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ONE-SIDED EXTRAPOLATION IN THE CLASSICAL AND GRAND BALL BANACH FUNCTION SPACES

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Abstract. In this note, we present Rubio de Francía's extrapolation results in one-sided setting, in grand Lebesgue, Lorentz and ball Banach function spaces. These results can be applied, for example, to obtain the boundedness for one-sided operators of Harmonic Analysis in these spaces.

1. Introduction

We present a one-sided variant of Rubio de Francía's extrapolation statements, generally speaking, in grand ball Banach function spaces (BBFS, briefly). In particular, we formulate the results in the classical Lebesgue and Lorentz spaces and, more generally, in grand BBFS. The BBFS were introduced in [29]. Grand X^p spaces based on the Banach spaces of function lattices X were introduced in [14] (see also [31]). Rubio de Francía's extrapolation properties in various grand function spaces were studied in [15, 17–20], and in general grand Banach function spaces in [25] (see also [21]). The results of one-sided extrapolation in various situations were derived in [10,16]. The same problem in BBFS was investigated in the recent paper [12]. Grand Lebesgue spaces (GLS, briefly) $L^{p)}(\Omega)$ were introduced in [13], where the authors studied the Jacobian integrability problem. Initially, it was introduced on a bounded open set. Later, GLS on unbounded domains was introduced in [33] and [28] using the so-called "grandizers". Recently, in [3], the authors introduced GLS on unbounded sets without "grandizers". In this note, we study grand function spaces, generally speaking, defined on an interval which may be unbounded. We formulate the results for the diagonal cases, i.e., when the exponents in the domain and value spaces are the same. However, we investigated the extrapolation problem in off-diagonal cases, as well; the results will be published later. The creator of the weighted extrapolation theory is J.L. Rubio de Francía. His method is so much universal that it has become the basis for a number of studies, new ideas and further extensions. In this direction, monograph [6] is particularly noteworthy. This monograph contains several important topics: weighted norm inequalities, Rubio de Francía extrapolation and its extension, Calderón–Zygmund decomposition, a new approach to weak inequalities, etc.

In monograph [6] (Theorem 4.6) and in papers [14,19,27] the extrapolation problems were studied in general-type Banach function spaces (BFC, briefly).

Another A_1 off-diagonal extrapolation theorem in the so-called ball quasi-Banach function spaces (BQBFC, briefly) was derived in [7].

Various extrapolation results in general weighted Banach function spaces in the on-diagonal case were considered in [2].

It should be mentioned that extrapolation theorems in BBFC and quasi-Banach function spaces were studied in [34] and [27], respectively, (see also [29]).

We formulate the main statements only in the right-hand side case and only for a diagonal case. We mention here that the appropriate results in the off-diagonal case are also obtained.

Let $\mathcal{M}$ and $L^1_{\rm loc}$ denote the space of Lebesgue measurable functions and the space of locally integrable functions on $\mathbb{R}$, respectively. For any $x \in \mathbb{R}$ and $r \in (0,\infty)$, define

$$B(x,r) = \{y \in \mathbb{R} : |y-x| < r\} \quad \text{and} \quad \mathbb{B} = \{B(x,r) : x \in \mathbb{R},\, r \in (0,\infty)\}.$$

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We recall the definition of a BQBFS from [29, Definition 2.2].

Definition 1.1. A quasi-Banach function space $X \subset \mathcal{M}$ is said to be a BQBFS on $\mathbb{R}$ if it satisfies the following conditions:

1. $\|f\|_X = 0 \iff f = 0$ a.e.;
2. $|g| \le |f|$ a.e. $\implies \|g\|_X \le \|f\|_X$;
3. $0 \le f_n \uparrow f$ a.e. $\implies \|f_n\|_X \uparrow \|f\|_X$;
4. $B \in \mathbb{B} \implies \chi_B \in X$.

If $\|\cdot\|_X$ is a norm, X is a Banach space that satisfies (1)–(4) and

$$B \in \mathbb{B} \implies \int\limits_B |f(x)|dx < C_B\|f\|_X, \quad \forall f \in X, \tag{1.1}$$

for some $C_B > 0$, then we call X a BBFS.

