Transactions of A. Razmadze
Mathematical Institute
Vol. **180** (2026), issue 1, 167–174

GROWTH OF LEBESGUE CONSTANTS

ELIJAH LIFLYAND[1*] AND ANATOLY PODKORYTOV[2]

Abstract. Let

$$\Big\{k \in \mathbb{Z}^n : 1 \le \prod_{j=1}^{n} |k_j|^{\gamma_j} \le R^{\gamma_1+\cdots+\gamma_n}\Big\},$$

where $\gamma_1, \dots, \gamma_n > 0$, be a "hyperbolic cross" dilated homothetically as $R \to +\infty$. In our study, their Lebesgue constants are, as expected, always of power growth R^p, $p > 0$, sometimes several times larger than a logarithmic factor. Surprisingly, contrary to the expected $p = \frac{n-1}{2}$ in any case, p may become, for an appropriate choice of $\gamma_1, \dots, \gamma_n$, arbitrarily larger than that fraction. In many cases, the estimates of the Lebesgue constants are exact in the sense that the upper and lower bounds differ from each other only by their coefficients.

1. Introduction

The behavior of the partial sums of Fourier expansions is an important component of research in many problems. On the other hand, it is an independent topic of harmonic analysis. The norms of the corresponding operators are called the Lebesgue constants. They are analytic characteristics in studies of convergence and summability of Fourier series, approximation and interpolation, and many other problems (see [1, 3, 5, 14, 15, 25] and references therein). Frequently, for instance, in studying uniform convergence, such operators are considered in C (or in L^1, the Lebesgue constants are the same in these spaces). Everything is clear in one-dimensional space: for the R-th partial sum, the norm differs from $\frac{4}{\pi^2}\ln R$ by a bounded value (see, e.g., [27, Sect. 2.12]).

In the multivariate setting, the situation becomes significantly more complicated. The main reason is obvious: unlike the one-dimensional case, there are different ways of ordering partial sums in multidimensional space. This leads to different types of convergence or divergence, for example, in cubes, spheres or polyhedra, and the difference may be drastic. Very often a partial sum of the Fourier series is generated by a finite set $B \cap \mathbb{Z}^n$:

$$S_B(f) = S_B(f;x) := \sum_{k\in B\cap\mathbb{Z}^n} \widehat{f}_k e^{i\langle k,x\rangle},$$

where $f \in L^1(\mathbb{T}^n)$, $\mathbb{T}^n = [-\pi,\pi)^n$, $\langle k,x\rangle = k_1x_1 + \cdots + k_nx_n$, and $\widehat{f}_k$ is the k-th Fourier coefficient of f. The norm of the operator $f \mapsto S_B(f)$ (in the space of continuous or integrable functions on $\mathbb{T}^n$) is called the B-th Lebesgue constant. It can be represented as

$$\mathscr{L}(B) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \Big| \sum_{k\in B\cap\mathbb{Z}^n} e^{i\langle k,x\rangle} \Big| \, dx. \tag{1.1}$$

The sum of the harmonics on the right-hand side is called the B-th Dirichlet kernel.

If the intersection $B \cap \mathbb{Z}^n$ is infinite, then the operator is defined only for good enough functions f, say, for polynomials. Considering it in the space of all polynomials, one may define the Lebesgue constant $\mathscr{L}(B)$ in a usual way as the least number C satisfying the inequality $\|S_B(f)\| \le C\|f\|$ for any polynomial f. This does not exclude the case $\mathscr{L}(B) = +\infty$. The works [20, Ch.IX § 4] or [4], for example, provide insight into such situations.

In order to get complete information, a family of sets B is usually used that exhausts the integer lattice. One of the most natural ways of ordering and exhausting is the R-dilation of a fixed set

2020 *Mathematics Subject Classification.* 42B05, 42B08, 42B10, 42B15.
Key words and phrases. Fourier series; Lebesgue constant; Hyperbolic cross.
*Corresponding author.

$B \subset \mathbb{R}^n$. Here, B is replaced by the set $RB = \{Ru : u \in B\}$. Of course, the generating set B is assumed to be known. The main feature of the one-dimensional case is the fact that the Dirichlet kernel can be expressed in a relatively simple manner ("rolled up"), whereas in the multi-dimensional case this is possible only in certain rare and trivial situations. This leads to the need of dealing with the trigonometric sums, in general.

