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LYAPUNOV FUNCTIONS AND THEIR COMPUTATION BY LINEAR OPTIMIZATION

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Abstract. Lyapunov functions characterize the stability of attractors in dynamical systems. Their analytical construction for general systems, however, remains a formidable task. Therefore, numerous numerical methods for their computation have been developed. We give a short overview of the theory of Lyapunov functions and discuss how they can be parameterized using linear programming in the so-called CPA method. We discuss a few of the many extensions of the CPA method, in particular to differential inclusions, and we discuss how the linear programming problem generated in the CPA method can often be solved very efficiently. We give numerous concrete examples to illustrate the technique of the CPA method.

1. Lyapunov Functions

We consider the ordinary differential equation (ODE)

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^n. \tag{1.1}$$

Sufficient conditions for (1.1) to define a dynamical system are that $\mathbf{f}$ is locally Lipschitz and bounded, where the second condition is not essential because one can always replace $\mathbf{f}$ with $(1 + \|\mathbf{f}\|)^{-1}\mathbf{f}$ to obtain an ODE with the same trajectories (see, e.g., Chapter 3 in [41]). It is well-known that obtaining an analytical solution to (1.1) is in general very difficult or impossible. However, one is often more interested in long-term results, than in exact solutions of the equation, that is, in the qualitative behavior of the dynamical systems, rather than quantitative one. For this kind of analysis, the Lyapunov theory has proven itself to be extremely useful [29, 34, 37, 42, 44]. The centrepiece of the Lyapunov theory is the (complete) Lyapunov function, which is a continuous, real-valued function from the state space $\mathbb{R}^n$, that is non-increasing along solution trajectories of the system and decreasing if possible. Solution trajectories of the system. The only case when it is non-decreasing along a trajectory, is if the trajectory is periodic, or infinitesimally close to being periodic; the technical term is *chain-recurrent*.

The usefulness of the Lyapunov theory is that it bypasses the solution of the system. That is, to show that the Lyapunov function is decreasing along solution trajectories, one does not need to know the solution itself. In more detail, let $t \mapsto \boldsymbol{\phi}(t, \mathbf{x})$ be the solution to (1.1) starting at $\mathbf{x}$ at time $t = 0$, let $V\colon \mathbb{R}^n \to \mathbb{R}$ be a differentiable function that has a strict minimum on the compact set $A \subset \mathbb{R}^n$, and let D be a neighbourhood of A. Then

$$\frac{d}{dt} V(\boldsymbol{\phi}(t,\mathbf{x}))\Big|_{t=0} = \nabla V(\boldsymbol{\phi}(t,\mathbf{x})) \cdot \dot{\boldsymbol{\phi}}(t,\mathbf{x})\Big|_{t=0} = \nabla V(\boldsymbol{\phi}(t,\mathbf{x})) \cdot \mathbf{f}(\boldsymbol{\phi}(t,\mathbf{x}))|_{t=0} = \nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}).$$

So, if $\dot{V}(\mathbf{x}) := \nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) < 0$ for all $\mathbf{x} \in D \setminus A \subset \mathbb{R}^n$, then V is decreasing along all solution trajectories passing through $D \setminus A$. In particular, if $D \setminus A$ is bounded and forward invariant (solutions cannot leave it), then all solutions starting in $D \setminus A$ must asymptotically approach the minimum of V at A. In applications, the most common setting is that system (1.1) has an equilibrium $A = \{\mathbf{x}_0\}$, which can without loss of generality be assumed to be located at the origin. In this setting, a Lyapunov function $V \in C^1(\mathbb{R}^n)$ for system (1.1) is the function satisfying the conditions

$$\alpha(\|\mathbf{x}\|) \le V(\mathbf{x}) \ \text{ and } \ \dot{V}(\mathbf{x}) \le -\beta(\|\mathbf{x}\|) \ \text{ for all } \mathbf{x} \in D,$$

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where $D \subset \mathbb{R}^n$ is a neighbourhood of the equilibrium at the origin, and $\alpha, \beta \colon [0,\infty) \to [0,\infty)$ are strictly increasing, continuous functions with $\alpha(0) = \beta(0) = 0$ and $\lim_{x\to\infty} \alpha(x) = \infty$; the so-called comparison functions of class $\mathscr{K}_\infty$ and class $\mathscr{K}$, respectively, [29, 33]. Any compact sublevel set $L_c := \{\mathbf{x} \in \mathbb{R}^n \colon V(\mathbf{x}) \le c\} \subset D$, $c > 0$, is forward invariant, and every solution starting in it is asymptotically attracted to the equilibrium at the origin. Hence, sublevel sets of V act as *traps* for the dynamics; $\mathbf{x} \in L_c$ implies $\boldsymbol{\phi}(t,\mathbf{x}) \in L_c$ for all $t \ge 0$.

The Lyapunov function was introduced by Aleksandr Lyapunov in 1892 to show the stability of equilibrium points [37], but the theory has since be extended to show that the qualitative behaviour of a dynamical system is characterized by the so-called complete Lyapunov function for the system [3,6,30]. Indeed, for the dynamical system defined by (1.1) there exists a smooth complete Lyapunov function $V \colon \mathbb{R}^n \to \mathbb{R}$ [27], and for a smooth $\mathbf{f}$ one can even prescribe its orbital derivative on a compact subset, where the flow is of a gradient-like [17] character.

