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CAPUTO DYNAMICAL SYSTEMS

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Abstract. Recent developments in the theory of dynamical systems and their attractors for the Caputo fractional differential equations are reviewed.

1. Introduction

The theory of fractional analysis and fractional differential equations is vast and has deep historical roots. The introduction of Caputo fractional differential equations (FDEs) in the 1960s allowed the initial value problems to be handled more naturally and the asymptotic behaviour of models based on them to be investigated. More recently, mathematically defined dynamical systems generated by the Caputo FDEs and their attractors have been introduced.

Dissipative Caputo FDEs have vector fields satisfying a dissipativity property. For ordinary differential equations (ODEs), this property implies the existence of an absorbing set containing all the long-term dynamical behaviour of the system such as the existence of an attractor. The situation is more complicated for the Caputo FDEs, since these are essentially integral equations, and the dissipative inequalities cannot be so easily exploited.

These developments are presented in the new Springer Briefs monograph *Attractors of Caputo Fractional Differential Equations* by Doan Thai Son, Hoang The Tuan and the author [9]. The ideas and results there will be presented here.

2. Caputo Fractional Differential Equations

Consider an autonomous Caputo fractional differential equation (FDE) of order $\alpha \in (0,1)$ on $\mathbb{R}^d$,

$$^{C}D^{\alpha}_{0+}x(t) = g(x(t)), \tag{2.1}$$

where $g : \mathbb{R}^d \to \mathbb{R}^d$ satisfies the global Lipschitz property. (This property can be weakened later). The Caputo FDE (2.1) with the initial condition $x(0) = x_0$ is essentially the integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(x(s))ds =: (\mathscr{T}x)(t), \tag{2.2}$$

where $\Gamma(\alpha) := \int_0^\infty t^{\alpha-1}e^{-t}dt$ is the Gamma function. Here, $\mathscr{T}$ is an operator from $C([0,T],\mathbb{R}^d)$ into itself.

The proof of the existence of a unique solution of (2.2) is based on the fixed point argument of the contraction map of the operator $\mathscr{T}$. This yields only local existence when using the usual supremum norm [5, Theorem 6.2]. Instead, the proof in [9, Theorem 3.1] uses a weighted norm

$$\|\xi\|_\gamma := \sup_{t\in[0,T]} \frac{\|\xi(t)\|}{E_\alpha(\gamma t^\alpha)} \quad \text{for all } \xi \in C([0,T],\mathbb{R}^d),$$

where $\gamma > 0$ is a suitable constant chosen to obtain a contraction map. The weight function $E_\alpha(\cdot)$ is the Mittag–Leffler function [5, 9]. This is essentially a generalisation of the exponential function

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defined with one or two parameters as follows:

$$E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha + 1)}, \quad E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha + \beta)}, \quad t \in \mathbb{R},$$

with α, $\beta > 0$, not necessarily integers, and $t \in \mathbb{R}$. Clearly, $E_\alpha(t) = E_{\alpha,1}(t)$.

The solutions of Caputo FDEs depend on their history and so cannot be combined, i.e., patched together, as for ODEs. They can even intersect at finite time in two or more dimensions [1, 2, 9]. In particular, they do not generate a semi-group, i.e., a semi-dynamical system on $\mathbb{R}^d$.

3. Scalar Caputo FDEs

In the scalar case, Cong & Tuan [1, 2] showed that any two solutions of (2.1) satisfy the lower bound

$$|x_2(t) - x_1(t)| \geq |x_2(0) - x_1(0)|\, E_\alpha\big(-Lt^\alpha\big).$$

Hence two solutions with different initial values cannot intersect at finite time. This is also true for higher dimensional systems with a triangular vector field, i.e., with

$$g(x_1, x_2, \ldots, x_d) = (g_1(x_1), g_2(x_1, x_2), \ldots, g_d(x_1, x_2, \ldots, x_d)).$$

Otherwise, the solutions may intersect (see [1]) for a two-dimensional counter-example.

