

ON THE TANGENT FIBER SURFACE OF THE SPACE $T(Vn)$

GOCHA TODUA

Abstract. The paper considers the theory of the surface of metric tangent fibers for the space $T(Vn)$. Analogues of the Gauss–Weingarten derivation formulas, as well as analogues of the generalized Gauss, Peterson–Codazzi–Mainardi equations, are found.

Let us consider the tangent fiber space $T(Vn)$, where the local coordinates of a point transform by the law

$$\bar{x}^i = \bar{x}^i(x^k), \quad \bar{y}^i = \bar{x}_k^i y^k, \quad i, j, k, \bar{i}, \bar{j}, \bar{k} = 1, 2, \dots, n.$$

Assume that the tensor field G_{AB} ($A, B, C = 1, 2, \dots, 2n$) is given on the space $T(Vn)$, i. e.,

$$\overline{G_{AB}} = \bar{x}_A^C \bar{x}_B^D G_{CD},$$

where

$$\bar{x}_B^A = \left\| \frac{\partial \bar{x}^A}{\partial \bar{x}^B} \right\| = \left\| \begin{array}{cc} \frac{\partial \bar{x}^i}{\partial x^j} & \frac{\partial \bar{x}^i}{\partial y^j} \\ \frac{\partial \bar{y}^i}{\partial x^j} & \frac{\partial \bar{y}^i}{\partial y^j} \end{array} \right\| = \left\| \begin{array}{cc} x_j^i & 0 \\ x_{pj}^i y^p & x_j^i \end{array} \right\|.$$

An inverse matrix of the matrix has the form

$$\bar{x}_B^A = \left\| \frac{\partial \bar{x}^A}{\partial \bar{x}^B} \right\| = \left\| \begin{array}{cc} \frac{\partial x^i}{\partial \bar{x}^k} & \frac{\partial x^i}{\partial \bar{y}^k} \\ \frac{\partial y^i}{\partial \bar{x}^k} & \frac{\partial y^i}{\partial \bar{y}^k} \end{array} \right\| = \left\| \begin{array}{cc} x_k^i & 0 \\ x_{jk}^i x_p^j y^p & x_k^i \end{array} \right\|.$$

Since $\bar{G}^{\bar{i}\bar{j}} = x_k^i x_p^j G^{\bar{k}\bar{p}}$, where $G_{\bar{i}\bar{j}} = \delta_i^j$ and $G_{\bar{l}\bar{i}} = x_l^m x_i^k G_{\bar{m}\bar{k}} + x_l^m x_q^h x_s^i y^s G_{\bar{m}\bar{h}}$, we can use them to construct the values $\Gamma_{\bar{i}}^k$ as follows: $\Gamma_{\bar{i}}^k = G^{\bar{k}\bar{j}} G_{\bar{j}\bar{i}}$.

Furthermore,

$$G^{\bar{j}\bar{p}} G_{\bar{p}\bar{i}} = x_q^j x_i^p G^{\bar{q}\bar{l}} x_p^s x_i^k G_{\bar{s}\bar{k}} + x_p^j x_q^s G^{\bar{p}\bar{q}} x_l^i x_{mi}^k x_h^m y^h G_{\bar{l}\bar{k}}.$$

Since $x_j^i x_k^j = \delta_k^i$, $x_{pk}^j x_q^p + x_l^j x_{qi}^l x_k^i = 0$, $-x_i^k x_{pk}^j x_s^p = x_l^j x_{si}^l$, we observe that the values $\Gamma_{\bar{i}}^k$ form a linearly connected object with the following transformation law:

$$\Gamma_{\bar{i}}^j = x_p^j x_i^k \Gamma_{\bar{k}}^p - x_i^k x_{pk}^j y^p,$$

and the values

$$g_{ij} = G_{ij} - \Gamma_i^k G_{\bar{k}\bar{j}} - \Gamma_j^k G_{\bar{k}\bar{i}} + \Gamma_i^p \Gamma_j^q G_{\bar{p}\bar{q}}$$

are a double covariant symmetric tensor, so we can construct an affine connection object $\Gamma_{\bar{j}\bar{k}}^i$ in the following manner:

$$\Gamma_{\bar{j}\bar{k}}^i = \frac{1}{2} g^{ip} (\nabla_k g_{pj} + \nabla_j g_{kp} - \nabla_p g_{jk}).$$

Note that the linear connectedness $\Gamma_{\bar{i}}^j$ induces the vertical affine connectedness defined by the object $\nabla_{\bar{k}} \Gamma_{\bar{i}}^j \equiv \Gamma_{\bar{k}\bar{i}}^j$ with the following transformation law:

$$\overline{\Gamma_{\bar{p}\bar{i}}^j} = \bar{x}_i^k \bar{x}_p^q x_l^j \Gamma_{\bar{q}\bar{k}}^l - x_{pi}^j.$$

2020 Mathematics Subject Classification. 53B05.

Key words and phrases. Tangent fibration; Linear connectedness; Tensor.

