}(\nu-k)(l-s+1)a_{ki+1j+1l+1s}+2A_{16}(\nu-k)(s+1)a_{ki+1j+1l+1s+1}$$
$$+A_{22}(k-i+1)(k-i)a_{k+1ijls}+2A_{23}(k-i)(i-j+1)a_{k+1i+1jls}+\cdots$$
$$+A_{66}(s+2)(s+1)a_{k+1i+2j+2l+2s+2}\big)x_1^{\nu-k-1}x_2^{k-i-1}x_3^{i-j}x_4^{j-l}x_5^{l-s}x_6^{s}=0.$$

Thus, for $0\le s\le l\le j\le i<k\le\nu-1$, we obtain

$$A_{11}(\nu-k+1)(\nu-k)a_{k-1ijls}+2A_{12}(\nu-k)(k-i)a_{kijls}$$
$$+2A_{13}(\nu-k)(i-j+1)a_{ki+1jls}+2A_{14}(\nu-k)(j-l+1)a_{ki+1j+1ls}$$
$$+2A_{15}(\nu-k)(l-s+1)a_{ki+1j+1l+1s}+2A_{16}(\nu-k)(s+1)a_{ki+1j+1l+1s+1}$$
$$+A_{22}(k-i+1)(k-i)a_{k+1ijls}+2A_{23}(k-i)(i-j+1)a_{k+1i+1jls}+\cdots$$
$$+A_{66}(s+2)(s+1)a_{k+1i+2j+2l+2s+2}=0.$$

It follows that condition (1.1), in matrix notation has the following form

$$S\cdot L=0,$$

where the matrix S is of the form

$$S=\begin{pmatrix} A_{11}\nu(\nu-1) & 2A_{12}(\nu-1) & 2A_{13}(\nu-1) & 2A_{14}(\nu-1) & \dots & \dots & \dots & 0\\ 0 & A_{11}(\nu-1)(\nu-2) & 0 & 0 & \dots & \dots & \dots & 0\\ 0 & 0 & A_{11}(\nu-1)(\nu-2) & 0 & \dots & \dots & \dots & 0\\ 0 & 0 & 0 & A_{11}(\nu-1)(\nu-2) & \dots & \dots & \dots & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & 0 & \dots & 2A_{11} & \dots & A_{66}(\nu-1)\nu \end{pmatrix}$$

and is the $\binom{\nu+3}{5}\times\binom{\nu+5}{5}$ matrix (the number of rows of the matrix S is equal to the number of (k,i,j,l,s) with $0\le s\le l\le j\le i<k\le\nu-1$, the number of columns is equal to the number of coefficients a_{kijls}, i.e., to the number of (k,i,j,l,s) with $0\le s\le l\le j\le i\le k\le\nu$).

We partition the matrix S into two matrices S_1 and S_2, where S_1 is the left-hand square nondegenerate $\binom{\nu+3}{5}\times\binom{\nu+3}{5}$ matrix. It consists of the first $\binom{\nu+3}{5}$ columns of the matrix S; the matrix S_2 consists of the last $\binom{\nu+5}{5}-\binom{\nu+3}{5}$ columns of the matrix S.

Similarly, we partition the matrix L into two matrices L_1 and L_2, where L_1 is the $\binom{\nu+3}{5}\times 1$ matrix consisting of the upper $\binom{\nu+3}{5}$ elements of L; the matrix L_2 consists of the lower $\binom{\nu+5}{5}-\binom{\nu+3}{5}$ elements of the matrix L.

According to the new notation, the matrix equality takes the form

$$S_1L_1+S_2L_2=0,$$

i.e.,

$$L_1=-S_1^{-1}S_2L_2.$$

It follows from this equality that the matrix L_1 is expressed in terms of the matrix L_2. Consequently, the first $\binom{\nu+3}{5}$ elements of the matrix L can be expressed in terms of its remaining elements. Since the matrix L consists of the coefficients of the spherical polynomial $P(X)$, its first $\binom{\nu+3}{5}$ coefficients can be expressed through its last $\binom{\nu+5}{5}-\binom{\nu+3}{5}$ coefficients.

