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THE CHARACTERISTIC PROBLEM FOR ONE CLASS OF HIGH-ORDER NONLINEAR EQUATION OF COMPOSITE TYPE

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Abstract. The multidimensional characteristic problem in the conical domain for a class of high-order nonlinear equations of composite type is considered. The theorems on the existence, uniqueness and nonexistence of solutions of this problem are proved.

In the Euclidean space $\mathbb{R}^{n+1}$ of variables $x=(x_1,\dots,x_n)$ and t, we consider the following nonlinear equation of composite type:

$$\Box^2\Delta u+f(u)=F, \tag{1}$$

where $\Box:=\frac{\partial^2}{\partial t^2}-\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}$ is the wave operator, $\Delta:=\frac{\partial^2}{\partial t^2}+\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, f and F are the given and u is an unknown real functions, $n\geq 2$.

By $D_T:|x|<t<T-|x|,\ |x|<\frac{1}{2}T$ we denote the conical domain in $\mathbb{R}^{n+1}$, bounded below by the characteristic cone of the future $S_+:t=|x|,\ x\in\mathbb{R}^n$, with the vertex at the point $O(o,\dots,o,o)$, and above by the characteristic cone of the past $S_-:t=T-|x|,\ x\in\mathbb{R}^n$, with the vertex at the point $O_1(o,\dots,o,T)$.

For equation (1) in domain D_T, we consider the following characteristic problem: find a solution u of equation (1) in domain D_T according to the boundary conditions

$$u|_{\partial D_T}=0,\qquad \Box u|_{\partial D_T}=0. \tag{2}$$

Note that for equation (1), boundary value problems in domains of different geometric structure were investigated in the works [1,5–7], and for composite equations of a type, different from (1), were considered in the works [2–4,8,9,12].

Let

$$\overset{o}{C}{}^k\left(\overline{D}_T\right):=\left\{u\in C^k\left(\overline{D}_T\right):u|_{\partial D_T}=0,\quad \Box u|_{\partial D_T}=0\right\},\quad k\geq 2. \tag{3}$$

Introduce the Hilbert space $\overset{o}{W}{}^2_{\Box}(D_T)$ as a completion of the classical space $\overset{o}{C}{}^6\left(\overline{D}_T\right)$ with respect to the norm

$$\begin{aligned}\|u\|^2_{\overset{o}{W}{}^2_{\Box}(D_T)}&=\|u\|^2_{W_2^1(D_T)}+\|\Box u\|^2_{W_2^1(D_T)}\\&=\int\limits_{D_T}\left[u^2+\sum_{i=1}^n\left(\frac{\partial u}{\partial x_i}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2\right]dxdt\\&+\int\limits_{D_T}\left[(\Box u)^2+\sum_{i=1}^n\left(\frac{\partial \Box u}{\partial x_i}\right)^2+\left(\frac{\partial \Box u}{\partial t}\right)^2\right]dxdt.\end{aligned} \tag{4}$$

Before introducing a notion of a weak generalized solution of problem (1), (2) from the space $\overset{o}{W}{}^2_{\Box}(D_T)$, let us suppose that $u\in\overset{o}{C}{}^6\left(\overline{D}_T\right)$ is a classical solution of this problem. Multiplying both parts of equation (1) by an arbitrary function $\varphi\in\overset{o}{C}{}^3\left(\overline{D}_T\right)$ and integrating the obtained equality by

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parts over the domain D_T, due to (2), (3), we obtain

$$\int\limits_{D_T}\left[\sum_{i=1}^{n}\frac{\partial\Box u}{\partial x_i}\frac{\partial\Box\varphi}{\partial x_i}+\frac{\partial\Box u}{\partial t}\frac{\partial\Box\varphi}{\partial t}\right]dxdt-\int\limits_{D_T}f\left(u\right)\varphi dxdt$$

$$=-\int\limits_{D_T}F\varphi dxdt\quad\forall\varphi\in\overset{o}{C}{}^{6}\left(\overline{D}_T\right).\tag{5}$$