Structural equations of the space $T(Vn)$ have the form:

$$\begin{cases} D\omega^i = \omega^k \wedge \tilde{\omega}_k^i, \\ D\tilde{\theta}^{\bar{l}} = \tilde{\theta}^{\bar{k}} \wedge \tilde{\omega}_{\bar{k}}^{\bar{l}} + R_{ik}^{\bar{l}} \omega^i \wedge \omega^k, \\ D\tilde{\omega}_{\bar{p}}^{\bar{l}} = \tilde{\omega}_{\bar{p}}^{\bar{l}} \wedge \tilde{\omega}_{\bar{l}}^{\bar{l}} + R_{\bar{p}ik}^{\bar{l}} \omega^i \wedge \omega^k + R_{\bar{p}i\bar{l}}^{\bar{l}} \omega^i \wedge \tilde{\theta}^{\bar{l}}, \\ D\tilde{\omega}_{\bar{j}}^{\bar{i}} = \tilde{\omega}_{\bar{j}}^{\bar{k}} \wedge \tilde{\omega}_{\bar{k}}^{\bar{i}} + R_{j\bar{p}q}^{\bar{i}} \omega^p \omega^q + R_{j\bar{p}\bar{k}}^{\bar{i}} \omega^p \wedge \tilde{\theta}^{\bar{k}}. \end{cases}$$

Assume that a hypersurface \mathfrak{M} is given on the space $T(Vn)$,

$$\omega^i = W_a^i \Psi^a, \quad (1)$$

and the 1-forms Ψ^a are such that

$$\begin{cases} D\Psi^a = \Psi^b \wedge \Psi_b^a, \\ D\Psi_b^a = \Psi_b^c \wedge \Psi_c^a + \Psi^c \wedge \Psi_{bc}^a. \end{cases}$$

Note that

$$D\tilde{\theta}^{\bar{j}} = \tilde{\theta}^{\bar{l}} \wedge \tilde{\omega}_{\bar{l}}^{\bar{j}} + R_{ik}^{\bar{j}} \omega^i \wedge \omega^k = \tilde{\theta}^{\bar{l}} \wedge \tilde{\omega}_{\bar{l}}^{\bar{j}} + R_{j\bar{p}q}^{\bar{i}} \omega^p \omega^q + R_{ik}^{\bar{j}} W_a^i \Psi^a \wedge W_a^k \Psi^a = R_{ab}^{\bar{j}} \Psi^a \wedge \Psi^b,$$

where $R_{ab}^{\bar{j}} = R_{ik}^{\bar{j}} W_a^i W_b^k$.

The extension of system (1) is given by

$$\begin{cases} \nabla W_a^i = W_{ab}^i \Psi^b, & \nabla W_{ab}^i + W_c^i \Psi_{ab}^c = W_{abc}^i \Psi^c, \\ \nabla W_{abc}^i + 2W_{(a|d}^i \Psi_{b)c}^d - W_d^i \Psi_{bc}^d = W_{abd}^i \Psi^d, \end{cases}$$

where

$$\begin{cases} \nabla W_{[a,b]}^i = 0, & \nabla W_{a[bc]}^i = -R_{j\bar{p}q}^i W_a^p W_b^q W_c^j, \\ \nabla W_{ab[cd]}^i = -R_{\bar{p}qj}^i W_{ab}^p W_c^q W_d^j. \end{cases}$$

The values W_a^i , W_{ab}^i and W_{abc}^i form a fundamental third-order difference-geometric object of the surface \mathfrak{M} .

The normal vector of the hypersurface \mathfrak{M} at the point T satisfies the equations

$$g_{ij} n^i W_a^j = 0, \quad g_{ij} n^i n^j = 1.$$

A metric tensor of the hypersurface \mathfrak{M} is written in the form

$$g_{ab} = g_{ij} W_a^i W_b^j \quad \text{and} \quad \nabla g_{ab} = g_{abc} \Psi^c,$$

where $g_{abc} = g_{ij} W_a^i W_{bc}^j + g_{ij} W_{ac}^i W_b^j$.

The vectors $W_{ab}^i e_i$ and $n_a^i e_i$ admit representations in the form of a linear combination of vectors of the reference point $\{T, W_a, n\}$:

$$W_{ab}^i e_i = \mathbb{U}_{ab}^c + \mathcal{L}_{ab} n, \quad (2)$$

$$n_a^i e_i = \mathcal{L}_a^b W_b + n_a n, \quad (3)$$

where

$$\mathbb{U}_{ab}^c = g^{cd} g_{ik} W_{ab}^i W_d^k, \quad \mathcal{L}_{ab} = g_{ki} n^k W_{ab}^i, \quad \mathcal{L}_a^b = -g^{cb} \mathcal{L}_{ca}, \quad n_a = g_{ki} n^k n_a^i. \quad (4)$$

Hence the following theorem follows.