We give a few simple examples of Lyapunov functions. For a physical system (1.1), in which the total energy E is conserved, we can set $V = E$, then $\dot{V}(\mathbf{x}) = \dot{E}(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{R}^n$. If the system is dissipative, which is most often the case in practice, since the amount of energy used to sustain movement is asymptotically attracted to equilibrium, then $\dot{V}(\mathbf{x}) = \dot{E}(\mathbf{x}) < 0$, unless $\mathbf{x}$ is an equilibrium, and it follows that the state of the system is asymptotically attracted to an equilibrium, since the system cannot lose energy indefinitely. Indeed, the Lyapunov function can be viewed as a mathematical extension of the physical concept of (dissipative) energy.

For a gradient system $\dot{\mathbf{x}} = -\nabla U(\mathbf{x})$, $U \colon \mathbb{R}^n \to \mathbb{R}$, $V = U$ is a natural Lyapunov function if U has a strict local minimum at some $\mathbf{x}_0 \in \mathbb{R}^n$, because

$$\begin{aligned}\frac{d}{dt} U(\boldsymbol{\phi}(t,\mathbf{x}))\Big|_{t=0} &= \nabla U(\boldsymbol{\phi}(t,\mathbf{x})) \bullet \mathbf{f}(\boldsymbol{\phi}(t,\mathbf{x}))|_{t=0} \\ &= -\nabla U(\boldsymbol{\phi}(t,\mathbf{x})) \bullet \nabla U(\boldsymbol{\phi}(t,\mathbf{x}))|_{t=0} = -\|\nabla U(\mathbf{x})\|^2 < 0\end{aligned}$$

for all $\mathbf{x} \in D \setminus \{\mathbf{x}_0\}$, where $D \subset \mathbb{R}^n$ is some neighbourhood of $\mathbf{x}_0$. Hence, all solutions starting close enough to the equilibrium $\mathbf{x}_0$ are asymptotically attracted to it.

For a linear system $\dot{\mathbf{x}} = A\mathbf{x}$, $A \in \mathbb{R}^{n\times n}$, the equilibrium at the origin is asymptotically stable if and only if for a symmetric and positive definite matrix $Q \in \mathbb{R}^{n\times n}$ there exists a symmetric and positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$A^{\mathrm{T}} P + PA = -Q. \tag{1.2}$$

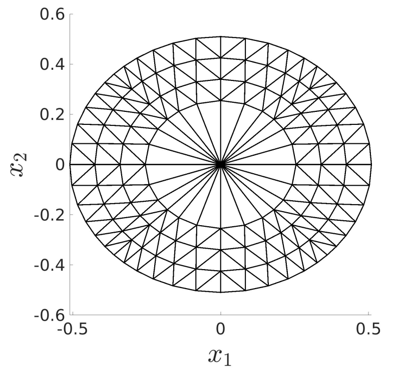
Then a global Lyapunov function for the system is given by $V(\mathbf{x}) = \mathbf{x}^{\mathrm{T}} P \mathbf{x}$, since V has a strict global minimum at the origin, and

$$\begin{aligned}\frac{d}{dt} V(\boldsymbol{\phi}(t,\mathbf{x}))\Big|_{t=0} &= \frac{d}{dt} \boldsymbol{\phi}(t,\mathbf{x})^{\mathrm{T}} P \boldsymbol{\phi}(t,\mathbf{x})\Big|_{t=0} = \dot{\boldsymbol{\phi}}(t,\mathbf{x})^{\mathrm{T}} P \boldsymbol{\phi}(t,\mathbf{x}) + \boldsymbol{\phi}(t,\mathbf{x})^{\mathrm{T}} P \dot{\boldsymbol{\phi}}(t,\mathbf{x})\Big|_{t=0} \\ &= \boldsymbol{\phi}(t,\mathbf{x})^{\mathrm{T}} (A^{\mathrm{T}} P + PA) \boldsymbol{\phi}(t,\mathbf{x})\Big|_{t=0} = \mathbf{x}^{\mathrm{T}} (A^{\mathrm{T}} P + PA)\mathbf{x} = -\mathbf{x}^{\mathrm{T}} Q \mathbf{x} < 0\end{aligned}$$

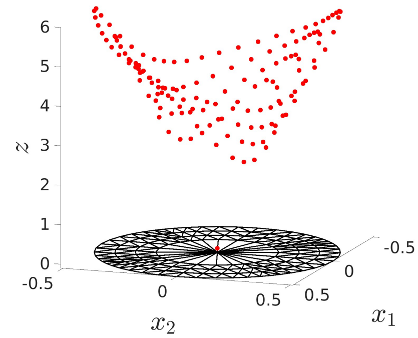
for $\mathbf{x} \ne \mathbf{0}$. Hence, every solution is asymptotically attracted to the origin and, moreover, one can show that there are constants $C \ge 1$ and $\alpha > 0$ such that $\|\boldsymbol{\phi}(t,\mathbf{x})\| \le C\mathrm{e}^{-\alpha t}\|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$ and all $t \ge 0$; the equilibrium at the origin is said to be globally exponentially stable. Equation (1.2) is called the *continuous-time Lyapunov equation* and is a special case of the Sylvester equation [43]. It has a positive definite solution $P = P^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ if and only if A is *Hurwitz*, i.e., the real parts of all eigenvalues of A are negative.