A detailed survey on the Lebesgue constants is given in [18]; it deals equally both with the linear methods of summability of the Fourier series and with partial sums. There are other survey type texts, but they (see, e.g., [8]) consider the Lebesgue constants only in one section and, moreover, they are outdated now. As for the Lebesgue constants for just partial sums, a concise, though almost comprehensive, overview of the basic results can be found in [11]. Of course, the most recent publications, like [12], cannot be mentioned in any of these surveys.

Let us outline briefly the picture of behavior of various Lebesgue constants arisen prior to our study. A trivial L^2 estimate leads to the bound $\sqrt{\operatorname{card}(B \cap \mathbb{Z}^n)}$ for $\mathscr{L}(B)$. However, crude this may seem, it is non-refinable in a sense that for any B, there is a subset $B_0 \subset B$ such that $\mathscr{L}(B_0) \geq C_1\sqrt{\operatorname{card}(B)}$, where C_1 is a positive absolute constant (see [20, Problem IX.4.54]). This reveals the fact that not only the number of lattice points involved is of importance, but also the arithmetic and geometric properties of B. Thus, one of the main tasks is to find natural assumptions on B such that the above bound can be improved. It has long been known that the Lebesgue constants for a bounded set B with nonempty interior tend to infinity as R grows. There are two main types of growth rate of the Lebesgue constants $\mathscr{L}(RB)$: power law and logarithmic. On the one hand, it was shown in [19] (see also [17]) that if a sufficiently smooth boundary of B contains at least one point with non-vanishing Gaussian curvature, then the lower bound $R^{\frac{n-1}{2}}$ is guaranteed. The spherical Lebesgue constants, as well as similar ones, but generated by a more general set, satisfy the same estimate as above. On the other hand, lack of the points with non-vanishing curvature on the boundary of the generating set B may lead to a lower growth of the Lebesgue constants than $R^{\frac{n-1}{2}}$. For example, if B is a polyhedron (say, cube), than the growth rate is $\ln^n R$. Note that for some time it was unknown whether the sets B generate an intermediate growth (see, e.g., [22]). Thus, for "normal" situations, that is, for the bounded B, the rate $R^{\frac{n-1}{2}}$ seemed to be a natural maximal limit of the growth of the Lebesgue constants for the R-dilated set B.

In this work, we are mainly interested in the hyperbolic Lebesgue constants. Since the appearance of Babenko's paper [2], interest has continued in various topics of Approximation Theory and Fourier Analysis in $\mathbb{R}^n$ related to the study of linear means with harmonics in the "hyperbolic crosses" has persisted. It is meaningful to immediately describe our basic object in detail. Given a vector $\gamma \in \mathbb{R}^n_+$, we associate with it the hyperbolic cross $H_\gamma = \left\{t \in \mathbb{R}^n : |t_1|^{\gamma_1} \dots |t_n|^{\gamma_n} \leq 1\right\}$. Our further reasoning will mostly be concerned with its positive part

$$H_\gamma^+ = \left\{t \in \mathbb{R}^n_+ : t_1^{\gamma_1} \dots t_n^{\gamma_n} \leq 1\right\}.$$

Without loss of generality, we assume that $\gamma_1 + \cdots + \gamma_n = 1$. We are interested in how large the Lebesgue constants corresponding to $RH_\gamma^+ = \left\{t \in \mathbb{R}^n_+ : t_1^{\gamma_1} \dots t_n^{\gamma_n} \leq R\right\}$ are, as $R \gtrsim 1$, that is, the values, with $k \in \mathbb{N}^n$,

$$\mathscr{L}(RH_\gamma^+) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{T}^n} \left| \sum_{k \in RH_\gamma^+} e^{i\langle k,x\rangle} \right| dx.$$

When studying the Lebesgue constants one can also consider the crosses with $k_j = 0$, moreover, the set H_γ is dilated. However, this case somewhat differs from the one considered above, since they differ from one another by a bounded operator $f(\cdot) \mapsto \int_{\mathbb{T}} f(\cdot + te_j)\,dt$, where e_j is the j-th basis vector in $\mathbb{R}^n$. Secondly, we would then loss the benefit of the representation (1.1), since the intersection $B \cap \mathbb{Z}^n$ is infinite for $B = RH_\gamma$.