Hence, the polynomials

$$\begin{aligned}
&P_{\nu-1,0000}(a^{(1)}_{00000}, a^{(1)}_{10000}, \dots, a^{(1)}_{\nu-2,\nu-2,\nu-2,\nu-2,\nu-2}, 1, 0, 0, \dots, 0),\\
&P_{\nu-1,1000}(a^{(2)}_{00000}, a^{(2)}_{10000}, \dots, a^{(2)}_{\nu-2,\nu-2,\nu-2,\nu-2,\nu-2}, 0, 1, 0, \dots, 0),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&P_{\nu,\nu,\nu,\nu,\nu}(a^{(t)}_{00000}, a^{(t)}_{10000}, \dots, a^{(t)}_{\nu-2,\nu-2,\nu-2,\nu-2,\nu-2}, 0, 0, 0, \dots, 1),
\end{aligned} \tag{2.1}$$

form the basis of the space $\mathcal{P}(\nu, Q)$, where the first $\binom{\nu+3}{5}$ coefficients, from a_{00000} to $a_{\nu-2,\nu-2,\nu-2,\nu-2,\nu-2}$, are calculated in terms of the remaining $\binom{\nu+5}{5} - \binom{\nu+3}{5}$ coefficients. The coefficients of the basis polynomial P_{kijls} are given in brackets, where $k = \nu - 1$ or $k = \nu$. Moreover, the polynomial P_{kijls} is such that its coefficient a_{kijls} is equal to 1, and all other coefficients for which k is equal to $\nu - 1$ or ν are equal to 0 .

3. On the Dimension of $T(\nu, Q)$ for Certain Quadratic Forms of Six Variables

Let

$$Q(X) = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + \cdots + b_{rr}x_r^2$$

with $0 < b_{11} < b_{22} < \cdots < b_{rr}$ be a diagonal quadratic form of r variables. We have shown [11, p.91], that the basis of the space $T(\nu, Q)$ is among the theta-series $\vartheta(\tau, P, Q)$ with spherical polynomials $P = P_{kij\dots s}$, where all indices $k = \nu, i, j, \dots s$ are even. Moreover, the dimension of the space $T(\nu, Q)$ satisfies estimate (1.2).

Now, consider the quadratic form of six variables

$$Q_2 = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + b_{44}x_4^2 + b_{55}(x_5^2 + x_6^2),$$

where $0 < b_{11} < b_{22} < b_{33} < b_{44} < b_{55} = b_{66}$. Using (1.2), we have

$$\dim T(\nu, Q_2) \le \binom{\frac{\nu}{2} + r - 2}{r - 2} = \binom{\frac{\nu}{2} + 4}{4}. \tag{3.1}$$

We improve this estimate.

We construct the integral automorphisms U of the quadratic form Q_2. Since

$$\begin{aligned}
&b_{11} = Q_2(\pm1, 0, 0, 0, 0, 0), \quad b_{22} = Q_2(0, \pm1, 0, 0, 0, 0), \quad b_{33} = Q_2(0, 0, \pm1, 0, 0, 0),\\
&b_{44} = Q_2(0, 0, 0, \pm1, 0, 0), \quad b_{55} = b_{66} = Q_2(0, 0, 0, 0, \pm1, 0) = Q_2(0, 0, 0, 0, 0, \pm1),
\end{aligned}$$

it is easy to verify that the integral automorphisms of the quadratic form Q_2 are equal to

$$\begin{pmatrix} e_1 & 0 & 0 & 0 & 0 & 0\\ 0 & e_2 & 0 & 0 & 0 & 0\\ 0 & 0 & e_3 & 0 & 0 & 0\\ 0 & 0 & 0 & e_4 & 0 & 0\\ 0 & 0 & 0 & 0 & e_5 & 0\\ 0 & 0 & 0 & 0 & 0 & e_6 \end{pmatrix}, \quad \begin{pmatrix} e_1 & 0 & 0 & 0 & 0 & 0\\ 0 & e_2 & 0 & 0 & 0 & 0\\ 0 & 0 & e_3 & 0 & 0 & 0\\ 0 & 0 & 0 & e_4 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & e_5\\ 0 & 0 & 0 & 0 & e_6 & 0 \end{pmatrix}, \quad (e_i = \pm1, \quad i = 1, 2, 3, 4, 5, 6).$$

Consider all possible polynomials $P = P_{kijls}(UX)$, where $P_{kijls}(X) \in P(\nu, Q_2)$ are the spherical basis polynomials (2.1) of order ν with respect to $Q_2(X)$, and $U \in G$ is an integral automorphism of the quadratic form $Q_2(X)$.