For any $0 < r < \infty$ and a BQBFS X, the r-convexification of X, X^r is defined as

$$X^r = \{f : |f|^r \in X\}.$$

The vector space X^r is equipped with the quasi-norm $\|f\|_{X^r} = \||f|^r\|_X^{1/r}$. The reader is referred to [23, Volume II, p. 53–54]. If X is a BQBFS, then X^r is a BQBFS whenever $r \in (0, 1)$.

Let X be a ball BBFS. The associate space X' of X, consists of all measurable functions f satisfying

$$\|f\|_{X'} = \sup\left\{ \int\limits_{\mathbb{R}} |f(x)g(x)|\, dx : g \in X,\ \|g\|_X \le 1\right\} < \infty.$$

Recall that (see [29, Proposition 2.3]) if X is a BBFS, then X' is also a BBFS.

1.1. Grand Lebesgue and Lorentz Spaces.

In 1992 T. Iwaniec and C. Sbordone [13] introduced a grand Lebesgue space over the bounded open set Ω in $\mathbb{R}^n$. This space is defined with respect to the norm

$$\|f\|_{L^{p)}(\Omega)} := \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{|\Omega|}\right)^{\frac{1}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}(\Omega)} < \infty, \ \ p > 1.$$

For further generalizations of this norm, we refer to [11].

Later, in [33] (see also [28]), the grand Lebesgue spaces were introduced on an unbounded domain Ω by using the "grandizers". Extrapolation results on such grand Lebesgue spaces were obtained in [32].

Recently, in [3], the authors introduced (see also [26, Definition 4.1]) the norm of the grand Lebesgue space for any domain Ω without using "grandizers".

Definition 1.2. [3] Let $1 < r < p < \infty$, $\theta > 0$, and let $\Omega \subset \mathbb{R}^n$ be a Lebesgue measurable set of nonzero measure, possibly infinite. Suppose that w is a weight function on Ω (i.e., w is a.e. positive locally integrable function on Ω). By the weighted grand Lebesgue space $L^{(r,p),\theta}(\Omega)$ we mean the space of the (Lebesgue) measurable functions defined over Ω such that

$$\|f\|_{L_w^{(r,p),\theta}(\Omega)} := \sup_{0<\varepsilon<p-r} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{L_w^{p-\varepsilon}(\Omega)} := \sup_{0<\varepsilon<p-r} \varepsilon^{\frac{\theta}{p-\varepsilon}} \left(\int\limits_\Omega |f(x)|^{p-\varepsilon} w(x)dx \right)^{\frac{\theta}{p-\varepsilon}} < \infty.$$

If w is constant, then we denote $L_w^{(r,p)}(\Omega)$ by $L^{(r,p)}(\Omega)$. It is obvious that when Ω is bounded, then $L_w^{(1,p)}(\Omega)$ is the classical weighted grand Lebesgue space denoted by $L_w^{p)}(\Omega)$.

Using the Hölder inequality, it is easy to see that for bounded Ω, the spaces $L_w^{(r,p),\theta}(\Omega)$ and $L_w^{p),\theta}(\Omega)$ coincide.

We also need the following weighted grand Lebesgue space defined with respect to the norm

$$\|f\|_{\mathcal{L}_w^{(r,p),\theta}(\Omega)} := \sup_{0<\varepsilon<p-r} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|fw\|_{L^{p-\varepsilon}(\Omega)}.$$

Let w be a weight function on Ω. In [19], the authors introduced a generalized weighted grand Lorentz space with respect to the norm

$$\|f\|_{L_w^{p),s,\theta}(\Omega)} = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{L_w^{p-\varepsilon,s}(\Omega)},$$

where $1 < p < \infty$, $1 \le s \le \infty$, $\theta > 0$, and the symbol $L_w^{q,s}(\Omega)$ denotes the weighted Lorentz space with parameters q, s for which the quantity

$$\|f\|_{L_w^{q,s}} = \begin{cases} \left(s \int\limits_0^\infty \left(w\{x \in \Omega : |f(x)| > \tau\} \right)^{s/q} \tau^{s-1} d\tau \right)^{1/s}, & \text{if } 1 \le s < \infty, \\ \sup_{s>0} s \Big(w(\{x \in \Omega : |f(x)| > s\} \Big)^{1/q}, & \text{if } s = \infty \end{cases}$$

is finite, where $wE := \int\limits_E w(x)dx$.