Theorem 1. *If for a hypersurface \mathfrak{M} , (1) is defined on the space $T(Vn)$, then the Gauss–Weingarten derivation formulas have the form (2) and (3).*

We call equations (2) and (3) the Gauss–Weingarten formulas of the hypersurface \mathfrak{M} . From (4), we obtain

$$\nabla \mathbb{U}_{ab}^c + W_{ab}^c = \mathbb{U}_{abd} \Psi^d, \quad (5)$$

$$\nabla \mathcal{L}_{abc} \Psi^c, \quad (6)$$

where

$$\begin{cases} \mathbb{U}_{abd}^c = g_{\dots,d}^{ec} g_{ik} W_{ab}^i W_e^k + g^{ei} g_{ik} W_{abd}^i W_e^k + g^{ec} g_{ik} W_{ab}^i W_{ed}^k, \\ \mathcal{L}_{abc} = g_{ki} n_c^k W_{ab}^i + g_{ki} n^k W_{abc}^i, \\ g_{\dots,d}^{ec} = -g^{ea} g^{bc} g_{abd}. \end{cases}$$

From (5) and (6) it follows that \mathbb{U}_{ab}^c is the object of affine connectedness and \mathcal{L}_{ab} is the tensor.

We call the object \mathbb{U}_{ab}^c the object of the induced affine connectedness of the hypersurface \mathfrak{M} . It is easy to prove that the induced affine connectedness and the internal affine connectedness coincide. The 1-forms of this connectedness have the form:

$$\tilde{\Psi}_b^a = \Psi_b^a + \mathbb{U}_{bc}^a \Psi^c.$$

Obviously,

$$D\Psi^a = \Psi^b \wedge \tilde{\Psi}_b^a, \quad D\tilde{\Psi}_b^a = \tilde{\Psi}_b^c \wedge \tilde{\Psi}_c^a + W_{bcd}^a \Psi^c \wedge \Psi^d,$$

where

$$W_{bcd}^a = \mathbb{U}_{b[cd]}^a - \mathbb{U}_{e[c}^a \mathbb{U}_{b|d]}^e.$$

The values W_{bcd}^a form the tensor which we call the curvature tensor of the hypersurface \mathfrak{M} .

Expanding equation (2), we obtain

$$R_{j pq}^i W_a^k W_b^p W_c^q W_e^i = \left(W_{abc}^d - \mathcal{L}_{a[b} \mathcal{L}_{c]}^d \right) - \left(\overset{k}{\nabla}_{[c} \mathcal{L}_{|a|b]} - W_{bc}^d \mathcal{L}_{ad} + \mathcal{L}_{a[b} n_{c]} \right) n^i,$$

where $\overset{k}{\nabla}_c$ is the symbol of nonholomorphic covariant differentiation.

From the above equalities, we obtain the generalized Gauss equations

$$R_{j pq} W_a^j W_b^p W_c^q W_e^i = W_{abce}^i + \mathcal{L}_{a[b} \mathcal{L}_{c]d} \tag{7}$$

and the generalized Peterson–Codazzi–Mainardi equations

$$R_{kpqr} W_a^k W_b^p W_c^q W_r^i n^i = W_{bc}^d \mathcal{L}_{ad} - \overset{k}{\nabla}_{[c} \mathcal{L}_{|a|b]} - \mathcal{L}_{a[b} n_{c]} \tag{8}$$

where

$$R_{ipqr} = g_{ij} R_{pqr}^j, \quad W_{abce} = g_{de} W_{abc}^d.$$

Hence the following theorem holds.

Theorem 2. *If for a hypersurface \mathfrak{M} , (1) is defined on the space $T(Vn)$, then the generalized Gauss equations and the generalized Peterson–Kodashi–Maitardi equations have the form (7) and (8).*

Equations (7) and (8) establish the connection between the curvature tensor of the space $T(Vn)$ and the curvature tensor of the hypersurface \mathfrak{M} in $T(Vn)$ [1–3, 7].

REFERENCES

1. F. I. Kagan, Affine connections on a tangent bundle. (Russian) *Izv. Vysš. Učebn. Zaved. Matematika* **1975**, no. 2(153), 31–42.
2. G. Todua, On the internal tensor structures of the fibration $T(\text{Lm}(Vn))$. *Trans. A. Razmadze Math. Inst.* **172** (2018), no. 1, 126–132.
3. G. Todua, On the vector fiber surface of the space $Lm(Vn)$ of triplet connectedness. *Trans. A. Razmadze Math. Inst.* **174** (2020), no. 3, 377–379.
4. G. Todua, Partial equipment of the manifold tangent space $TT(T(Vn))$. *Trans. A. Razmadze Math. Inst.* **177** (2023), no. 3, 471–474.
5. G. Todua, Structural equations of the tangent space $T(T(Vn))$. *Trans. A. Razmadze Math. Inst.* **178** (2024), no. 2, 345–348.
6. G. Todua, Full equipment of the manifold tangent fiber space $TT(T(Vn))$. *Trans. A. Razmadze Math. Inst.* **178** (2024), no. 1, 175–177.

7. K. Yano, Sh. Kobayashi, Prolongations of tensor fields and connections to tangent bundles. *I. General theory. J. Math. Soc. Japan* **18** (1966), 194–210.

(Received: 31.05.2025; Published online: 10.02.2026)

EUROPEAN UNIVERSITY, 76 DAVID GURAMISHVILI AVENUE, TBILISI 0141, GEORGIA
Email address: `todua.gocha@eu.edu.ge`