Cong & Tuan [1] showed that the evolution mappings

$$\Phi_{s,t}(\cdot) : \mathbb{R} \to \mathbb{R}, \qquad s, t \geq 0,$$

generated by the solutions of a scalar Caputo FDE with $\Phi_{0,t}x_0 = x(t, x_0)$ are a two-parameter family of bijections satisfying

$$\Phi_{s,t} := \Phi_{0,t} \circ \Phi_{0,s}^{-1} \quad \text{for all} \quad s, t \geq 0.$$

This means that to continue the solution at $x(s)$ beyond time $s > 0$, we have to find its starting point $x_0 = \Phi_{0,s}^{-1}x(s)$ and then to map forward from there to obtain

$$x(t) = \Phi_{0,t}x_0 = \Phi_{0,t} \circ \Phi_{0,s}^{-1}x(s), \quad t \geq s \geq 0.$$

Cong & Tuan called this a *nonlocal dynamical system*. It differs significantly from the two-parameter families generated by nonsutonic ODEs via the concatenation of solutions (see [11]).

4. Caputo Semi-dynamical Systems

The representation of the integral equation (2.2) for the Caputo FDE (2.1) is a special case of the Volterra integral equation

$$x_f(t) = f(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(x_f(s)\, ds,$$

where $f \in \mathfrak{C} := C(\mathbb{R}^+, \mathbb{R}^d)$.

Sell [12] showed that the family of Volterra operators $\{\mathfrak{T}_t\}_{t\geq 0}$ on $\mathfrak{C}$ is defined by

$$(\mathfrak{T}_t f)(\theta) = f(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t + \theta - s)^{\alpha-1} g(x(s))\, ds, \qquad \theta \in \mathbb{R}^+,$$

with $x(t)$ given by (7.3). So, $(\mathfrak{T}_t f)(0) = x(t)$, $t \geq 0$, is a semi-group on $\mathfrak{C}$ with the topology of uniform convergence on compact subsets. This topology is induced by the metric

$$\rho(f, h) := \sum_{n=1}^{\infty} \frac{1}{2^n} \rho_n(f, h), \qquad \rho_n(f, h) := \frac{\sup_{t\in[0,n]} \|f(t) - h(t)\|}{1 + \sup_{t\in[0,n]} \|f(t) - h(t)\|}.$$

With the restriction to $f(t) \equiv id_{x_0}$, Doan & Kloeden [6] called this the *Caputo semi-group.* This semi-group represents the Caputo FDE (2.1) as an autonomous semi-dynamical system on the space $\mathfrak{C}$.

Of course, the $\mathfrak{T}_t f$ do not remain equal to id_{x_0} unless x_0 is a zero of the vector field g, i.e., a steady state solution of the Caputo FDE. Two solutions of the Caputo FDE for different initial values may intersect at finite time here, but in the above semi-group formulation they differ in the "flag" $(\mathfrak{T}_t id_{x_0})(\theta)$, $\theta > 0$, that they "carry".

5. Dissipative Caputo FDEs

According to a fractional counterpart of the Tuan & Trinh [13] chain rule, the solutions of the Caputo FDE (2.1) satisfy

$$^{C}D^{\alpha}_{0+}\|x(t)\|^2 \leq 2\left\langle x(t), {}^{C}D^{\alpha}_{0+}x(t)\right\rangle.$$

Therefore, if the vector field g of equation (2.1) satisfies the dissipativity condition

$$\langle x, g(x)\rangle \leq a - b\|x\|^2 \tag{5.1}$$

for $a, b > 0$, then along the solutions of equation (2.1), we have

$$^{C}D^{\alpha}_{0+}\|x(t)\|^2 \leq 2\left\langle x(t), g(x(t))\right\rangle \leq 2a - 2b\|x(t)\|^2.$$

This can be integrated (see Kloeden [10]) to obtain

$$\|x(t)\|^2 \leq \|x_0\|^2 E_\alpha(-2bt^\alpha) + \frac{a}{b}\left(1 - E_\alpha(-2bt^\alpha)\right). \tag{5.2}$$

The global existence and uniqueness of solutions of a Caputo FDE can be shown by using this inequality when the vector field satisfies the local Lipschitz condition and the above dissipativity condition [10].

It follows from (5.2) that $\|x(t,x_0)\| \leq R$ for all $t \geq 0$ when $\|x_0\| \leq R$ and $R^2 \geq 1 + \frac{a}{b} =: R_*^2$. Thus the set

$$\mathscr{B} := \left\{x \in \mathbb{R}^d \,:\, \|x\|^2 \leq 1 + \frac{a}{b} =: R_*^2\right\}$$

is an absorbing set for the solutions of the Caputo FDE (2.1). In particular, there exists $T_R \geq 0$ such that $\|x(t,x_0)\| \in \mathscr{B}^*$, i.e., $\|x(t,x_0)\| \leq R^*$ for all $\|x_0\| \leq R$ and $t \geq T_R$. The set $\mathscr{B}$ is also positive invariant, i.e., solutions starting in it remain in it for all future time.