Similarly to [3], we introduce the space $L_w^{(\delta,p),s,\theta}(\Omega)$ on any domain Ω defined with respect to the norm

$$\|f\|_{L_w^{(\delta,p),s,\theta}(\Omega)} = \sup_{0<\varepsilon<p-\delta} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{L_w^{p-\varepsilon,s}(\Omega)}, \quad 0 < \delta < p.$$

Let X be a BBFS on Ω, and let Φ_δ be the class of positive non-increasing functions $\varphi(\cdot)$ on $(0, \delta]$, $\delta < p - 1$, such that $\lim_{x\to 0+} \varphi(x) = 0$. We introduce a grand BBFS similarly to [25] (see also [14] and [31]): for a bounded domain Ω and $\varphi(\cdot) \in \Phi_{p-1}$ we define the norm as follows:

$$\|f\|_{X^{p),\varphi(\cdot)}(\Omega)} := \sup_{0<\varepsilon<p-1} \varphi(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{X^{p-\varepsilon}}, \quad p > 1.$$

If $\varphi(\varepsilon) \equiv \varepsilon^\theta$, where $\theta > 0$, then we denote $X^{p),\varphi(\cdot)}$ by $X^{p),\theta}$.

For $X = L^1$, the space $X^{p),\varphi(\cdot)}$ coincides with the space introduced in [4] for measurable φ (for $\varphi(\varepsilon) \equiv \varepsilon^\theta$, $\theta > 0$, it is Iwaniec-Sbordone space $L^{p),\theta}$).

If Ω is an arbitrary domain, then by the symbol $X^{(r,p),\varphi(\cdot)}(\Omega)$ we denote the space defined with respect to the norm

$$\|f\|_{X^{(r,p),\varphi(\cdot)}(\Omega)} := \sup_{0<\varepsilon<p-r} \varphi(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{X^{p-\varepsilon}}.$$

For any $f \in L^1_{\text{loc}}$, the one-sided Hardy–Littlewood maximal operators M^+f and M^-f are defined as follows:

$$M^+f(x) = \sup_{t>0} \frac{1}{t} \int\limits_x^{x+t} |f(y)|\, dy, \quad M^-f(x) = \sup_{t>0} \frac{1}{t} \int\limits_{x-t}^{x} |f(y)|\, dy, \quad x \in \mathbb{R},$$

respectively.

We now recall the definition of *one-sided Muckenhoupt classes.*

Let $1 < p < \infty$ and let w be a weight function on $\mathbb{R}$. We say that $w \in A_p^+$ if

$$[w]_{A_p^+} := \sup_{a\in\mathbb{R}, h>0} \left(\frac{1}{h} \int\limits_{a-h}^{a} w(t)dt \right) \left(\frac{1}{h} \int\limits_a^{a+h} w(t)^{-\frac{1}{p-1}} dt \right)^{p-1} < \infty.$$

We say that $w \in A_1^+$ if there exists a constant $C > 0$ such that

$$M^-w(t) \le Cw(t), \quad t \in \mathbb{R}. \tag{1.2}$$

For any $\omega \in A_1^+$, the smallest constant C in (1.2) is denoted by $[\omega]_{A_1^+}$.

Define

$$A_\infty^+ = \bigcup_{p\in[1,\infty)} A_p^+.$$

It is easy to see that the class A_p^+ has the property

$$[w]_{A^+_{p-\varepsilon_1}} \le [w]_{A^+_{p-\varepsilon_2}}; \quad 0 < \varepsilon_1 < \varepsilon_2 \le p - 1.$$

These classes have also the so-called openness property: if $w \in A_p^+$, then these exists a constant $\varepsilon_p^+ > 0$ such that $w \in A_{p-\varepsilon_p^+}^+$ (see [24,30]).

Definition 1.3. Denote by σ_p^+ the best possible constant among those constants ε_p^+ for which the openness property holds.