We have to find which polynomials P satisfy the equality

$$\sum_{U_h} P_{kijls}(U_h X) = 0, \text{ for some } U_h \in G.$$

For such polynomials, according to Lemma 1.1, we have $\vartheta(\tau, P, Q) = 0$.

For example, if $U_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$, then

$$P_{kijls}(X) + P_{kijls}(U_1X) = \sum_{k=0}^{\nu}\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{l=0}^{j}\sum_{s=0}^{l}(1+(-1)^s)a_{kijls}x_1^{\nu-k}x_2^{k-i}x_3^{i-j}x_4^{j-l}x_5^{l-s}x_6^{s}.$$

The equality

$$P_{kijls}(X) + P_{kijls}(U_1X) = 0$$

holds if and only if the condition

$$(1+(-1)^s)\,a_{kijls} = 0$$

is satisfied. This means that the index s of the coefficient equal to one must be odd. Similarly, it follows that if among the last coefficients of P, at least one of the indices k, i, j, l, s of the coefficient (which is equal to one) is odd, then, by Lemma 1.1, for the spherical polynomial $P = P_{kijls}$, the theta-series satisfies the condition $\vartheta(\tau, P, Q) = 0$. Hence, if theta-series are linearly independent, then the indices k, i, j, l, s of the corresponding spherical polynomial P must all be even. According to (3.1), such k, i, j, l, s indices are equal to $\binom{\frac{\nu}{2}+4}{4}$.

For automorphism $U_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$ we have

$$\begin{aligned} P(U_2X) &= \sum_{k=0}^{\nu}\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{l=0}^{j}\sum_{s=0}^{l} a_{kijls}^{(h)}x_1^{\nu-k}x_2^{k-i}x_3^{i-j}x_4^{j-l}x_6^{l-s}x_5^{s} \\ &= \sum_{k=0}^{\nu}\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{l=0}^{j}\sum_{s=0}^{l} a_{kijls}^{(h)}x_1^{\nu-k}x_2^{k-i}x_3^{i-j}x_4^{j-l}x_5^{l-(l-s)}x_6^{l-s}, \end{aligned}$$

whence it follows that if all the last coefficients of the basis polynomial $P(X)$ are zero except for one $a_{kijls}^{(h)} = 1$, then all the last coefficients of the polynomial $P(U_2X)$ are zero except for one $a_{k,i,j,l,l-s}^{(t)} = 1$. Hence, $P_{kijls}(U_2X) = P_{kijll-s}(X)$ is a basis polynomial of the space $\mathcal{P}(\nu, Q_2)$. Further, it is known [5, p. 38], that

$$\vartheta(\tau, P(X), Q_2) = \vartheta(\tau, P(U_2X), Q_2).$$

Thus, the theta-series $\vartheta(\tau, P(X), Q_2)$ and $\vartheta(\tau, P(U_2X), Q_2)$, corresponding to different basis polynomials $P(X) = P_{kijls}(X)$ and $P(U_2X) = P_{kijll-s}(X)$, are linearly dependent.

Calculate how many such linearly dependent theta-series we have. Let k, i, j, l and s be even (otherwise, it can be shown that $\vartheta(\tau, P, Q_2) = 0$), i.e., $k = \nu, 2 \mid i,\ 2 \mid j,\ 2 \mid l,\ 2 \mid s$ and s takes

$$\sum_{2|s,s=0}^{l} 1 = \frac{l}{2} + 1$$

even values for every even l. Consequently, we have

$$\left[\frac{1}{2}\left(\frac{l}{2}+1\right)\right] = \begin{cases} \frac{l}{4} & \text{if } l \equiv 0(\text{mod}4), \\ \frac{l+2}{4} & \text{if } l \equiv 2(\text{mod}4) \end{cases}$$