This set $\mathscr{B}$ is compact in $\mathbb{R}^d$ and contains the corresponding omega limit set

$$\Omega := \left\{y \in\in \mathbb{R}^d \,:\, \exists \text{ all bnd'd}\, x_{0,n} \in \mathbb{R}^d, t_n \to \infty \text{ such that } x(t_n, x_{0,n}) \to y\right\},$$

which is a nonempty compact subset of $\mathscr{B}$ and contains the future dynamics of the autonomous Caputo FDE (2.1). However, Ω cannot, strictly speaking, be called the attractor of the Caputo FDE, since this does not generate a semi-group in $\mathbb{R}^d$. Perhaps, it could be called an *attracting set*.

Note that Ω contains the steady state solutions of the Caputo FDE (2.1) (see [9]).

6. Attractors of Caputo Semi-dynamical Systems

For some time, a main difficulty was how to apply the dissipativity condition (5.1) to the vector field g inside the integral equations (7.4) for the $(\mathfrak{T}_t f)(\theta)$ terms in order to establish the existence of an absorbing set in the space $\mathfrak{C}$.

The Tuan & Trinh chain rule inequality can be used for the case $\theta = 0$, i.e., for the Caputo FDE (2.1). As shown above, this leads to a compact positively invariant absorbing set $\mathscr{B}$ for its solutions R^d. Let $R^2 \geq 1 + \frac{a}{b} := R_*^2$. Then the dissipativity condition (5.1) and the solution estimates yield

$$B_R := \sup_{t\geq 0, \|x_0\|\leq R} \|x(t,x_0)\| < \infty, \qquad B_R^g := \sup_{\|x\|\leq R} \|g(x)\| < \infty,$$

where the continuity of the vector field g has been used in the second bound. Such bounds can then be used to estimate the $(\mathfrak{T}_t id_{x_0})(\theta)$ terms for $\theta > 0$. However, due to the growth nature of the obtained bounds, Doan & Kloeden [6] found it necessary to restrict a subspace $\mathfrak{C}_\alpha$ of $\mathfrak{C}$ with the weighted norm

$$\|f\|_\alpha := \|f(0)\| + \sum_{N=1}^{\infty} \frac{1}{2^N N^\alpha}\|f\|_N, \qquad \|f\|_N := \sup_{t\in[1/N,N]} \|f(t)\|, \quad N = 1, 2, \ldots.$$

Then they showed that the closed and bounded subset

$$\mathfrak{B} := \left\{\chi \in \mathfrak{C} \,:\, \|\chi\|_\alpha \leq 2R^* + \frac{B^g_{R^*}}{\alpha\Gamma(\alpha)} =: \widehat{R}^*\right\}$$

of $\mathfrak{C}_\alpha$ is an absorbing set for the Caputo semi-group $(\mathfrak{T}_t id_{x_0})(\cdot)$ and absorbs the bounded sets of constant initial data id_{x_0} with $\|x_0\| \leq R$ at time $t \geq T_R$

Doan & Kloeden [6] also showed that the operators $(\mathfrak{T}_t id_{x_0})(\cdot)$ are asymptotically compact, i.e., for every sequence $t_n \to \infty$ there is a subsequence $t_n \to \infty$ such that $(\mathfrak{T}_{t_n} id_{x_0})(\cdot) \to \infty\ \chi^*(\cdot) \in \mathfrak{C}_\alpha$.

The theory of autonomous semi-dynamical systems on a Banach space [11] can then be applied to prove the existence of a global attractor.

Theorem 6.1 ([7,9]). *Suppose that the vector field g is locally Lipschitz and satisfies the dissipativity condition* (5.1). *Then the semi-group $\{\mathfrak{T}_t\}_{t\geq 0}$ on the space $\mathfrak{C}_\alpha$ has a global attractor $\mathfrak{A} \subset \mathfrak{C}_\alpha$, which has the structure*

$$\mathfrak{A} = \bigcap_{t\geq 0} \mathfrak{T}_t \mathfrak{B}$$

and attracts all bounded subsets id_D of $\mathfrak{C}_\alpha$ consisting of constant initial functions id_{x_0} for $x_0 \in D$ for the bounded subsets D of $\mathbb{R}^d$ in the sense that

$$\lim_{t\to\infty} \operatorname{dist}_{\mathfrak{C}_\alpha} (\mathfrak{T}_t id_D, \mathfrak{A}) = 0.$$

In particular, $\mathfrak{A}$ is a compact subset of $\mathfrak{C}_\alpha$ and is invariant in the sense that

$$\mathfrak{T}_t \mathfrak{A} = \mathfrak{A} \qquad t \geq 0.$$

A distinctive feature of the attractor $\mathfrak{A}$ here is that its universe of attraction consists of bounded subsets id_D of $\mathfrak{C}_\alpha$ as described in the theorem. The solutions do not usually remain in this universe.