What is currently known about this problem and what motivated us to return to it? The exact degree of growth for them $\mathscr{L}(RH_\gamma) \asymp R^{\frac{1}{2}}$ was established in the two-dimensional case independently by Belinsky [3] and by A. and V. Yudins [26], and then generalized to the case of arbitrary dimension in [16], where the bilateral bound $R^{\frac{n-1}{2}}$ was asserted. It is worth noting that the results obtained were applied to the problem of approximation on the classes of functions with continuous partial and mixed derivatives in [3] and [16] and to the study of the uniform convergence in [10]. However, the works by Temlyakov are apparently better adjoin to our problem (see [23] and [24, Ch.4] or [6] to mention a few).

The above gives the impression that everything is going well and the hyperbolic case does not slip out of the general picture, though possesses certain technical difficulties. It turned out that such an impression is erroneous, and just these Lebesgue constants destroy the picture drawn above for certain parameters of hyperbolic crosses. In a sense, $R^{\frac{n-1}{2}}$ can no more be considered as "Everest" among the "Lebesgue-constants-mountains". Well, the two-dimensional case is the most studied and presents no discomfort, but a larger and more rapid growth rate is already possible in the three-dimensional case. It turned out that not all the cases were considered in [16], but only those that conform to the established stereotype. It is not difficult to understand that the latter is violated even in the three-dimension case. Indeed,

$$\int\limits_{\mathbb{T}^3}\Big|\sum_{1\le|k_1|^{\gamma_1}|k_2|^{\gamma_2}|k_3|^{\gamma_3}\le R} e^{i\langle k,x\rangle}\Big|\,dx \ge 2\pi\int\limits_{\mathbb{T}^2}\Big|\sum_{1\le|k_1|^{\gamma_1}|k_2|^{\gamma_2}\le R} e^{i(k_1x_1+k_2x_2)}\Big|\,dx_1\,dx_2.$$

Applying now the known two-dimensional estimate and recalling that $\gamma_1+\gamma_2+\gamma_3=1$, we immediately arrive at the lower bound $\mathscr{L}(RH_\gamma^+)\gtrsim R^{\frac{1}{2(\gamma_1+\gamma_2)}}$ (of course, any other pair could be in the denominator). It is not a big deal to choose γ_1,γ_2, and γ_3 such that the power is arbitrarily greater than one. Similar arguments for any dimension are not much more complicated.

Of course, these new bounds have to be supplemented with the corresponding upper ones. Our main goal is to establish all possible growth rates for the Lebesgue constants with $R^{\frac{n-1}{2}}$, being the least one. All this will be attained according to the parameters $\gamma_1,\dots,\gamma_n$ of the hyperbolic cross. Besides precise power estimates, certain crosses are subject to additional logarithmic factors. It is an open problem whether such factors can be removed or not. Moreover, the value of these factors is questionable. For instance, in the three-dimension case, we can prove that the upper bound can be reduced to $R\ln^{\frac{1}{2}}R$ rather than to $R\ln R$, as shown by Theorem 2.2. A corresponding picture in higher dimensions is still open, and we shall search for it elsewhere.

Also, $A\asymp B$ denotes $A\lesssim B\lesssim A$, where from here on we use the notation "$\lesssim$" as abbreviation for "$\le C$" with C being a positive constant that may be different in different occurrences (and, of course, different in $A\lesssim B\lesssim A$). Similarly, "$\gtrsim$" substitutes for "$\ge C$". The constants C in this work never depend on the coefficient of dilation R, but may depend on $\gamma_1,\dots,\gamma_n$.

2. Results

We start with general estimates for the Lebesgue constants. Being well-suited to the constants generated by convex domains, they will nevertheless be actively used in the hyperbolic case, as well.

2.1. **General estimates.**

For the set $V\subset\mathbb{R}^n$ of finite volume, we assume that

$$I_V=\int\limits_{\mathbb{T}^n}\big|\widehat{\chi}(x)\big|\,dx,$$

where the function χ is 1 on V and 0 otherwise (the characteristic function of V), and $\widehat{\chi}$ is its Fourier transform:

$$\widehat{\chi}(x)=\int\limits_{\mathbb{R}^n}\chi(y)\,e^{i\langle x,y\rangle}dy=\int\limits_{V}e^{i\langle x,y\rangle}dy.$$

Our goal is to make the trivial estimate $I_V\lesssim\sqrt{\operatorname{mes}(V)}$ more precise by means of certain restrictions on the geometry of V.