This linear analysis is also often used for nonlinear systems to obtain local stability certificates. If $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$, then system (1.1) has an equilibrium at $\mathbf{x}_0$ and this equilibrium is locally exponentially stable, i.e., the estimate $\|\boldsymbol{\phi}(t,\mathbf{x}) - \mathbf{x}_0\| \le C\mathrm{e}^{-\alpha t}\|\mathbf{x} - \mathbf{x}_0\|$, holds true for all $\mathbf{x}$ in some neighbourhood of $\mathbf{x}_0$ if and only if the Jacobian $D\mathbf{f}(\mathbf{x}_0) \in \mathbb{R}^{n\times n}$ is Hurwitz. In this case, $V(\mathbf{x}) = (\mathbf{x} - \mathbf{x}_0)^{\mathrm{T}} P (\mathbf{x} - \mathbf{x}_0)$ is a local Lyapunov function for the nonlinear system (1.1), where P is the solution to (1.2) with $A = D\mathbf{f}(\mathbf{x}_0)$, valid in some neighbourhood of $\mathbf{x}_0$.

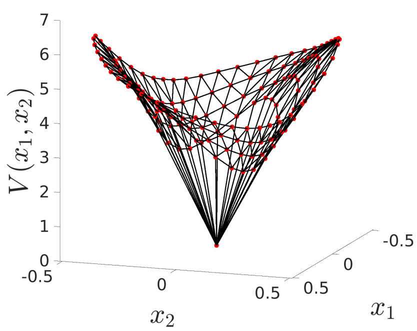
In applications of dynamical systems in engineering and the natural sciences, the equilibrium point of (1.1) is often the intended operating point of the system. Local stability is usually the bare minimum for the system to be of any use. Usually, one is also interested in how much disturbances the system



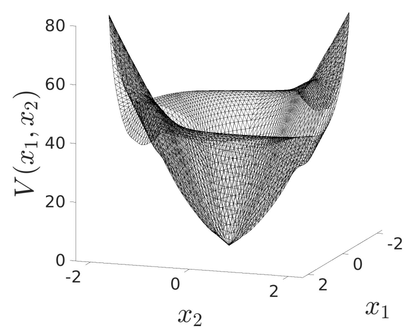
(A) Triangulation we use to compute a CPA Lyapunov function for system (2.1). This triangulation is used in Figs. 1b and 1c.



(B) On the z-axis, we plot the values $V_{\mathbf{x}}$ of a feasible solution to the LP problem (2.6) for system (2.1), where the $\mathbf{x} \in \mathscr{V}$ are the vertices of the triangulation plotted in the x_1, x_2-plane.



(C) By interpolating the values $V_{\mathbf{x}}$, $\mathbf{x} \in \mathscr{V}$, from Fig. 1b using formula (2.2), we get a CPA Lyapunov function for system (2.1).



(D) A CPA Lyapunov function for system (2.1) computed as in Fig. 1c, but using a denser triangulation of a larger domain.

can take before stability is lost. That is, one is interested in the existence of a Lyapunov function for the system in a significant neighbourhood of the equilibrium point, and not only in local stability.

2. CPA Method: Lyapunov Functions Via Linear Optimization

In the CPA method for computing Lyapunov functions, a continuous and piecewise affine (CPA) Lyapunov function is parameterized using linear programming (LP) [11, 31, 32, 38, 39]. It works for general nonlinear systems (1.1) in n-dimensional space if $\mathbf{f}$ is a C^2 vector field. For pedagogical reasons, we illustrate in detail how it works using a concrete example of the time-reversed van der Pol oscillator

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \begin{pmatrix} -x_2 \\ -(1 - x_1^2)x_2 + x_1 \end{pmatrix} \tag{2.1}$$

where $n = 2$, while explaining the method in n-dimensions.

First, the domain of interest $D \subset \mathbb{R}^n$, where the CPA method attempts to parameterize the Lyapunov function, is triangulated, that is, subdivided into n-simplices (triangles for $n = 2$), that are pairwise disjoint or intersect in a common face (side or vertex for $n = 2$). In Figure 1a we plot a triangulation, on which we will parameterize a CPA Lyapunov function for system (2.1).

Denote by $\mathscr{T} = \{\mathfrak{S}_\nu\}_{\nu \in I}$, where I is an index set, the triangulation of D. Each $\mathfrak{S}_\nu$ is the convex hull of its vertices,

$$\mathfrak{S}_\nu = \operatorname{co}(\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \dots, \mathbf{x}_n^\nu) = \left\{ \sum_{i=0}^n \lambda_i \mathbf{x}_i^\nu \in \mathbb{R}^n \colon \sum_{i=0}^n \lambda_i = 1, \quad \lambda_i \geq 0 \text{ for } i = 0, 1, \dots, n \right\}.$$

We denote the set of all vertices of all simplices in $\mathscr{T}$ by $\mathscr{V}$. The variables of the LP problem constructed in the CPA method are the values $V_\mathbf{x}$, $\mathbf{x} \in \mathscr{V}$, of the Lyapunov function $V \colon D \to [0, \infty)$ we wish to parameterize. From the values $V_\mathbf{x}$, $\mathbf{x} \in \mathscr{V}$, the function V is defined through

$$V(\mathbf{x}) = \sum_{i=0}^n \lambda_i V_{\mathbf{x}_i^\nu} \quad \text{for} \quad \mathbf{x} = \sum_{i=0}^n \lambda_i \mathbf{x}_i^\nu \in \mathfrak{S}_\nu. \tag{2.2}$$