Main Results

1.2. One-sided Extrapolation in Grand Lebesgue Spaces.

Theorem 1.4 (Bounded Interval)**.** *Let* $p_0 \in [1,\infty)$ *and let* $\mathcal{F}$ *be the class of pairs of non-negative functions defined on* $\mathbb{R}$*. Suppose that for all* $(f,g) \in \mathcal{F}$ *and all* $w \in A_1^+$*,*

$$\Big(\int\limits_{\mathbb{R}} g^{p_0} w dx\Big)^{\frac{1}{p_0}} \le CN\big([w]_{A_1^+}\big)\Big(\int\limits_{\mathbb{R}} f^{p_0} w dx\Big)^{\frac{1}{p_0}} \tag{1.3}$$

holds, where the positive constant C *is independent of* (f,g) *and* w*, and the constant* $N\big([w]_{A_1^+}\big)$ *is such that the mapping* $\cdot \mapsto N(\cdot)$ *is non-decreasing. Let* I *be a bounded interval in* $\mathbb{R}$*. Then for* $1 < p < \infty$*,* $\theta > 0$*,* $w \in A_p^+$ *and* $(f,g) \in \mathcal{F}$ *with supports in* I*, we have*

$$\|g\|_{L_w^{p),\theta}} \le \overline{C}\|f\|_{L_w^{p),\theta}}, \tag{1.4}$$

where $\overline{C}$ *is the positive constant independent of* $(f,g) \in \mathcal{F}$*.*

Theorem 1.5 (Unbounded Interval)**.** *Suppose that* $p_0 \in [1,\infty)$*. Let* $\mathcal{F}$ *be the class of pairs of non-negative functions defined on* $\mathbb{R}$*. Suppose that for all* $(f,g) \in \mathcal{F}$ *and all* $w \in A_{p_0}^+$*, we have*

$$\Big(\int\limits_{\mathbb{R}} g^{p_0} w dx\Big)^{\frac{1}{p_0}} \le CN\big([w]_{A_1^+}\big)\Big(\int\limits_{\mathbb{R}} f^{p_0} w dx\Big)^{\frac{1}{p_0}}, \tag{1.5}$$

where the positive constant C *is independent of* (f,g) *and* w*, and* N *is the constant independent of* (f,g) *and depending on* $[w]_{A_1^+(X)}$ *such that the mapping* $\cdot \mapsto N(\cdot)$ *is non-decreasing. Then for* $1 < p < \infty$*,* $\varphi \in \Phi_{\sigma_p^+}$*,* $w \in A_p^+$ *and* $(f,g) \in \mathcal{F}$*, we have*

$$\|g\|_{L_w^{(\sigma_p^+,p),\varphi(\cdot)}(\mathbb{R})} \le \overline{C}\|f\|_{L_w^{(\sigma_p^+,p),\varphi(\cdot)}(\mathbb{R})}, \tag{1.6}$$

where $\overline{C}$ *is the positive constant independent of* $(f,g) \in \mathcal{F}$*, and the constant* σ_p^+ *is defined in Definition* 1.3*.*

1.3. **Grand BBFS.** We say that a BBFS denoted by X belongs to $\mathbb{M}^+$ (resp., X belongs to $\mathbb{M}^-$) if the operator M^+ (resp., M^-) is bounded in X.

Theorem 1.6 (Bounded interval)**.** *Let* $\mathcal{F}$ *be a family of pairs* (f,g) *of measurable non-negative functions* f,g *defined on* $\mathbb{R}$*. Suppose that for some* $1 \le p_0 < \infty$*, for every* $w \in A_1^+$ *and all* $(f,g) \in \mathcal{F}$*, the one-weight inequality*

$$\Big(\int\limits_{\mathbb{R}} g^{p_0}(x) w(x)\, dx\Big)^{\frac{1}{p_0}} \le CN\big([w]_{A_1^+}\big)\Big(\int\limits_{\mathbb{R}} f^{p_0}(x) w(x)\, dx\Big)^{\frac{1}{p_0}} \tag{1.7}$$

holds, where C *and* $N\big([w]_{A_1^+}\big)$ *are positive constants such that* C *is independent of* (f,g) *and* w*, and* $N\big([w]_{A_1^+}\big)$ *is independent of* (f,g) *and depends on* $[w]_{A_1^+}$ *so that the mapping* $\cdot \mapsto N(\cdot)$ *is non-decreasing. Let* X *be a BBFS and let there exist* $1 < q_0 < \infty$ *such that* X^{1/q_0} *is again a BBFS.*