Theorem 2.1. *Let the set V satisfy the conditions:*

1) *Its projection E onto the space of the first $n-2$ variables is of finite volume (in $\mathbb{R}^{n-2}$);*

2) *The intersection of V with any line, parallel to the $(n-1)$-st or n-th coordinate axis, is an interval, maybe empty;*

3) *There exist numbers Δ and Δ', $1 \le \Delta \le \Delta'$ such that for each point $(x_1 \dots, x_{n-2})$ in E, the corresponding section of V*

$$\{(x_{n-1}, x_n) \in \mathbb{R}^2 : (x_1, \dots, x_{n-1}, x_n) \in V\}$$

is located in the rectangle with the sides Δ and Δ'.
Then there exists a coefficient C, depending only on the dimension n such that

$$I_V \le C\sqrt{\Delta \operatorname{mes}(E)}\Big(1 + \ln \frac{\Delta'}{\Delta}\Big).$$

In particular, $I_V \le CR^{\frac{n-1}{2}}$ if V is contained in a cube with side R and satisfies condition 2). Moreover, any convex subset of this cube admits this estimate.

Remark 2.1. An attentive reader may notice that an assumption $n > 2$ is hidden in the formulation of the theorem. Nevertheless, if $n = 2$, the situation becomes even simpler. The changes needed are minimal: remove 1), simplify 3) to "the set V is located in the rectangle with the sides Δ and Δ'", and reduce the last inequality to $I_V \le C\sqrt{\Delta}\,\big(1 + \ln \frac{\Delta'}{\Delta}\big)$.

There is an alternative way to formally reduce the plane problem to a volumetric one by applying the theorem with $n = 3$ to the product $[0, 1] \times V$.

The following less general corollary is proved in [21] for two dimensions regardless of Theorem 2.1. Just this result will be used in the proof of the main result in an indirect and non-immediate way.

Corollary. *Let $B \subset [0, A_1] \times [0, A_2] \times \cdots \times [0, A_n]$, where $1 \le A_1 \le A_2 \le \cdots \le A_n$, so that the intersection of B with an arbitrary line parallel to a coordinate axis is an interval, maybe empty. Then*

$$\mathscr{L}(B) \lesssim \sqrt{A_1\, A_2 \dots A_{n-1}}\,\Big(1 + \ln \frac{A_n}{A_{n-1}}\Big).$$

This estimate is simpler in dimension two: $\mathscr{L}(B) \lesssim \sqrt{A_1}\,\big(1 + \ln \frac{A_2}{A_1}\big)$.

It is worth mentioning that a similar result was independently obtained in [26, Corollary 2]. On the other hand, it was proved in [7] that if the finite set $B \subset \mathbb{Z}_+^n$ is such that for every point $k = (k_1, \dots, k_n) \in B$, the formula

$$\prod_{j=1}^{n} [1, k_j] \cap \mathbb{Z}_+^n \subseteq B, \tag{2.1}$$

holds, then

$$\mathscr{L}(B) \le 50n^3 |\operatorname{card}(B \cap \mathbb{Z}^n)|^{\frac{n-1}{2n}}. \tag{2.2}$$

All these results are not applicable directly to the hyperbolic cross; for example, (2.2) yields an additional and unnecessary factor $\ln R$ even for $n = 2$ and the simplest cross $B = \{(x_1, x_2) : 1 \le |x_1 x_2| \le R\}$.

2.2. **Hyperbolic case.** We are going to formulate our main result. All the formulations will be given for the case $\gamma_1 \le \cdots \le \gamma_n$, whereas any other case reduces to that merely by the renumeration of the variables.

Theorem 2.2. *Let $0 < \gamma_1 \leqslant \cdots \leqslant \gamma_n$ and $\gamma_1 + \cdots + \gamma_n = 1$. Then*

$$R^{\frac{1}{2}\theta} \lesssim \mathscr{L}(RH_\gamma) \lesssim R^{\frac{1}{2}\theta} \ln^{k-1} R,$$

where

$$\theta = \max_{1 \le \ell < n} \frac{\ell}{\gamma_1 + \cdots + \gamma_{\ell+1}},$$

and k is the number of ℓ for which $\theta = \frac{\ell}{\gamma_1 + \cdots + \gamma_{\ell+1}}$.