Note that the representation of $\mathbf{x} \in \mathfrak{S}_\nu$ as a convex combination of the vertices $\mathbf{x}_i^\nu$ of $\mathfrak{S}_\nu$ is unique, since $\mathfrak{S}_\nu$ is an n-simplex. It is not difficult to see that V, defined in this way, is continuous on D and has a constant gradient $\nabla V_\nu \in \mathbb{R}^n$ within each simplex $\mathfrak{S}_\nu$. For V to be a Lyapunov function, it should have a minimum at the equilibrium point at the origin. This is very easy to enforce using linear constraints, as long as we ensure that the origin is a vertex of simplices in $\mathscr{T}$, i.e., $\mathbf{0} \in \mathscr{V}$. The linear constraints

$$V_\mathbf{0} = 0 \quad \text{and} \quad V_\mathbf{x} \geq \|\mathbf{x}\| \text{ for all } \mathbf{x} \in \mathscr{V} \setminus \{\mathbf{0}\},$$

then enforce $V(\mathbf{0}) = 0$ and that for every $\mathbf{x} \in D$, we have

$$V(\mathbf{x}) = \sum_{i=0}^n \lambda_i V_{\mathbf{x}_i^\nu} \geq \sum_{i=0}^n \lambda_i \|\mathbf{x}_i^\nu\| \geq \left\| \sum_{i=0}^n \lambda_i \mathbf{x}_i^\nu \right\| = \|\mathbf{x}\|,$$

with V defined as in (2.2).

As the piecewise affine function V is merely Lipschitz continuous, and not differentiable, the condition

$$D^+V(\mathbf{x}) := \limsup_{h \to 0+} \frac{V(\mathbf{x} + h\mathbf{f}(\mathbf{x})) - V(\mathbf{x})}{h} \leq -\|\mathbf{x}\| \text{ for all } \mathbf{x} \in D^\circ \text{ (upper, right Dini-derivative)} \tag{2.3}$$

is used instead of $\dot{V}(\mathbf{x}) \leq -\|\mathbf{x}\|$ to enforce the decrease of V along solution trajectories (D° denotes the interior of D). Condition (2.3) holds true for all $\mathbf{x} \in D^\circ$ if for every $\nu \in I$, we have

$$\nabla V_\nu \bullet \mathbf{f}(\mathbf{x}_i^\nu) + \|\nabla V_\nu\|_1 E_i^\nu(\mathbf{f}) \leq -\|\mathbf{x}_i^\nu\| \quad \text{for } i = 0, 1, \dots, n, \tag{2.4}$$

where

$$E_i^\nu(\mathbf{f}) := \max_{j=0,1,\dots,n} \sum_{r,s=1}^n \frac{B_{r,s}^\nu(\mathbf{f})}{2} |[\mathbf{x}_i^\nu - \mathbf{x}_d^\nu]_r| (|[\mathbf{x}_j^\nu - \mathbf{x}_d^\nu]_s| + |[\mathbf{x}_i^\nu - \mathbf{x}_d^\nu]_s|), \tag{2.5}$$

with $[\mathbf{y}]_i$ as the ith component of the vector $\mathbf{y}$, $d \in \{0, 1, \dots, n\}$ is arbitrary, and $B_{r,s}^\nu(\mathbf{f})$ are system specific constants satisfying the condition

$$B_{r,s}^\nu(\mathbf{f}) \geq \max_{\substack{i=1,2,\dots,n \\ \mathbf{x} \in \mathfrak{S}_\nu}} \left| \frac{\partial^2 f_i}{\partial x_r \partial x_s}(\mathbf{x}) \right|.$$

This is proved in [39, Lemma 4.16, Theorem II]. Since the components of the vectors ∇V_ν are linear in the variables $V_{\mathbf{x}_i^\nu}$, $i = 0, 1, \dots, n$, condition (2.4) can be enforced using the auxiliary variables C_j^ν, $\nu \in I$ and $j = 1, 2, \dots, n$. The LP (linear programming) problem that parameterizes a CPA Lyapunov function for system (1.1) is then defined as follows:

$$\begin{aligned} &V_\mathbf{0} = 0 \quad \text{and} \quad V_\mathbf{x} \geq \|\mathbf{x}\| \text{ for all } \mathbf{x} \in \mathscr{V} \setminus \{\mathbf{0}\}, \\ &-C_j^\nu \leq [\nabla V_\nu]_j \leq C_j^\nu \text{ for all } \nu \in I \text{ and } j = 1, 2, \dots, n, \text{ and} \\ &\nabla V_\nu \bullet \mathbf{f}(\mathbf{x}_i^\nu) + E_i^\nu(\mathbf{f}) \sum_{j=1}^n C_j^\nu \leq -\|\mathbf{x}_i^\nu\| \quad \text{for all } \nu \in I \text{ and } i = 0, 1, \dots, n. \end{aligned} \tag{2.6}$$

For our concrete example of system (2.1), it is not difficult to see that we can set

$$B_{1,1}^{\nu}(\mathbf{f}) = 2 \cdot \max_{\mathbf{x}\in\mathfrak{S}_\nu} |x_2|, \quad B_{1,2}^{\nu}(\mathbf{f}) = B_{2,1}^{\nu}(\mathbf{f}) = 2 \cdot \max_{\mathbf{x}\in\mathfrak{S}_\nu} |x_1|, \quad \text{and} \quad B_{2,2}^{\nu}(\mathbf{f}) = 0.$$