Note that the corresponding omega limit set Ω in $\mathbb{R}^d$ satisfies

$$\Omega = \left\{\chi(0) \in \mathbb{R}^d \,:\, \chi \in \mathfrak{A}\right\}.$$

This set is determined solely by the limiting dynamics of the solution $x(t, x_0) = (\mathfrak{T}_t id_{x_0})(0)$ of the Caputo FDE. The solution then determines the other parts of the $\mathfrak{T}_t id_{x_0}$ and hence the attractor $\mathfrak{A}$ itself.

6.1. **Scalar and triangular systems.** The above discussion also applies to Caputo FDEs with scalar and triangular vector fields. Their special structure allows more to be said about the dynamics inside the omega limit set Ω.

In the scalar case, a dissipative vector field must have at least one zero, which corresponds to a steady state solution. When these are hyperbolic, i.e., the eigenvalue of the linear part has a non-zero real part, then the omega limit set consists of the steady state solutions and the heteroclinic trajectories joining them.

It has essentially the same geometric structure as the omega limit set (in this case, it is also the global attractor) of the corresponding ODE, with the major exception that the rates of attraction are not exponential, but rather subexponential of the form $E_\alpha(-t^\alpha) \sim t^{-\alpha}$.

A similar situation holds for the triangular vector fields with certain restrictions (see [7,9]).

7. Non-autonomous Caputo FDEs

The non-autonomous situation with time dependent vector fields is considerably more complex both for fractional and ordinary differential equations [11]. More can be said when the time change in the vector field is given by a control system.

Consider the non-autonomous Caputo fFDE of order $\alpha \in (0, 1)$ on $\mathbb{R}^d$ of the form

$$^{C}D^{\alpha}_{0+}x(t) = g(x(t), \vartheta_t(p)) \quad \text{ for } t \in [0, T[, \tag{7.1}$$

with a control system ϑ_t in the vector field $g(x, p)$. Specifically, $\vartheta_t : P \to P$, $t \in \mathbb{R}$, is a group of operators, i.e., an autonomous dynamical system, and P is a compact metric space. The solution of the Caputo FDE (7.1) with initial condition $x(0) = x_0$ and $p_0 \in P$ satisfies the integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-)^{\alpha-1} g(x(s), \vartheta_s(p_0))ds. \tag{7.2}$$

Define the operators $\mathfrak{T}_t : \mathfrak{C}_\alpha \times P \to \mathfrak{C}_\alpha$ as follows:

$$(\mathfrak{T}_t(id_{x_0}, p_0))(0) := x(t, x_0, p_0) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(x(s, x_0, p_0), \vartheta_s(p_0))\, ds \tag{7.3}$$

and

$$(\mathfrak{T}_t(id_{x_0}, p_0))(\theta) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t+\theta-s)^{\alpha-1} g(x(s, x_0, p_0), \vartheta_s(p_0))\, ds \tag{7.4}$$

for $\theta > 0$.

The resulting Caputo semi-group $\{(\mathfrak{T}_t, \vartheta_t)\}_{t\in\mathbb{R}^+}$ on $\mathfrak{C}_\alpha \times P$ is called a *Caputo skew-product flow.*

As in the autonomous case, the attractor in the skew-product flow case also attracts only a restricted class of initial values in $\mathfrak{C}_\alpha$, which is not invariant under the dynamics. The following theorem is due to Cui & Kloeden [3, Theorem 2].