In particular, in a "generic" situation where all the fractions $\frac{\ell}{\gamma_1+\ldots+\gamma_{\ell+1}}$ *are different* $(k=1)$*, a bilateral estimate*

$$\mathscr{L}(RH_\gamma) \asymp R^{\frac{1}{2}\theta}$$

holds.

Observe that here and in what follows we use the notation $\mathscr{L}(RH_\gamma)$ with a certain abuse of rigidity, since it cannot be identified with (1.1). However, the difference between them is a constant, as described in Introduction. We hope that such a licence will not cause confusion. In connection with this theorem, we mention another result due to Dyachenko. In [9], under the assumption (2.1), a more precise result than (2.2) is obtained. It gives a correct upper estimate in the case $B = \{(x_1, \ldots, x_n) : 1 \le |x_1 \cdots x_n| \le R\}$, but not in "anisotropic" cases.

3. Specifications and Discussion

It is desirable to have direct estimates of the constants $\mathscr{L}(RH_\gamma)$ via the coordinates of the vector γ rather than via the related numbers $\theta_1, \ldots, \theta_{n-1}$. We cannot say that this goal is achieved in full, since there is no formula for calculating θ and k immediately through $\gamma_1, \ldots, \gamma_n$. It is quite possible that such a formula is not achievable.

For $n = 2$, everything is simple: $\theta = \theta_1 = 1$ and $k = 1$, which leads to the previously known bilateral estimate $\mathscr{L}(RH_\gamma) \asymp R^{\frac{1}{2}}$. For higher dimensions, the situation becomes more complicated.

Let us discuss the case of three dimensions. Then either $k = 1$ or $k = 2$, which yields

$$\theta = \max\Big\{\frac{1}{\gamma_1+\gamma_2}, \frac{2}{\gamma_1+\gamma_2+\gamma_3}\Big\} = \max\Big\{\frac{1}{1-\gamma_3}, 2\Big\}.$$

Therefore, $k = 1$, provided $\gamma_3 \ne \frac{1}{2}$; hence, a bilateral estimate

$$\mathscr{L}(RH_\gamma) \asymp \begin{cases} R, & \text{if } \gamma_3 < \frac{1}{2}; \\ R^{\frac{1}{2(1-\gamma_3)}}, & \text{if } \gamma_3 > \frac{1}{2} \end{cases} \tag{3.1}$$

holds. Observe that for γ_3 close to 1, the power $\frac{1}{2(1-\gamma_3)}$ becomes arbitrarily large, in particular, greater than 1 if $\gamma_3 > \frac{1}{2}$.

If $\gamma_3 = \frac{1}{2}$, then $k = 2$ and a logarithmic factor appears in our upper estimate

$$R \lesssim \mathscr{L}(RH_\gamma) \lesssim R \ln R. \tag{3.2}$$

There is a geometric interpretation for the conditions on γ_1, γ_2 and γ_3: if these are the lengths of the sides of a triangle, for instance, $\gamma_1 = \gamma_2 = \gamma_3 = \frac{1}{3}$, then the first relation in (3.1) is valid. If the triangle degenerates into an interval, then (3.2) holds. Finally, if the sum of the first two numbers is smaller than the third one, then the second relation in (3.1) is valid.

Let now $n = 4$, then $k \in \{1, 2, 3\}$ and

$$\theta = \max\Big\{\frac{1}{\gamma_1+\gamma_2}, \frac{2}{\gamma_1+\gamma_2+\gamma_3}, 3\Big\} = \max\Big\{\frac{1}{1-\gamma_3-\gamma_4}, \frac{2}{1-\gamma_4}, 3\Big\}. \tag{3.3}$$