Figure 1b shows a feasible solution $V_{\mathbf{x}}$, $\mathbf{x} \in \mathscr{V}$, to this LP problem for system (2.1); that is, all conditions in (2.6) hold true for these values of $V_{\mathbf{x}}$. This solution was obtained using the Gurobi linear solver [22], which is free for academic use. These values are then interpolated to the CPA Lyapunov function V for the system as in Figure 1c, using formula (2.2). Note that the constants $B_{r,s}^{\nu}(\mathbf{f})$ are the only inputs to the LP problem (2.6), apart from the triangulation $\mathscr{T}$ and the vector field $\mathbf{f}$ on the right-hand-side of (1.1). Any sublevel set of V, $\{\mathbf{x} \in D \colon V(\mathbf{x}) \leq c\} \subset D^\circ$, $c > 0$, that is a compact subset of the open set D°, is forward invariant for the dynamics; see Figure 2b for an illustrative example.

Note that exactly the same procedure as for the planar system (1.1) can be used in an n-dimensional space, and in [11] it was proved that if the equilibrium is exponentially stable, then one can always use the LP problem (2.6) to parameterize the CPA Lyapunov function for the system.

3. Extensions of the CPA Method

The CPA Method can be extended and adapted to different kinds of systems and different scenarios in various ways, for example, for differential inclusions and switched systems [1, 2, 4, 36], for discrete-time systems [10, 23, 26, 40], for contraction metrics, which are a kind of the Lyapunov function on the tangent space [14–16], and for nonautonomous systems [20, 25].

As an example, consider the extension to differential inclusions. As a concrete example, consider the system

$$\dot{\mathbf{x}} \in \operatorname{co}\{\mathbf{f}(\mathbf{x}), \mathbf{g}(\mathbf{x})\}, \quad \mathbf{f}(\mathbf{x}) = \begin{pmatrix} -x_2 \\ -(1-x_1^2)x_2 + x_1 \end{pmatrix} \quad \text{and} \quad \mathbf{g}(\mathbf{x}) = \begin{pmatrix} -x_2 \\ x_1 - x_2(1 - x_1^2 + 0.1x_1^4) \end{pmatrix}. \tag{3.1}$$

Note that $\mathbf{f}$ in (3.1) is equal to $\mathbf{f}$ in (2.1). A solution to (3.1) with initial value $\boldsymbol{\xi} \in \mathbb{R}^2$ is a continuous function $\mathbf{x} \colon [0, \infty) \to \mathbb{R}^2$ such that $\mathbf{x}(0) = \boldsymbol{\xi}$ and $\dot{\mathbf{x}}(t) \in \operatorname{co}\{\mathbf{f}(\mathbf{x}(t)), \mathbf{g}(\mathbf{x}(t))\}$ for almost all $t \in [0, \infty)$. The extension of the CPA method to compute a Lyapunov function for the differential inclusion (3.1) is simple, it suffices to add to the LP problem (2.6) the linear constants

$$\nabla V_\nu \bullet \mathbf{g}(\mathbf{x}_i^\nu) + E_i^\nu(\mathbf{g}) \sum_{j=1}^{n} C_j^\nu \leq -\|\mathbf{x}_i^\nu\| \quad \text{for all } \nu \in I \text{ and } i = 0, 1, \ldots, n,$$

with $E_i^\nu(\mathbf{g})$ defined as in (2.5) with

$$B_{r,s}^{\nu}(\mathbf{g}) \geq \max_{\substack{i=1,2,\ldots,n \\ \mathbf{x}\in\mathfrak{S}_\nu}} \left| \frac{\partial^2 g_i}{\partial x_r \partial x_s}(\mathbf{x}) \right|.$$

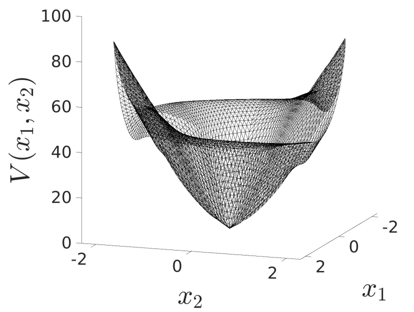
For system (3.1), we can set

$$B_{1,1}^{\nu}(\mathbf{g}) = \max_{\mathbf{x}\in\mathfrak{S}_\nu} |x_2|(2 + 1.2x_1^2), \quad B_{1,2}^{\nu}(\mathbf{g}) = B_{2,1}^{\nu}(\mathbf{g}) = \max_{\mathbf{x}\in\mathfrak{S}_\nu} |x_1|(2 + 0.4x_1^2), \quad \text{and} \quad B_{2,2}^{\nu}(\mathbf{g}) = 0.$$

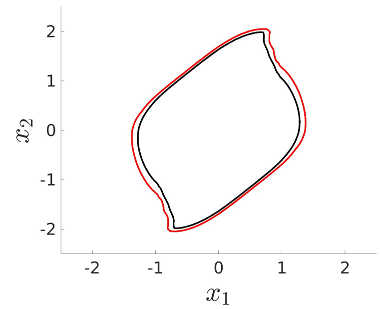
In Figure 2a, we plot a CPA Lyapunov function computed for the difference inclusion (3.1), and in Figure 2b, we compare a sublevel set of this function to a sublevel set of the CPA Lyapunov function for system (2.1) depicted in Figure 1d. Note the simplicity of extending the CPA method to differential inclusions!

