

ON INTERSECTIONS OF MAZURKIEWICZ TYPE SETS WITH PLANE ALGEBRAIC CURVES

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Abstract. In the present note, several results concerning the richness of intersections of Mazurkiewicz type sets in \mathbb{R}^2 with various irreducible plane algebraic curves are discussed. Mazurkiewicz type sets are considered here with respect to the family of all straight lines in \mathbb{R}^2 and also with respect to the family of all circles in \mathbb{R}^2 .

Every Mazurkiewicz type set with respect to the family of all straight lines in the Euclidean plane \mathbb{R}^2 and every irreducible plane algebraic curve in \mathbb{R}^2 of degree at least 2 has a finite intersection with any straight line in the plane. Therefore it is natural to investigate how the above-mentioned finiteness property affects these two classes of sets and their close interrelation. In this context, several results concerning the intersections of Mazurkiewicz type sets with plane algebraic curves are examined. Some theorems regarding similar questions for Mazurkiewicz type sets with respect to the family of all circles in the plane, are also presented.

Let us recall some definitions that will be used below.

Definition 1. For a natural number $n > 1$, a set $B \subset \mathbb{R}^2$ is called a Mazurkiewicz n -type set (or an n -point set) in \mathbb{R}^2 if $\text{card}(B \cap l) = n$ for any straight line l lying in \mathbb{R}^2 .

In connection with Definition 1, note that in 1914, Mazurkiewicz proved that there exists a set $X \subset \mathbb{R}^2$ such that every straight line in \mathbb{R}^2 meets X at exactly two points (see [15]). Such a set X is called a Mazurkiewicz set. Later on, Sierpinski proved that for every natural number $n > 1$, there exists a set $Y \subset \mathbb{R}^2$ such that every straight line in \mathbb{R}^2 meets Y at exactly n points. The proof of the existence of n -point sets initiated extensive studies of their various aspects, leading to the discovery of many significant properties of these sets. Actually, n -point sets remain extensively studied objects of contemporary mathematics (see [3–12]).

Definition 2. For a natural number $k > 2$, a set $B \subset \mathbb{R}^2$ is called a Mazurkiewicz k -type set in \mathbb{R}^2 with respect to the family of all circles if $\text{card}(B \cap C) = k$ for any circle C lying in \mathbb{R}^2 .

For instance, some theorems related to the existence and various properties of Mazurkiewicz type sets with respect to the family of all circles of \mathbb{R}^2 are presented in [1] and [13].

Recall that by the definition, a plane algebraic curve C is any subset of the plane \mathbb{R}^2 formed by all solutions (x, y) of a non-constant polynomial equation f in two variables, i.e.,

$$C = \{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\}.$$

Similarly to any Mazurkiewicz type set with respect to the lines, every irreducible algebraic curve of degree at least two intersects every line at a finite number of points. Likewise, if one considers a Mazurkiewicz type set with respect to the family of all circles in \mathbb{R}^2 , then any such set has a finite intersection with every circle in \mathbb{R}^2 (in this regard, note that any irreducible algebraic curve of degree at least three also meets every circle of \mathbb{R}^2 at a finite number of points).

In view of the said above, it is natural to ask how a Mazurkiewicz type set with respect to straight lines (with respect to circles) and an irreducible algebraic curve of degree at least two (at least three) may be disposed relative to each other in the plane.

In this context, Larman's result is known.

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Larman's result. No Mazurkiewicz set in \mathbb{R}^2 contains a simple arc (i.e., does not contain a homeomorphic image of the closed unit interval $[0, 1]$) (see [14]). Also, no three-point set contains a simple arc.

On the other hand, it was proved in [2] that for every natural number $n \geq 4$ there exists an n -point set which contains a simple arc.

Theorem 1. *Let $C \subset \mathbb{R}^2$ be any irreducible algebraic curve of degree at least two and let n be an arbitrary fixed natural number such that $n \geq 2$. Then there exist Mazurkiewicz n -type sets: $M_1 \subset \mathbb{R}^2$, $M_2 \subset \mathbb{R}^2$, $M_3 \subset \mathbb{R}^2$, $M_4 \subset \mathbb{R}^2$ each of which is with respect to the family of all straight lines in \mathbb{R}^2 and possesses the following properties:*

- 1) *The intersection of M_1 with any nondegenerate simple arc contained in C is of cardinality continuum, has Lebesgue linear measure zero, and is of the Baire first category;*
- 2) *The intersection of M_2 with any nondegenerate simple arc contained in C is of cardinality continuum, has Lebesgue linear measure zero and is of the Baire second category;*
- 3) *The intersection of M_3 with any nondegenerate simple arc contained in C is of cardinality continuum, is non-measurable with respect to the Lebesgue linear measure (moreover, is thick with respect to this measure), and is of the Baire first category;*
- 4) *The intersection of M_4 with any nondegenerate simple arc contained in C is of cardinality continuum, is non-measurable with respect to the Lebesgue linear measure (moreover, is thick with respect to this measure), and is of the Baire second category.*

Before formulating a result concerning the intersections of plane irreducible algebraic curves of degree at least three and Mazurkiewicz type sets with respect to the family of all circles in \mathbb{R}^2 , let us note that no Mazurkiewicz set with respect to the family of all circles in \mathbb{R}^2 contains a simple arc. Note also that a very particular case of the proposition that no Mazurkiewicz set with respect to the family of all circles in \mathbb{R}^2 contains a non-degenerate line segment can be proved within the frameworks of elementary Euclidean geometry.

