

## TOPOLOGICAL INSULATORS AND STABLE ISOMORPHISM VERSUS ISOMORPHISM OF VECTOR BUNDLES

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**Abstract.** Topological insulators are materials whose electronic properties are governed by topological invariants rather than local geometric features. In this note, we review the mathematical foundations underlying topological insulators, emphasizing the connection with vector bundles and  $K$ -theory. We discuss how “real” and “quaternionic” vector bundles arise naturally in the presence of time-reversal symmetry, and summarize our recent results about the conditions under which stable isomorphism implies isomorphism for these bundles. Finally, we describe ongoing work concerning the  $G$ -equivariant  $K$ -theory for finite groups, highlighting new phenomena that occur in this context.

### 1. INTRODUCTION

Topological insulators have emerged as a central topic in condensed matter physics, since they exhibit remarkable conductivity properties: despite being insulators in the bulk, they may have very robust conducting states on boundaries. The distinguishing feature of these materials is that their electronic properties are determined by global topological invariants rather than local parameters, making them resilient to perturbations. Mathematically, these systems can be described using the language of operator algebras, vector bundles, and  $K$ -theory. In this note, we briefly review the mathematical formalism that models topological insulators, emphasizing connections with  $C^*$ -algebras and vector bundles. We describe how time-reversal symmetry introduces “real” and “quaternionic” structures. We explain the physical motivation for the question whether stably isomorphic vector bundles with extra structure are isomorphic, and we discuss the recent results on this question in our joint work with Malkhaz Bakuradze (see [2]). We conclude with some remarks on ongoing research in  $G$ -equivariant  $K$ -theory, which exhibits noticeably different phenomena compared to the non-equivariant case.

### 2. LATTICE MODELS AND THE HILBERT SPACE FORMALISM

We consider an electron moving in a  $d$ -dimensional crystal with  $k$  internal degrees of freedom per lattice site. The Hilbert space of the system is

$$\mathcal{H} = \ell^2(\mathbb{Z}^d, \mathbb{C}^k).$$

The dynamics are governed by a bounded, self-adjoint Hamiltonian  $H$  acting on  $\mathcal{H}$ . We assume translation invariance, that is,

$$S_n H = H S_n, \quad \text{for all } n \in \mathbb{Z}^d$$

for the translation operator  $S_n$  defined by  $(S_n f)(x) = f(x - n)$ . Under this assumption, there are matrices  $H_a \in M_k(\mathbb{C})$  such that

$$(Hg)(m) = \sum_{a \in \mathbb{Z}^d} H_a g(m - a). \tag{2.1}$$

We assume further that  $H$  has finite range, that is,  $H_a = 0$  for all but finitely many  $a$ .

The Hamiltonian  $H$  describes an insulator if and only if it is invertible. We may also consider the restriction of  $H$  to a half-space by compressing  $H$  to the subspace

$$\ell^2(\mathbb{N} \times \mathbb{Z}^{d-1}, \mathbb{C}^k) \subset \ell^2(\mathbb{Z}^d, \mathbb{C}^k).$$

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The restricted operator may fail to be invertible. This means physically that conducting states appear on the boundary of a finite-size fragment of the material. Next, we sketch how the presence of these states may be topologically protected by a non-vanishing index in a suitable  $K$ -theory.

### 3. $C^*$ -ALGEBRAIC FORMULATION AND THE INDEX MAP

The allowed Hamiltonians generate the  $C^*$ -algebra

$$C^*(\mathbb{Z}^d) \otimes M_k(\mathbb{C}),$$

where  $C^*(\mathbb{Z}^d) \cong C(\mathbb{T}^d)$  is the group  $C^*$ -algebra of  $\mathbb{Z}^d$ . On the half-space, one unitary generator is replaced by a unilateral shift, so this corresponds to

$$\mathcal{T} \otimes C^*(\mathbb{Z}^{d-1}) \otimes M_k(\mathbb{C}),$$

where  $\mathcal{T}$  is the Toeplitz  $C^*$ -algebra. These  $C^*$ -algebras fit in a short exact sequence

$$0 \rightarrow \mathcal{K} \otimes C^*(\mathbb{Z}^{d-1}) \otimes M_k(\mathbb{C}) \rightarrow \mathcal{T} \otimes C^*(\mathbb{Z}^{d-1}) \otimes M_k(\mathbb{C}) \rightarrow C^*(\mathbb{Z}^d) \otimes M_k(\mathbb{C}) \rightarrow 0,$$

where  $\mathcal{K}$  is the  $C^*$ -algebra of compact operators on  $\ell^2(\mathbb{N})$ . Roughly speaking, a half-space Hamiltonian in  $\mathcal{T} \otimes C^*(\mathbb{Z}^{d-1}) \otimes M_k(\mathbb{C})$  produces a bulk Hamiltonian in  $C^*(\mathbb{Z}^d) \otimes M_k(\mathbb{C})$  by identifying  $\mathbb{N} \times \mathbb{Z}^{d-1}$  with  $(\mathbb{N} - s) \times \mathbb{Z}^{d-1}$  and letting  $s$  go to  $\infty$ .