When deriving (5), it was taken into account that on the characteristic surface ∂D_T the derivative with respect to the conormal $\frac{\partial}{\partial\mathcal{N}}=v_t\frac{\partial}{\partial t}-\sum_{i=1}^{n}v_{x_i}\frac{\partial}{\partial x_i}$ is an inner differential operator, where $v=(v_{x_1},\dots,v_{x_n},v_t)$ is the unit outward normal vector to ∂D_T, and thus $\frac{\partial\varphi}{\partial\mathcal{N}}|_{\partial D_T}=0$, since $\varphi|_{\partial D_T}=0$, and it was also taken into account that by virtue of (2) and the formulas for integration by parts, we get

$$\int\limits_{D_T}\Box^2\Delta u\cdot\varphi dxdt=\int\limits_{\partial D_T}\left[\varphi\frac{\partial}{\partial\mathcal{N}}\Box\Delta u-\Box\Delta u\cdot\frac{\partial}{\partial\mathcal{N}}\varphi\right]ds$$

$$+\int\limits_{D_T}\Box\Delta u\cdot\Box\varphi dxdt=\int\limits_{D_T}\Delta\Box u\cdot\Box\varphi dxdt=\int\limits_{\partial D_T}\left[\frac{\partial\Box u}{\partial v}\cdot\Box\varphi-\Box u\cdot\frac{\partial\Box\varphi}{\partial v}\right]ds$$

$$-\int\limits_{D_T}\left[\sum_{i=1}^{n}\frac{\partial\Box u}{\partial x_i}\frac{\partial\Box\varphi}{\partial x_i}+\frac{\partial\Box u}{\partial t}\frac{\partial\Box\varphi}{\partial t}\right]dxdt=-\int\limits_{D_T}\left[\sum_{i=1}^{n}\frac{\partial\Box u}{\partial x_i}\frac{\partial\Box\varphi}{\partial x_i}+\frac{\partial\Box u}{\partial t}\frac{\partial\Box\varphi}{\partial t}\right]dxdt.$$

In a certain sense, the converse statement is also true, i.e., if $u\in\overset{o}{C}{}^{6}\left(\overline{D}_T\right)$ satisfies the integral equality (5) for any $\varphi\in\overset{o}{C}{}^{3}\left(\overline{D}_T\right)$, then standard reasoning implies that u is a solution of equation (1) in the domain D_T and, by definition (3) of the space $\overset{o}{C}{}^{6}\left(\overline{D}_T\right)$, satisfies the boundary conditions (2). Below, we will use equality (5) as a basis for defining a weak generalized solution of problem (1), (2) in the Hilbert space $\overset{o}{W}{}^{2}_{\Box}(D_T)$ from (4) under certain conditions imposed on the growth rate of the nonlinearity of the function $f\left(u\right)$.

Now, we present the conditions imposed on the function $f\left(u\right)$ from (1):

$$f\in C\left(\mathbb{R}\right),\quad\left|f\left(u\right)\right|\le M_1+M_2\left|u\right|^{\alpha},\quad\alpha=\text{const}\ge0,\quad u\in\mathbb{R},\tag{6}$$

where $M_i=\text{const}\ge0$, $i=1,2$, and

$$0\le\alpha=\text{const}<\frac{n+1}{n-1}.\tag{7}$$

Remark 1. The embedding operator $I:W_2^1\left(D_T\right)\to L_q\left(D_T\right)$ is a linear compact operator for $1<q<\frac{2(n+1)}{n-1}$ and $n>1$ [11]. At the same time, the Nemitski operator $K:L_q\left(D_T\right)\to L_2\left(D_T\right)$, acting according to the formula $Ku=f\left(u\right)$, where $u\in L_q\left(D_T\right)$ and the function f satisfies conditions (6) and (7), is continuous and bounded for $q\ge2\alpha$ [12]. Therefore, if $\alpha<\frac{n+1}{n-1}$, then there exists a number q such that $1<q<\frac{2(n+1)}{n-1}$ and $q\ge2\alpha$. In this case, the operator

$$K_0=KI:W_2^1\left(D_T\right)\to L_2\left(D_T\right)$$

is continuous and compact. It follows in particular that if $u\in\overset{o}{W}{}^{2}_{\Box}(D_T)\subset W^{1}_{2}(D_T)$, then $f\left(u\right)\in L_2\left(D_T\right)$.