Theorem 2. *Let $C \subset \mathbb{R}^2$ be any irreducible algebraic curve of degree at least three and let k be an arbitrary fixed natural number such that $k \geq 3$. Then there exist Mazurkiewicz k -type sets: $M_1 \subset \mathbb{R}^2$, $M_2 \subset \mathbb{R}^2$, $M_3 \subset \mathbb{R}^2$, $M_4 \subset \mathbb{R}^2$ each of which is with respect to the family of all circles in \mathbb{R}^2 , and possesses the following properties:*

- 1) *The intersection of M_1 with any nondegenerate simple arc contained in C is of cardinality continuum, has Lebesgue linear measure zero, and is of the Baire first category;*
- 2) *The intersection of M_2 with any nondegenerate simple arc contained in C is of cardinality continuum, has Lebesgue linear measure zero, and is of Baire second category;*
- 3) *The intersection of M_3 with any nondegenerate simple arc contained in C is of cardinality continuum, is non-measurable with respect to the Lebesgue linear measure (moreover, is thick with respect to this measure), and is of the Baire first category;*
- 4) *The intersection of M_4 with any nondegenerate simple arc contained in C is of cardinality continuum, is non-measurable with respect to the Lebesgue linear measure (moreover, is thick with respect to this measure), and is of the Baire second category.*

Note that there exist algebraic curves of degree two or less, for which the analogue of Theorem 2 remains valid.

As usual, we denote by \mathfrak{c} the cardinality of the continuum.

Comment 1. For any given irreducible plane algebraic curve C of degree at least 2 and any cardinal number $0 \leq \tau \leq \mathfrak{c}$ and any natural number $n \geq 2$, there exists a Mazurkiewicz n -type set with respect to the family of all straight lines in the Euclidean plane, whose intersection with C has cardinality τ .

Comment 2. For any given irreducible plane algebraic curve C of degree at least 3, any cardinal number $0 \leq \tau \leq \mathfrak{c}$ and any natural number $k \geq 3$, there exists a Mazurkiewicz k -type set with respect to the family of all circles in the Euclidean plane, whose intersection with C has cardinality τ .

It should be noted that, besides the plane algebraic curves, there are many transcendental plane curves for which the analogues of Theorems 1 and 2 hold true.

Comment 3. Let us note that Theorems 1 and 2, as well as Comments 1 and 2, remain valid for subsets of the Euclidean plane that are more general than algebraic and transcendental curves. However, these subsets may no longer possess the “nice” properties of algebraic curves and may have a very pathological structure.

REFERENCES

1. L. Beraia, T. Tetunashvili, Some properties of Mazurkiewicz type sets. *Trans. A. Razmadze Math. Inst.* **177** (2023), no. 3, 491–493.
2. K. Bouhjar, J. J. Dijkstra, J. van Mill, Three-point sets. *Topology Appl.* **112** (2001), no. 2, 215–227.
3. K. Bouhjar, J. J. Dijkstra, On the structure of n -point sets. *Israel J. Math.* **137** (2003), 321–354.
4. J. Cobb, Are there generalizations of n -point sets? *Questions Answers Gen. Topology* **22** (2004), no. 2, 81–90.
5. J. J. Dijkstra, J. van Mill, Two point set extensions—a counterexample. *Proc. Amer. Math. Soc.* **125** (1997), no. 8, 2501–2502.
6. J. J. Dijkstra, K. Kunen, J. van Mill, Hausdorff measures and two point set extensions. *Fund. Math.* **157** (1998), no. 1, 43–60.
7. D. L. Fearnley, L. Fearnley, J. W. Lamoreaux, On the dimension of n -point sets. *Topology Appl.* **129** (2003), no. 1, 15–18.
8. A. Kharazishvili, *Set Theoretical Aspects of Real Analysis*. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2015.
9. A. Kharazishvili, Measurability properties of Mazurkiewicz sets. *Bull. TICMI* **20** (2016), no. 2, 44–46.
10. A. Kharazishvili, *Elements of Combinatorial Geometry*. Part 1. Georgian Nat. Acad. of Sciences, Tbilisi 2016.
11. A. Kharazishvili, On Mazurkiewicz sets from the measure-theoretical point of view. *Bull. TICMI* **21** (2017), no. 1, 45–54.
12. A. Kharazishvili, Mazurkiewicz sets of universal measure zero. *Trans. A. Razmadze Math. Inst.* **176** (2022), no. 1, 139–141.
13. A. Kharazishvili, T. Tetunashvili, On some coverings of the Euclidean plane with pairwise congruent circles. *Amer. Math. Monthly* **117** (2010), no. 5, 414–423.
14. D. G. Larman, A problem of incidence. *J. London Math. Soc.* **43** (1968), 407–409.
15. S. Mazurkiewicz, Sur un ensemble plan qui a avec chaque droite deux et seulement deux points communs. *CR Varsovie* **7** (1914), 382–384.

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