4. CPA Method Combined with Other Methods

The main limitation of the CPA method is that it solves a system-dependent LP problem, which can be very large for higher-dimensional systems. A way around this is to construct the LP problem in the CPA method, but instead of solving it with a linear solver, to use a different method for assigning the values to the variables $V_{\mathbf{x}}$, $\mathbf{x} \in \mathscr{V}$. Subsequently, the linear constraints of the LP problem are



(A) A CPA Lyapunov function for the differential inclusion (3.1). Note that this function is additionally a Lyapunov function for the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and for the system $\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x})$, with $\mathbf{f}$ and $\mathbf{g}$ defined in (3.1).



(B) Level-sets of the CPA Lyapunov functions computed for system (2.1) (red) and for the differential inclusion (3.1) (black) for level 40. The corresponding sublevel sets are forward invariant for the systems, i.e., a solution entering the set can never leave. Further, the solution is asymptotically attracted to the equilibrium at the origin.

verified to determine where the function interpolated from the variables satisfies the conditions of the Lyapunov function. For example, one can use the formulas

$$V_1(\mathbf{x}) = \int_0^T \|\boldsymbol{\phi}(t,\mathbf{x})\| dt \quad \text{or} \quad V_2(\mathbf{x}) = \int_0^T \|\mathbf{f}(\boldsymbol{\phi}(t,\mathbf{x}))\| dt, \tag{4.1}$$

for Lyapunov functions from the inverse theorems (see, e.g., [5, 7, 9, 35]); $V_2(\mathbf{x})$ is the length of the solution trajectory $\boldsymbol{\phi}([0,T],\mathbf{x})$, since $\|\mathbf{f}(\boldsymbol{\phi}(t,\mathbf{x}))\| = \|\dot{\boldsymbol{\phi}}(t,\mathbf{x})\|$. Note that these formulas depend on the solution trajectories of the system, which must be numerically computed for each vertex $\mathbf{x} \in \mathscr{V}$, and $T > 0$ must be sufficiently large. In particular, one can only use the formulas for V_1 and V_2 to numerically approximate values at a finite number of points $\mathbf{x}$, and they cannot be used to compute the Lyapunov function on a whole domain. Hence, the CPA method is used to interpolate this finite number of approximate values of the Lyapunov functions to a true Lyapunov function defined on the domain of interest.

In Figure 3, we plot sublevel sets of CPA Lyapunov functions computed for the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \begin{pmatrix} -x_1 - x_2 - x_3 \\ \sin(x_1) - 2x_2(1+x_1) + x_3 \\ x_1(1+x_1) + x_2 - \sin(x_3) \end{pmatrix} \tag{4.2}$$

using a triangulation with $18^3 \cdot 3! = 34,992$ tetrahedra (3-simplices) when solving the LP problem, but a triangulation with $100^2 \cdot 50 \cdot 3! = 3,0000,000$ tetrahedra when using the formulas for V_1 and V_2. In the LP problems, we used the easily derived crude upper bounds $B^{\nu}_{r,s}(\mathbf{f}) = 2$ for all $\nu \in I$ and $r, s = 1, 2, 3$. Each computation is completed in less than 10 seconds on a standard PC (i285K, 64 GB, Linux Mint). Note that each such sublevel set is a lower bound on the equilibrium's basin of attraction. The formulas V_1 and V_2 fail usually to deliver good values for the Lyapunov function very close to the origin, but this problem is easily fixable by proving local stability of the equilibrium using linearization and quadratic Lyapunov functions as discussed in Section 1.

Paper [28] studies integral formulas other than V_1 and V_2 that yield larger lower bounds on the basin of attraction. Yoshizawa's construction formula [45] is used in [8, 24], and in [12], a generalized interpolation in Hilbert spaces with a reproducing kernel is used to solve Zubov's equation [46] and assign values to the variables. For computing Lyapunov functions on the tangent space, the so-called contraction metrics, this combined approach also works and is even more valuable, since it avoids

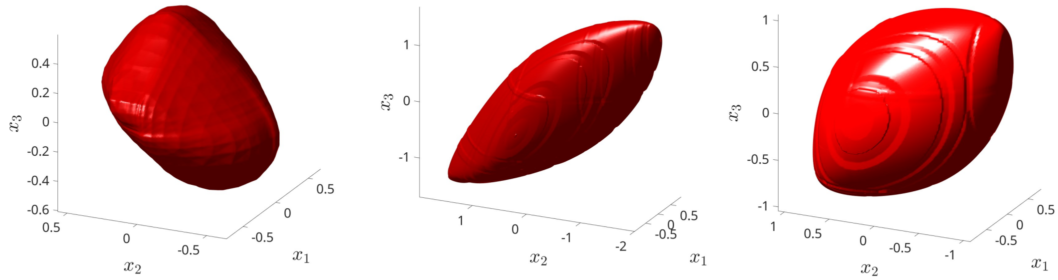


FIGURE 3. Sublevel-set of CPA Lyapunov functions computed for the system (4.2) using a LP solver (left), the formula for V_1 in (4.1) with subsequent certification (middle), and the formula for V_2 in (4.1) with subsequent certification (right).

solving huge semi-definite optimization problems explicitly (see [18] for an overview). For both, ordinary Lyapunov functions and tangent-space Lyapunov functions, it can be proved that these combined approaches yield true Lyapunov functions (see, e.g., [13, 19, 21] in addition to the references above).