Definition 1. Let conditions (6), (7) and $F\in L_2\left(D_T\right)$ be satisfied. The function $u\in\overset{o}{W}{}^{2}_{\Box}(D_T)$ is called a weak generalized solution of problem (1), (2), if the integral equality (5) is valid for any function $\varphi \in \overset{o}{W}{}^2_{\square}(D_T)$, i.e.,

$$\int_{D_T} \left[\sum_{i=1}^{n} \frac{\partial \square u}{\partial x_i} \frac{\partial \square \varphi}{\partial x_i} + \frac{\partial \square u}{\partial t} \frac{\partial \square \varphi}{\partial t} \right] dxdt$$

$$- \int_{D_T} f(u)\,\varphi dxdt = - \int_{D_T} F \varphi dxdt \qquad \forall \varphi \in \overset{o}{H}{}^2_{\square}(D_T). \tag{8}$$

Note that by Remark 1, the integral $\int_{D_T} f(u)\,\varphi dxdt$ in the left-hand side of equality (8) is well-defined, since $u \in \overset{o}{W}{}^2_{\square}(D_T)$ implies that $f(u) \in L_2(D_T)$ and, consequently, $f(u)\,\varphi \in L_1(D_T)$.

Let us consider the following condition imposed on the function f :

$$\lim_{|u| \to \infty} \inf \frac{f(u)}{u} \leq 0. \tag{9}$$

Theorem 1. *Let conditions* (6), (7) *and* (9) *be fulfilled. Then for any function* $F \in L_2(D_T)$, *the boundary value problem* (1), (2) *has at least one weak generalized solution in the space* $\overset{o}{W}{}^2_{\square}(D_T)$ *in the sense of Definition* 1.

Theorem 2. *Let conditions* (6), (7) *be fulfilled and the function* f *be non−inereasing, i.e.,* $(f(u)\text{-}f(v))(u-v) \leq 0 \quad \forall u, v \in \mathbb{R}$. *Then for any function* $F \in L_2(D_T)$, *the boundary value problem* (1), (2) *cannot have more than one weak generalized solution* $u \in \overset{o}{W}{}^2_{\square}(D_T)$ *in the sense of Definition* 1.

Corollary. Let conditions (6), (7), (9) be fulfilled and the function f be non-increasing. Then for any function $F \in L_2(D_T)$, the boundary value problem (1), (2) has a unique weak generalized solution in the space $\overset{o}{W}{}^2_{\square}(D_T)$ in the sense of Definition 1.

Note that if condition (9) is violated, then a sufficiently wide class of functions $F \in L_2(D_T)$ can be defined when problem (1), (2) does not have a weak generalized solution in the space $\overset{o}{W}{}^2_{\square}(D_T)$ in the sense of Definition 1. Indeed, the following theorem holds.

Theorem 3. *Let conditions* (6), (7) *be fulfilled and*

$$f(u) \leq -|u|^{\gamma} \qquad \forall u \in \mathbb{R}, \qquad \gamma = \text{const} > 1. \tag{10}$$

If $F = \lambda F_0$ *with* $\lambda = \text{const} > 0$, $F_0 > 0$ *and* $F_0 \in L_2(D_T)$, *then there exists a number* $\lambda_0 = \lambda_0(F_0, \gamma)$ *such that for* $\lambda > \lambda_0$, *problem* (1), (2) *does not have a weak generalized solution in the space* $\overset{o}{W}{}^2_{\square}(D_T)$ *in the sense of Definition* 1.

Note that when condition (10) is satisfied, condition (9) is violated.

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