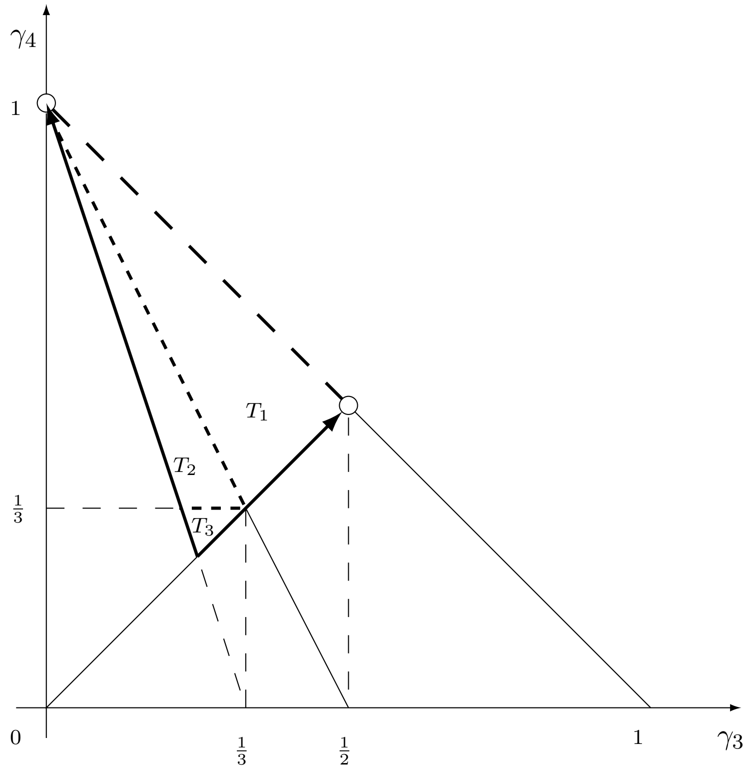
Thus θ is a function of the variables γ_3 and γ_4. Where is it defined? Besides the obvious conditions $0 < \gamma_3 \le \gamma_4$ and $\gamma_3 + \gamma_4 < 1$, one should take into account the inequality $1 - \gamma_3 - \gamma_4 = \gamma_1 + \gamma_2 \le 2\gamma_3$. Under all these restrictions, we can see that the point (γ_3, γ_4) belongs to the triangle

$$T = \big\{(\gamma_3, \gamma_4) : 0 < \gamma_3 \le \gamma_4,\ \gamma_3 + \gamma_4 < 1,\ 3\gamma_3 + \gamma_4 \ge 1\big\},$$

with vertices $(0, 1)$, $\big(\frac{1}{4}, \frac{1}{4}\big)$ and $\big(\frac{1}{2}, \frac{1}{2}\big)$. Since $\gamma_3 + \gamma_4 < 1$ for any pair (γ_3, γ_4) in T, there exists a pair (γ_1, γ_2) such that $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ satisfies all the restrictions for the parameters of the cross H_γ:

$$0 < \gamma_1 \le \gamma_2 \le \gamma_3 \le \gamma_4 \quad \text{and} \quad \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 = 1.$$

In other words, the function θ is defined everywhere on T.



Omitting elementary calculations, we point out the parts of T on which

- all the fractions in (3.3) are equal ($k = 3$);
- only two of them are equal ($k = 2$);
- all the fractions are different ($k = 1$).

Equality $k = 3$ takes place only at the point $(\frac{1}{3}, \frac{1}{3})$, so

$$R^{\frac{3}{2}} \lesssim \mathscr{L}(RH_\gamma) \lesssim R^{\frac{3}{2}} \ln^2 R \quad \text{if} \quad \gamma = \Big(\gamma_1, \frac{1}{3} - \gamma_1, \frac{1}{3}, \frac{1}{3}\Big),$$

where $0 < \gamma_1 \le \frac{1}{6}$.

Equality $k = 2$ is valid on two intervals connecting the point $\big(\frac{1}{3}, \frac{1}{3}\big)$ with the points $(0, 1)$ and $\big(\frac{2}{9}, \frac{1}{3}\big)$, where only the last one $\big(\frac{2}{9}, \frac{1}{3}\big)$ among those three belongs to the corresponding (horizontal) interval. In these cases, the estimates

$$R^{\frac{1}{2(\gamma_1+\gamma_2)}} \lesssim \mathscr{L}(RH_\gamma) \lesssim R^{\frac{1}{2(\gamma_1+\gamma_2)}} \ln R$$

and

$$R^{\frac{3}{2}} \lesssim \mathscr{L}(RH_\gamma) \lesssim R^{\frac{3}{2}} \ln R$$

hold, respectively.