Let I *be a bounded interval in* $\mathbb{R}$*. Then for any* $p > 1$*,* $\varphi(\cdot) \in \Phi_{p-1}$*, there exists a positive constant* C *such that for all pairs of functions* $(f,g) \in \mathcal{F}$ *with compact support in* I*, the inequality*

$$\|g\|_{X^{p),\varphi(\cdot)}} \le C\|f\|_{X^{p),\varphi(\cdot)}}, \quad (f,g) \in \mathcal{F},$$

holds, provided $(X^{(p-\varepsilon)/q_0})' \in \mathbb{M}$*,* $\varepsilon \in (0,\sigma)$*, and that* $L := \sup_{0<\varepsilon<\sigma} \|M\|_{(X^{(p-\varepsilon)/q_0})'} < \infty$*, where* σ *is some small positive constant.*

Theorem 1.7 (Unbounded interval)**.** *Let $\mathcal{F}$ be a family of all pairs (f,g) of measurable non-negative functions f,g defined on $\mathbb{R}$. Suppose that for some $1 \le p_0 < \infty$, for every $w \in A_1^+$ and all $(f,g) \in \mathcal{F}$, the one-weight inequality*

$$\Bigg(\int\limits_{\mathbb{R}} g^{p_0}(x)w(x)\,dx\Bigg)^{\frac{1}{p_0}} \le CN\big([w]_{A_1^+}\big)\Bigg(\int\limits_{\mathbb{R}} f^{p_0}(x)w(x)\,dx\Bigg)^{\frac{1}{p_0}}$$

holds, where C and $N\big([w]_{A_1^+}\big)$ are positive constants such that C is independent of (f,g) and w, and $N\big([w]_{A_1^+}\big)$ is independent of (f,g) and depends on $[w]_{A_1}$ so that the mapping $\cdot \mapsto N(\cdot)$ is non-decreasing. Let X be a BFS and assume that there exists $1 < q_0 < \infty$ such that X^{1/q_0} is again a BFS.

Then for any $p > 1$, for all $\varphi(\cdot) \in \Phi_\delta$, where δ is sufficiently small positive constant $\delta \in (0, p-1)$, there exists a positive constant C such that for all pairs of functions $(f,g) \in \mathcal{F}$, the inequality

$$\|g\|_{X^{(\delta,p),\varphi(\cdot)}(\mathbb{R})} \le C\|f\|_{X^{(\delta,p),\varphi(\cdot)}(\mathbb{R})}, \quad (f,g) \in \mathcal{F},$$

holds, provided $(X^{(p-\varepsilon)/q_0})' \in \mathbb{M}$, $\varepsilon \in (0,\delta)$, and that $L := \sup_{0<\varepsilon<\delta} \|M\|_{(X^{(p-\varepsilon)/q_0})'} < \infty$ for some small positive number δ.

Remark 1.8. Taking $g = Tf$ (i.e., $(f,g) = (f,Tf)$) in the statements of this section, as a particular case, we can formulate appropriate extrapolation statements for T, where T is one of the operators of Harmonic Analysis such that it is bounded in weighted Lebesgue spaces under the one-sided Muckenhoupt condition on weights. Such operators are, for example, one-sided Hardy–Littlewood maximal and one-sided Calderón–Zygmund singular integral operators, commutators of one-sided singular integrals, one-sided fractional integral operators, etc. (see e.g., [1, 24, 30]).

1.4. **One-sided Extrapolation in the Classical and Grand Lorentz Spaces.** Now, we present one-sided extrapolation results in the classical and grand Lorentz spaces. For simplicity, we formulate them only in the right-hand side case.