linearly dependent theta-series for every even l. Similarly, for every even j, we have

$$\sum_{\substack{l=0\\ l\equiv 0(\mathrm{mod}4)}}^{j} \frac{l}{4} + \sum_{\substack{l=0\\ l\equiv 2(\mathrm{mod}4)}}^{j} \frac{l+2}{4} = \begin{cases} (1+\frac{j}{4})\frac{j}{4} & \text{if } j\equiv 0(\mathrm{mod}4),\\ \left(\frac{j+2}{4}\right)^2 & \text{if } j\equiv 2(\mathrm{mod}4)\end{cases}$$

linearly dependent theta-series. Also, for every even i, we have

$$\sum_{\substack{j=0\\ j\equiv 0(\mathrm{mod}4)}}^{i} (1+\frac{j}{4})\frac{j}{4} + \sum_{\substack{j=0\\ j\equiv 2(\mathrm{mod}4)}}^{i} \left(\frac{j+2}{4}\right)^2$$

$$= \begin{cases} \frac{1}{24}(\frac{i}{4}+1)i(i+5) & \text{if } i\equiv 0(\mathrm{mod}4),\\ \frac{1}{24}(\frac{i}{2}+1)(\frac{i}{2}+3)(i+1) & \text{if } i\equiv 2(\mathrm{mod}4)\end{cases}$$

linearly dependent theta-series. The number of linearly dependent theta-series for even ν is

$$\sum_{\substack{i=0\\ i\equiv 0(\mathrm{mod}4)}}^{\nu} \frac{1}{24}(\frac{i}{4}+1)i(i+5) + \sum_{\substack{i=0\\ i\equiv 2(\mathrm{mod}4)}}^{\nu} \frac{1}{24}(\frac{i}{2}+1)(\frac{i}{2}+3)(i+1)$$

$$= \begin{cases} \frac{1}{3\cdot 2^8}\nu(\nu+4)^2(\nu+8) & \text{if } \nu\equiv 0(\mathrm{mod}4),\\ \frac{1}{3\cdot 2^8}(\nu+2)(\nu+6)(\nu^2+8\nu+4) & \text{if } \nu\equiv 2(\mathrm{mod}4).\end{cases} \tag{3.2}$$

Hence, from (3.1), for the maximal number of linearly independent theta-series, we obtain (a similar result is given in [3])

$$\dim T(\nu,Q_2) \le \begin{cases} \binom{\frac{\nu}{2}+4}{4} - \frac{1}{3\cdot 2^8}\nu(\nu+4)^2(\nu+8) & \text{if } \nu\equiv 0(\mathrm{mod}4),\\ \binom{\frac{\nu}{2}+4}{4} - \frac{1}{3\cdot 2^8}(\nu+2)(\nu+6)(\nu^2+8\nu+4) & \text{if } \nu\equiv 2(\mathrm{mod}4).\end{cases}$$

Thus we have the following

Theorem 3.1. *Let $Q_2(X)$ be the diagonal quadratic form of six variables, given by $Q_2(X) = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + b_{44}x_4^2 + b_{55}(x_5^2+x_6^2)$, then*

$$\dim T(\nu,Q_2) \le \begin{cases} \frac{1}{3\cdot 2^8}(\nu+4)(\nu+8)(\nu^2+12\nu+24) & \textit{if } \nu\equiv 0(\mathrm{mod}4),\\ \frac{1}{3\cdot 2^8}(\nu+2)(\nu+6)^2(\nu+10) & \textit{if } \nu\equiv 2(\mathrm{mod}4).\end{cases}$$

Similarly, consider the non-diagonal quadratic form of six variables

$$Q_3 = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + b_{44}x_4^2 + b_{55}(x_5^2+x_6^2) + b_{12}x_1x_2,$$

where $0<|b_{12}|<b_{11}<b_{22}<b_{33}<b_{44}<b_{55}=b_{66}$.