Theorem 7.1. *Suppose that the vector field g is locally Lipschitz in both variables and satisfies the dissipativity condition* (5.1) *uniformly in $p \in P$. Then the semi-group $\{(T_t, \vartheta_t)\}_{t\in\mathbb{R}^+}$ on the space $\mathfrak{C}_\alpha \times P$ corresponding to the integral equations* (7.2) *has an attractor $\mathfrak{A} \subset \mathfrak{C}_\alpha \times P$, which attracts the bounded subsets of $\mathfrak{C}_\alpha \times P$ consisting of constant initial functions in $\mathfrak{C}_\alpha$ and having the structure*

$$\mathfrak{A} = \bigcup_{p\in P} \mathfrak{A}(p) \times \{p\},$$

where the $\mathfrak{A}(p)$ are the compact subsets of $\mathfrak{C}_\alpha$. Moreover, the sets $\mathfrak{A}(p)$ are positively invariant in the sense that

$$T_t(\mathfrak{A}(p), p) = \mathfrak{A}(\vartheta_t(p)), \qquad t \geq 0, p \in P,$$

and attract the inverse map in the sense that

$$\lim_{t\to\infty} \mathrm{dist}_{\mathfrak{C}_\alpha} (T_t(\mathfrak{D}, \vartheta_{-t}(p)), \mathfrak{A}(p)) = 0, \qquad p \in P,$$

attracts all bounded subsets id_D of $\mathfrak{C}_\alpha$ consisting of constant initial functions id_{x_0} for $x_0 \in D$ for bounded subsets D of $\mathbb{R}^d$.

7.1. **An explicit example.** As a motivating example, consider the Caputo scalar FDE

$$^C D^\alpha_{0+} x(t) = -x(t) + \cos t. \tag{7.5}$$

Here, P is the *hull* [11, 12] of the functions $\cos(\cdot)$, i.e.,

$$P = \bigcup_{0\leq\tau\leq 2\pi} \cos(\tau + \cdot),$$

which is a compact metric space with the metric induced by the supremum norm $d_P(p_1, p_2) = \sup_{t\in\mathbb{R}} |p_1(t) - p_2(t)|$. In addition, let $\vartheta_t : P \to P$ be the left shift operator $\vartheta_t(\cos(\cdot)) = \cos(t + \cdot)$. This shift operator is continuous in the above metric. Indeed, it is an isometry with $d_P(\vartheta_t(p_1), \vartheta_t(p_2)) = d_P(p_1, p_2)$, $\quad p_1, p_2 \in P$.

Note that the linear Caputo equation (7.5) is strictly contractive. Let $x(t) = x(t, x_0, p_0)$ and $y(t) = y(t, y_0, p_0)$. Then

$$^C D^\alpha_{0+} z(t) = -z(t), \qquad z(t) := x(t) - y(t).$$

Hence $z(t) = E_\alpha(-t^\alpha) z(0)$ that yields the strictly contractive property

$$|z(t)| = E_\alpha(-t^\alpha)|z(0)|. \tag{7.6}$$

In particular, $z(t) \to 0$ as $t \to \infty$ with the non-exponential rate $t^{-\alpha}$ (see [9]).

To see what happens to individual solutions, note that the linear Caputo FDE (7.5) has explicit solutions given by the variation of constants formula [9, Lemma 1.4]

$$x(t, x_0, p_0) = E_\alpha(-t^\alpha) x_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha) p(s, p_0) ds.$$

Let $\tau > 0$ and let $q_{-\tau} = p(-\tau, p_0)$, then $p_0 = p(\tau, q_{-\tau})$. Following [8],

$$x(\tau, x_0, q_{-\tau}) = E_\alpha(-\tau^\alpha)x_0 + \int_{-\tau}^{0} (-\nu)^{\alpha-1} E_{\alpha,\alpha}(-(-\nu)^\alpha)p(\nu, p_0)d\nu,$$

since $p(\nu + \tau, q_{-\tau}) = p(\nu, p_0)$. Hence by the pullback limit (see [3, 11]),

$$\lim_{\tau\to\infty} x(\tau, x_0, q_{-\tau}) = a(p_0) := \int_{-\infty}^{0} (-\nu)^{\alpha-1} E_{\alpha,\alpha}(-(-\nu)^\alpha)p(\nu, p_0)d\nu.$$

The strictly contractive condition (7.6) gives

$$|x(\tau, x_0, q_{-\tau}) - y(\tau, y_0, q_{-\tau})| = E_\alpha(-\tau^\alpha)|x_0 - y_0|,$$

which means all such solutions converge in the pullback sense to the same limit, i.e., $a(p_0)$. In particular, the pullback attractor sets $\mathfrak{A}(p)$ in Theorem 7.1 are the singleton sets $\mathfrak{A}(p) = \{a(p)\}$. In fact, this corresponds to a single entire solution $\chi^*(t) = a(p^*(t))$ in $\mathbb{R}$ of the Caputo FDE (7.5).

In its original form, the strictly contractive condition (7.6) also shows that all solutions of the Caputo FDE (7.5) converge forwards in time to this entire pullback solution.

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