Theorem 1.9. *Let $\mathcal{F}$ be a family of pairs (f,g) of measurable non-negative functions f,g defined on $\mathbb{R}$. Suppose that for some $1 \le p_0 < \infty$, for every $w \in A_1^+$ and all $(f,g) \in \mathcal{F}$, the one-weight inequality*

$$\Bigg(\int\limits_{\mathbb{R}} g^{p_0}(x)w(x)\,dx\Bigg)^{\frac{1}{p_0}} \le CN\big([w]_{A_1^+}\big)\Bigg(\int\limits_{\mathbb{R}} f^{p_0}(x)w(x)\,dx\Bigg)^{\frac{1}{p_0}} \tag{1.8}$$

holds with the positive constant C independent of (f,g) and w, and the positive constant $N\big([w]_{A_1^+}\big)$ independent of (f,g) and depending on $[w]_{A_1^+}$ so that the mapping $\cdot \to N(\cdot)$ is non-decreasing. Then for any $1 < p, s < \infty$, for all $(f,g) \in \mathcal{F}$ and any $w \in A_p^+$,

$$\|g\|_{L^{p,s}_w(\mathbb{R})} \le C_0\|f\|_{L^{p,s}_w(\mathbb{R})},$$

where the positive constant C_0 is independent of (f,g).

Now, we pass to the case of grand Lorentz spaces.

Theorem 1.10. *Let w be a weight function on $\mathbb{R}$ and let $\mathcal{F}$ be a family of pairs (f,g) of measurable non-negative functions f,g defined on $\mathbb{R}$. Suppose that for some $1 \le p_0 < \infty$, for every $w \in A_1^+$ and all $(f,g) \in \mathcal{F}$, the one-weight inequality*

$$\Bigg(\int\limits_{\mathbb{R}} g^{p_0}(x)w(x)\,dx\Bigg)^{\frac{1}{p_0}} \le CN\big([w]_{A_1^+}\big)\Bigg(\int\limits_{\mathbb{R}} f^{p_0}(x)w(x)\,dx\Bigg)^{\frac{1}{p_0}}$$

holds with some positive constant C which does not depend on (f,g) and w, and the positive constant $N\big([w]_{A_1^+}\big)$ such that the mapping $\cdot \mapsto N(\cdot)$ is non-decreasing. Let I be a bounded interval in $\mathbb{R}$. Then for any $1 < p < \infty$, $1 \le s < \infty$, $\theta > 0$, $w \in A_p^+$ and for all measurable $(f,g) \in \mathcal{F}$ having support on I,

$$\|g\|_{L^{p),s,\theta}_w} \le C\|f\|_{L^{p),s,\theta}_w},$$

with the positive constant C, independent of (f,g).

Further, the next statement holds.

Theorem 1.11. *Let w be a weight function on $\mathbb{R}$ and let $\mathcal{F}$ be a family of pairs (f,g) of measurable non-negative functions f,g defined on $\mathbb{R}$. Suppose that for some $1 \leq p_0 < \infty$, for every $w \in A_1^+$ and all $(f,g) \in \mathcal{F}$, the one-weight inequality*

$$\bigg(\int\limits_{\mathbb{R}} g^{p_0}(x)w(x)\,dx\bigg)^{\frac{1}{p_0}} \leq CN\big([w]_{A_1^+}\big)\bigg(\int\limits_{\mathbb{R}} f^{p_0}(x)w(x)\,dx\bigg)^{\frac{1}{p_0}}$$

holds with some positive constant C which does not depend on (f,g) and w, and the positive constant $N\big([w]_{A_1^+}\big)$ such that the mapping $\cdot \mapsto N(\cdot)$ is non-decreasing. Then for any $1 < p < \infty$, $1 \leq s < \infty$, $\theta > 0$, $w \in A_p^+$ and for all measurable $(f,g) \in \mathcal{F}$, the inequality

$$\|g\|_{L_w^{(\sigma_p^+,p),s,\theta}(\mathbb{R})} \leq C\|f\|_{L_w^{(\sigma_p^+,p),s,\theta}(\mathbb{R})}$$

holds with the positive constant C, independent of (f,g), where σ_p^+ is the constant from Definition 1.3.

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