We construct the integral automorphisms U of the quadratic form Q_3. It is easy to verify that the integral automorphisms of the quadratic form Q_3 are

$$\begin{pmatrix} e_1&0&0&0&0&0\\ 0&e_1&0&0&0&0\\ 0&0&e_2&0&0&0\\ 0&0&0&e_3&0&0\\ 0&0&0&0&e_4&0\\ 0&0&0&0&0&e_5\end{pmatrix}, \quad \begin{pmatrix} e_1&0&0&0&0&0\\ 0&e_1&0&0&0&0\\ 0&0&e_2&0&0&0\\ 0&0&0&e_3&0&0\\ 0&0&0&0&0&e_4\\ 0&0&0&0&e_5&0\end{pmatrix}, \quad (e_i=\pm1, \quad i=1,2,3,4,5).$$

Consider all possible polynomials $P = P_{kijls}(UX)$, where $P_{kijls}(X)\in P(\nu,Q_3)$ are the spherical basis polynomials (2.1) of order ν with respect to $Q_3(X)$, and $U\in G$ is an integral automorphism of the quadratic form $Q_3(X)$.

We have to find which polynomials P satisfy the equality

$$\sum_{U_h} P_{kijls}(U_h X) = 0 \quad \text{for some} \quad U_h \in G.$$

As in the case of the quadratic form Q_2, here too, P_{kijls} satisfies this condition if, among the last coefficients of P for which k is $\nu-1$ or ν, at least one of the indices i, j, l, s of the coefficient equal to one, is odd. For such s spherical polynomial $P = P_{kijls}$, by Lemma 1.1, $\vartheta(\tau, P, Q) = 0$. Hence, if the theta-series are linearly independent, then the indices i, j, l, s of the corresponding spherical polynomial P must be even. According to (1.3), such k, i, j, l, s indices are equal:

$$\frac{1}{r-2}\binom{\frac{\nu}{2}+r-3}{r-3}(\nu+r-2) = \frac{\nu+4}{4}\binom{\frac{\nu}{2}+3}{3}. \tag{3.3}$$

For the automorphism U_2, we have

$$P(U_2X) = \sum_{k=0}^{\nu}\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{l=0}^{j}\sum_{s=0}^{l} a^{(h)}_{kijls} x_1^{\nu-k} x_2^{k-i} x_3^{i-j} x_4^{j-l} x_6^{l-s} x_5^{s}$$

$$= \sum_{k=0}^{\nu}\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{l=0}^{j} a^{(h)}_{kijls} x_1^{\nu-k} x_2^{k-i} x_3^{i-j} x_4^{j-l} x_5^{l-(l-s)} x_6^{l-s}.$$

Thus $P(U_2X)$ is a basis polynomial of the space $\mathcal{P}(\nu, Q_3)$. Furthermore, it is known [5, p. 38], that

$$\vartheta(\tau, P(X), Q_3) = \vartheta(\tau, P(U_2X), Q_3).$$

So, the theta-series $\vartheta(\tau, P(X), Q_3)$ and $\vartheta(\tau, P(U_2X), Q_3)$, corresponding to different basis polynomials $P(X) = P_{kijls}(X)$ and $P(U_2X) = P_{kijll-s}(X)$, are linearly dependent.

Calculate how many such linearly dependent theta-series we have. Let i, j, l, s be even (otherwise, it can be shown that $\vartheta(\tau, P, Q_3) = 0$), i.e., $2 \mid i$, $2 \mid j$, $2 \mid l$, $2 \mid s$.

For $k = \nu - 1$, the number of linearly dependent theta-series is

$$\sum_{\substack{i=0 \\ i\equiv 0 (\mathrm{mod} 4)}}^{\nu-2} \frac{1}{24}(\frac{i}{4}+1)i(i+5) + \sum_{\substack{i=0 \\ i\equiv 2 (\mathrm{mod} 4)}}^{\nu-2} \frac{1}{24}(\frac{i}{2}+1)(\frac{i}{2}+3)(i+1)$$

$$= \begin{cases} \frac{1}{3\cdot 2^8}\nu(\nu+4)(\nu^2+4\nu-8) & \text{if } \nu \equiv 0 (\mathrm{mod} 4), \\ \frac{1}{3\cdot 2^8}(\nu-2)(\nu+2)^2(\nu+6) & \text{if } \nu \equiv 2 (\mathrm{mod} 4). \end{cases}$$

For $k = \nu$, the number of linearly dependent theta-series is given by estimate (3.2).