Equality $k = 1$ holds true at the rest of the points of T, which form three smaller triangles (see Figure):

$T_1 = \{(\gamma_3, \gamma_4) : \gamma_3 \le \gamma_4,\ \gamma_3 + \gamma_4 < 1,\ 2\gamma_3 + \gamma_4 > 1\}$, where $\theta = \theta_1 = \frac{1}{1-\gamma_3-\gamma_4}$;

$T_2 = \{(\gamma_3, \gamma_4) : \gamma_4 > \frac{1}{3},\ 2\gamma_3 + \gamma_4 < 1 \le 3\gamma_3 + \gamma_4\}$, where $\theta = \theta_2 = \frac{2}{1-\gamma_4}$

and

$T_3 = \{(\gamma_3, \gamma_4) : \gamma_3 \le \gamma_4 < \frac{1}{3},\ 3\gamma_3 + \gamma_4 \ge 1\}$, where $\theta = \theta_3 = 3$.

Each of them preserves its own power growth of the Lebesgue constants:

$$\mathscr{L}(RH_\gamma) \asymp \begin{cases} R^{\frac{1}{2(\gamma_1+\gamma_2)}}, & (\gamma_3,\gamma_4)\in T_1,\\ R^{\frac{1}{\gamma_1+\gamma_2+\gamma_3}}, & (\gamma_3,\gamma_4)\in T_2,\\ R^{\frac{3}{2}}, & (\gamma_3,\gamma_4)\in T_3. \end{cases}$$

As for arbitrary dimensions, searching appropriate values for the parameters θ and k, basic to Theorem 2.2, in low dimensions has shown that even there the only modus operandi is by brute force. Thus, in every concrete situation, that is, for given $\gamma_1,\dots,\gamma_n$, it is clear how to find θ and k. However, there are simple general situations where it is quite easy to indicate θ omitting lengthy calculations. For example, if $\gamma_1+\gamma_2\le\gamma_3$, then the numbers $\{\theta_\ell\}_{1\le\ell<n}$ do not increase, and hence $\theta=\theta_1=\frac{1}{\gamma_1+\gamma_2}$. From this, it follows that the estimates are almost obvious.

Acknowledgement

This is just an overview of the results already obtained and presented as a plenary talk at the annual meeting of the Georgian Math. Union in Batumi, September 2025. The authors are indebted to the organizers of this meeting for the opportunity to further share the essence of their work with a wider audience in published form.

The authors are grateful to K. P. Kokhas for his help in preparing the figure.