5. Conclusions

We discussed the Lyapunov stability theory and Lyapunov functions, which are real-valued functions defined on the state space of a dynamical system, non-increasing along all solution trajectories and decreasing whenever possible. Lyapunov functions are an extension of the physical concept of (dissipative) energy. We then discussed the CPA method to compute Lyapunov functions for ordinary differential equations (ODEs) using linear optimization and provided examples of its use to compute continuous and piecewise affine (CPA) Lyapunov functions. Further, we discussed extensions of the CPA method for various kinds of systems and setups and presented an example for differential inclusions. Finally, we considered how to avoid solving the linear programming problem in the CPA method and instead use another method to *guess* the values for the variables of the LP problem such that these values represent a feasible solution. Subsequently, we confirmed where the conditions for the Lyapunov function are satisfied by verifying the validity of the constraints of the linear programming problem.

References

1. S. Andersen, E. August, S. Hafstein, J. Piccini, Simulation and Modeling Methodologies, Technologies and Applications. In: *13th International Conference, SIMULTECH 2023 Rome, Italy, July 12-14, 2023 Revised Selected Papers*, 192–213 Springer, 2024.
2. A. Andersen, P. Giesl, S. Hafstein, Common Lyapunov functions for switched linear systems: linear programming-based approach. *IEEE Control Syst. Lett.* **7** (2023), 901–906.
3. J. Auslander, Generalized recurrence in dynamical systems. *Contr. to Diff. Equ.* **3** (1964), 65–74.
4. R. Baier, L. Grüne, S. Hafstein, Linear programming based Lyapunov function computation for differential inclusions. *Discrete Contin. Dyn. Syst. Ser. B* **17** (2012), no. 1, 33–56.
5. J. Björnsson, P. Giesl, S. Hafstein, C. Kellett, Computation of Lyapunov functions for systems with multiple local attractors. *Discrete Contin. Dyn. Syst.* **35** (2015), no. 9, 4019–4039.
6. C. Conley, *Isolated Invariant Sets and the Morse Index*. CBMS Regional Conference Series in Mathematics, 38. American Mathematical Society, Providence, RI, 1978.
7. A. Doban, *Stability Domains Computation and Stabilization of Nonlinear Systems: Implications for Biological Systems*. PhD thesis: Eindhoven University of Technology, 2016.
8. A. Doban, M. Lazar, Computation of Lyapunov functions for nonlinear differential equations via a Yoshizawa-type construction. *IFAC-PapersOnLine* **49** (2016), no. 18, 29–34.
9. A. Doban, M. Lazar, Computation of Lyapunov functions for nonlinear differential equations via a Massera-type construction. *IEEE Trans. Automat. Control* **63** (2018), no. 5, 1259–1272.