Thus, the total number of linearly dependent theta-series for $k = \nu - 1$ and $k = \nu$ is

$$\frac{1}{3\cdot 2^7}\nu(\nu+2)(\nu+4)(\nu+6).$$

For the maximal number of linearly independent theta-series, from (3.3), we obtain

$$\dim T(\nu, Q_3) \le \frac{\nu+4}{4}\binom{\frac{\nu}{2}+3}{3} - \frac{1}{3\cdot 2^7}\nu(\nu+2)(\nu+4)(\nu+6) = \binom{\frac{\nu}{2}+4}{4}.$$

Thus, we have the following

Theorem 3.2. *Let $Q_3(X)$ be the non-diagonal quadratic form of six variables, given by $Q_3(X) = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + b_{44}x_4^2 + b_{55}(x_5^2 + x_6^2) + b_{12}x_1x_2$, then*

$$\dim T(\nu, Q_3) \le \binom{\frac{\nu}{2}+4}{4}.$$

We now construct the basis of the space $T(\nu, Q_3)$ for $\nu = 2$. For the quadratic form $Q_3(X)$, we have

$$|A| = \det A = 2^4(4b_{11}b_{22} - b_{12}^2)b_{33}b_{44}b_{55}^2, \quad a_{11}^* = \frac{2b_{22}}{4b_{11}b_{22} - b_{12}^2},$$
$$a_{12}^* = a_{21}^* = -\frac{b_{12}}{4b_{11}b_{22} - b_{12}^2}, \quad a_{22}^* = \frac{2b_{11}}{4b_{11}b_{22} - b_{12}^2}, \quad a_{33}^* = \frac{1}{2b_{33}},$$
$$a_{44}^* = \frac{1}{2b_{44}}, \quad a_{55}^* = a_{66}^* = \frac{1}{2b_{55}}, \quad \text{and other} \quad a_{ij}^* = 0 \text{ for } i \neq j.$$

It is easy to verify that the spherical polynomials (2.1) of second order:

$$P_{10000} = \frac{b_{12}}{4b_{22}}x_1^2 + x_1x_2, \quad P_{11000} = x_1x_3,$$
$$P_{11100} = x_1x_4, \quad \ldots, \quad P_{20000} = -\frac{b_{11}}{b_{22}}x_1^2 + x_2^2,$$
$$P_{21000} = x_2x_3, \quad P_{22000} = -\frac{4b_{11}b_{22} - b_{12}^2}{4b_{22}b_{33}}x_1^2 + x_3^2, \quad P_{22100} = x_3x_4,$$
$$P_{22200} = -\frac{4b_{11}b_{22} - b_{12}^2}{4b_{22}b_{44}}x_1^2 + x_4^2, \quad \ldots, \quad P_{22220} = P_{22222} = -\frac{4b_{11}b_{22} - b_{12}^2}{4b_{22}b_{55}}x_1^2 + x_5^2$$

form the basis of the space of spherical second order polynomials with respect to $Q_3(x)$.

Now, we construct the corresponding generalized theta-series. Suppose the quadratic form $Q_3(x)$ is such that $b_{22} \neq b_{11}x_1^2$, $b_{mm} \neq \sum_{i=1}^{m-1} b_{ii}x_i^2 + b_{12}x_1x_2$, $m = 3, 4, 5$. Consider all possible polynomials P_{kijls} with even indices i, j, l, s and $k = \nu - 1, \nu$; their number is 5 :