Next we introduce  $K$ -theory and the index map. The Hamiltonian of an insulator is a self-adjoint invertible element of the  $C^*$ -algebra  $C^*(\mathbb{Z}^d) \otimes M_k(\mathbb{C})$ . Functional calculus allows us to deform it among self-adjoint invertible operators to a self-adjoint involution, namely,

$$F = \text{sign}(H).$$

This satisfies the conditions  $F^2 = 1$  and  $F = F^*$  and contains the same information as the corresponding projection

$$p = \frac{1 + F}{2} \in C^*(\mathbb{Z}^d) \otimes M_k(\mathbb{C}).$$

The latter represents the class  $[H] \in K_0(C^*(\mathbb{Z}^d))$ . The boundary map for the extension above maps this class to an index in  $K_1(\mathcal{K} \otimes C^*(\mathbb{Z}^{d-1}))$ . If this index is nonzero, then  $H$  cannot lift to an invertible operator in  $\mathcal{T} \otimes C^*(\mathbb{Z}^{d-1}) \otimes M_k(\mathbb{C})$ . This means that any Hamiltonian on the half-space that behaves like  $H$  in the bulk is a conductor. That is, there are conducting states on the boundary, and their existence is determined by the nonvanishing index. Since the index is homotopy invariant, the existence of these boundary states does not depend on small perturbations of the Hamiltonian.

### 4. VECTOR BUNDLES AND THE BLOCH BUNDLE

Using the Fourier transform,  $C^*(\mathbb{Z}^d)$  is identified with  $C(\mathbb{T}^d)$ , so that the  $K_0$ -class  $[H]$  defines a class in  $K^0(\mathbb{T}^d)$ . The associated vector bundle is called the *Bloch bundle*. Its fibre at  $z \in \mathbb{T}^d$  is the image of the Fourier transform of  $p$  at  $z$ ; the latter is a projection in  $M_k(\mathbb{C})$ .

It is physically interesting to know whether or not this vector bundle is trivial because this is equivalent to the existence of “exponentially localised Wannier functions” (see [3, Proposition 4.3]), which are a tool used for computations in physics. It is usually much easier to decide whether the Bloch bundle is stably trivial, that is, becomes trivial after adding a trivial bundle. This means that its class in reduced  $K$ -theory vanishes. If the reduced  $K$ -theory is torsion-free, this happens if and only if its Chern numbers vanish. Physicists have long studied the Chern numbers of the Bloch bundle as topological invariants related to conductivity phenomena. As a result, it is physically relevant to know whether the triviality of the Bloch bundle follows from its stable triviality. It is well known that vector bundles of sufficiently high rank that are stably trivial are automatically trivial; more generally, stably isomorphic bundles of sufficiently high rank are isomorphic (see [4]). Generalizations of these classical results are needed to treat physical systems with certain extra symmetries.

## 5. “REAL” AND “QUATERNIONIC” BUNDLES

In quantum mechanics, a time-reversal symmetry is represented by an anti-unitary operator commuting with the Hamiltonian. The square of this anti-unitary operator is  $\pm 1$ , where  $+1$  occurs for bosons and  $-1$  for fermions. In the Hilbert space  $\ell^2(\mathbb{Z}^d, \mathbb{C}^k)$ , we assume that time-reversal symmetry is given by applying a certain anti-unitary operator pointwise to  $\mathbb{C}^k$ . Depending on the sign, this has the effect that the coefficients  $H_a$  in (2.1) belong now to  $M_k(\mathbb{R}^k)$  or  $M_k(\mathbb{H}^k)$  instead of  $M_k(\mathbb{C}^k)$ . The same happens with the matrix coefficients of the projection  $p$ . However, if we take the Fourier transform, this does not correspond to the Bloch bundle over the torus being a real or quaternionic bundle. Instead, the Bloch bundle is still a complex vector bundle, but with an extra symmetry operator, namely, a conjugate-linear involution  $\theta$  mapping  $E_z$  to  $E_{\bar{z}}$  for all  $z \in \mathbb{T}^d$ . If  $\theta^2 = 1$  (bosons), this is a “real” vector bundle; if  $\theta^2 = -1$ , this is a “quaternionic” vector bundle. Such vector bundles, especially the “real” kind, have long been known in index theory (see [1]), but have not received much attention. In particular, it has not been shown that “real” or “quaternionic” bundles of sufficiently high rank are isomorphic if they are stably isomorphic. The recent article [2] filled this gap. A crucial step in proving this is to show that any bundle of sufficiently high rank contains a trivial vector bundle of rank 1 as a direct summand. Both results can be combined to the statement that the stabilisation map from bundles of rank  $k$  to bundles of rank  $k + 1$ , adding the trivial bundle of rank 1 induces a bijection on isomorphism classes for sufficiently large  $k$ . The following theorems from [2] provide the details of these statements:

**Theorem 5.1.** *Let  $d_1, d_0, k \in \mathbb{N}$ . Let  $X$  be a  $\mathbb{Z}/2$ -CW-complex. Assume that the free cells in  $X$  have dimension at most  $d_1$  and that the trivial cells have dimension at most  $d_0$ . Let*

$$k_0 := \max \left\{ d_0, \left\lceil \frac{d_1 - 1}{2} \right\rceil \right\}, \quad k_1 := \max \left\{ d_0 + 1, \left\lceil \frac{d_1}{2} \right\rceil \right\}.$$

- (1) *Let  $E$  be a “real” vector bundle over  $X$  of rank  $k \geq k_0$ . There is an isomorphism  $E \cong E_0 \oplus (X \times \mathbb{C}^{k-k_0})$  for some “real” vector bundle  $E_0$  over  $X$  and the trivial “real” vector bundle  $X \times \mathbb{C}^{k-k_0}$  of rank  $k - k_0$ .*
- (2) *Let  $E_1$  and  $E_2$  be two “real” vector bundles over  $X$  of rank  $k \geq k_1$ . If  $E_1$  and  $E_2$  are stably isomorphic, that is,  $E_1 \oplus E_3 \cong E_2 \oplus E_3$  for some “real” vector bundle  $E_3$ , then they are isomorphic.*

**Theorem 5.2.** *Let  $d_1, d_0, k \in \mathbb{N}$ . Let  $X$  be a  $\mathbb{Z}/2$ -CW-complex. Assume that the free cells in  $X$  have dimension at most  $d_1$  and that the trivial cells have dimension at most  $d_0$ . Let*

$$k_0 := \max \left\{ \left\lceil \frac{d_0 - 3}{2} \right\rceil, \left\lceil \frac{d_1 - 1}{2} \right\rceil \right\}, \quad k_1 := \max \left\{ \left\lceil \frac{d_0 - 2}{2} \right\rceil, \left\lceil \frac{d_1}{2} \right\rceil \right\}.$$

- (1) *Let  $E$  be a “quaternionic” vector bundle over  $X$  of rank  $k \geq k_0$ . There is an isomorphism  $E \cong E_0 \oplus \theta_X^{2\lfloor (k-k_0)/2 \rfloor}$  for some “quaternionic” vector bundle  $E_0$  over  $X$  and the trivial “quaternionic” vector bundle  $\theta_X^{2\lfloor (k-k_0)/2 \rfloor}$  of rank  $2\lfloor (k-k_0)/2 \rfloor$ .*
- (2) *Let  $E_1$  and  $E_2$  be two “quaternionic” vector bundles over  $X$  of rank  $k \geq k_1$ . If  $E_1$  and  $E_2$  are stably isomorphic, that is,  $E_1 \oplus E_3 \cong E_2 \oplus E_3$  for some “quaternionic” vector bundle  $E_3$ , then they are isomorphic.*

 6. EQUIVARIANT  $K$ -THEORY AND ONGOING WORK

When the system has classical crystallographic symmetries, then the Bloch bundle becomes an equivariant vector bundle for a certain group. In  $G$ -equivariant  $K$ -theory, say, for a finite group  $G$ , new phenomena may arise. The main new issue is that there may be several non-isomorphic trivial vector bundles, corresponding to inequivalent representations of  $G$ . This is not the case for “real” and “quaternionic” bundles, although they at first glance may appear more complicated than equivariant vector bundles, since they involve the group  $\mathbb{Z}/2$  acting on the vector bundle via anti-unitary maps.

For example, consider a  $\mathbb{Z}/2$ -equivariant complex vector bundle over the circle  $\mathbb{T}^1 := \{z \in \mathbb{C} \mid |z| = 1\}$  with  $\mathbb{Z}/2$  acting by  $z \mapsto \bar{z}$ . Its fibres at  $\pm 1$  are complex representations of the group  $\mathbb{Z}/2$ , and both may be arbitrary. It may happen that for  $+1$ , we have a trivial representation and for  $-1$ ,

we have a nontrivial representation of some rank  $k$ . This vector bundle has high rank  $k$ , but has no trivial subbundles, since the representations for  $\pm 1$  are disjoint. When one tries to prove analogues of the main theorems above, one has to control multiplicities of representations instead of just ranks.

## 7. CONCLUSION

Topological insulators provide a rich interplay between condensed matter physics and  $K$ -theory. The mathematical description involving  $C^*$ -algebras, vector bundles, and indices not only explains the existence of boundary states, but also motivates questions about “real”, “quaternionic”, and equivariant vector bundles. In particular, the classical question whether stably isomorphic vector bundles are actually isomorphic has physical applications.

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