References

1. L. Aizenberg, N. Tarkhanov, Stable expansions in homogeneous polynomials. In: *Complex analysis and dynamical systems III*, 1–22, Contemp. Math., 455, Israel Math. Conf. Proc., Amer. Math. Soc., Providence, RI, 2008.
2. K. I. Babenko, Approximation by trigonometric polynomials in a certain class of periodic functions of several variables. (Russian) *translated from Dokl. Akad. Nauk SSSR* **132** (1960), 982–985; *Soviet Math. Dokl.* **1** (1960), 672–675.
3. E. Belinsky, Behavior of the Lebesgue constants of certain methods of summation of multiple Fourier series. (Russian) In: *Metric Questions of the Theory of Functions and Mappings, Naukova Dumka, Kiev*, 1977, 19–39.
4. E. Belinsky, I. R. Liflyand, The behavior of Lebesgue constants of hyperbolic partial sums. (Russian) *translated from Mat. Zametki* **43** (1988), no. 2, 192–196; *Math. Notes* **43** (1988), no. 1-2, 107–109.
5. P. Dencker, W. Erb, Yu. Kolomoitsev, T. Lomako, Lebesgue constants for polyhedral sets and polynomial interpolation on Lissajous-Chebyshev nodes. *J. Complexity* **43** (2017), 1–27.
6. D. Dung, V. Temlyakov, T. Ullrich, *Hyperbolic Cross Approximation.* Edited and with a foreword by Sergey Tikhonov. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser/Springer, Cham, 2018.
7. M. I. Dyachenko, The order of growth of Lebesgue constants of Dirichlet kernels of monotone type. (Russian) *translated from Vestnik Moskov. Univ. Ser. I Mat. Mekh.* **1989**, no. 6, 33–37, 98; *Moscow Univ. Math. Bull.* **44** (1989), no. 6, 41–46.
8. M. I. Dyachenko, Some problems in the theory of multiple trigonometric series. (Russian) *translated from Uspekhi Mat. Nauk* **47** (1992), no. 5(287), 97–162, 208; *Russian Math. Surveys* 47 (1992), no. 5, 103–171.
9. M. I. Dyachenko, Norms of Dirichlet kernels and of some other trigonometric polynomials in L_p spaces. (Russian) *translated from Mat. Sb.* **184** (1993), no. 3, 3–20; *Russian Acad. Sci. Sb. Math.* **78** (1994), no. 2, 267–282.
10. M. I. Dyachenko, Uniform convergence of hyperbolic partial sums of multiple Fourier series. (Russian) *translated from Mat. Zametki* **76** (2004), no. 5, 723–731; *Math. Notes* **76** (2004), no. 5-6, 673–681.
11. M. Ganzburg, I. R. Liflyand, The Lebesgue constants of Fourier partial sums. In: *Topics in classical and modern analysis—in memory of Yingkang Hu*, 147–158, Appl. Numer. Harmon. Anal., Birkhäuser/Springer, Cham, 2019.
12. Yu. Kolomoitsev, E. Liflyand, Asymptotics of the Lebesgue constants for a d-dimensional simplex. *Proc. Amer. Math. Soc.* **149** (2021), no. 7, 2911–2926.
13. S. V. Konyagin, On Pringsheim convergence of a subsequence of partial sums of a multiple trigonometric Fourier series. (Russian) *translated from Tr. Mat. Inst. Steklova* **323** (2023), 167–180; *Proc. Steklov Inst. Math.* **323** (2023), no. 1, 159–172.
14. S. V. Konyagin, H. Queffélec, E. Saksman, K. Seip, Riesz projection and bounded mean oscillation for Dirichlet series. *Studia Math.* **262** (2022), no. 2, 121–149.
15. S. Konyagin, M. Skopina, Comparison of the L^1-norms of total and truncated exponential sums. (Russian) *translated from Mat. Zametki* **69** (2001), no. 5, 699–707; *Math. Notes* **69** (2001), no. 5-6, 644–651.
16. I. R. Liflyand, The exact order of Lebesgue constants of hyperbolic partial sums of multiple Fourier series. (Russian) *Mat. Zametki* **39** (1986), no. 5, 674–683.
17. E. Liflyand, Sharp estimates of the Lebesgue constants of partial sums of multiple Fourier series. *Trudy Mat. Inst. im. V. A. Steklova* **180** (1987), 151–152.
18. E. R. Liflyand, Lebesgue constants of multiple Fourier series. *Online J. Anal. Comb.* no. 1 (2006), Art. 5, 112 pp.

19. E. R. Liflyand, A. G. Ramm, A. I. Zaslavsky, Estimates from below for Lebesgue constants. *J. Fourier Anal. Appl.* **2** (1996), no. 3, 287–301.
20. B. M. Makarov, M. G. Goluzina, A. A. Lodkin, A. N. Podkorytov, *Problèmes D'analyse Réele.* Cassini, Paris, 2010.
21. A. N. Podkorytov, On the Lebesgue constants for double Fourier series. *Vestnik Leningrad. Univ. Matem.* **7** (1977), 79–84.
22. A. N. Podkorytov, Intermediate rates of growth of Lebesgue constants in the two-dimensional case. (Russian) Numerical methods and questions in the organization of calculations, 7. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **139** (1984), 148–155.
23. V. N. Temlyakov, Approximation of functions with bounded mixed derivative. (Russian) *translation of Trudy Mat. Inst. Steklov* **178** (1986). *English transl. Proc. Steklov Inst. Math.* **1989**, no. 1(178), vi+121 pp.
24. V. Temlyakov, *Multivariate Approximation.* Cambridge Monographs on Applied and Computational Mathematics, 32. Cambridge University Press, Cambridge, 2018.
25. R. M. Trigub, E. S. Belinsky, *Fourier Analysis and Approximation of Functions.* Kluwer Academic Publishers, Dordrecht, 2004.
26. A. A. Yudin, V. A. Yudin, Discrete imbedding theorems and Lebesgue constants. (Russian) *Mat. Zametki* **22** (1977), no. 3, 381–394.
27. A. Zygmund, *Trigonometric Series: Vols. I, II.* Second edition, reprinted with corrections and some additions. Cambridge University Press, London-New York, 1968.

(Received: 25.11.2025; Published online: 10.02.2026)

[1]Department of Mathematics, Bar-Ilan University, 5290002 Ramat-Gan, Israel

[2]St. Petersburg State University, Saint Petersburg, Russia
Email address: liflyand@gmail.com
Email address: a.podkorytov@gmail.com