$$\vartheta(\tau, P_{10000}, Q_3) = \sum_{n=1}^{\infty}\Big(\sum_{Q_3(x)=n} P_{10000}(x)\Big)z^n = \sum_{n=1}^{\infty}\Big(\sum_{Q_3(x)=n}\Big(\frac{b_{12}}{4b_{22}}x_1^2 + x_1x_2\Big)\Big)z^n$$
$$= \frac{b_{12}}{2b_{22}}z^{b_{11}} + \cdots + 0z^{b_{22}} + \cdots + 0z^{b_{33}} + \cdots + 0z^{b_{44}} + \cdots + 0z^{b_{55}} + \cdots,$$
$$\vartheta(\tau, P_{20000}, Q_3) = \sum_{n=1}^{\infty}\Big(\sum_{Q_3(x)=n} P_{20000}(x)\Big)z^n = \sum_{n=1}^{\infty}\Big(\sum_{Q_3(x)=n}\Big(-\frac{b_{11}}{b_{22}}x_1^2 + x_2^2\Big)\Big)z^n$$
$$= -\frac{2b_{11}}{b_{22}}z^{b_{11}} + \cdots + 2z^{b_{22}} + \cdots + 0z^{b_{33}} + \cdots + 0z^{b_{44}} + \cdots + 0z^{b_{55}} + \cdots,$$
$$\vartheta(\tau, P_{22000}, Q_3) = \sum_{n=1}^{\infty}\Big(\sum_{Q_3(x)=n} P_{22000}(x)\Big)z^n = \sum_{n=1}^{\infty}\Big(\sum_{Q_3(x)=n}\Big(-\frac{4b_{11}b_{22} - b_{12}^2}{4b_{22}b_{33}}x_1^2 + x_3^2\Big)\Big)z^n$$
$$= -\frac{4b_{11}b_{22} - b_{12}^2}{2b_{22}b_{33}}z^{b_{11}} + \cdots + 0z^{b_{22}} + \cdots + 2z^{b_{33}} + \cdots + 0z^{b_{44}} + \cdots + 0z^{b_{55}} + \cdots,$$
$$\vartheta(\tau, P_{22200}, Q_3) = \sum_{n=1}^{\infty}\Big(\sum_{Q_3(x)=n} P_{22200}(x)\Big)z^n = \sum_{n=1}^{\infty}\Big(\sum_{Q_3(x)=n}\Big(-\frac{4b_{11}b_{22} - b_{12}^2}{4b_{22}b_{44}}x_1^2 + x_4^2\Big)\Big)z^n$$
$$= -\frac{4b_{11}b_{22} - b_{12}^2}{2b_{22}b_{44}}z^{b_{11}} + \cdots + 0z^{b_{22}} + \cdots + 0z^{b_{33}} + \cdots + 2z^{b_{44}} + \cdots + 0z^{b_{55}} + \cdots.$$
$$\vartheta(\tau, P_{22220}, Q_3) = \sum_{n=1}^{\infty}\Big(\sum_{Q_3(x)=n} P_{22220}(x)\Big)z^n = \sum_{n=1}^{\infty}\Big(\sum_{Q_3(x)=n}\Big(-\frac{4b_{11}b_{22} - b_{12}^2}{4b_{22}b_{55}}x_1^2 + x_5^2\Big)\Big)z^n$$
$$= -\frac{4b_{11}b_{22} - b_{12}^2}{2b_{22}b_{55}}z^{b_{11}} + \cdots + 0z^{b_{22}} + \cdots + 0z^{b_{33}} + \cdots + 0z^{b_{44}} + \cdots + 2z^{b_{55}} + \cdots.$$

These generalized theta-series are linearly independent, since the determinant constructed from the coefficients of these theta-series is not equal to zero. By Theorem 3.2, we have $\dim T(2, Q_3) \leq 5$. Hence, these theta-series form the basis of the space $T(2, Q_3)$. We have the following

Theorem 3.3. *Let $Q_3(X)$ be the non-diagonal quadratic form of six variables given by* $Q_3(X) = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + b_{44}x_4^2 + b_{55}(x_5^2 + x_6^2) + b_{12}x_1x_2$, *and let* $b_{22} \neq b_{11}x_1^2$, $b_{mm} \neq \sum_{i=1}^{m-1} b_{ii}x_i^2 + b_{12}x_1x_2$, $m = 3, 4, 5$, *then* $\dim T(2, Q_3) = 5$ *and the generalized theta-series:*

$$\vartheta(\tau, P_{10000}, Q_3); \vartheta(\tau, P_{20000}, Q_3); \vartheta(\tau, P_{22000}, Q_3); \vartheta(\tau, P_{22200}, Q_3), \vartheta(\tau, P_{22220}, Q_3)$$

form the basis of the space $T(2, Q_3)$.

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