10. P. Giesl, S. Hafstein, Computation of Lyapunov functions for nonlinear discrete time systems by linear programming. *J. Difference Equ. Appl.* **20** (2014), no. 4, 610–640.
11. P. Giesl, S. Hafstein, Revised CPA method to compute Lyapunov functions for nonlinear systems. *J. Math. Anal. Appl.* **410** (2014), no. 1, 292–306.
12. P. Giesl, S. Hafstein, Computation and verification of Lyapunov functions. *SIAM J. Appl. Dyn. Syst.* **14** (2015), no. 4, 1663–1698.
13. P. Giesl, S. Hafstein, Perspectives in Dynamical Systems II - Numerical and Analytical Approaches. In: *Springer Proceedings in Mathematics and Statistics, chapter Lyapunov Functions by Interpolating Numerical Quadrature: Proof of Convergence*, 205–228, vol. 454, Springer, 2024.
14. P. Giesl, S. Hafstein, I. Mehrabinezhad, Computation and verification of contraction metrics for exponentially stable equilibria. *J. Comput. Appl. Math.* **390** (2021), Paper no. 113332, 21 pp.
15. P. Giesl, S. Hafstein, I. Mehrabinezhad, Computation and verification of contraction metrics for periodic orbits. *J. Math. Anal. Appl.* **503** (2021), no. 2, Paper no. 125309, 32 pp.
16. P. Giesl, S. Hafstein, I. Mehrabinezhad, Computing Contraction Metrics for three-dimensional systems. *IFAC-PapersOnLine* **54** (2021), no. 9, 297–303.
17. P. Giesl, S. Hafstein, S. Suhr, Existence of complete Lyapunov functions with prescribed orbital derivative. *Discrete Contin. Dyn. Syst. Ser. B* **27** (2022), no. 11, 6927–6941.
18. P. Giesl, S. Hafstein, C. Kawan, Review on contraction analysis and computation of contraction metrics. *J. Comput. Dyn.* **10** (2023), no. 1, 1–47.
19. P. Giesl, S. Hafstein, I. Mehrabinezhad, Contraction metrics by numerical integration and quadrature: Uniform error estimate. In: *Proceedings of the 20th International Conference on Informatics in Control, Automation and Robotics-(Volume 1)*, vol. 1, 196–205. Science and Technology Publications, Lda, 2023.
20. P. Giesl, S. Hafstein, J. Jackiewicz, CPA Lyapunov functions for time-periodic systems in one spatial dimension. *J. Comput. Dyn.* **12** (2025), no. 3, 393–417.
21. P. Giesl, S. Hafstein, I. Mehrabinezhad, Informatics in Control, Automation and Robotics. vol. 1436 of *Lecture Notes in Electrical Engineering*, chapter Positively Invariant Sets for ODEs and Predictor-Corrector Multi-step Numerical Solvers., pages 43–68, Springer, 2025.
22. *Gurobi Optimization, LLC*. Gurobi Optimizer Reference Manual, 2026. `https://www.gurobi.com`.
23. S. Hafstein, C. Kellett, H. Li, Computation of Lyapunov functions for discrete-time systems using the Yoshizawa construction. In: *53rd IEEE conference on decision and control*, 5512–5517, IEEE, 2014.
24. S. Hafstein, C. Kellett, H. Li, Continuous and piecewise affine Lyapunov functions using the Yoshizawa construction. In: *2014 American control conference*, 548–553, IEEE, 2014.
25. S. Hafstein, H. Li, Computation of Lyapunov functions for nonautonomous systems on finite time-intervals by linear programming. *J. Math. Anal. Appl.* **447** (2017), no. 2, 933–950.
26. S. Hafstein, J.-J. Palacios Roman, Computation of Lyapunov functions for discrete-time switched linear systems: proof of convergence in the plane. *IFAC-PapersOnline* **59** (2025), no. 19, 579–584.
27. S. Hafstein, S. Suhr, Smooth complete Lyapunov functions for ODEs. *J. Math. Anal. Appl.* **499** (2021), no. 1, Paper no. 125003, 15 pp.
28. S. Hafstein, A. Valfells, H "Efficient computation of Lyapunov functions for nonlinear systems by integrating numerical solutions. *Nonlinear Dynamics* **97** (2019), no. 3, 1895–1910.
29. W. Hahn, *Stability of Motion*. Translated from the German manuscript by Arne P. Baartz. Die Grundlehren der mathematischen Wissenschaften, Band 138. Springer-Verlag New York, Inc., New York, 1967.
30. M. Hurley, Lyapunov functions and attractors in arbitrary metric spaces. *Proc. Amer. Math. Soc.* **126** (1998), no. 1, 245–256.
31. P. Julian, *A High-level Canonical Piecewise Linear Representation: Theory and Applications*. Thesis (Ph.D.)–Universidad Nacional del Sur (Argentina). ProQuest LLC, Ann Arbor, MI, 1999.
32. P. Julian, J. Guivant, A. Desages, A parametrization of piecewise linear Lyapunov functions via linear programming. Multiple model approaches to modelling and control. *Internat. J. Control* **72** (1999), no. 7-8, 702–715.
33. C. Kellett, A compendium of comparison function results. *Math. Control Signals Systems* **26** (2014), no. 3, 339–374.
34. H. Khalil, *Nonlinear Systems*. Third Edition. Prentice Hall, Inc. Upper Saddle River, NJ. 2002.
35. J. Kurzweil, On the inversion of Ljapunov's second theorem on stability of motion. *AMS Translations Series 2* **24** (1963), 19–77.
36. H. Li, S. Hafstein, C. Kellett, Computation of continuous and piecewise affine Lyapunov functions for discrete-time systems. *J. Difference Equ. Appl.* **21** (2015), no. 6, 486–511.
37. A. M. Lyapunov, The general problem of the stability of motion. Translated by A. T. Fuller from Édouard Davaux's French translation (1907) of the 1892 Russian original. With an editorial (historical introduction) by Fuller, a biography of Lyapunov by V. I. Smirnov, and the bibliography of Lyapunov's works collected by J. F. Barrett. Lyapunov centenary issue. *Internat. J. Control* **55** (1992), no. 3, 521–790.
38. S. Marinósson, Lyapunov function construction for ordinary differential equations with linear programming. *Dyn. Syst.* **17** (2002), no. 2, 137–150.
39. S. Marinósson, *Stability Analysis of Nonlinear Systems with Linear Programming: A Lyapunov Functions Based Approach.* PhD thesis: Gerhard-Mercator-University Duisburg, Duisburg, Germany, 2002.

40. J.-J. Palacious Roman, S. Hafstein, Convex Lyapunov Functions for Switched Discrete-Time Systems by Linear Programming. In: *2024 25th International Carpathian Control Conference (ICCC)*, 01–06, IEEE, 2024.
41. L. Perko, *Differential Equations and Dynamical Systems*. Third edition. Texts in Applied Mathematics, 7. Springer-Verlag, New York, 2001.
42. S. Sastry, *Nonlinear Systems. Analysis, Stability, and Control*. Interdisciplinary Applied Mathematics, 10. Springer-Verlag, New York, 1999.
43. J. Sylvester, Sur l'équation en matrices $px = xq$. *CR Acad. Sci. Paris* **99** (1884), no. 2, 67–71.
44. M. Vidyasagar, *Nonlinear Systems Analysis*. Reprint of the second (1993) edition. Classics in Applied Mathematics, 42. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002.
45. T. Yoshizawa, *Stability Theory by Liapunov's Second Method*. Publications of the Mathematical Society of Japan, 9. Mathematical Society of Japan, Tokyo, 1966.
46. V. I. Zubov, *Methods of A. M. Lyapunov and their Application*. Translation prepared under the auspices of the United States Atomic Energy Commission; edited by Leo F. Boron. P. Noordhoff Ltd., Groningen, 1964.

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